Improved Relativistic Hydrodynamics from AdS/CFT

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based on M.L. and Edward Shuryak, arXiv:0905.4069; arXiv:0704.1647

Motivation: Experiments (RHIC) probe systems with finite gradients.
Phenomenologically observed low viscosity is an "effective" viscosity measured at momentum typical for a process in study.
New phenomena: Conical flows linear perturbations on top of global explosion. These are small size perturbations sensitive to high gradients.

Main Idea:
Introduce all order dissipative terms in the gradient expansion of $T^{\mu\nu}$.

$(\nabla\nabla\mathbf{u})$ we keep $\qquad\qquad$ $(\nabla\mathbf{u})^2$ we neglect

Extract momenta dependent viscosities by matching two-point correlation functions of stress tensor with correlation functions computed from BH AdS/CFT.

Outlook of the talk:

- Old Life on the boundary: relativistic hydro (NS and IS)
- Life in the bulk: gravity perspective
- New Life on the boundary: all order (linearized) hydro
- The bulk meets the boundary

Relativistic Hydrodynamics

Energy momentum tensor

$$\langle \mathbf{T}^{\mu\nu} \rangle = (\epsilon + \mathbf{P})\, \mathbf{u}^{\mu}\, \mathbf{u}^{\nu} + \mathbf{P}\, \mathbf{g}^{\mu\nu} + \mathbf{\Pi}^{\langle\mu\nu\rangle}$$

$\Pi^{\mu\nu}$ **- tensor of dissipations (ideal fluid:** $\Pi^{\mu\nu} = 0$**)**

$$\mathbf{\Pi}^{\langle\mu\nu\rangle} = \frac{\mathbf{1}}{\mathbf{2}} \mathbf{\Delta}^{\mu\alpha}\, \mathbf{\Delta}^{\mu\beta} \left(\mathbf{\Pi}_{\alpha\beta} + \mathbf{\Pi}_{\beta\alpha} \right) - \frac{\mathbf{1}}{\mathbf{3}} \mathbf{\Delta}^{\mu\nu}\, \mathbf{\Delta}^{\alpha\beta}\, \mathbf{\Pi}_{\alpha\beta}$$

$$\mathbf{\Delta}^{\mu\nu} = \mathbf{g}^{\mu\nu} + \mathbf{u}^{\mu}\, \mathbf{u}^{\nu} \qquad\qquad \mathbf{u}^{\mathbf{2}} = -\mathbf{1}$$

Navier Stokes hydro (expanding in the velocity gradient)

$$\mathbf{\Pi}_{\alpha\beta} = -\, \eta\, \nabla_{\alpha}\, \mathbf{u}_{\beta}$$

$$\nabla_{\mu} \langle \mathbf{T}^{\mu\nu} \rangle = \mathbf{0} \qquad \longrightarrow \qquad \mathrm{Navier - Stokes\ Eq.}$$

Conformal invariance

$$T^{\mu}_{\mu} \;=\; 0 \qquad \longrightarrow \qquad \epsilon \;=\; 3\,P \quad \text{and} \quad \xi \;=\; 0$$

Entropy density and EoS

$$s \;=\; \frac{\epsilon \;+\; P}{T} \;=\; 4\,k_{SB}\,T^3$$

No dissipation no entropy production:

$$\frac{ds}{dt} \;=\; 0 \qquad if \qquad \Pi^{\mu\nu} \;=\; 0$$

Plane wave perturbation:

$$\delta u \;=\; \delta u_0\, e^{-i\,\omega\,t \;+\; i\,k\,z} \qquad\qquad \delta P \;=\; \delta P_0\, e^{-i\,\omega\,t \;+\; i\,k\,z}$$

Linearized Hydro leads to the dispersion relation

$$\omega \;=\; c\,k \;-\; i\,\frac{2\,\eta}{s\,T}\,k^2$$

Sound velocity $c \;=\; 1/\sqrt{3}$ **Sound attenuation** $\sim\; \eta$

Retarded Correlators

$$\mathbf{G}^{\mu\nu\alpha\beta}(\mathbf{k},\omega) \;=\; -\,\mathbf{i}\int_0^\infty \mathbf{dt}\int \mathbf{d^3x}\,\mathbf{e}^{-\mathbf{i}\,\omega\,\mathbf{t}\,+\,\mathbf{ikz}}\,\langle[\mathbf{T}^{\mu\nu}(\mathbf{x},\mathbf{t}),\;\mathbf{T}^{\alpha\beta}(\mathbf{0})]\rangle$$

The sound:

$$\mathbf{G^S}(\mathbf{k},\omega) \;\equiv\; \mathbf{G^{tztz}} \;=\; (\epsilon\,+\,\mathbf{P})\,\frac{\mathbf{k^2}\,-\,\mathbf{4\,i}\,\bar{\eta}\,\omega\,\mathbf{k^2}}{\mathbf{k^2}\,-\,\mathbf{3}\,\omega^{\mathbf{2}}\,-\,\mathbf{4\,i}\,\bar{\eta}\,\omega\,\mathbf{k^2}}$$

The shear:

$$\mathbf{G^D}(\mathbf{k},\omega) \;\equiv\; \mathbf{G^{txtx}} \;=\; (\epsilon\,+\,\mathbf{P})\,\frac{\bar{\eta}\,\mathbf{k^2}}{-\mathbf{i}\,\omega\,+\,\bar{\eta}\,\mathbf{k^2}}$$

The scalar:

$$\mathbf{G^T}(\mathbf{k},\omega) \;\equiv\; \mathbf{G^{xyxy}} \;=\; -\,\mathbf{i}\,(\epsilon\,+\,\mathbf{P})\,\omega\,\bar{\eta}$$

$2\,\pi\,T \;=\; 1$ and $\bar{\eta} \;\equiv\; 2\,\pi\,\eta/s$

Israel-Stewart second order Hydrodynamics

Solves causality problems present in Navier-Stokes

Add extra term in the gradient expansion + non-linear terms in (∇u)

$$\mathbf{\Pi}^{\mu\nu} = (\mathbf{1} - \tau_{\mathbf{R}}\, \mathbf{u}_\lambda \nabla^\lambda)\ \mathbf{\Pi}^{\mu\nu}_{\mathbf{NS}}$$

Iterate the equation

$$(\mathbf{1} + \tau_{\mathbf{R}}\, \mathbf{u}_\lambda \nabla^\lambda)\ \mathbf{\Pi}^{\mu\nu} = \mathbf{\Pi}^{\mu\nu}_{\mathbf{NS}}$$

When thinking about small perturbations $\mathbf{u}_\lambda \nabla^\lambda \to \nabla_{\mathbf{t}} \to -\mathbf{i}\,\omega$

The IS second order hydro is equivalent (in the linear approximation) to

$$\eta \to \frac{\eta}{\mathbf{1} - \mathbf{i}\,\tau_{\mathbf{R}}\,\omega}$$

Sound dispersion

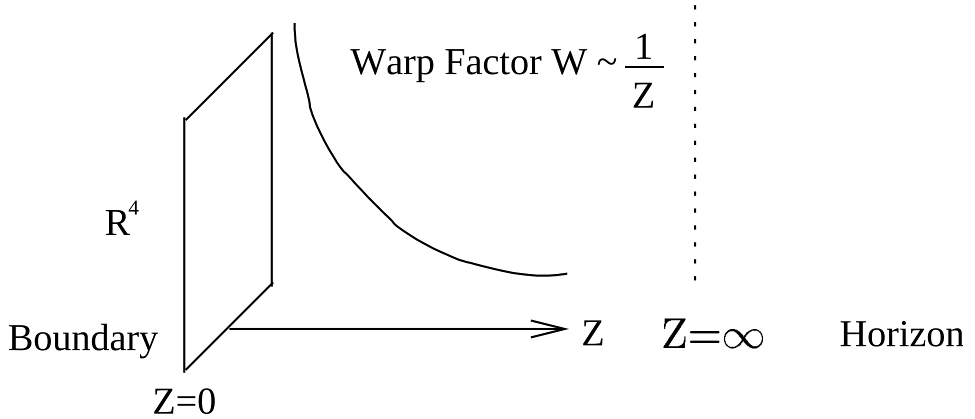
$$\omega = \mathbf{c}\,\mathbf{k}\,[\mathbf{1} + \bar{\eta}\,\mathbf{c}^2\,\mathbf{k}^2\,(\mathbf{2}\,\tau_{\mathbf{R}} - \bar{\eta})] - \mathbf{i}\,\mathbf{c}^2\,\bar{\eta}\,\mathbf{k}^2\,[\mathbf{1} + \mathbf{c}^2\,\mathbf{k}^2\,\bar{\eta}\,\tau_{\mathbf{R}}\,(\mathbf{2}\,\bar{\eta} - \tau_{\mathbf{R}})]$$

AdS/CFT correspondence: weakly coupled super-gravity in $AdS_5 \times S_5$ is "dual" to strongly coupled $\mathcal{N} = 4$ SYM gauge theory in 4d

't Hooft coupling

$$\lambda \equiv g_{YM}^2 N_c = 4\pi g_s N_c \gg 1 \qquad R^4 \equiv L^4 = \lambda \alpha'^2$$

$$ds^2 = R^2 \frac{dx^2 + dz^2}{z^2}$$



Retarded Correlators from gravity

P. Kovtun and A. Starinets, Phys.Rev.Lett.96:131601,2006

For three channels a=S,D,T

$$\frac{d^2}{dr^2}Z_a(r) + p_a(r)\frac{d}{dr}Z_a(r) + q_a(r)Z_a(r) = 0\,,$$

Absorptive boundary condition (incoming wave) at the horizon $r = 1$:

$$Z_a(r \to 1) \;\sim\; e^{-i\omega/2}$$

Two independent local solutions at $r = 0$,

$$Z_a(r) \;=\; \mathcal{A}_a\, Z_a^I(r) \;+\; \mathcal{B}_a\, Z_a^{II}(r)\,,$$

Z_a^I **is irregular in the origin while** Z_a^{II} **is a regular solution.**

$$\tilde{G}^a(\omega, k) \;=\; -\,8\,P\,\frac{\mathcal{B}_a(\omega, k)}{\mathcal{A}_a(\omega, k)}$$

Small momenta perturbation theory

Extending R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804:100,2008

$$\frac{1}{(\epsilon + \mathbf{P})} \mathbf{G}^{\mathrm{T}} = -\mathbf{i}\,\frac{\mathbf{1}}{\mathbf{2}}\,\omega - \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{k}^{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{2}}\,(\ln \mathbf{2} - \mathbf{1})\,\omega^{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{4}}\,(\mathbf{3} - \mathbf{4}\ \ln \mathbf{2})\ \mathbf{k}^{\mathbf{4}} + \mathbf{i}\ \ln \mathbf{2}\,\omega\,\mathbf{k}^{\mathbf{2}} - \ln^{\mathbf{2}} \mathbf{2}\,\mathbf{k}^{\mathbf{6}} \ldots$$

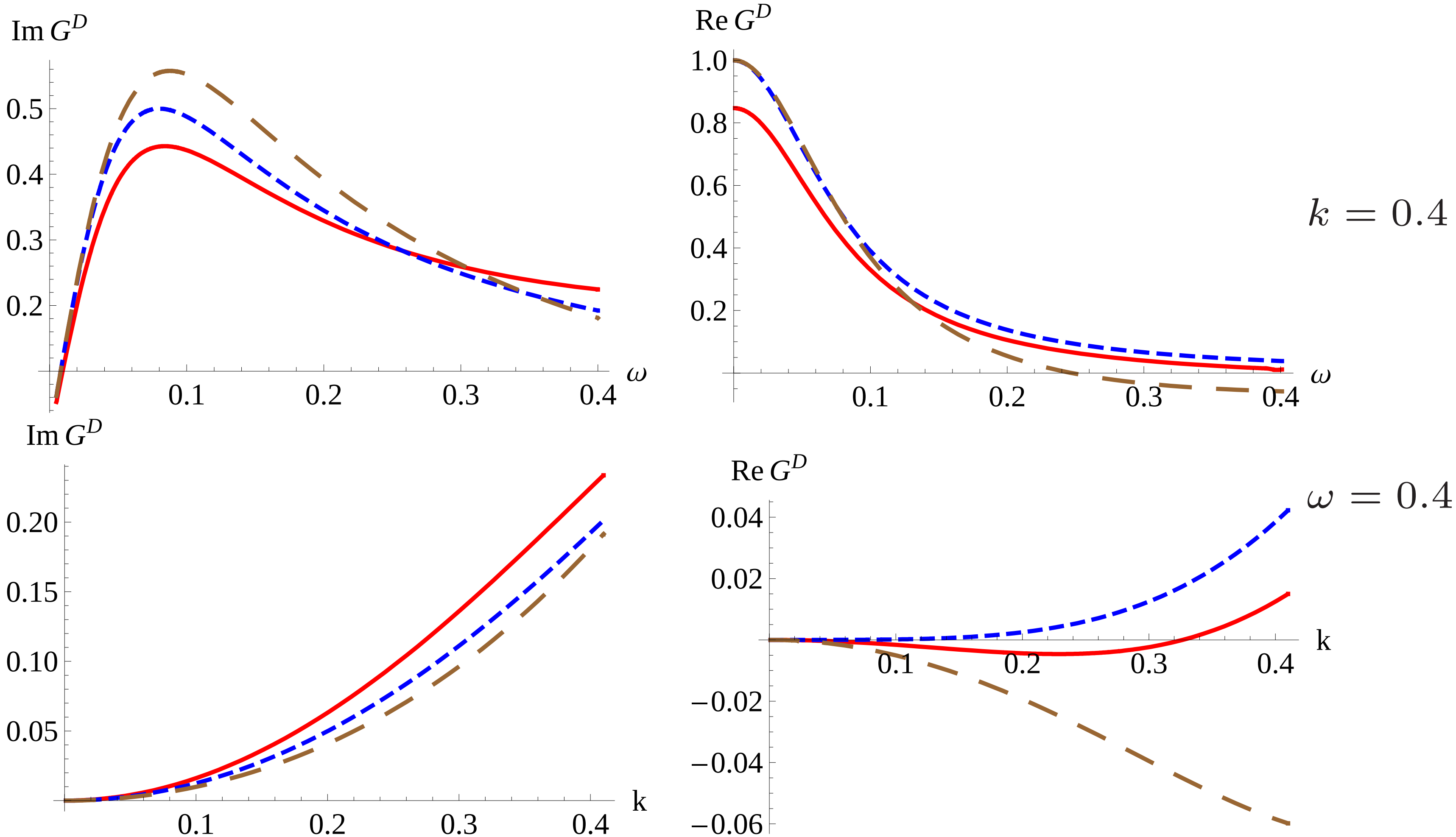
$$\frac{1}{(\epsilon + \mathbf{P})} \mathbf{G}^{\mathrm{D}} = \frac{\mathbf{i}\,\mathbf{k}^{\mathbf{2}}/\mathbf{2}\,[\mathbf{1} + \mathbf{i}\,(\mathbf{2} - \ln \mathbf{2})\,\omega - \mathbf{k}^{\mathbf{2}}/\mathbf{2} \ldots] + \omega\,\mathbf{k}^{\mathbf{2}}/\mathbf{2} + \cdots}{\omega + \mathbf{i}\,\mathbf{k}^{\mathbf{2}}/\mathbf{2}\,[\mathbf{1} + \mathbf{i}\,(\mathbf{2} - \ln \mathbf{2})\,\omega - \mathbf{k}^{\mathbf{2}}/\mathbf{2} + \cdots]}$$

$$\frac{1}{(\epsilon + \mathbf{P})} \mathbf{G}^{\mathrm{S}} = \frac{-\mathbf{k}^{\mathbf{2}} + \mathbf{i}\,\mathbf{2}\,[\mathbf{1} - \mathbf{i}\,\omega\,(\ln \mathbf{2} - \mathbf{2}) + \cdots]\,\omega\,\mathbf{k}^{\mathbf{2}} + \mathbf{2}\,\omega^{\mathbf{2}}\,\mathbf{k}^{\mathbf{2}} \ldots}{\mathbf{3}\,\omega^{\mathbf{2}} - \mathbf{k}^{\mathbf{2}} + \mathbf{i}\,\mathbf{2}\,\omega\,\mathbf{k}^{\mathbf{2}}\,[\mathbf{1} - \mathbf{i}\,\omega\,(\ln \mathbf{2} - \mathbf{2}) + \cdots]}$$

Sound mode: $$\omega = \frac{k}{\sqrt{3}} - i\,\frac{k^2}{3} + \frac{3 - \ln 4}{6\sqrt{3}}\,k^3 + \cdots$$

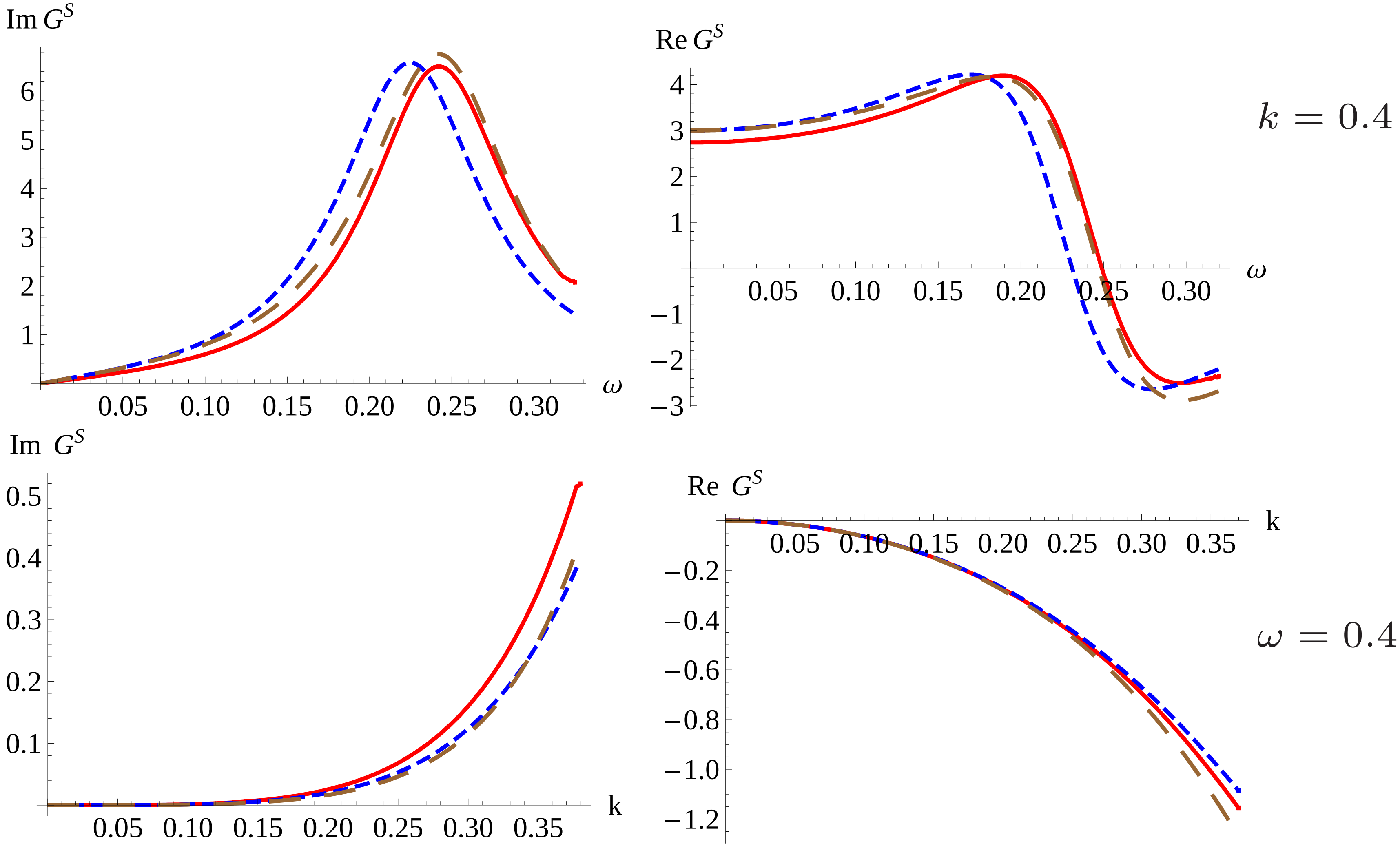
Shear mode: $$\omega = -i\,\frac{k^2}{2} - i\,\frac{1 - \ln 2}{4}\,k^4 + \cdots$$

$\tau_R = 2 - \ln 2$ S. Bhattacharyya, V. E Hubeny, S. Minwalla, M. Rangamani, JHEP 0802:045,2008



red - BH AdS/CFT   P. Kovtun, A. Starinets, Phys.Rev.Lett.96:131601,2006

blue dash - Navier Stokes   brown dash - IS second order hydro.

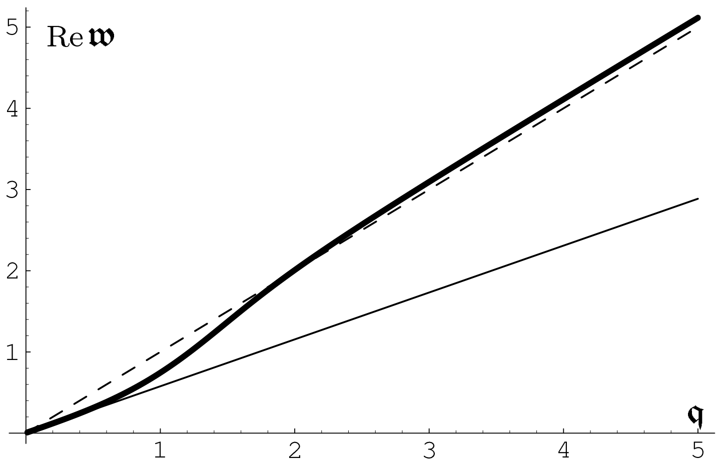


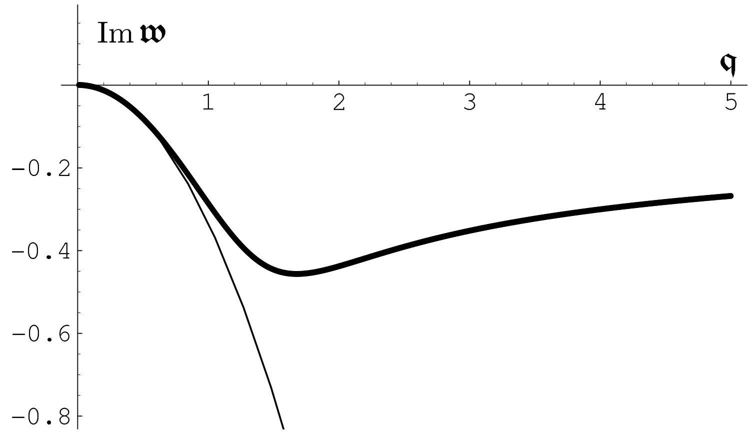
Im G^S
6
5
4
3
2
1
0.05
0.10
0.15
0.20
0.25
0.30
ω
Re G^S
4
3
2
1
−1
−2
−3
0.05
0.10
0.15
0.20
0.25
0.30
ω
$k = 0.4$
Im G^S
0.5
0.4
0.3
0.2
0.1
0.05
0.10
0.15
0.20
0.25
0.30
0.35
k
Re G^S
0.05
0.10
0.15
0.20
0.25
0.30
0.35
k
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
$\omega = 0.4$

Sound and Holography

P. Kovtun and A. Starinets, Phys.Rev.D72:086009,2005

Quasi-normal mode analysis in the AdS BH background - the sound channel





$$\Re e[\omega] = c\, k + \sum_{n=1}^{\infty} r_n\, k^{2n+1} \qquad \Im m[\omega] = -\bar{\eta}\left[c^2 k^2 + \sum_{n=2}^{\infty} \beta_n\, k^{2n}\right]$$

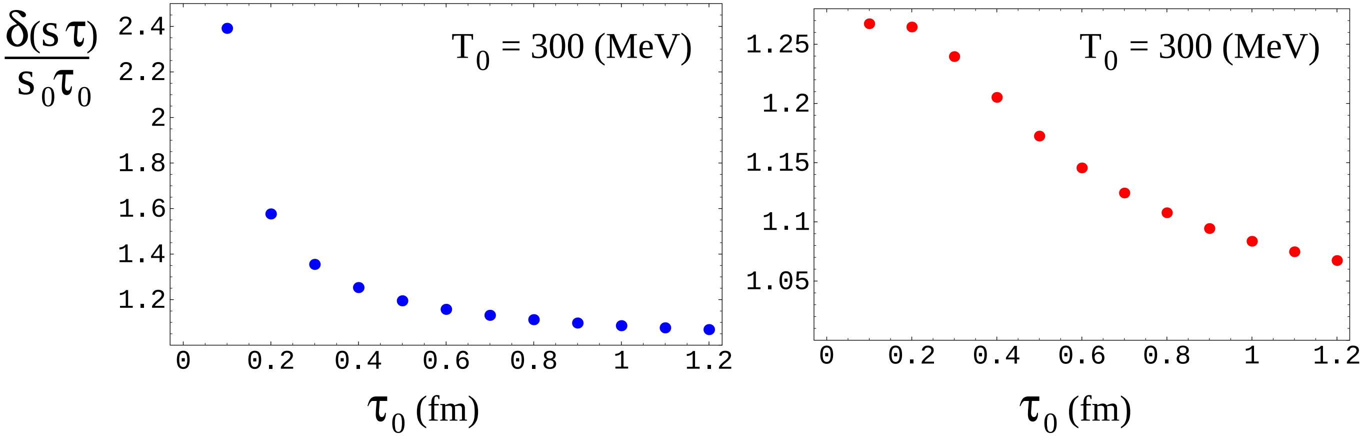
$\beta_2 < 0$ while the IS second order hydro leads to $\beta_2 > 0$

Momenta dependent viscosity (naive)

M.L. and E. Shuryak, Phys.Rev.C76:021901,2007.

Effective viscosity
$$\eta(k) \;=\; \bar{\eta}\left[1 \;+\; c^{-2}\sum_{n=1}^{\infty}\beta_{n+1}\,k^{2\,n}\right]$$

Entropy production in the Bjorken Hydro:



Too naive: Bjorken hydro is very sensitive to the non-linear effects (so far neglected)

Life on the boundary: Linearized Hydro to all orders

Invariance under the local Weyl transformation $\quad g_{\mu\nu} \;\longrightarrow\; e^{-2\,\Omega(x,t)}\,g_{\mu\nu}$

$$T^{\mu\nu} \;\longrightarrow\; e^{6\,\Omega(x,t)}\,T^{\mu\nu}\,; \qquad u^{\mu} \;\longrightarrow\; e^{\Omega(x,t)}\,u^{\mu} \qquad C^{\mu}_{\alpha\nu\beta} \;\longrightarrow\; C^{\mu}_{\alpha\nu\beta}$$

$$C^{\lambda}_{\mu\nu\alpha} \;=\; R^{\lambda}{}_{\mu\nu\alpha} - \frac{1}{2}\,(g^{\lambda}_{\nu}\,R_{\mu\alpha} - g^{\lambda}_{\alpha}\,R_{\mu\nu} - g_{\mu\nu}\,R^{\lambda}{}_{\alpha} + g_{\mu\alpha}\,R^{\lambda}{}_{\nu}) + \frac{1}{6}\,R\,(g^{\lambda}_{\nu}\,g_{\mu\alpha} - g^{\lambda}_{\alpha}\,g_{\mu\nu}),$$

Introduce all order gradient expansion of $T^{\mu\nu}$:

$$\mathbf{\Pi}^{\mu\nu} \;=\; -\,\mathbf{2}\,\eta\,\nabla^{\mu}\,\mathbf{u}^{\nu} \;+\; \mathbf{2}\,\kappa\,\mathbf{u}_{\alpha}\,\mathbf{u}_{\beta}\,\mathbf{C}^{\mu\alpha\nu\beta} \;+\; \rho\,(\mathbf{u}_{\alpha}\,\nabla_{\beta} \;+\; \mathbf{u}_{\beta}\,\nabla_{\alpha})\,\mathbf{C}^{\mu\alpha\nu\beta} \;+\; \xi\,\nabla_{\alpha}\,\nabla_{\beta}\,\mathbf{C}^{\mu\alpha\nu\beta}$$

$$\eta \;=\; \eta[\nabla^2, (u\nabla)]\,; \qquad \kappa \;=\; \kappa[\nabla^2, (u\nabla)]\,; \qquad \rho \;=\; \rho[\nabla^2, (u\nabla)]\,; \qquad \xi \;=\; \xi[\nabla^2, (u\nabla)]\,;$$

$\nabla^2 \;\longrightarrow\; \omega^2 \;-\; k^2$ **and** $(u\,\nabla) \;\longrightarrow\; -\,i\,\omega$**.**

Retarded Correlators from Hydrodynamics

Linear response

$$G^{\alpha\beta\mu\nu} \;=\; \frac{\delta T^{\alpha\beta}}{\delta h^{\mu\nu}}\Big|_{h=0}$$

The scalar (h^{xy}):

$$\mathbf{G^{T}(k,\omega) \;=\; -\,i\,\omega\,\eta \;-\; \kappa\,\frac{1}{2}\,(\omega^2 \;+\; k^2) \;-\; \rho\,\frac{i\,\omega}{2}\,(\omega^2 \;-\; k^2) \;+\; \xi\,\frac{1}{4}\,(\omega^2 \;-\; k^2)^2}$$

The shear (h^{tx}):

$$\mathbf{G^{D}(k,\omega) \;=\; (\epsilon+P)\,\frac{\bar{\eta}\,k^2 \;-\; i\bar{\kappa}\,\omega\,k^2/2 \;-\; \bar{\rho}\,k^2\,(k^2 \;-\; 2\,\omega^2)/4 \;+\; i\,\bar{\xi}\,\omega\,k^2\,(\omega^2 \;-\; k^2)/4}{-i\,\omega \;+\; \bar{\eta}\,k^2}}$$

The sound (h^{tz}):

$$\mathbf{G^{S}(k,\omega) \;=\; (\epsilon \;+\; P)\,\frac{k^2 \;-\; 4\,i\,\bar{\eta}\,\omega\,k^2 \;-\; 2\,\bar{\kappa}\,\omega^2\,k^2 \;-\; 2\,i\,\bar{\rho}\,\omega^3\,k^2 \;+\; \bar{\xi}\,\omega^4\,k^2}{k^2 \;-\; 3\,\omega^2 \;-\; 4\,i\,\bar{\eta}\,\omega\,k^2}}$$

4 vs 3 Puzzle

There should be one to one correspondence between linearized $T^{\mu\nu}$ and the full set of its correlators.

Our program is to equate the "hydro" correlators with the correlators computed from the bulk gravity. The goal is to invert these equations in order to determine the four transport coefficient functions.

We end up having only 3 equations for 4 unknown functions!

This system does not seem to have a unique solution. Despite our failure to simultaneously determine all transport coefficient functions, we are able to extract them perturbatively in the long-wave limit approximation.

In the long-wave limit all coefficient functions are expandable in power series

$$\eta \;=\; \eta_0\,(1+i\eta_{0,1}\,\omega+\eta_{2,0}\,k^2+\eta_{0,2}\,w^2+i\,\eta_{2,1}\,\omega\,k^2+i\,\eta_{0,3}\,\omega^3+\eta_{4,0}\,k^4+\eta_{2,2}\,\omega^2\,k^2+\eta_{0,4}\,\omega^4+\cdots);$$

$$\kappa \;=\; \kappa_0\,(1\;+\;i\,\kappa_{0,1}\,\omega\;+\;\kappa_{2,0}\,k^2\;+\;\kappa_{0,2}\,w^2\;+i\,\kappa_{2,1}\,\omega\,k^2\;+\;i\,\kappa_{0,3}\,\omega^3\;+\;\cdots)\,;$$

$$\rho \;=\; \rho_0\,(1\;+\;i\,\rho_{0,1}\,\omega\;+\;\rho_{2,0}\,k^2\;+\;\rho_{0,2}\,w^2\;+\;\cdots)$$

$$\xi \;=\; \xi_0\,(1\;+\;i\,\xi_{0,1}\,\omega\;+\;\cdots)$$

1st and 2nd order hydro

$$\eta_0 \;=\; 1/2; \qquad \tau_R \equiv \eta_{0,1} \;=\; 2-\ln 2\,; \qquad \kappa_0 \;=\; 2\,\eta_0$$

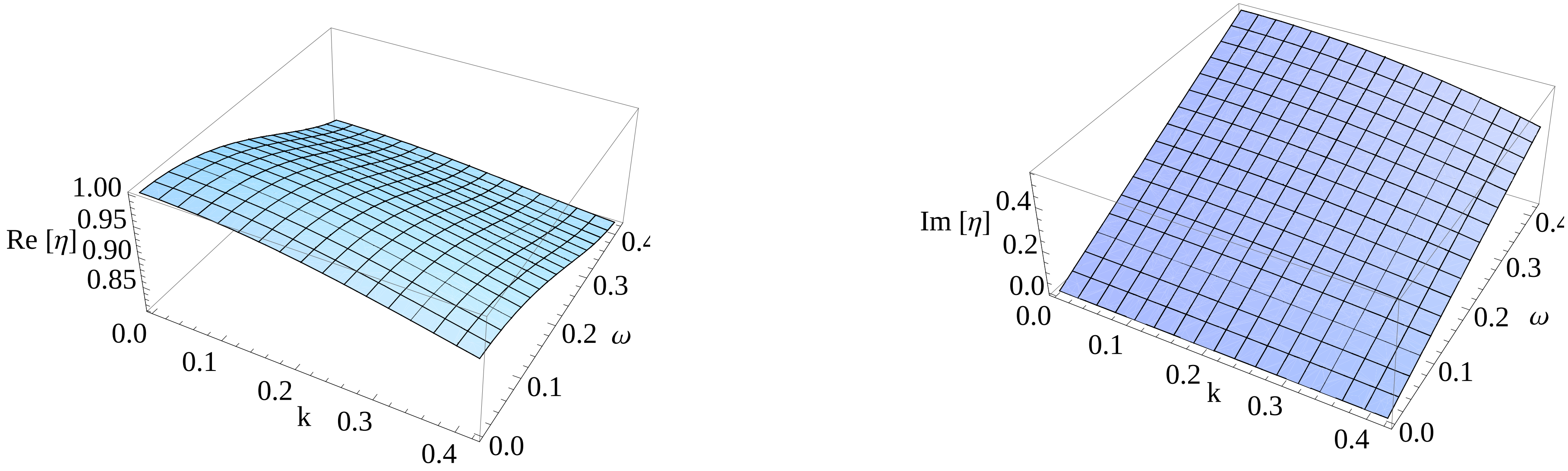
3rd order hydro

$$\lambda \equiv \eta_{2,0} \;=\; -\,1/2; \qquad \eta_{0,2} \simeq -\,1.379 \pm 0.001 \simeq -\frac{3}{2}+\frac{\ln^2 2}{4}$$

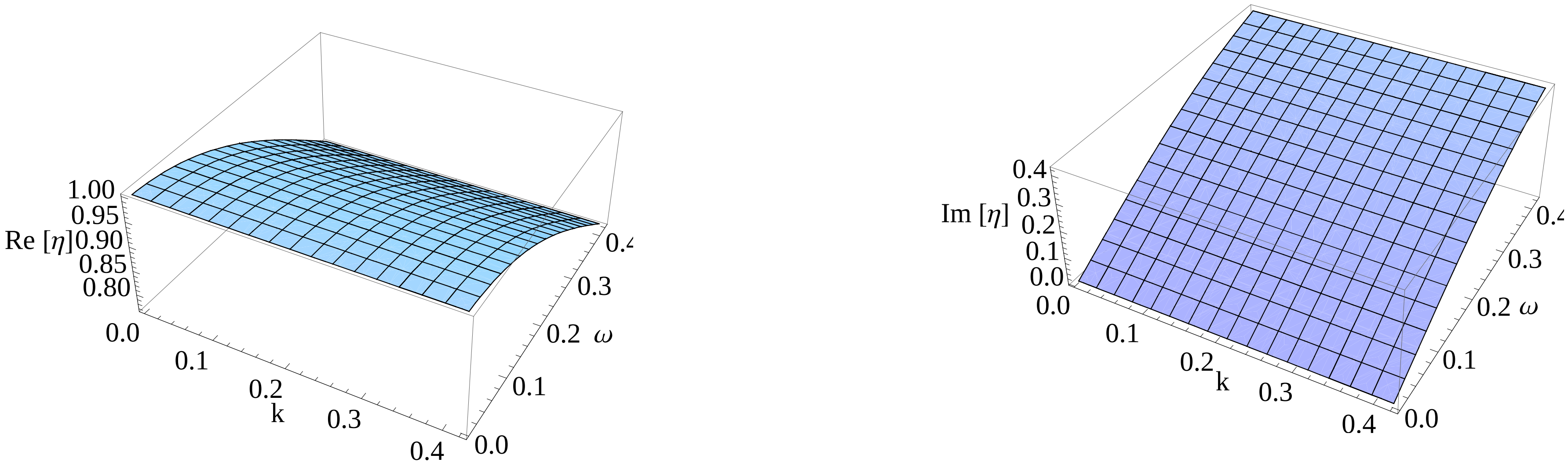
$$\kappa_{0,1} \;=\; 5/2 - 2\,\ln 2\,; \qquad \rho_0 \;=\; 4\,\eta_0$$

4th order hydro

$$\eta_{2,1} \;=\; -\,2.275 \pm 0.005\,; \qquad \eta_{0,3} \;=\; -\,0.082 \pm 0.003$$



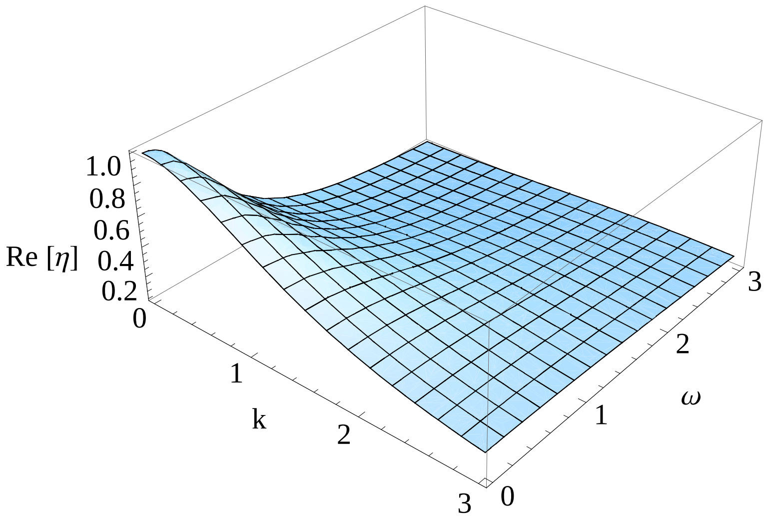
$$\eta \;=\; 1 + i\,\tau_R\,\omega + \lambda\,k^2 + \gamma\,w^2\;\ldots \qquad \tau_R \;=\; 2 - \ln[2]\,, \qquad \lambda \;=\; -1/2\,, \qquad \gamma \;\simeq\; -1.38$$

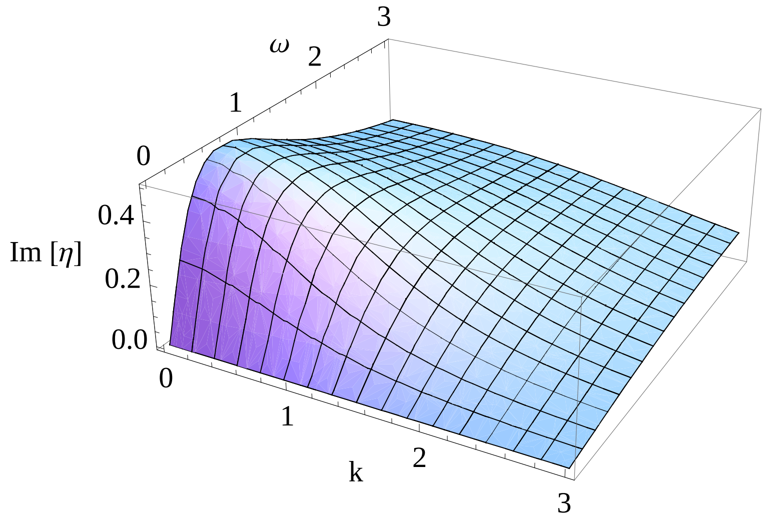


$$\bar{\eta} \;=\; 1 \;+\; i\,\tau_R\,\omega \;-\; \tau_R^2\,\omega^2\;\ldots$$

Improved Causal Hydrodynamics

$$\eta_{\mathbf{model}} = \frac{\eta_0}{\mathbf{1} - \lambda\,\mathbf{k^2} - \mathbf{i}\,\omega\,\tau_{\mathbf{R}}}$$





Conclusions

- We have initiated study of all order (linearized) hydrodynamics.
- The 4 vs 3 puzzle remains unsolved.
  Possible solutions may involve either the membrane paradigm approach or prove that the iterative procedure works to any order
- We have determined few new transport coefficients
- We cautiously suggest that the results based on IS might be less reliable than it was previously thought. We have proposed an improved phenomenological model.
- The effective viscosity is a decreasing function both of frequency and momentum. This behavior might be the reason behind the low viscosity observed at RHIC. It may also explain the exceptionally good survival of various hydrodynamic flows, particularly the sound waves.