# GLUON CONDENSATES AS SUSCEPTIBILITY RELATIONS

CP³ Origins
Cosmology & Particle Physics

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## Main result

to be derived and discussed throughout

$$\bar{g}\frac{\partial}{\partial \bar{g}} M_H^2 = -\frac{1}{2}\langle H(E_H)|\frac{1}{\bar{g}^2}\bar{G}^2|H(E_H)\rangle_c$$

$$\bar{g}\frac{\partial}{\partial \bar{g}} \Lambda_{\rm GT} = -\frac{1}{2}\langle 0|\frac{1}{\bar{g}^2}\bar{G}^2|0\rangle$$

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Del Debbio and RZ 1306.4274 (PLB'2014)

- advocate: LHS provides a definition of the RHS
- **direct** computation of gluon condensates (RHS) plagued by power divergences — no definite result known
  $\langle 0|G^2|0\rangle = 0 \pm 0.012\,\text{GeV}^4$ c.f. also Ioffe'05 **indirect** determinations

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* barred quantities correspond to renormalised quantities & c stands for connected part

N.B. $\frac{\partial}{\partial g} E_H = \frac{\partial}{\partial g} M_H$ $\qquad \langle 0|T_\mu{}^\mu|0\rangle = D\Lambda_{\rm GT}$

# Overview

- Derivations (A) Feynman-Hellmann & Trace anomaly & RG-Eqs
  (B) Hamiltonian formalism (direct use of FH-thm)

- Illustration in exactly solvable models

- How to implement coupling derivative

- Where it cam from: corrections to scaling of the mass(-operator)

- Epilogue (applications)

- Backup slides: - comment energy momentum tensor on lattice
  - issue of Konishi-anomaly

# two derivations

# (A) trace anomaly, Feynman-Hellmann-thm & RGE

Del Debbio and RZ 1306.4274 (PLB'2014)

**Feynman-Hellmann thm**: $$\frac{\partial E_\lambda}{\partial \lambda} = \langle \psi(\lambda) | \frac{\partial \hat{H}(\lambda)}{\partial \lambda} | \psi(\lambda) \rangle$$ idea: $\frac{\partial \langle \psi(\lambda) | \psi(\lambda) \rangle}{\partial \lambda} = 0$

- useful provided H(λ) known (QFT different normalisation has to be taken into account)

example H(m) = $mN_F\bar{q}q$+..

$$\frac{\partial}{\partial \bar{m}} M_\Psi^2 = N_F \langle \psi | \bar{q}q | \psi \rangle_c$$

- For λ=g (gauge coupling) complicated since $A_0$ not dynamical.
  **Show:** if use all ingredients in the title then we can get relations!

- Fix notation |H(adron)⟩: $$\langle H(E', \vec{p'}) | H(E, \vec{p}) \rangle = 2E(\vec{p})(2\pi)^{D-1} \delta^{(D-1)}(\vec{p} - \vec{p'}) \;,$$

$$\langle X \rangle_{E_H} \equiv \langle H(E, \vec{p}) | X | H(E, \vec{p}) \rangle_c \;,$$

$$Q \equiv N_f m \bar{q} q \;, \quad G \equiv g^{-2} G^A_{\alpha\beta} G^{A\,\alpha\beta} \;,$$

**three step procedure ….**

## 1. EM-tensor & trace anomaly :

$$T_\mu{}^\mu|_{\mathrm{on-shell}} = \frac{\bar\beta}{2\bar g}\bar G + (1+\bar\gamma_m)\bar Q \ ,$$

for gauge theory
(bar renormalised quantities important!)

Adler et al, Collins et al
N.Nielsen '77 Fujikawa '81

Evaluate on physical state |H⟩ one gets:

$$2M_H^2 = \frac{\bar\beta}{2\bar g}\bar G_{E_H} + (1+\bar\gamma_m)\bar Q_{E_H} \ , \quad D\Lambda_{\mathrm{GT}} = \frac{\bar\beta}{2\bar g}\langle \bar G\rangle_0 + (1+\bar\gamma_m)\langle\bar Q\rangle_0 \ .$$

trace anomaly (1)

## 2. Feynman-Hellmann-thm (mass)

$$\bar m \frac{\partial}{\partial \bar m} E_H^2 = \langle \bar Q\rangle_{E_H} \ , \quad \bar m\frac{\partial}{\partial \bar m}(D\Lambda_{\mathrm{GT}}) = \langle \bar Q\rangle_0 \ .$$

FH-thm (2)

## 3. Renormalization group equation

$$\left(\bar\beta\frac{\partial}{\partial \bar g} - (1+\bar\gamma_m)\bar m\frac{\partial}{\partial\bar m} + 2\right) M_H^2 = 0 \ , \quad \left(\bar\beta\frac{\partial}{\partial \bar g} - (1+\bar\gamma_m)\bar m\frac{\partial}{\partial\bar m} + D\right)\Lambda_{\mathrm{GT}} = 0 \ ,$$

RG (3)

**Combining our results takes on the form:**

$$\bar{g}\frac{\partial}{\partial \bar{g}} E_H^2 = -\frac{1}{2}\langle \bar{G} \rangle_{E_H} \ , \quad \bar{g}\frac{\partial}{\partial \bar{g}} \Lambda_{\mathrm{GT}} = -\frac{1}{2}\langle \bar{G} \rangle_0 \ .$$

# (B) After all from the Hamiltonian formalism

Prochazka and RZ JPA 2014
1312.5495

- guiding question: H-formalism is non-covariant! how Lorentz invariance emerge?

$\vec{\pi} = \vec{E},\ \vec{A}\ \textit{indep.}$ canonical variables ($\pi_0 = 0,\ A_0$ Lagrangian multiplier)

$$[A^k(x_0, \vec{x}), E_l(x_0, \vec{y})] = i\delta^k_l \delta^{(D-1)}(\vec{x} - \vec{y})$$

$$\mathcal{H} = \mathcal{H}_g + \mathcal{H}_C + \mathcal{H}_\mathcal{G}$$

$\mathcal{H}_g$: $\frac{1}{2}(\vec{E}^2 + \vec{B}^2) - \overline{q}(i\vec{\gamma}\cdot(\vec{\partial} + ig\vec{A}) - m)q$

$\mathcal{H}_C$: primary, secondary constraints

$\mathcal{H}_\mathcal{G}$: Gauss constraint $A_0^a((\vec{D}\cdot\vec{E})^a + \bar{q}t^a\gamma_0 q)$

$$2B_k = \epsilon_{kij}G_{ij} = \epsilon_{kij}(\partial_i A_j - \partial_j A_i + ig[A_i, A_j])$$

- step 1: only $H_g$ non-vanishing on physical states - drop $H_{C,G}$

- step 2: put g's into right place by
  performing canonical transformation: $\vec{A} \to \frac{1}{g}\vec{A}\ , \quad \vec{E} \to g\vec{E}$

- a) leaves can. commutator invariant
  b) no rescaling (Konishi) anomaly (non-trivial)

$$\mathcal{H}_g = \frac{1}{2}(g^2 \vec{E}^2 + \frac{1}{g^2}\vec{B}^2) - \overline{q}(i\vec{\gamma}\cdot\vec{\partial} + i\vec{A} + m)q$$

- the pathway to a Lorentz-invariant result is now straightforward …

$$g\frac{\partial}{\partial g}\mathcal{H}_g = g^2\vec{E}^2 - \frac{1}{g^2}\vec{B}^2 = -\frac{1}{2}\frac{1}{g^2}G_{\mu\nu}G^{\mu\nu}$$

- very same relations (as before) emerge

$$\bar{g}\frac{\partial}{\partial \bar{g}}E_H^2 = -\frac{1}{2}\langle \bar{G}\rangle_{E_H} \ , \quad \bar{g}\frac{\partial}{\partial \bar{g}}\Lambda_{\rm GT} = -\frac{1}{2}\langle \bar{G}\rangle_0 \ .$$

- **advantage:** the Hamiltonian derivation makes it clear that relation valid for product groups e.g. G = U(1)xSU(2)xSU(3)

# illustration in exactly solvable models

- Schwinger model (QED2 massless fermions) photon mass $e^2/\pi$
- massive flavoured Schwinger model cosmological constant
- N=2 SYM (Seiberg-Witten) monopole mass

# Photon mass Schwinger model

- Schwinger model: QED2 $m_f=0$ - generation of photon mass: **$M_\gamma=e/\sqrt{\pi}$**

$$e\frac{\partial}{\partial e}M_\gamma^2 = -\frac{1}{2}\langle\gamma|F^2|\gamma\rangle_c$$

adaption 2D [e]=1

- Lowenstein-Swieca operator solution can compute RHS —

$$F_{\mu\nu} = \frac{\sqrt{\pi}}{e}\epsilon_{\mu\nu}\Box\Sigma$$

$\Sigma$ free field mass $e^2/\pi$

$$\langle\gamma|F^2|\gamma\rangle_c = \frac{\pi}{e^2}\epsilon_{\mu\nu}\epsilon^{\mu\nu}2(-M_\gamma^2)^2 = -4\frac{e^2}{\pi}$$

evaluate on photon state

- Insert into equation above and solve

$$e\frac{\partial}{\partial e}M_\gamma^2 = 2\frac{e^2}{\pi} \quad \Rightarrow \quad M_\gamma^2 = \frac{e^2}{\pi} + C$$

boundary condition C=0 and this completes the illustration!

# N=2 SYM (Seiberg-Witten)

- BPS states obey: **$M_{(e,m)} = 2\,|n_e\, a + n_m\, a_D|^2$** where $a, a_D$ part of SW-solution

- BPS-Hamiltonian magnetic monopoles ($n_e=0$, $\bar{B}$ static ⇒ $\bar{E}=0$ & no fermions as BPS)

$$\mathcal{H}_{\rm BPS} = \frac{1}{g^2}\vec{D}\phi\cdot\vec{D}\phi + \frac{1}{2}\frac{1}{g^2}\vec{B}^2$$

- BPS-eqn: $\vec{D}\phi|{\rm BPS}\rangle = \frac{1}{\sqrt{2}}\vec{B}|{\rm BPS}\rangle \quad \Rightarrow \quad \mathcal{H}_{\rm BPS} = \frac{1}{g^2}\vec{B}^2$

$$g\frac{\partial}{\partial g}\mathcal{H}_{\rm BPS} = -2\frac{1}{g^2}\vec{B}^2 \overset{\vec{E}=0}{=} -\frac{1}{g^2}G^2$$

N.B. additional factor 2 because of supersymmetry

⇒ shown main Eqn obeyed BPS-subspace

- Unlike Schwinger model, can't compute RHS directly
  used LHS to get RHS=$\langle {\rm BPS}|G^2|{\rm BPS}\rangle$
  RHS governed by magnetic coupling $g_D$ e.g. RHS →0 for $g_D$ →0

# Implementation of derivative coupling

# Implementation of $\frac{\partial \mathbf{M_H}}{\partial \mathbf{g}}$

- "… is not so immediate in a theory with running coupling" (depends on your background)

- lattice: every QCD parameter g,$m_u$, $m_d$ …is associated with hadronic observable

- step 1: $M_\rho = 770\text{MeV}$ measure coupling in scheme $A$:

  $g_A(\alpha M_\rho) = c\ , \quad c, \alpha \in \mathbf{R}$

- step 2: $M_\rho^\epsilon = 770(1+\epsilon)\text{MeV}$ measure coupling in scheme $A$:

  $g_A^\epsilon(\alpha M_\rho) = c_\epsilon \ \ (= g_A^\epsilon(\alpha_\epsilon M_\rho^\epsilon)\ , \text{where } \alpha_\epsilon = \frac{\alpha}{1+\epsilon})^*$

- finally:

$$\frac{\partial M_\rho}{\partial g_A} = \lim_{\epsilon \to 0} \frac{M_\rho^\epsilon - M_\rho}{g_A^\epsilon - g_A}$$

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$^*\ g_A(x) = g_A^\epsilon(x(1+\epsilon))$

# Where it all came from

# scaling correction to hadronic mass in near conformal phase

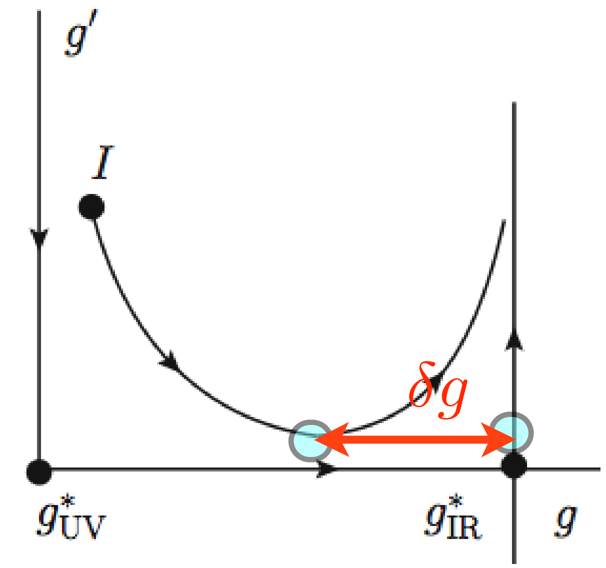
Del Debbio and RZ 1306.4038 (PRD'2013)

- consider conformal theory with mass deformation expand around fixed pt coupling g*

$$\beta = \beta_1 \delta g + \mathcal{O}(\delta g^2) \;, \qquad \delta g \equiv g - g^* \;,$$

$$\gamma_m = \gamma_m^* + \gamma_m^{(1)} \delta g + \mathcal{O}(\delta g^2) \;,$$

$$\gamma_{ij} = \gamma_{ij}^* + \gamma_{ij}^{(1)} \delta g + \mathcal{O}(\delta g^2) \;, \qquad (\gamma_{ij} \equiv (\gamma_O)_{ij}) \;.$$




- each local operator O investigate Callan-Symanzik-Weinberg-'tHooft type RGE

$$\left( \Lambda \frac{\partial}{\partial \Lambda} \delta_{ij} + \beta(g) \frac{\partial}{\partial g} \delta_{ij} - \gamma_m m \frac{\partial}{\partial m} \delta_{ij} - \gamma_{ij} \right) O_j(g, m, \Lambda) = 0$$

UV-cut off: $\Lambda$ , $\gamma_m = -\Lambda \frac{d}{d\Lambda} \ln m$ ..

- correction to scaling to hadronic mass through
  a) RGE above applied to O = $M_H$
  b) or apply scaling to all four quantities in trace anomaly

$$2M_H^2 = \left( \frac{\bar{\beta}}{2\bar{g}} \right) \bar{G}_{M_H} + (1 + \bar{\gamma}_m) \bar{Q}_{M_H} \;,$$

a) and b) agree only if **susceptibility relation** hold

# Epilogue

$$\bar{g}\frac{\partial}{\partial \bar{g}} E_H^2 = -\frac{1}{2}\langle \bar{G} \rangle_{E_H} \;, \quad \bar{g}\frac{\partial}{\partial \bar{g}} \Lambda_{\rm GT} = -\frac{1}{2}\langle \bar{G} \rangle_0 \;.$$

- Scheme dep. of RHS inherited from scheme dependence of coupling g
- LHS defines RHS - suggest total change of viewpoint
  *(Recall: direct computation of G-condensate fails because power divergences mixing with lower dimensional operators e.g. identity (quartic UV-divergence))*
- Practice computation of **❮H|G²|H❯** (should be) straightforward
  a) lattice
  b) approaches like AdS/QCD or Dyson-Schwinger Eqn which produce $M_H$

  - opens up opportunities to define β and $\gamma_m$ through interplay with trace anomaly:

$$2M_H^2 = \frac{\bar{\beta}}{2\bar{g}}\langle H|\bar{G}^2|H\rangle + (1+\bar{\gamma}_m)\bar{m}\langle H|qq|H\rangle$$

For example if $\bar{m}$=0 then $$(\bar{\beta}_{\rm YM})^{-1} = -\tfrac{\partial}{\partial \bar{g}} \ln M_H$$

- Computation of **⟨0|G²|0⟩** is more difficult per se
  - on lattice demands mastering EMT
  problems due to breaking of translation symmetry (additional renormalisation)
  recent progress using Wilson flow del Debbio,Patella, Rago JHEP(2014)

  - check PCD(ilaton)C hypothesis for gauge theory dilation candidate (Higgs imposter)
    (analogue PCAC soft pion reduction)

$$2m_D^2 = \frac{\beta}{2g}\langle D|G^2|D\rangle + O(m_q) \overset{\text{soft dilaton}}{\simeq} \frac{\beta}{2g}\frac{1}{f_D^2}\langle 0|G^2|0\rangle + O(m_q) \qquad \langle D|G^2|0\rangle = f_D$$

  - compute QCD contribution to cosmological constant
    N.B. practice mastering EMT already enough -
    yet relations useful in eliminating constant which is independent of g

  - one could do conversion calculation to MS-bar and compare with value extracted from OPE (e.g. charmonium sum rules or SVZ sum rules)

**THANKS FOR YOUR ATTENTION**

# backup slides

# renormalisation of energy momentum tensor (EMT)

- **continuum EMT** does **not renormalise** ($Z_{T\alpha\beta}=1$) since **conserved quantity** (still need to renormalise parameters of theory of course)

- **lattice** break Lorentz symmetry to hyper cubic symmetry
  hence the EMT is not conserved anymore $Z_{T\alpha\beta}=1$ does not apply
  or in other words we can write down further invariant with which the EMT mixes

**Problem**: how to **tune counterterms**

**translation Ward identity** to probe EMT

Caracciolo, Curci, Menotti, Pelissetto'90

$$\langle 0| \int d^3x T_{0\mu}(x)\phi(x_1)...\phi(x_n)|0\rangle = -\sum_{i=0}^{n} \frac{\partial}{\partial(x_i)^{\mu}} \langle 0|\phi(x_1)...\phi(x_n)|0\rangle$$

**Problem**: each probe contact term no gain

- using **Wilson flow** can avoid contact terms (probes are in bulk….)

del Debbio,Patella, Rago JHEP(2014)

# issue of konishi anomaly

- rescale field coupled to a gauge field by a constant then term appears $G^2$
  like chiral transformation gives rise to G*G-term (supersymmetry same footing)

- perform transformation

$$\vec{A} \to \frac{1}{f(g)} \vec{A} \, ,$$
$$\vec{E} \to f(g) \vec{E} \, .$$

- p-integral measure transforms as (same as Fujikawa chiral anomaly computation)

$$\ln \det \frac{\delta Q'(x)}{\delta Q(y)} = \ln \det \begin{pmatrix} f(g)^{-1} \delta(x-y) & 0 \\ 0 & f(g) \delta(x-y) \end{pmatrix} =$$
$$\ln \det \begin{pmatrix} f(g)^{-1} & 0 \\ 0 & f(g) \end{pmatrix} \delta(x-y) = \ln \det \delta(x-y) \, ,$$

$$Q \equiv (\vec{A}, \vec{E})$$