Bound states in perturbation theory

Bad Honnef, February 12-15, 2017

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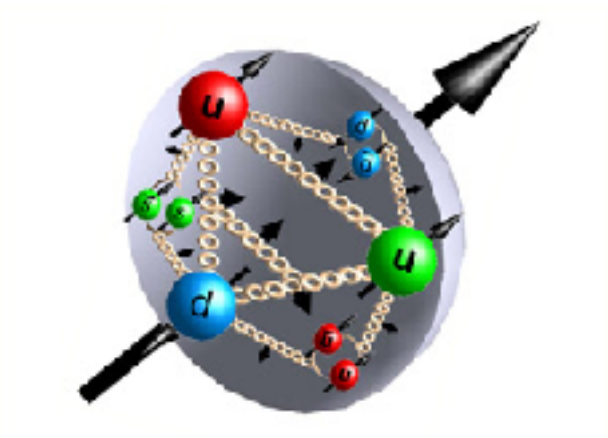
Positronium has both perturbative $\mathcal{O}(\alpha)$ and non-perturbative $\mathcal{O}(\alpha^\infty)$ aspects.
Atoms are accurately described by QED.



Hadrons have confinement and CSB.
They also have perturbative features.
Is there a QCD approach analogous to QED?

arXiv:1612.09463
arXiv:1605.01532

QED works for atoms

The PQED expansion (usually) starts from the $\mathcal{O}(\alpha^\infty)$ Schrödinger atom.

Example: Hyperfine splitting in Positronium at $\mathsf{O}(\alpha^7 \log\alpha)$ using NRQED

Orthopositronium: $J^{PC} = 1^{--}$ Parapositronium: $J^{PC} = 0^{-+}$

$\Delta E = E(\text{ortho}) - E(\text{para})$ $\Delta\nu = \Delta E/2\pi\hbar$

$$\Delta\nu_{QED} = m_e\alpha^4 \left\{ \frac{7}{12} - \frac{\alpha}{\pi}\left(\frac{8}{9} + \frac{\ln 2}{2}\right) \right.$$
$$+\frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2 \ln\alpha + \frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2}\right)\ln 2 - \frac{53}{32}\zeta(3)\right]$$
$$\left. -\frac{7\alpha^3}{8\pi}\ln^2\alpha + \frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2 - \frac{217}{90}\right) + \mathcal{O}\left(\alpha^3\right)\right\} = 203.39169(41)\ \text{GHz}$$

$\ln\alpha$ from would-be IR singularities

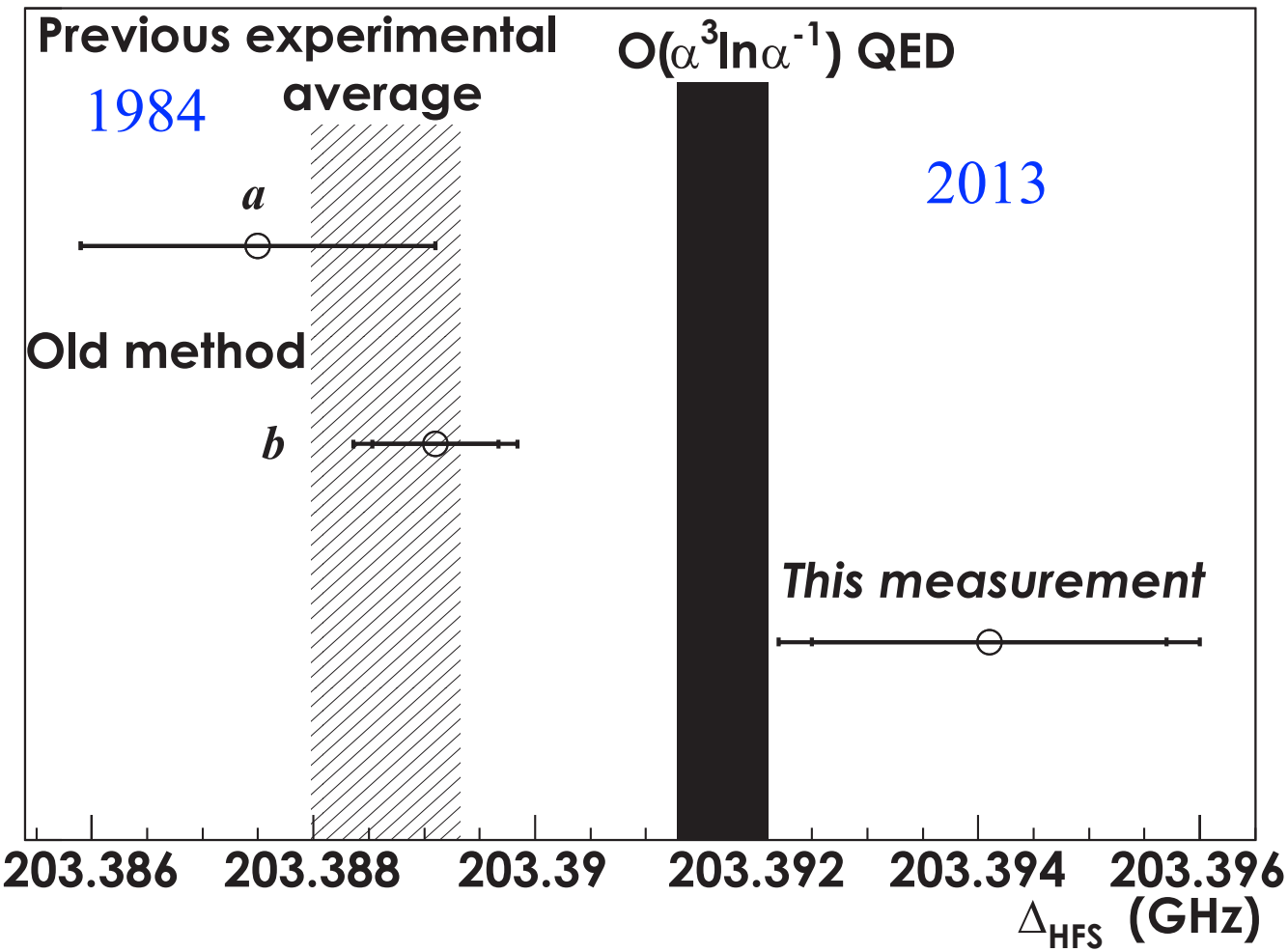
M. Baker et al, 1402.0876

QED vs Data: Hyperfine splitting in Positronium

$\Delta\nu_{QED} = 203.39169(41)$ GHz

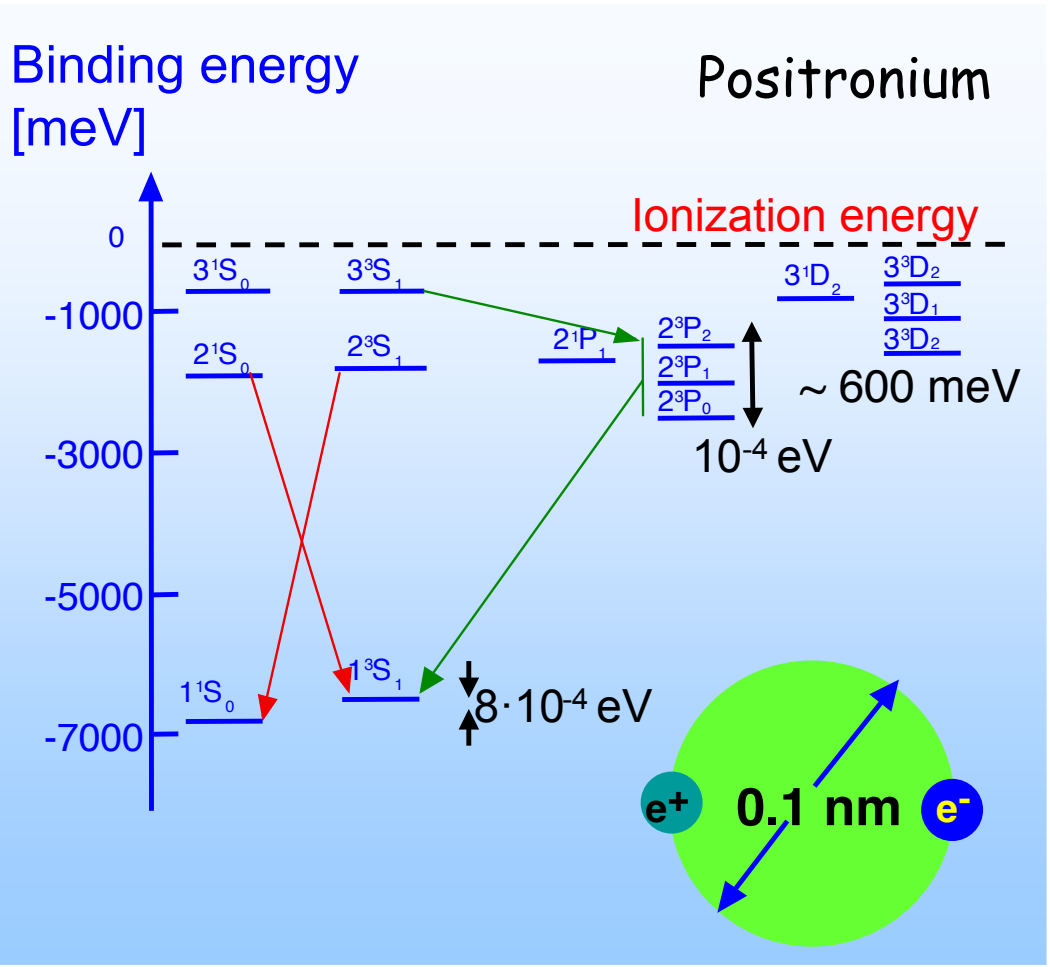
$\Delta\nu_{EXP} = 203.38865(67)$ GHz (1984) M. W. Ritter et al, Phys. Rev. A30 (1984) 1331

$\Delta\nu_{EXP} = 203.3941 \pm .003$ GHz (2013) A. Ishida et al, PLB 734 (2014) 338 [1310.6923]

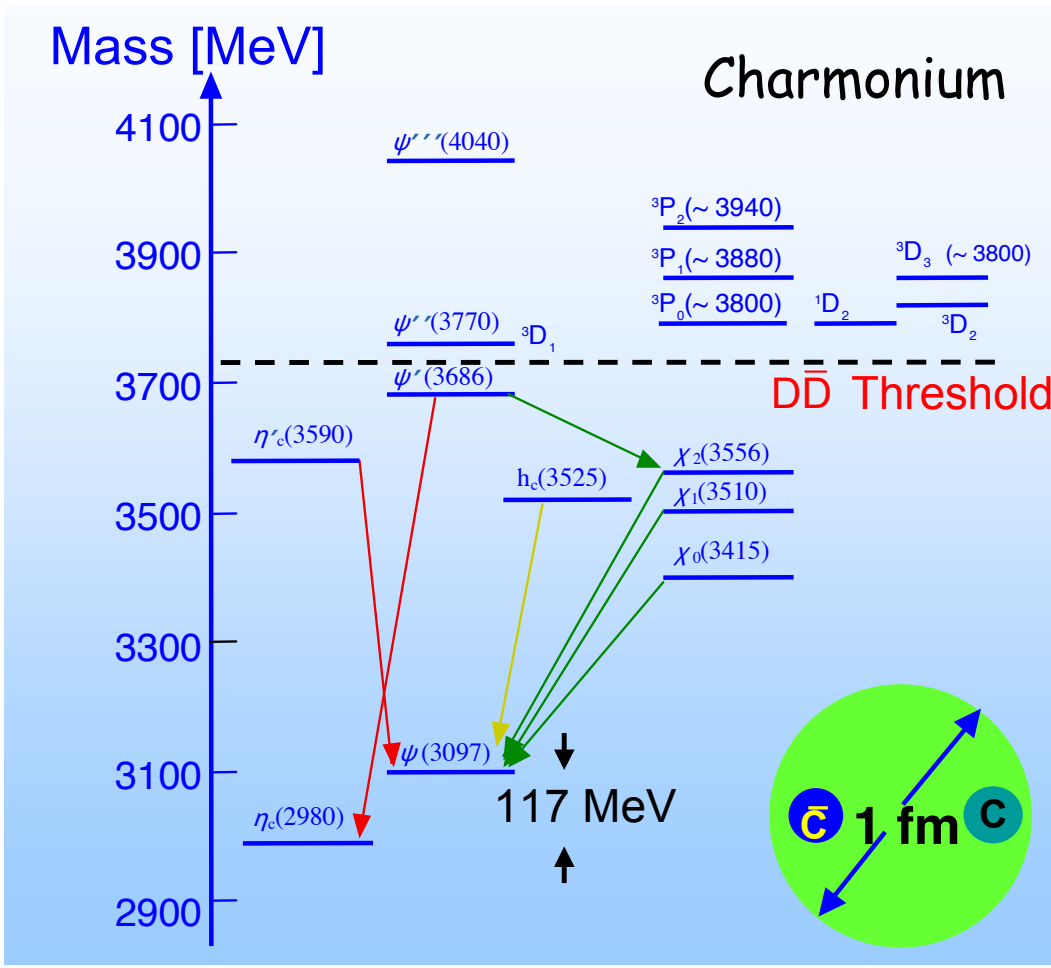


"The J/ψ is the Hydrogen atom of QCD"

QED



QCD



$$V(r) = -\frac{\alpha}{r}$$

$$V(r) = cr - \frac{4}{3}\frac{\alpha_s}{r}$$

QM I: The Positronium atom

The Schrödinger equation with a classical potential is postulated:

$$\left[-\frac{\boldsymbol{\nabla}^2}{2\mu} - \frac{\alpha}{|\boldsymbol{x}|} \right] \varphi(\boldsymbol{x}) = E_b \, \varphi(\boldsymbol{x})$$

Ground state binding energy:	$E_b = -\frac{1}{2}\mu\,\alpha^2$	O(α²)	+ PQED corrections
Electron velocity:	$v = \frac{p}{m} \simeq \alpha$	O(α)	Non-relativistic
Potential:	$V(r) = -\frac{\alpha}{r}$	O(α²)	Classical
Wave function:	$\Phi(\boldsymbol{x}) = N \exp(-\alpha\, m_e\, \lvert\boldsymbol{x}\rvert)$	O(α∞)	

QED: Atoms from Feynman diagrams

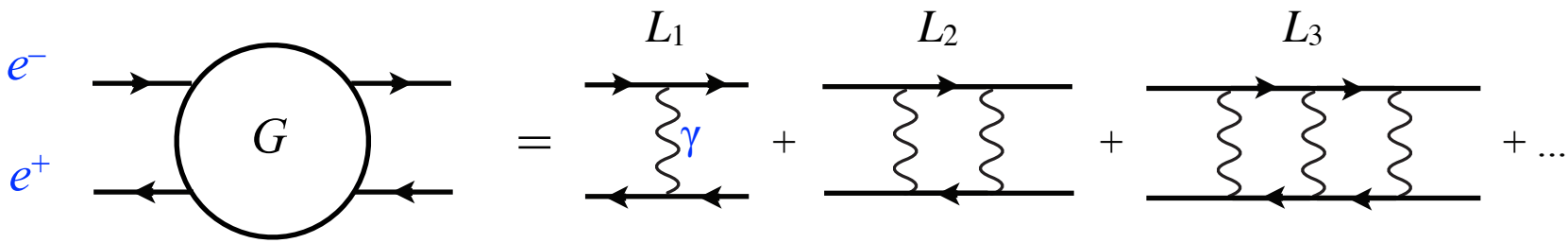
The S-matrix allows to calculate any scattering process to arbitrary order:

$$S_{fi} = {}_{\mathbf{out}}\langle f| \left\{ \mathrm{T}\exp\left[-i \int_{-\infty}^{\infty} dt\, H_I(t) \right] \right\} |i\rangle_{\mathbf{in}}$$

But: The free *in*- and *out*-states at $t = \pm\infty$ have no overlap with bound states.

No finite order Feynman diagram for $e^+e^- \to e^+e^-$ has a positronium pole.

Positronium poles arise from a divergent sum of ladder diagrams:



The divergence arises from a geometric sum of tree diagrams (no loops).

It gives the Schrödinger equation with classical potential $V(r) = -\alpha/r$.

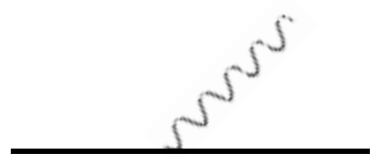
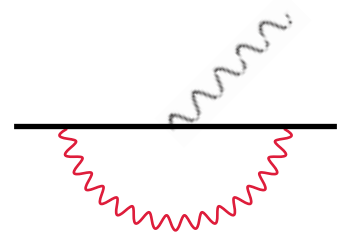
Classical fields correspond to $\hbar \to 0$

Recall functional integral formulation of QFT:

$$G(x_1, \ldots, x_n) = \int [d\varphi] \, \exp\left(iS[\varphi]/\hbar\right) \varphi(x_1) \ldots \varphi(x_n)$$

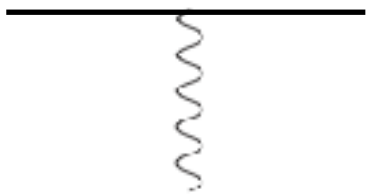
The limit $\hbar \to 0$ gives classical field eqs: $\dfrac{\delta S[\varphi]}{\delta \varphi} = 0$

Higher orders in $\hbar$ correspond to loop corrections in Feynman diagrams.

$O(e\hbar^0)$  $O(e\alpha\hbar)$ 

Each loop adds one power of $\alpha\hbar$: Expansions in α and $\hbar$ are equivalent.

Powers of α and $\hbar$ do not match in tree diagrams

$O(e\hbar^0)$ 

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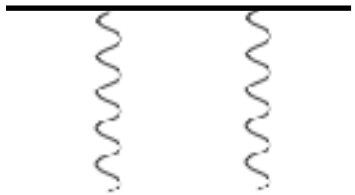
$O(e^2\hbar^0)$: 

The ladders form a geometric sum of $O(\hbar^0)$ diagrams: Gives classical field.

Define Born bound states as lowest order in $\hbar$ (no loops)

Consistent with the Schrödinger atom.

Not limited to non-relativistic dynamics.

Coupling is frozen in the absence of loops: $\alpha_s(0) \approx 0.5$

Instantaneous A^0 gauge field

The A^0 field does not propagate in time: no $\partial_t A^0$ term in the Lagrangian

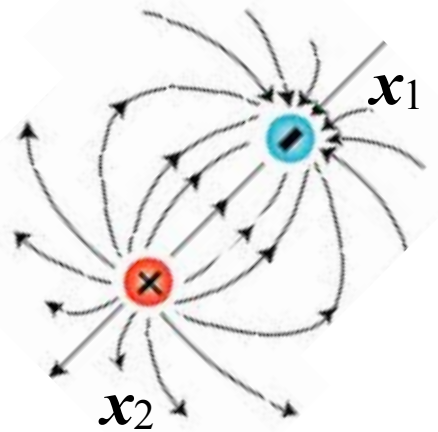
$\Longrightarrow$ Equations of motion allow to eliminate A^0 in terms of electron fields

$$\frac{\delta \mathcal{S}_{QED}}{\delta \hat{A}^0(t,\boldsymbol{x})} = 0 \qquad \Rightarrow \qquad -\boldsymbol{\nabla}^2 \hat{A}^0(t,\boldsymbol{x}) = e\psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x})$$

Gauss' law for the A^0 field of an $\left|e^-(\boldsymbol{x}_1)\, e^+(\boldsymbol{x}_2)\right\rangle$ Fock component is:

$$-\boldsymbol{\nabla}^2 A^0(\boldsymbol{x}) = e\left[\delta^3(\boldsymbol{x}-\boldsymbol{x}_1) - \delta^3(\boldsymbol{x}-\boldsymbol{x}_2)\right]$$

$$\Longrightarrow \quad eA^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$



Note: A^0 is determined instantaneously for all $\boldsymbol{x}$, and depends on $\boldsymbol{x}_1, \boldsymbol{x}_2$

The propagating vector components A^j are of higher order in α

Only fermion fields at lowest order: No gauge fixing necessary.

Positronium state at rest

$$|n;\boldsymbol{P}=0\rangle = \int \frac{d\boldsymbol{k}}{(2\pi)^3}\,\varphi_n(\boldsymbol{k})\,b^\dagger_\lambda(\boldsymbol{k})\,d^\dagger_{\bar\lambda}(-\boldsymbol{k})\;|0\rangle$$

$\varphi_n(\boldsymbol{k})$: Schrödinger wf

$$= \int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\,\bar\psi(t,\boldsymbol{x}_1)_\alpha\,\Phi_n^{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\psi(t,\boldsymbol{x}_2)_\beta\,|0\rangle$$

in mom. space the 4×4 wf. is:

$$\Phi_n^{\alpha\beta}(\boldsymbol{k}) \equiv \left[\gamma^0 u(\boldsymbol{k},\lambda)\right]_\alpha v^\dagger_\beta(-\boldsymbol{k},\bar\lambda)\varphi_n(\boldsymbol{k})$$

$$\propto \gamma^0(\not{k}+m)(1+\gamma^0)\Gamma(1-\gamma^0)(-\not{k}+m)\varphi_n(\boldsymbol{k})$$

where $(1+\gamma^0)\Gamma(1-\gamma^0) = \begin{pmatrix} 0 & \chi_\lambda\,\chi^\dagger_{\lambda'} \\ 0 & 0 \end{pmatrix}$ accounts for the 2×2 helicity d.o.f.'s of e^- and e^+

Bound state condition

$$\mathcal{H}_{QED}\ |n;\boldsymbol{P}=0\rangle = (2m_e + E_b)\ |n;\boldsymbol{P}=0\rangle$$

$$\mathcal{H}_{QED}(t) = \int d\boldsymbol{x}\,\psi^\dagger(t,\boldsymbol{x})\left[-i\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}} + m\gamma^0 + \tfrac{1}{2}e\mathcal{A}^0(\boldsymbol{x})\right]\psi(t,\boldsymbol{x})$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $e\mathcal{A}^0(\boldsymbol{x}) = \int d\boldsymbol{y}\,\dfrac{\alpha}{|\boldsymbol{x}-\boldsymbol{y}|}\,\psi^\dagger(t,\boldsymbol{y})\psi(t,\boldsymbol{y})$

The factor $\frac{1}{2}$ is due to the (negative) field energy,

$$\int d\boldsymbol{x}\,\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}\int d\boldsymbol{x}\,\mathcal{A}^0\cdot\boldsymbol{\nabla}^2\mathcal{A}^0 = -\tfrac{1}{2}\int d\boldsymbol{x}\,\bar\psi(\boldsymbol{x})e\mathcal{A}^0\psi(\boldsymbol{x})$$

Using $\left\{\psi^\dagger_\alpha(t,\boldsymbol{x}),\psi_\beta(t,\boldsymbol{y})\right\} = \delta_{\alpha,\beta}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$ and neglecting e^+e^- pair production/annihilation

the Schrödinger equation follows: $\left(-\dfrac{\boldsymbol{\nabla}^2}{m_e} - \dfrac{\alpha}{|\boldsymbol{x}|}\right)\varphi_n(\boldsymbol{x}) = E_b\,\varphi_n(\boldsymbol{x})$

Positronium in motion

The Born approximation is not limited to non-relativistic dynamics.

Positronium states with CM momentum $\boldsymbol{P} = (0, 0, P)$ are obtained by boosts $U(\xi)$ along the z-axis, which transforms the $(t = 0)$ fields as

$$U(\xi)\psi(0, \boldsymbol{x})U^\dagger(\xi) = e^{-\xi\alpha_3/2}\psi(z\sinh\xi, \boldsymbol{x})$$

Note: The transformed time depends on z.
States of any momentum $\boldsymbol{P}$ are defined at equal time.
Boost covariance emerges via the QED dynamics.

The time shift can be adjusted by the generator H of time translations.

An infinitesimal boost $U(d\xi) = 1 - id\xi\,\mathcal{K}$ generated by

$$\mathcal{K}(t=0) = -\int d\boldsymbol{x}\,\psi^\dagger(0,\boldsymbol{x})\left[z(-i\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}} + m\gamma^0 + \tfrac{1}{2}\gamma^0 e\not{A}) - \tfrac{1}{2}i\alpha_3\right]\psi(0,\boldsymbol{x})$$

applied to a $P = 0$ Positronium state gives a state with $P = d\xi\, M$.

Positronium state with momentum P

In a finite boost to $P = M\sinh\xi$ the state becomes:

$$|n;P\rangle \equiv U(\xi)\,|n;P=0\rangle =$$

$$= \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar{\psi}(0,\boldsymbol{x}_1)\, e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\, e^{-\xi\alpha_3/2}\, \Phi_n^{(\xi)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\, e^{\xi\alpha_3/2}\, \psi(0,\boldsymbol{x}_2)\,|0\rangle$$

where the wave function is Lorentz contracted:

$$\Phi_n^{(\xi)}(x,y,z/\cosh\xi) = \Phi_n^{(0)}(x,y,z)$$

$|n;P\rangle$ is an eigenstate of the momentum $\mathcal{P}$ and energy $\mathcal{H}$ operators, with the appropriate (boosted) eigenvalues.

The gauge field in the Hamiltonian applied to the boosted state,

$\mathcal{H}(\mathcal{A})U(\xi)\,|n;P=0\rangle$ is the boosted version of the the rest frame A^0 field.

Summary for Positronium

- The standard PQED expression for the S-matrix in the IP is inappropriate, since Positronium has zero overlap with the free *in* and *out* states.
- The $\mathcal{O}(\hbar^0)$ term of the PQED expansion is a (Born level) bound state.
- The Born term involves only the instantaneous A^0 field.
- The eqs. of motion express A^0 in terms of the electron field.
- Eigenstates of H with any momentum $\boldsymbol{P}$ can be found explicitly.

Järvinen (2005)

- In the rest frame the result agrees with the usual Schrödinger atom.
- The corresponding method for loop corrections remains to be developed.

Describing confinement in QCD?

The $\mathcal{A}_a^0$ gluon field is instantaneous: No $\partial_t \mathcal{A}_a^0$ term in $\mathcal{L}_{QCD}$.

Gauss' law for $\mathcal{A}_a^0$ has homogeneous solutions, which do not vanish at $|\boldsymbol{x}| = \infty$

The boundary condition makes $\mathcal{A}_a^0$ of $\mathcal{O}\left(\alpha_s^0\right)$, whereas $\mathcal{A}_a^j$ is of $\mathcal{O}\left(g\right)$.

At $\mathcal{O}\left(\alpha_s^0\right)$ the non-abelian contributions vanish: $f_{abc}\,\mathcal{A}_a^0\,\mathcal{A}_b^0 = 0$

and Gauss' law allows to express $\mathcal{A}_a^0$ in terms of the fermion fields:

$$-\boldsymbol{\nabla}^2 \mathcal{A}_a^0(t,\boldsymbol{x}) = g\,\psi_A^\dagger(t,\boldsymbol{x}) T_a^{AB} \psi_B(t,\boldsymbol{x})$$

The choice of homogeneous solutions ($\boldsymbol{\nabla}^2 \mathcal{A}_a^0(t,\boldsymbol{x}) = 0$)
is strongly constrained by Poincaré invariance.

A homogeneous solution for QCD

The following is a trivial solution of $\boldsymbol{\nabla}^2 \mathcal{A}^0_a(t,\boldsymbol{x}) = 0$ when $\kappa \neq \kappa(\boldsymbol{x})$:

$$A^0_a(t,\boldsymbol{x}) = \kappa \int d\boldsymbol{y}\, \psi^\dagger_A(t,\boldsymbol{y}) T^{AB}_a \psi_B(t,\boldsymbol{y})\, \boldsymbol{x}\cdot\boldsymbol{y}$$

To preserve Poincaré invariance we need to have color singlet $q\bar{q}$ states:

$$|n, P=0\rangle = \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar{\psi}_A(\boldsymbol{x}_1)\, \Phi^{AB}_n(\boldsymbol{x}_1-\boldsymbol{x}_2)\, \psi_B(\boldsymbol{x}_2)\, |0\rangle$$

with $\Phi^{AB}_n(\boldsymbol{x}) = \dfrac{\delta^{AB}}{\sqrt{N_c}}\, \Phi_n(\boldsymbol{x})$ is a (globally) color singlet state.

Color covariance ensures that the expectation value of the field vanishes

$$\langle n, P | A^0_a(\boldsymbol{x})\, |n, P\rangle = 0$$

Color singlet states cannot generate a color field.

The color field is invisible to an external observer (unlike in QED!), but confines each color component AA: $\bar{\psi}_A(t,\boldsymbol{x}_1)\, \psi_A(t,\boldsymbol{x}_2)\, |0\rangle$

Remark on $q\bar{q}$ pairs

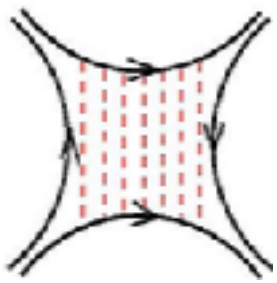
(Light) quarks in hadrons are ultra-relativistic and there is a sea of $q\bar{q}$ pairs.

Nevertheless, the hadron spectrum reflects only the valence quark dof's.

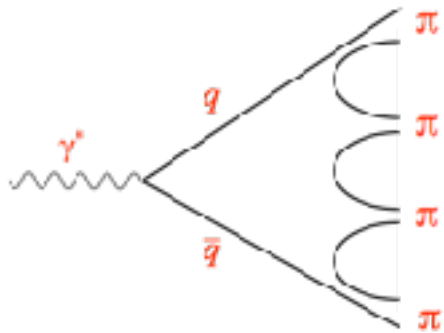
Relativistic Dirac states also have this feature (see below).

Hadron phenomenology shows that multi-particle states are dual to $q\bar{q}$:

Duality:

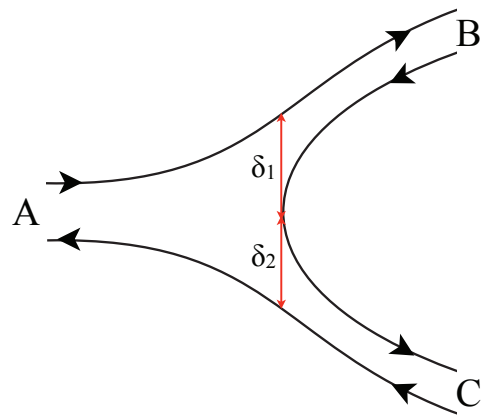


String-breaking:



In the following, the color octet $q\bar{q}$ pairs are included in the vacuum, an analogy to the Dirac case.

The singlet $q\bar{q}$ states allow to calculate (singlet) string breaking (hadron pair production, loops,..)
Such effects are required by unitarity at $\mathcal{O}(\hbar^0)$.



The homogeneous A^0 solution

$$A_a^0(t,\boldsymbol{x}) = \kappa \int d\boldsymbol{y}\, \psi_A^\dagger(t,\boldsymbol{y}) T_a^{AB} \psi_B(t,\boldsymbol{y})\, \boldsymbol{x}\cdot\boldsymbol{y} \quad \propto \boldsymbol{x}$$

The field energy density is independent of $\boldsymbol{x}$ for each color component AA:

$$\sum_a (\boldsymbol{\nabla} A_a^0)\cdot(\boldsymbol{\nabla}\hat{A}_a^0)\,\bar{\psi}_A(\boldsymbol{x}_1)\psi_A(\boldsymbol{x}_2)\,|0\rangle = C_F\kappa^2(\boldsymbol{x}_1-\boldsymbol{x}_2)^2\bar{\psi}_A(\boldsymbol{x}_1)\psi_A(\boldsymbol{x}_2)\,|0\rangle$$

The (infinite) total field energy $\propto \int d\boldsymbol{x}$ must be independent of $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$.
Hence

$$C_F\,\kappa^2 = \frac{\Lambda^4}{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2}$$

The QCD interaction Hamiltonian then gives an exactly linear potential:

$$H_{int}^{QCD}\bar{\psi}_A(\boldsymbol{x}_1)\psi_A(\boldsymbol{x}_2)\,|0\rangle = V(\boldsymbol{x}_1-\boldsymbol{x}_2)\bar{\psi}_A(t,\boldsymbol{x}_1)\psi_A(t,\boldsymbol{x}_2)\,|0\rangle$$

where $\quad V(\boldsymbol{x}) = \frac{1}{2}\sqrt{C_F}\,g\Lambda^2|\boldsymbol{x}|$

The Fock state structure of Dirac states

Relativistically bound states have a sea of $f\bar{f}$ pairs. As an example, consider the Dirac Hamiltonian for a fixed (strong) external field $A^\mu(\boldsymbol{x})$,

$$H = \int d^3\boldsymbol{x}\bar{\psi}(\boldsymbol{x})\big[-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m + e\not{A}(\boldsymbol{x})\big]\psi(\boldsymbol{x})$$

There are e^- and e^+ "valence" eigenstates, J.-P. Blaizot and PH (2015)

$$H\,|n\rangle = E_n\,|n\rangle \qquad |n\rangle = \int d\boldsymbol{x}\,\psi^\dagger_\alpha(\boldsymbol{x})\phi_{n\,\alpha}(\boldsymbol{x})\,|\Omega\rangle \equiv c^\dagger_n\,|\Omega\rangle$$

$$H\,|\bar{n}\rangle = \bar{E}_n\,|\bar{n}\rangle \qquad |\bar{n}\rangle = \int d\boldsymbol{x}\,\bar{\phi}^\dagger_{n\,\alpha}(\boldsymbol{x})\psi_\alpha(\boldsymbol{x})\,|\Omega\rangle \equiv \bar{c}^\dagger_n\,|\Omega\rangle$$

where the wave functions $\phi(\boldsymbol{x})$ and $\bar{\phi}(\boldsymbol{x})$ satisfy the Dirac equation

$$\big(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m + e\not{A}\big)\phi_n(\boldsymbol{x}) = E_n\gamma^0\phi_n(\boldsymbol{x}) \qquad E_n > 0$$

$$\big(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m + e\not{A}\big)\bar{\phi}_n(\boldsymbol{x}) = -\bar{E}_n\gamma^0\bar{\phi}_n(\boldsymbol{x}) \qquad \bar{E}_n > 0$$

The Dirac ground state $|\Omega\rangle$

The annihilation operators are Bogoliubov transforms of the free operators:

$$c_n = \sum_{\boldsymbol{p}} \phi_n^\dagger(\boldsymbol{p}) \left[u(\boldsymbol{p}) b_{\boldsymbol{p}} + v(-\boldsymbol{p}) d_{-\boldsymbol{p}}^\dagger \right] \equiv B_{np} b_p + D_{np} d_p^\dagger$$

$$\bar{c}_n = \sum_{\boldsymbol{p}} \left[b_{\boldsymbol{p}}^\dagger u^\dagger(\boldsymbol{p}) + d_{-\boldsymbol{p}} v^\dagger(-\boldsymbol{p}) \right] \bar{\phi}_n(\boldsymbol{p}) \equiv \bar{B}_{np} b_p^\dagger + \bar{D}_{np} d_p$$

They diagonalize the Dirac Hamiltonian: $H = \sum_n \left[E_n c_n^\dagger c_n + \bar{E}_n \bar{c}_n^\dagger \bar{c}_n \right]$

The ground state $|\Omega\rangle = N_0 \exp\left[- b_p^\dagger \left(B^{-1}\right)_{pm} D_{mq} d_q^\dagger \right] |0\rangle$

contains the e^+e^- pairs and satisfies

$$c_n |\Omega\rangle = \bar{c}_n |\Omega\rangle = H |\Omega\rangle = 0$$

Example

The Dirac states

in case of a linear potential in D=1+1 dimensions

The linear potential confines e^-, repels e^+ : $V(e^+) = -V(e^-) = -\frac{1}{2}e^2|x|$

Positrons are allowed at large |x| provided they have large momenta.

The accelerating/decelerating positrons have a continuous energy spectrum.

Hence also the Dirac states have a continuous energy spectrum.

AUGUST 1, 1932 PHYSICAL REVIEW VOLUME 41

The Dirac Electron in Simple Fields*

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(Received June 6, 1932)

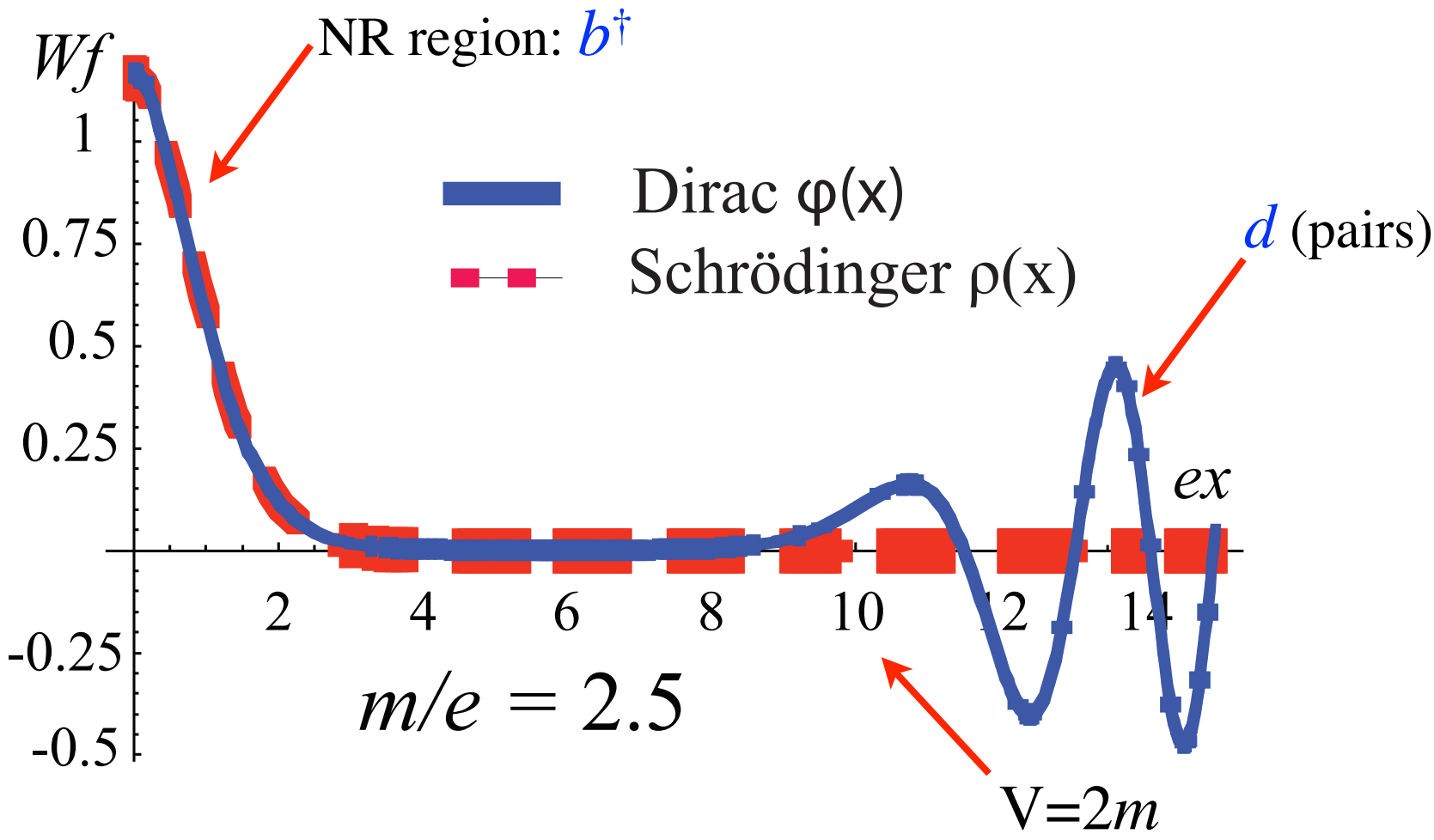
The relativity wave equations for the Dirac electron are transformed in a simple manner into a symmetric canonical form. This canonical form makes readily possible the investigation of the characteristics of the solutions of these relativity equations for simple potential fields. If the potential is a polynomial of any degree in x, a continuous energy spectrum characterizes the solutions. If the potential is a polynomial of any degree in $1/x$, the solutions possess a continuous energy spectrum when the energy is numerically greater than the rest-energy of the electron: values of the energy numerically less than the rest-energy are barred. When the potential is a polynomial of any degree in r, all values of the energy are allowed. For potentials which are polynomials in $1/r$ of degree higher than the first, the energy spectrum is again continuous. The quantization arising for the Coulomb potential is an exceptional case.

See also: E. C. Titchmarsh, Proc. London Math. Soc. (3) 11 (1961) 159 and 169; Quart. J. Math. Oxford (2), 12 (1961), 227.

Paul Hoyer Bad Honnef 2017

Example of Dirac states: $V(x) = \frac{1}{2}e^2|x|$ in D=1+1

$$|M \geq 0\rangle = \int \frac{dp}{2\pi 2E} \int dx \left[b_p^\dagger u^\dagger(p) e^{-ipx} + d_p v^\dagger(p) e^{ipx} \right] \begin{bmatrix} \varphi(x) \\ \chi(x) \end{bmatrix} |\Omega\rangle$$



The "single particle" Dirac wave function contains pair contributions (duality)

$q\bar{q}$ bound states

As previously noted, take

$$|q\bar{q};\boldsymbol{P}=0\rangle = \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar{\psi}_A(t,\boldsymbol{x}_1)\, \Phi^{AB}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_B(t,\boldsymbol{x}_2)\ |0\rangle$$

with $$\Phi^{AB}(\boldsymbol{x}_1-\boldsymbol{x}_2) = \frac{\delta^{AB}}{\sqrt{N_C}}\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

and $$A^0_a(t,\boldsymbol{x}) = \kappa \int d\boldsymbol{y}\, \psi^\dagger_A(t,\boldsymbol{y}) T^{AB}_a \psi_B(t,\boldsymbol{y})\, \boldsymbol{x}\cdot\boldsymbol{y}$$

The bound state condition $\mathcal{H}_{QCD}\,|q\bar{q}\rangle = M\,|q\bar{q}\rangle$ requires

$$i\boldsymbol{\nabla}\cdot\left\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\right\} + m\left[\gamma^0,\Phi(\boldsymbol{x})\right] = \left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

with $V(\boldsymbol{x}) = \frac{1}{2}\sqrt{C_F}\, g\Lambda^2|\boldsymbol{x}|$ (as above).

$q\bar{q}$ wave functions

The separation of angular and radial coordinates in the BSE

$$i\boldsymbol{\nabla}\cdot\left\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\right\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[E-V(r)\right]\Phi(\boldsymbol{x})$$

for any radial potential $V = V(r)$ and
equal fermion masses $m_1 = m_2 = m$ is in:

Geffen and Suura, PRD 16 (1977) 3305

The solutions of given spin j and j_z are classified according to their charge conjugation C and parity P quantum numbers:

pion trajectory: $P=(-1)^{j+1}$ $C=(-1)^{j}$

a_1 trajectory: $P=(-1)^{j+1}$ $C=(-1)^{j+1}$

rho trajectory: $P=(-1)^{j}$ $C=(-1)^{j}$

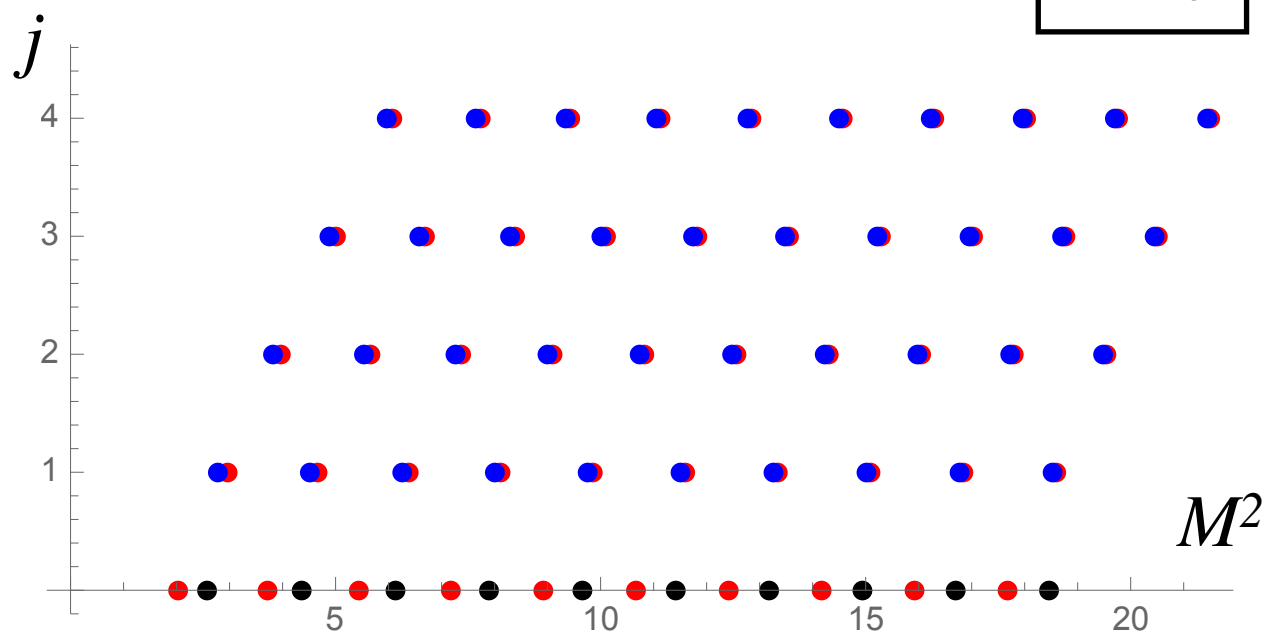
There are no "quark model exotics" with $P=(-1)^{j}$ and $C=(-1)^{j+1}$

π, a_1 and ρ spectra (m = 0)

The π, a_1 and ϱ trajectories are nearly degenerate for $j \geq 1$.

There are no $j = 0$ states on the a_1 trajectory.

The $j = 0$ states are non-degenerate due to the boundary conditions



There are also $M = 0$ states.

The massless 0^{++} (σ) state has vacuum quantum numbers.

Its mixing with the chirally symmetric vacuum would cause chiral symmetry breaking.

Summary

The similarities between quarkonia and atoms raise the question of if/how the QED bound state methods can be applied to QCD.

The perturbative expansion must be developed around an interacting state.

The Schrödinger atom is a Born level contribution: no loops, $\mathcal{O}(\hbar^0)$.

A^0 dominates A^j , and is expressed in terms of the electron field (Gauss' law).

The Hamiltonian approach allows to define atomic, equal-time states of any $\boldsymbol{P}$.

Gauss' law has homogeneous solutions, with A^0 non-vanishing at $|\boldsymbol{x}| \to \infty$.

The boundary condition introduces a dimensionful constant Λ_{QCD} .

The preservation of Poincare invariance requires to consider only color singlet states and $\nabla A^0 \neq \nabla A^0(\boldsymbol{x})$, giving a linear potential.

The resulting $q\bar{q}$ spectrum seems promising: Linear Regge trajectories, duality.

With Poincaré invariance, the approach is open for further studies: DIS, CSB, ..