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Master Thesis

The Low-Energy Constants of the Extended Linear Sigma Model at Tree-Level

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Notation

Units, Indices, and Summation

We will always work in natural units, $\hbar = c = \epsilon_0 = 1$.

In order to ensure a uniform notation throughout the whole work, we agree that the Latin indices $i, j, k, \ldots$ always run over the "spatial" coordinate labels $1, 2, 3$. In addition to that, the Latin indices $a, b, c, \ldots$ will always run from 0 to 3. Furthermore, the Greek indices $\mu, \nu, \ldots$ generally run over the coordinate labels $0, 1, 2, 3$ of four-dimensional Minkowski space-time. Deviations from these conventions will be defined when necessary.

Unless indicated otherwise, we will always sum over repeated co- and contravariant indices. The co- and contravariant notation will also be used for tensorial objects which are defined in Cartesian coordinates.

Metric, Differential Operators, and Tensors

Throughout the whole work, we will use the convention in which the metric of Minkowski space-time is given by

$$(g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ,$$

so that the d'Alembertian is given by

$$\Box = \partial_\mu \partial^\mu = \partial_t^2 - \triangle ,$$

in which

$$(\partial_\mu) = \left(\frac{\partial}{\partial x^\mu} \right) = (\partial_t, \boldsymbol{\nabla}) , \qquad (\partial^\mu) = (g^{\mu\nu} \partial_\nu) = (\partial_t, -\boldsymbol{\nabla})^T$$

denote the co- and contravariant 4-gradient. It should also be noted that the short-hand notations

$$\partial_i = \frac{\partial}{\partial x^i}, \qquad \partial^i = \frac{\partial}{\partial x_i}$$

will be used frequently.

While the vectors of n-dimensional position and momentum space

$$\mathbf{r} = (x_1, x_2, \ldots, x_n)^T \ , \qquad \mathbf{k} = (k_1, k_2, \ldots, k_n)^T$$

are always indicated by boldface letters, the vectors defined in internal vector spaces, such as isospin space, are specified in the usual arrow notation

$$\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \ .$$

The three- and four-dimensional Levi-Civita tensor is defined as the totally antisymmetric tensor with $\epsilon_{123} = 1$ and $\epsilon_{0123} = 1$.

Contents

Chapter 1

Motivation

Today, physicists believe that all phenomena in the universe can be traced back to four fundamental interactions. Two of them are very well known, since everyday life is strongly influenced and shaped by the effects of gravitation and electromagnetism. While these two forces are also present at macroscopic scales, weak and strong interactions are nuclear forces and therefore only take place at microscopic scales. Even if one naively could suspect something else, the weakest interaction is given by gravitation. At the mass scale of a typical hadron, like the proton or the neutron, gravitation couples at a magnitude of about 10^{-36} to matter. Since gravitation is very well described at the classical level by general relativity, a quantum theory of gravitation was not found up to now. The next strongest interaction is given by the weak interaction, which couples at a magnitude of about 10^{-5} to matter. While the gravitational interaction has an infinite range, the range of weak interactions is given by $\sim 10^{-3}$ fm. This very short range is due to the fact that the gauge bosons of weak interactions are very heavy. Then, with a coupling of about 10^{-2} to matter, the next strongest interaction is given by the electromagnetic interaction. Similar to gravitation, the electromagnetic force has an infinite range and therefore influences nature at many different scales. Finally, the strongest interaction in our universe is given by the strong interaction. This fundamental force couples to typical hadronic matter at a magnitude of about 1. The range of this interaction is given by about 1 fm.

In contrast to gravitation, physicists believe that they found a mathematical description of the other three fundamental interactions in the framework of Quantum Field Theory. A fundamental feature of these theories is given by the occurrence of so-called gauge bosons which arise in the adjoint representation of the gauge symmetry of the respective fundamental theory. These gauge bosons serve as "messengers" of the different forces and realize the interactions in those so-called gauge theories. As already mentioned, the three gauge bosons of the weak interaction have non-vanishing masses, while the photon and the eight gluons are massless. In order to explain these masses and include them in a gauge-invariant way, one needs a mechanism which is able to generate them in a dynamical way, the so-called Higgs mechanism. This mechanism was included in the so-called Glashow-Weinberg-Salam theory of electroweak interactions, which unified the weak and the electromagnetic interaction. It was also possible to summarize the electroweak and the strong interactions in only one theory, the so-called Standard Model of Particle Physics.

The main focus of this work lies on the properties of the strong interaction. As already mentioned, physicists found a Quantum Field Theory which describes strong interactions. This theory is called Quantum Chromodynamics (QCD). This name derives from the fact that the fundamental strong interaction is realized by the exchange of a quantum number which is called color. The fundamental degrees of freedom in QCD are quarks which arise in the fundamental representation of the $SU(3)_C$ color gauge symmetry, and gluons which arise in the adjoint representation of this symmetry. Due to a feature of QCD, which is called confinement, it is not possible to observe objects which contain open color. In nature, we are only able to observe singlets with respect to the color symmetry of QCD. These singlets arise as bound states of quarks and/or anti-quarks and are referred to as hadrons. Also so-called glueballs, i.e., bound states of gluons are discussed in the literature.

The usual way of studying the basic properties of a Quantum Field Theory is given by a perturbative expansion of the different interaction terms of the theory in powers of the respective coupling constant. However, this approach is only valid as long as the coupling constant remains small, so that the interaction terms may be treated as a perturbation of the free theory. In the framework of renormalization, it turns out that the coupling constants of a Quantum Field Theory are not constant, but depend on a momentum scale. In the case of QCD, the coupling constant is given by the so-called strong coupling α_S. While at large energies this coupling remains small, so that perturbation theory can easily be applied, the situation changes completely at low energies. At small energies, the strong coupling becomes arbitrarily large, so that the perturbative approach fails. In order to investigate the low-energy spectrum of QCD, one has to use methods which do not rely on a perturbative expansion of the strong coupling, so-called non-perturbative methods.

One possibility for such a method are effective or hadronic models. In these models, one does not consider the strong interaction at the level of quarks and gluons. There, the fundamental degrees of freedom are mesons ($\bar{q}q$-states) and baryons (qqq- or $\bar{q}\bar{q}\bar{q}$-states). In this work, the main focus lies on the mesonic part of the so-called Extended Linear Sigma Model (eLSM). The eLSM describes a hadronic model which incorporates not only scalar and pseudoscalar mesons, but also vector and axial-vector mesons. The "term" linear derives from the fact that the fundamental fields which describe the different types of mesons enter in a linear realization of the chiral symmetry of QCD.

The importance of chiral symmetry in the low-energy regime of QCD also derives from the fact that it is considered to be spontaneously broken. Due to the Goldstone Theorem, this spontaneous symmetry breakdown gives rise to the occurrence of massless pseudoscalar particle excitations, so-called Nambu-Goldstone bosons, which dominate the low-energy spectrum. The aim of this work is to reduce the eLSM in a way, so that we finally obtain a theory which only contains the interactions of these Nambu-Goldstone bosons among themselves. Then, the low-energy couplings of the resulting theory can be compared to those of Chiral Perturbation Theory (ChPT) which represents a model-independent method in order to describe the low-energy properties of the strong interaction.

This thesis is arranged as follows: In the second Chapter [2], we want to introduce the basic methods and ideas of Quantum Field Theory, such as the functional approach to second quantization and the framework of spontaneously broken global symmetries. Furthermore, we want to introduce the basic properties of QCD in vacuum. Then, the third Chapter [3] is dedicated to Chiral Perturbation Theory. There, we want to introduce this framework as an Effective Field Theory which is based on a systematic analysis of the hadronic n-point functions of strong interactions. Then, in Chapter [4], we finally turn to the eLSM and introduce the basic term structure of this model. Furthermore, we consider different cases of this model and present the calculation of the low-energy couplings of these models. Finally, in Chapter [5], we compare our final results with those obtained from Chiral Perturbation Theory and discuss the meaning of these results.

Chapter 2

Introduction

The basic idea of this first chapter is to introduce some important methods which are crucial in the construction and the understanding of elementary and effective quantum field theories. Therefore, in Sec. [2.1], we begin at the classical level and derive the Euler-Lagrange equations using Hamilton's principle of the stationary action. Then, we focus on symmetry transformations of the classical action functional S and derive Noether's theorem which we will use frequently in the upcoming sections in order to identify the conserved currents of several theories. Another important concept which has to be addressed in this Chapter, is the notion of a spontaneously broken symmetry. The spontaneous breakdown of a continuous symmetry and the famous Goldstone theorem are important tools in modern theoretical physics and will be discussed in Sec. [2.2]. The last Section [2.3] of this introductory chapter is dedicated to the theory of strong interactions, Quantum Chromodynamics (QCD), and its basic features in vacuum.

2.1 Classical Field Theory and Symmetries

As mentioned above, this section serves as a brief reminder of the very basic concepts of the Lagrangian formulation of classical field theory. Due to the great importance of conservation laws in physics, we will also derive Noether's theorem which gives a direct connection between the symmetry of the classical action S under a transformation of the space-time variables and/or the fields of the theory and a conserved quantity.

2.1.1 The Classical Equations of Motion

The Lagrangian formulation of classical field theory can be motivated as the transition of the generalized coordinates $\mathbf{q}(t) = (q_1(t), \ldots, q_N(t))^T$ and the generalized velocities $\dot{\mathbf{q}}(t) = (\dot{q}_1(t), \ldots, \dot{q}_N(t))^T$ to continuous field variables

$$q_j(t) \to \phi(t, \mathbf{r}), \qquad \dot{q}_j \to \dot{\phi}(t, \mathbf{r}) \tag{2.1}$$

for $j = 1, \ldots, N$. In Eq. (2.1) the spatial dependence of the field variables ϕ and $\dot{\phi}$ is, in this context, not to be understood as spatial coordinates, but as a continuous index. Since the values of the field variables become the dynamical variables of the theory, we obtain a theory with an infinite number of degrees of freedom. Therefore, the initial Lagrangian $L(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$ becomes a functional of the field variables ϕ and $\dot{\phi}$. This functional can be written as a volume integral over the so-called Lagrangian density

$$L\big[\phi, \dot{\phi}\big] = \int_V \mathrm{d}^3\mathbf{r}\, \mathscr{L} \ , \tag{2.2}$$

where V denotes the volume of the physical system. In general, the Lagrangian density is not only a function of the fields ϕ and $\dot{\phi}$, but also of $\boldsymbol{\nabla}\phi$. Therefore, in relativistic notation, the Lagrangian density can be written as

$$\mathscr{L} = \mathscr{L}\left(\phi, \partial_\mu \phi\right) \ . \tag{2.3}$$

The basic quantity of the Lagrangian formulation of classical field theory is the so-called action functional S, defined as the sum of all contributions of the Lagrangian $L\big[\phi, \dot{\phi}\big]$ in the time interval $[t_i, t_f] \in \mathbb{R}$

$$S = \int_{t_i}^{t_f} \mathrm{d}t\, L = \int_\Omega \mathrm{d}^4x\, \mathscr{L}\left(\phi, \partial_\mu \phi\right) \ , \tag{2.4}$$

where we used Eq. (2.2) and introduced the arbitrary space-time volume $\Omega = [t_i, t_f] \times V \subset \mathbb{R} \times \mathbb{R}^3$. At this point, it should be mentioned that the Lagrangian density (2.3) has to be a scalar under transformations of the Poincaré[1] group in order to obtain a Poincaré-invariant action functional S. This can be seen by considering the transformation behavior of the space-time measure which obviously transforms as a scalar

$$\mathrm{d}^4x' = \left|\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\right| \mathrm{d}^4x = \left|\det\left(\Lambda^\mu{}_\nu\right)\right| \mathrm{d}^4x = \mathrm{d}^4x \ , \tag{2.5}$$

where we used that the determinant of a Lorentz transformation is equal to ± 1.

In general, a theory contains more than only one field. To this end, we generalize our considerations to a theory containing N fields $\phi_a(x),\ a = 1, \ldots, N$, defined by the action

$$S = \int_\Omega \mathrm{d}^4x \,\mathscr{L}\left(\phi_a, \partial_\mu\phi_a\right) \ . \tag{2.6}$$

The equations of motion for the fields $\phi_a(x)$ can be obtained by using Hamilton's principle of stationary action. In this context, it has to be taken into account that only the fields inside the space-time volume Ω are varied. The space-time variables and also the fields on the surface of the space-time volume $\partial\Omega$ are held fixed at a constant value. Therefore, it is possible to interchange the variation and the partial derivatives. Using this consideration, one finds

$$\begin{aligned} 0 = \delta S &= \delta \int_\Omega \mathrm{d}^4x \,\mathscr{L}\left(\phi_a, \partial_\mu\phi_a\right) \\ &= \int_\Omega \mathrm{d}^4x \left[\frac{\partial\mathscr{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)\right] \\ &= \int_\Omega \mathrm{d}^4x \left[\frac{\partial\mathscr{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a \ , \end{aligned} \tag{2.7}$$

where we integrated the second term by parts in the last step. Finally we use that the variation of the fields $\delta\phi_a$ was arbitary, so that the square bracket in Eq. (2.7) has to vanish by itself. This condition yields the Euler-Lagrange equations for the N fields $\phi_a(x)$

$$\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)} - \frac{\partial\mathscr{L}}{\partial\phi_a} = 0 \quad \text{for } a = 1, \ldots, N \ . \tag{2.8}$$

2.1.2 Noether's Theorem

In the second part of this short section about classical field theory, we will establish a connection between conserved quantities and the symmetry of the classical action functional S with respect to continuous transformations of the fields and the space-time variables. To this end, we only consider infinitesimal transformations of the space-time variables and the fields. The starting point of these considerations will be Eq. (2.6), i.e., the action functional of a theory containing N fields $\phi_a(x),\ a = 1, \ldots, N$. The space-time variables transform according to

$$x'^\mu = x^\mu + \delta x^\mu \ , \tag{2.9}$$

while the fields transform as

$$\phi'_a(x') = \phi_a(x) + \delta\phi_a(x) \ . \tag{2.10}$$

At this point it is important to mention that the variation $\delta\phi_a(x)$ contains one part proportional to the variation of the field itself and another part arising from the fact that also the space-time variables are varied. This can be seen as follows

$$\begin{aligned} \delta\phi_a(x) &= \phi'_a(x') - \phi_a(x) \\ &= \phi'_a(x') - \phi_a(x') + \phi_a(x') - \phi_a(x) \\ &= \Delta\phi_a(x') + \partial_\mu\phi_a(x)\delta x^\mu \\ &= \Delta\phi_a(x) + \partial_\mu\phi_a(x)\delta x^\mu \ , \end{aligned} \tag{2.11}$$

in which we used that the difference $\Delta\phi_a(x') \equiv \phi'_a(x') - \phi_a(x')$ corresponds to the variation of the field at a fixed space-time coordinate x'. This kind of variation, in which we only consider the variation of the field, is often denoted as the total variation of the field. This total variation has the advantage to be commutable with the derivatives. Furthermore, we expanded the field $\phi_a(x')$ in a Taylor polynomial up to first order in

[1] However, it turns out that weak interactions break the symmetry under parity and time-reversal transformations, so that in nature the realized symmetry group reduces to the proper orthochronous Poincaré group.

δx^μ. Finally, in the last step, we used that the total variation $\Delta\phi_a(x')$ is, up to first order, identical with the total variation $\Delta\phi_a(x)$ at x. Now, we consider the transformed action functional, which is of the form

$$S\left[\phi'_a(x')\right] = S\left[\phi_a(x)\right] + \delta S\ . \tag{2.12}$$

In order to have a theory which is invariant under the transformations (2.9) and (2.10), we have to require that the last term δS in Eq. (2.12) vanishes. Therefore, we rewrite this term as follows

$$\begin{aligned}
\delta S &= S\left[\phi'_a(x')\right] - S\left[\phi_a(x)\right] \\
&= S\left[\phi'_a(x')\right] - S\left[\phi_a(x')\right] + S\left[\phi_a(x')\right] - S\left[\phi_a(x)\right] \\
&= \int_{\Omega'} \mathrm{d}^4x'\ \mathscr{L}\left(\phi'_a(x'), \partial'_\mu\phi'_a(x')\right) - \int_{\Omega'} \mathrm{d}^4x'\ \mathscr{L}\left(\phi_a(x'), \partial'_\mu\phi_a(x')\right) \\
&\quad + \int_{\Omega'} \mathrm{d}^4x'\ \mathscr{L}\left(\phi_a(x'), \partial'_\mu\phi_a(x')\right) - \int_{\Omega} \mathrm{d}^4x\ \mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)\ .
\end{aligned} \tag{2.13}$$

In the following step, we expand the first term in Eq. (2.13) around $\phi_a(x')$. Using that $\phi'_a(x') = \phi_a(x') + \Delta\phi_a(x')$, we find

$$\begin{aligned}
\int_{\Omega'} \mathrm{d}^4x'\ \mathscr{L}\left(\phi'_a(x'), \partial'_\mu\phi'_a(x')\right) &= \int_{\Omega'} \mathrm{d}^4x'\ \Big\{\mathscr{L}\left(\phi_a(x'), \partial'_\mu\phi_a(x')\right) + \frac{\partial\mathscr{L}\left(\phi_a(x'), \partial'_\mu\phi_a(x')\right)}{\partial\phi_a(x')}\Delta\phi_a(x') \\
&\quad + \frac{\partial\mathscr{L}\left(\phi_a(x'), \partial'_\mu\phi_a(x')\right)}{\partial\left(\partial'_\mu\phi_a(x')\right)}\partial'_\mu\Delta\phi_a(x')\Big\} \\
&= \int_{\Omega'} \mathrm{d}^4x'\ \mathscr{L}\left(\phi_a(x'), \partial'_\mu\phi_a(x')\right) \\
&\quad + \int_{\Omega} \mathrm{d}^4x\ \left\{\frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\phi_a(x)}\Delta\phi_a(x) + \frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\left(\partial_\mu\phi_a(x)\right)}\partial_\mu\Delta\phi_a(x)\right\}\ ,
\end{aligned} \tag{2.14}$$

where we immediately used that the total variation commutes with the 4-gradient. Next, we consider the third term in Eq. (2.13). In this term, the Lagrangian density can be expanded around x. In addition to that, we perform a change of variables from $x' \to x$ inside the integral. This leads to

$$\begin{aligned}
\int_{\Omega'} \mathrm{d}^4x'\ \mathscr{L}\left(\phi_a(x'), \partial'_\mu\phi_a(x')\right) &= \int_{\Omega} \mathrm{d}^4x\ \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\left\{\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right) + \partial_\mu\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)\delta x^\mu\right\} \\
&= \int_{\Omega} \mathrm{d}^4x\ \Big\{\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right) + \partial_\mu\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)\delta x^\mu \\
&\quad + \mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)\frac{\partial\delta x^\mu}{\partial x^\mu}\Big\}\ ,
\end{aligned} \tag{2.15}$$

where we used, that $\det\left(\partial x'^\mu/\partial x^\nu\right) = 1 + \partial\delta x^\mu/\partial x^\nu$ and neglected terms of second order in the infinitesimal quantities δx^μ and $\partial\delta x^\mu/\partial x^\mu$. The functional determinant can be calculated by taking the exponential of $\ln\left(\det(A)\right) = \mathrm{Tr}\left(\ln(A)\right)$. Now, we insert the results (2.14) and (2.15) into Eq. (2.13) to obtain

$$\begin{aligned}
\delta S &= \int_{\Omega} \mathrm{d}^4x\ \left\{\frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\phi_a(x)}\Delta\phi_a(x) + \frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\left(\partial_\mu\phi_a(x)\right)}\partial_\mu\Delta\phi_a(x)\right\} \\
&\quad + \int_{\Omega} \mathrm{d}^4x\ \left\{\partial_\mu\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)\delta x^\mu + \mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)\frac{\partial\delta x^\mu}{\partial x^\mu}\right\} \\
&= \int_{\Omega} \mathrm{d}^4x\ \left\{\frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\phi_a(x)}\Delta\phi_a(x) + \frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\left(\partial_\mu\phi_a(x)\right)}\partial_\mu\Delta\phi_a(x)\right\} \\
&\quad + \int_{\Omega} \mathrm{d}^4x\ \left\{\partial_\mu\left[\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)\delta x^\mu\right]\right\}\ .
\end{aligned} \tag{2.16}$$

By considering the second term of the first integral of Eq. (2.16), we recognize that this term can be rewritten as

$$\frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\left(\partial_\mu\phi_a(x)\right)}\partial_\mu\left(\Delta\phi_a(x)\right) = \partial_\mu\left[\frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\left(\partial_\mu\phi_a(x)\right)}\Delta\phi_a(x)\right] - \left(\partial_\mu\frac{\partial\mathscr{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)}{\partial\left(\partial_\mu\phi_a(x)\right)}\right)\Delta\phi_a(x). \tag{2.17}$$

This is essentially the same step as in the derivation of the Euler-Lagrange equations. But, in Eq. (2.7) we assumed that the variation of the fields vanishes on the surface $\partial\Omega$ and outside the space-time volume Ω.

This assumption requires that the first term, i.e., the surface term, in Eq. (2.17) vanishes. In this calculation we do not make this assumption. Inserting (2.17) into Eq. (2.16), dropping the arguments of the Lagrangian densities and rearranging the terms, we finally obtain

$$\begin{aligned}
\delta S &= \int_\Omega \mathrm{d}^4x \, \left\{ \frac{\partial \mathscr{L}}{\partial \phi_a(x)} - \partial_\mu \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \right\} \Delta\phi_a(x) + \int_\Omega \mathrm{d}^4x \, \partial_\mu \left\{ \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \Delta\phi_a(x) + \mathscr{L}\delta x^\mu \right\} \\
&= \int_\Omega \mathrm{d}^4x \, \partial_\mu \left\{ \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \Big(\delta\phi_a(x) - \partial_\nu \phi_a(x) \delta x^\nu \Big) + \mathscr{L}\delta x^\mu \right\} \\
&= \int_\Omega \mathrm{d}^4x \, \partial_\mu \left\{ \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \delta\phi_a(x) - \left[\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \partial_\nu \phi_a(x) - g^\mu_{\ \nu} \mathscr{L} \right] \delta x^\nu \right\} \\
&= \int_\Omega \mathrm{d}^4x \, \partial_\mu \left\{ \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \delta\phi_a(x) - \theta^\mu_{\ \nu} \delta x^\nu \right\} \ , \qquad (2.18)
\end{aligned}$$

where we used that the fields $\phi_a(x)$ fulfill the classical equations of motion (2.8). Furthermore, we used Eq. (2.11) to rewrite the total variation and defined the so-called energy-momentum tensor

$$\theta^\mu_{\ \nu} = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \partial_\nu \phi_a(x) - g^\mu_{\ \nu} \mathscr{L} \ . \qquad (2.19)$$

At the beginning of the above derivation, we started with the requirement that $\delta S = 0$, therefore Eq. (2.18) has to vanish. We recognize that the space-time volume Ω was chosen arbitrary, so that the integral vanishes, if

$$\partial_\mu \left\{ \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \delta\phi_a(x) - \theta^\mu_{\ \nu} \delta x^\nu \right\} = 0 \ , \qquad (2.20)$$

which obviously has the form of a continuity equation for the conserved Noether current

$$J^\mu = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a(x)\right)} \delta\phi_a(x) - \theta^\mu_{\ \nu} \delta x^\nu \ . \qquad (2.21)$$

Finally, we are able to evaluate the volume integral of Eq. (2.20) by using the Gaussian theorem to obtain the conserved Noether charge Q

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} Q, \qquad Q = \int_V \mathrm{d}^3\mathbf{r} \, J^0 \ . \qquad (2.22)$$

The derivation of this section shows what is generally known as Noether's Theorem: The invariance of the classical action functional with respect to a continuous transformation of the field variables and/or the space-time variables gives rise to the existence of a conserved current (2.21) and therefore to the existence of a conserved charge (2.22).

This section shall be completed by a discussion of the symmetry properties of the energy-momentum tensor (2.19). This energy-momentum tensor is, in general, not a symmetric tensor. In practice, it is sometimes advantageous to work with a symmetric energy-momentum tensor. In order to obtain a symmetric, physically absolutely equivalent energy-momentum tensor, often denoted as $T^{\mu\nu}$, one adds the 4-divergence of a third-rank tensor $\vartheta^{\lambda\mu\nu}$, satisfying $\vartheta^{\lambda\mu\nu} = -\vartheta^{\mu\lambda\nu}$,

$$T^{\mu\nu} = \theta^{\mu\nu} + \partial_\lambda \vartheta^{\lambda\mu\nu} \ . \qquad (2.23)$$

This tensor is referred to as Belinfante tensor. Such a Belinfante tensor always exists, if the difference of the unsymmetrized energy-momentum tensor and its transposed, i.e., the antisymmetric part of the energy-momentum tensor, is proportional to the 4-divergence of a third-rank tensor

$$\theta^{\nu\alpha} - \theta^{\alpha\nu} = -\partial_\mu \tilde{\vartheta}^{\mu\nu\alpha} \ , \qquad (2.24)$$

which is antisymmetric in the second and third index $\tilde{\vartheta}^{\mu\nu\alpha} = -\tilde{\vartheta}^{\mu\alpha\nu}$. Now, we choose

$$\vartheta^{\mu\nu\alpha} = \frac{1}{2} \left[\tilde{\vartheta}^{\mu\nu\alpha} + \tilde{\vartheta}^{\nu\alpha\mu} - \tilde{\vartheta}^{\alpha\mu\nu} \right] \ , \qquad (2.25)$$

so that

$$\begin{aligned}
T^{\nu\alpha} - T^{\alpha\nu} &= \theta^{\nu\alpha} - \theta^{\alpha\nu} + \partial_\mu \left(\vartheta^{\mu\nu\alpha} - \vartheta^{\mu\alpha\nu} \right) \\
&= \theta^{\nu\alpha} - \theta^{\alpha\nu} + \partial_\mu \tilde{\vartheta}^{\mu\nu\alpha} \\
&= 0 \ , \qquad (2.26)
\end{aligned}$$

where we used Eqs. (2.23), (2.25), and the symmetry properties of $\tilde{\vartheta}^{\mu\nu\alpha}$. Using Eq. (2.25), we are in the position to determine the explicit form of the Belinfante tensor $\vartheta^{\mu\nu\alpha}$. To this end, we consider the infinitesimal Lorentz transformations of a general theory, described by Eq. (2.6). A general ansatz for the transformation of the fields $\phi_a(x)$ under the Lorentz group is given by

$$\phi'_a(x') = \phi_a(x) + \frac{i}{2}\omega_{\mu\nu}\left(J^{\mu\nu}\right)_{ab}\phi^b(x) \ , \tag{2.27}$$

where the group parameters satisfy $\omega^{\mu\nu} = -\omega^{\nu\mu}$ and the $\left(J^{\mu\nu}\right)_{ab}$ are finite-dimensional representations of the generators of the Lorentz group. Under infinitesimal Lorentz transformations the space-time variables transform according to

$$x'^{\mu} = x^{\mu} + \omega^{\mu\nu}x_{\nu} \ . \tag{2.28}$$

Inserting the transformations (2.27) and (2.28) into the Noether current (2.21), we obtain

$$\begin{aligned} J^{\mu} &= \frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu}\phi_a(x)\right)}\left[\frac{i}{2}\omega_{\nu\alpha}\left(J^{\nu\alpha}\right)_{ab}\phi^b(x)\right] - \theta^{\mu\nu}\omega_{\nu\alpha}x^{\alpha} \\ &= \frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu}\phi_a(x)\right)}\left[\frac{i}{2}\omega_{\nu\alpha}\left(J^{\nu\alpha}\right)_{ab}\phi^b(x)\right] - \frac{1}{2}\left[\theta^{\mu\nu}\omega_{\nu\alpha}x^{\alpha} - \theta^{\mu\nu}\omega_{\alpha\nu}x^{\alpha}\right] \\ &= \frac{1}{2}\omega_{\nu\alpha}\left[\frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu}\phi_a(x)\right)}i\left(J^{\nu\alpha}\right)_{ab}\phi^b(x) + \theta^{\mu\alpha}x^{\nu} - \theta^{\mu\nu}x^{\alpha}\right] \\ &= \frac{1}{2}\omega_{\nu\alpha}M^{\mu\nu\alpha} \ , \end{aligned} \tag{2.29}$$

where we used the antisymmetry of the group parameters and renamed $\nu \leftrightarrow \alpha$ in the second term of the second step. Now, the conservation of this Noether current implies that $0 = \partial_{\mu}M^{\mu\nu\alpha}$ which can be rewritten as

$$\begin{aligned} 0 &= \partial_{\mu}M^{\mu\nu\alpha} \\ &= \partial_{\mu}\left[\frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu}\phi_a(x)\right)}i\left(J^{\nu\alpha}\right)_{ab}\phi^b(x)\right] + \theta^{\nu\alpha} - \theta^{\alpha\nu} \ , \quad \Rightarrow \quad \theta^{\nu\alpha} - \theta^{\alpha\nu} = -\partial_{\mu}\left[\frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu}\phi_a(x)\right)}i\left(J^{\nu\alpha}\right)_{ab}\phi^b(x)\right] \ . \end{aligned} \tag{2.30}$$

Finally, using Eqs. (2.24) and (2.25), we obtain the explicit form of the Belinfante tensor of a general theory (2.6)

$$\vartheta^{\mu\nu\alpha} = \frac{1}{2}\left[\frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu}\phi_a(x)\right)}i\left(J^{\nu\alpha}\right)_{ab}\phi^b(x) + \frac{\partial \mathscr{L}}{\partial\left(\partial_{\nu}\phi_a(x)\right)}i\left(J^{\alpha\mu}\right)_{ab}\phi^b(x) - \frac{\partial \mathscr{L}}{\partial\left(\partial_{\alpha}\phi_a(x)\right)}i\left(J^{\mu\nu}\right)_{ab}\phi^b(x)\right] \ . \tag{2.31}$$

2.2 Functional Methods, Spontaneously Broken Global Symmetries, and the Goldstone Theorem

The main focus of this second section will be the spontaneous breakdown of global, continuous symmetries. This important mechanism shall be introduced in two steps. In the first step, we consider a simple toy model with a discrete internal symmetry which will be broken spontaneously. This simple example will show us that a spontaneous symmetry breakdown requires two prerequisites: degenerate vacua and an infinitely large space. Then, we extend our considerations to a more complicated toy model with a continuous internal symmetry. The treatment of this model will show us that the spontaneous breakdown of a continuous symmetry is closely connected to the existence of massless pseudoscalar excitations. The occurrence of these so-called Nambu-Goldstone bosons is well described by the famous Goldstone theorem which we will prove at the end of this section. Since this proof requires functional methods, we will start this section with a very brief reminder of the functional approach to second quantization.

2.2.1 The Quantum Effective Action and its Symmetries

An important tool in the mathematical treatment of quantum many-particle problems is the process of second quantization. One approach to second quantization is the so-called canonical quantization. In this framework, one starts with the classical action of the theory in question. Then, one calculates the general solutions of the classical equations of motion in accordance with Eq. (2.8). These solutions can be represented as a decomposition in a complete and orthonormal set of functions, depending on the symmetry of the space. Usually, one uses a Fourier decomposition. Now, the Fourier amplitudes in this decomposition will be promoted to creation and annihilation operators, so that the classical field becomes a field operator. In addition to that, the canonical momentum densities associated with each classical field will also be promoted to operators. Then, we require that the field operator and the associated momentum density fulfill the following equal-time commutation or anticommutation relations

$$[\phi_a(t,\mathbf{r}),\pi_b(t,\mathbf{r}')]_\pm = i\delta_{ab}\delta^{(3)}\left(\mathbf{r}-\mathbf{r}'\right) \ , \tag{2.32}$$

where the symbol $[\cdot,\cdot]_+$ denotes the anticommutator and $[\cdot,\cdot]_-$ denotes the commutator. The decision whether to demand a commutator or an anticommutator relation in Eq. (2.32) is based on the statistics of the fields in question. For bosonic fields, we require an equal-time commutation relation, for fermionic fields, we claim an equal-time anticommutation relation. At this point, it should be taken into account that (2.32) has to be modified, if the fields contain additional degrees of freedom arising from internal symmetries of the field space.

Now, using Eq. (2.32), it is possible to find commutation or anticommutations relations for the creation and annihilation operators. These relations can be used to rewrite the physical observables like the 4-momentum operator or the angular-momentum operator in terms of the creation and annihilation operators. The particle picture is now realized as follows: The physical n-particle states are represented as vectors in a complex Hilbert space $\mathfrak{h}$. These vectors are implemented as square-integrable functions with n variables. The states which describe real physical systems are defined on the subspaces $\mathfrak{h}_B$ and $\mathfrak{h}_F$ of $\mathfrak{h}$. Here the subspace $\mathfrak{h}_B$ contains symmetric functions and the subspace $\mathfrak{h}_F$ contains antisymmetric functions. According to the spin-statistics theorem, the symmetric functions describe physical bosons and the antisymmetric functions describe physical fermions. Therefore, these subspaces are often called state spaces. Then, the n-particle states can be generated by the action of the creation operators on a unique vacuum state $|\Omega\rangle$.

Another approach to second quantization is given by functional integrals. Since the central studies of this work are based on functional methods, we have to repeat the important steps, which lead to the functional approach to Quantum Field Theory. To this end, we consider a general theory describing N spin-0 fields $\phi_a(x)$, $a=1,\ldots,N$, whose Lagrangian is given by

$$\mathscr{L}\left(\phi_a,\partial_\mu\phi_a\right) = \frac{1}{2}\left(\partial_\mu\phi_a\right)^2 - \frac{m^2}{2}\phi_a^2 - \mathscr{V}\left(\phi_a\right) \ . \tag{2.33}$$

From classical Lagrangian Field Theory, we know that each of the N classical fields is associated with a conjugate momentum density

$$\begin{aligned}
\pi^a(x) &= \frac{\partial\mathscr{L}}{\partial\left(\partial_0\phi_a\right)} \\
&= \frac{1}{2}\frac{\partial}{\partial\left(\partial_0\phi_a\right)}\left\{\left(\partial_0\phi_l\right)\left(\partial_0\phi^l\right) - \left(\boldsymbol{\nabla}\phi_l\right)\cdot\left(\boldsymbol{\nabla}\phi^l\right)\right\} \\
&= \frac{1}{2}\left\{\frac{\partial\left(\partial_0\phi_l\right)}{\partial\left(\partial_0\phi_a\right)}\left(\partial_0\phi^l\right) + \left(\partial_0\phi_l\right)\frac{\partial\left(\partial_0\phi^l\right)}{\partial\left(\partial_0\phi_a\right)}\right\} \\
&= \partial_0\phi^a(x) \ ,
\end{aligned} \tag{2.34}$$

where we used that $\partial\left(\partial_0\phi_l\right)/\partial\left(\partial_0\phi_a\right) = \delta_l{}^a$. Then, the Hamiltonian of the theory is given by

$$\begin{aligned}
H &= \int \mathrm{d}^3\mathbf{r}\ \mathscr{H} \\
&= \int \mathrm{d}^3\mathbf{r}\ \left[\pi_a(x)\partial_0\phi^a(x) - \mathscr{L}\left(\phi_a,\partial_\mu\phi_a\right)\right] \\
&= \int \mathrm{d}^3\mathbf{r}\ \left[\frac{1}{2}\pi_a(x)\pi^a(x) + \frac{1}{2}\left(\boldsymbol{\nabla}\phi_a(x)\right)\cdot\boldsymbol{\nabla}\phi^a(x) + \frac{m^2}{2}\phi_a^2(x) + \mathscr{V}\left(\phi_a(x)\right)\right] \ ,
\end{aligned} \tag{2.35}$$

where we used Eq. (2.34). As already mentioned at the beginning of this subsection, the canonical formalism relates the classical fields $\phi_a(x)$, $\pi_a(x)$ with field operators $\hat{\phi}_a(x)$ and $\hat{\pi}_a(x)$ which we consider in the Heisenberg picture. These operators have sets of time-dependent eigenstates which fulfill the following eigenvalue equations

$$\hat{\phi}_a(x)\left|\phi_1,\ldots,\phi_N,t\right\rangle = \phi_a(x)\left|\phi_1,\ldots,\phi_N,t\right\rangle \ , \tag{2.36}$$

$$\hat{\pi}_a(x)\left|\pi_1,\ldots,\pi_N,t\right\rangle = \pi_a(x)\left|\pi_1,\ldots,\pi_N,t\right\rangle \ . \tag{2.37}$$

The time-dependence of the above eigenstates is realized by the action of the so-called time evolution operator $\hat{U}$ on the states

$$|\phi_1,\ldots,\phi_N,t'\rangle = \hat{U}(t,t')\,|\phi_1,\ldots,\phi_N,t\rangle \ , \tag{2.38}$$

$$|\pi_1,\ldots,\pi_N,t'\rangle = \hat{U}(t,t')\,|\pi_1,\ldots,\pi_N,t\rangle \ , \tag{2.39}$$

where

$$\hat{U}(t',t) = e^{-i\hat{H}(\hat{\vec{\phi}},\hat{\vec{\pi}})(t'-t)} \ . \tag{2.40}$$

Furthermore, the sets of eigenstates of the Heisenberg field operators are complete and orthonormal, i.e., they satisfy

$$\int \prod_{\mathbf{r}} \mathrm{d}\vec{\phi}(\mathbf{r})\,|\phi_1,\ldots,\phi_N,t\rangle\,\langle\phi_1,\ldots,\phi_N,t| = \mathbb{1} \ , \tag{2.41}$$

$$\langle\phi_{1,i},\ldots,\phi_{N,i},t_i|\phi_{1,j},\ldots,\phi_{N,j},t_j\rangle = \delta\left[\vec{\phi}_i(x)-\vec{\phi}_j(x)\right] \ , \tag{2.42}$$

and

$$\int \prod_{\mathbf{r}} \frac{\mathrm{d}\vec{\pi}(\mathbf{r})}{2\pi}\,|\pi_1,\ldots,\pi_N,t\rangle\,\langle\pi_1,\ldots,\pi_N,t| = \mathbb{1} \ , \tag{2.43}$$

$$\langle\pi_{1,i},\ldots,\pi_{N,i},t_i|\pi_{1,j},\ldots,\pi_{N,j},t_j\rangle = \delta\left[\vec{\pi}_i(x)-\vec{\pi}_j(x)\right] \ , \tag{2.44}$$

where $\mathrm{d}\vec{\phi}(x) \equiv \mathrm{d}\phi_1(x)\ldots\mathrm{d}\phi_N(x)$ and $\mathrm{d}\vec{\pi}(x) \equiv \mathrm{d}\pi_1(x)\ldots\mathrm{d}\pi_N(x)$. At this point, it should be mentioned that the completeness relations (2.41) and (2.43) are also fulfilled for the time-independent eigenstates $|\phi_1,\ldots,\phi_N,0\rangle$ and $|\pi_1,\ldots,\pi_N,0\rangle$. Finally, the overlap of these eigenstates is given by

$$\langle\phi_1,\ldots,\phi_N,0|\pi_1,\ldots,\pi_N,0\rangle = \exp\left\{i\int \mathrm{d}^3\mathbf{r}\ \pi_a(\mathbf{r})\phi^a(\mathbf{r})\right\} \ . \tag{2.45}$$

After these preliminaries, we come to the starting point of the derivation of the functional integral for the theory (2.33), which is given by the transition amplitude of an initial field configuration at t_i to a final field configuration at t_f,

$$\langle\phi_{1,f},\ldots,\phi_{N,f},t_f|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle \ . \tag{2.46}$$

Now, we decompose the time interval $[t_i,t_f]$ into $n+1$ time slices of the same length

$$t_k = t_i + k\tau \ , \qquad k = 1,\ldots,n \ , \tag{2.47}$$

and insert a complete set of field operator eigenstates at each of the above grid points into Eq. (2.46)

$$\begin{aligned}
&\langle\phi_{1,f},\ldots,\phi_{N,f},t_f|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle \\
&= \int \prod_{j=1}^{n} \mathrm{d}\vec{\phi}_j(\mathbf{r})\ \langle\phi_{1,f},\ldots,\phi_{N,f},t_f|\phi_{1,n},\ldots,\phi_{N,n},t_n\rangle\cdots\langle\phi_{1,1},\ldots,\phi_{N,1},t_1|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle \\
&= \int \prod_{j=1}^{n} \mathrm{d}\vec{\phi}_j(\mathbf{r})\ \langle\phi_{1,f},\ldots,\phi_{N,f},0|\hat{U}(t_f,t_n)|\phi_{1,n},\ldots,\phi_{N,n},0\rangle\cdots\langle\phi_{1,1},\ldots,\phi_{N,1},0|\hat{U}(t_1,t_i)|\phi_{1,i},\ldots,\phi_{N,i},0\rangle \ ,
\end{aligned} \tag{2.48}$$

where we used the time evolution of the field operator eigenstates in the last line and the convolution property of the time evolution operator. The next step of the calculation shows an important property of the functional integral, therefore we perform this step in detail. To this end, we pick out one of the transition amplitudes of the above expression and rewrite it as follows

$$\begin{aligned}
&\langle\phi_{1,l+1},\ldots,\phi_{N,l+1},0|\hat{U}(t_{l+1},t_l)|\phi_{1,l},\ldots,\phi_{N,l},0\rangle \\
&= \langle\phi_{1,l+1},\ldots,\phi_{N,l+1},0|1-i\hat{H}(\hat{\vec{\phi}},\hat{\vec{\pi}})\tau|\phi_{1,l},\ldots,\phi_{N,l},0\rangle \\
&= \langle\phi_{1,l+1},\ldots,\phi_{N,l+1},0|\phi_{1,l},\ldots,\phi_{N,l},0\rangle \\
&\quad - i\tau\int\prod_{\mathbf{r}}\frac{\mathrm{d}\pi_l(\mathbf{r})}{2\pi}\ \langle\phi_{1,l+1},\ldots,\phi_{N,l+1},0|\pi_{1,l},\ldots,\pi_{N,l},0\rangle\,\langle\pi_{1,l},\ldots,\pi_{N,l},0|\hat{H}(\hat{\vec{\phi}},\hat{\vec{\pi}})|\phi_{1,l},\ldots,\phi_{N,l},0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \phi_{1,l+1}, \dots, \phi_{N,l+1}, 0 | \phi_{1,l}, \dots, \phi_{N,l}, 0 \rangle \\
&\quad - i\tau \int \prod_{\mathbf{r}} \frac{\mathrm{d}\pi_l(\mathbf{r})}{2\pi} \, \langle \phi_{1,l+1}, \dots, \phi_{N,l+1}, 0 | \pi_{1,l}, \dots, \pi_{N,l}, 0 \rangle \, \langle \pi_{1,l}, \dots, \pi_{N,l}, 0 | H(\vec{\phi}_l, \vec{\pi}_l) | \phi_{1,l}, \dots, \phi_{N,l}, 0 \rangle \\
&= \delta\left[\vec{\phi}_{l+1}(\mathbf{r}) - \vec{\phi}_l(\mathbf{r})\right] - i\tau \int \prod_{\mathbf{r}} \frac{\mathrm{d}\pi_l(\mathbf{r})}{2\pi} \, \exp\left\{ i \int \mathrm{d}^3\mathbf{r} \, \pi_{a,l} \left(\phi^a_{l+1} - \phi^a_l\right) \right\} H(\vec{\phi}_l, \vec{\pi}_l) \;, \qquad (2.49)
\end{aligned}
$$

where we inserted a complete set of momentum density eigenstates and used the overlap (2.45) and the fact that the field operator eigenstates are orthonormal. The important step of the above calculation happens in the penultimate line: There, we use that the Hamilton operator is a function of the field operator $\hat{\phi}_a$ and the conjugate momentum density operator $\hat{\pi}_a$. Then, using the eigenvalue equations (2.36) and (2.37) of these operators, we observe that the action of the Hamilton operator on the eigenstate $|\phi_{1,l}, \dots, \phi_{N,l}, 0\rangle$ projects the respective eigenvalues out of the Hilbert-space vector. This transforms the Hamilton operator into the classical Hamiltonian $H(\vec{\phi}_l, \vec{\pi}_l)$. In order to compactify the result (2.49), we use the exponential representation of the functional delta-distribution

$$
\delta\left[\vec{\phi}_i(\mathbf{r}) - \vec{\phi}_j(\mathbf{r})\right] = \int \prod_{\mathbf{r}} \frac{\mathrm{d}\pi(\mathbf{r})}{2\pi} \, \exp\left\{ i \int \mathrm{d}^3\mathbf{r} \, \pi_a \left(\phi^a_i - \phi^a_j\right) \right\} \;, \qquad (2.50)
$$

so that we finally obtain

$$
\begin{aligned}
&\langle \phi_{1,l+1}, \dots, \phi_{N,l+1}, 0 | \hat{U}(t_{l+1}, t_l) | \phi_{1,l}, \dots, \phi_{N,l}, 0 \rangle \\
&= \int \prod_{\mathbf{r}} \frac{\mathrm{d}\pi_l(\mathbf{r})}{2\pi} \, \exp\left\{ i \int \mathrm{d}^3\mathbf{r} \, \pi_{a,l} \left(\phi^a_{l+1} - \phi^a_l\right) \right\} \Big(1 - i\tau H(\vec{\phi}_l, \vec{\pi}_l) \Big) \qquad (2.51)
\end{aligned}
$$

Now, we can use the above result for each of the transition amplitudes in Eq. (2.48) and find

$$
\begin{aligned}
&\langle \phi_{1,f}, \dots, \phi_{N,f}, t_f | \phi_{1,i}, \dots, \phi_{N,i}, t_i \rangle \\
&= \int \prod_{j=1}^{n} \mathrm{d}\vec{\phi}_j(\mathbf{r}) \prod_{j=0}^{n} \frac{\mathrm{d}\vec{\pi}_j}{2\pi} \, \exp\left\{ i \int \mathrm{d}^3\mathbf{r} \sum_{j=0}^{n} \pi_{a,j} \left(\phi^a_{j+1} - \phi^a_j\right) \right\} \prod_{j=0}^{n} \Big(1 - i\tau H(\vec{\phi}_j, \vec{\pi}_j) \Big) \\
&= \int \prod_{j=1}^{n} \mathrm{d}\vec{\phi}_j(\mathbf{r}) \prod_{j=0}^{n} \frac{\mathrm{d}\vec{\pi}_j}{2\pi} \, \exp\left\{ i\tau \int \mathrm{d}^3\mathbf{r} \left[\sum_{j=0}^{n} \pi_{a,j} \frac{\phi^a_{j+1} - \phi^a_j}{\tau} - \mathscr{H}(\vec{\phi}_j, \vec{\pi}_j) \right] \right\} \\
&\overset{n\to\infty,}{\underset{\tau\to 0}{=}} \int \mathscr{D}\vec{\phi}(x) \mathscr{D}\vec{\pi}(x) \, \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}^4x \, \left[\pi_a(x) \partial_0 \phi^a(x) - \mathscr{H}(\vec{\phi}, \vec{\pi}) \right] \right\} . \qquad (2.52)
\end{aligned}
$$

The last line of Eq. (2.52) requires some explanation: The limits $n \to \infty$ and $\tau \to 0$ transform the sum over j in the exponential into an integral in the Riemannian sense. The latter limit also makes sure that the quotient of the $\vec{\phi}$ fields becomes a partial derivative with respect to the time variable. Furthermore, the functional integral measures are defined as

$$
\mathscr{D}\vec{\phi}(x) \equiv \lim_{\substack{n\to\infty \\ \tau\to 0}} \prod_{j=1}^{n} \prod_{\mathbf{r}} \mathrm{d}\vec{\phi}(\mathbf{r}) \;, \qquad \mathscr{D}\vec{\pi}(x) \equiv \lim_{\substack{n\to\infty \\ \tau\to 0}} \prod_{j=0}^{n} \prod_{\mathbf{r}} \frac{\mathrm{d}\vec{\pi}(\mathbf{r})}{2\pi} \;. \qquad (2.53)
$$

The result (2.52) is often called Phase-Space Path Integral. But the Phase-Space Path Integral of the scalar field theory (2.33) can also be brought into another form. To this end, we notice that the Hamiltonian density of the theory, Eq. (2.35), only contains terms which are at maximum of quadratic powers in the conjugate momenta $\pi_a(x)$. This means that the functional integral over the conjugate momenta is simple given by a functional generalization of a Gaussian integral, compare [6.2]. We find

$$
\begin{aligned}
&\langle \phi_{1,f}, \dots, \phi_{N,f}, t_f | \phi_{1,i}, \dots, \phi_{N,i}, t_i \rangle \\
&= \int \mathscr{D}\vec{\phi}(x) \mathscr{D}\vec{\pi}(x) \, \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}^4x \, \left[\pi_a(x) \partial_0 \phi^a(x) - \mathscr{H}(\vec{\phi}, \vec{\pi}) \right] \right\} \\
&= \int \mathscr{D}\vec{\phi}(x) \mathscr{D}\vec{\pi}(x) \, \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}^4x \, \left[-\frac{1}{2} \left(\pi_a(x) - \partial_0 \phi_a(x)\right)^2 + \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)) \right] \right\} \\
&= \mathcal{N} \int \mathscr{D}\vec{\phi}(x) \, \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}^4x \, \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)) \right\} , \qquad (2.54)
\end{aligned}
$$

where we absorbed the value of the Gaussian functional integral in the normalization constant $\mathcal{N}$. This result is often referred to as Feynman form of the functional integral of the scalar field theory.

In general, one is not only interested in the transition amplitudes of a quantum field theory, but also in the expectation values of the field operators. Especially, the time-ordered vacuum expectation values of arbitrary products of field operators are of particular importance, since they correspond to the n-point Greens functions of the theory. In the following, we want to derive an expression for the n-point functions of the scalar field theory (2.33). It will be shown that all n-point functions of the theory can be collected in only one so-called generating functional. The concept of a generating functional guarantees that the n-point functions of a theory take a rather simple form in the functional formulation of Quantum Field Theory. Then, for later purposes, we will also introduce two other kinds of generating functionals, which are able to generate only special types of n-point functions. In order to derive a functional integral expression of the n-point functions of our scalar field theory, we consider the following time-ordered expectation value

$$\langle\phi_{1,f},\ldots,\phi_{N,f},t_f|T\left\{\hat{\phi}^{k_1}_{a_1}\cdots\hat{\phi}^{k_m}_{a_m}\right\}|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle \ , \tag{2.55}$$

where T denotes the time-ordering operator. In the above expression, the indices a_j label the different scalar fields in field space, so that $a_j\in[1,N]$ for all $j=1,\ldots,m$. Furthermore, we decompose the time interval $[t_i,t_f]$ according to Eq. (2.47) and choose the time grid points in way, so that m of them match with the time arguments k_j, $j=1,\ldots,m$ of the field operators. In order to simplify the above expectation value, we choose a distinct time-ordering and assume $k_1>k_2>\ldots>k_m$, so that Eq. (2.55) becomes

$$\langle\phi_{1,f},\ldots,\phi_{N,f},t_f|\hat{\phi}^{k_1}_{a_1}\cdots\hat{\phi}^{k_m}_{a_m}|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle \ . \tag{2.56}$$

According to the previous derivation, we insert complete sets of time-dependent field operator eigenstates, so that the above expectation value becomes

$$\begin{aligned}
&\langle\phi_{1,f},\ldots,\phi_{N,f},t_f|\hat{\phi}^{k_1}_{a_1}\cdots\hat{\phi}^{k_m}_{a_m}|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle\\
&=\int\prod_{j=1}^{n}\mathrm{d}\vec{\phi}_j\ \langle\phi_{1,f},\ldots,\phi_{N,f},t_f|\phi_{1,n},\ldots,\phi_{N,n},t_n\rangle\cdots\langle\phi_{1,k_1+1},\ldots,\phi_{N,k_1+1},t_{k_1+1}|\hat{\phi}^{k_1}_{a_1}|\phi_{1,k_1},\ldots,\phi_{N,k_1},t_{k_1}\rangle\\
&\quad\cdots\langle\phi_{1,k_m+1},\ldots,\phi_{N,k_m+1},t_{k_m+1}|\hat{\phi}^{k_m}_{a_m}|\phi_{1,k_m},\ldots,\phi_{N,k_m},t_{k_m}\rangle\cdots\langle\phi_{1,1},\ldots,\phi_{N,1},t_1|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle\\
&=\int\prod_{j=1}^{n}\mathrm{d}\vec{\phi}_j\ \phi^{k_1}_{a_1}\ldots\phi^{k_m}_{a_m}\ \langle\phi_{1,f},\ldots,\phi_{N,f},0|\hat{U}(t_f,t_n)|\phi_{1,n},\ldots,\phi_{N,n},0\rangle\\
&\quad\cdots\langle\phi_{1,1},\ldots,\phi_{N,1},0|\hat{U}(t_1,t_i)|\phi_{1,i},\ldots,\phi_{N,i},0\rangle \ ,
\end{aligned}\tag{2.57}$$

where we used the eigenvalue equation (2.36). According to the discussion of Eq. (2.49), it should be emphasized that the fields in front of the matrix elements in Eq. (2.58) are classical field quantities which of course commute. An important consequence of this insight is that any other time-ordering of the operators in Eq. (2.56) would yield the same result (2.57). This means that the time-ordering of the operators is always encoded in the functional integral. Returning to the above result, we recognize that, apart from the product of the field operator eigenvalues, the expressions in Eqs. (2.48) and (2.57) are identical, so that we can follow the same calculational steps as in the previous derivation. Inserting conjugate momentum eigenstates into each of the matrix elements in Eq. (2.57), summarizing the exponentials and taking the limits $n\to\infty$ and $\tau\to 0$ yields

$$\begin{aligned}
&\langle\phi_{1,f},\ldots,\phi_{N,f},t_f|T\left\{\hat{\phi}^{k_1}_{a_1}\cdots\hat{\phi}^{k_m}_{a_m}\right\}|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle\\
&=\mathcal{N}\int\mathscr{D}\vec{\phi}(x)\ \phi_{a_1}(x_{k_1})\cdots\phi_{a_m}(x_{k_m})\ \exp\left\{i\int_{t_i}^{t_f}\mathrm{d}^4x\ \mathscr{L}(\phi_a(x),\partial_\mu\phi_a(x))\right\} .
\end{aligned}\tag{2.58}$$

In order to derive an expression for the n-point functions of the theory, we return to the expectation value (2.56) and rewrite it in a different way. Now, we insert two sets of energy eigenstates of the Hamilton operator of the theory

$$\begin{aligned}
&\langle\phi_{1,f},\ldots,\phi_{N,f},t_f|T\left\{\hat{\phi}^{k_1}_{a_1}\cdots\hat{\phi}^{k_m}_{a_m}\right\}|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle\\
&=\sum_{n,m}\langle\phi_{1,f},\ldots,\phi_{N,f},t_f|n\rangle\ \langle n|T\left\{\hat{\phi}^{k_1}_{a_1}\cdots\hat{\phi}^{k_m}_{a_m}\right\}|m\rangle\ \langle m|\phi_{1,i},\ldots,\phi_{N,i},t_i\rangle\\
&=\sum_{n,m}\langle\phi_{1,f},\ldots,\phi_{N,f},0|\hat{U}(t_f,0)|n\rangle\ \langle n|T\left\{\hat{\phi}^{k_1}_{a_1}\cdots\hat{\phi}^{k_m}_{a_m}\right\}|m\rangle\ \langle m|\hat{U}(0,t_i)|\phi_{1,i},\ldots,\phi_{N,i},0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n,m} e^{-i(E_n t_f - E_m t_i)} \langle \phi_{1,f}, \ldots, \phi_{N,f}, 0 | n \rangle \langle n | T \left\{ \hat{\phi}^{k_1}_{a_1} \cdots \hat{\phi}^{k_m}_{a_m} \right\} | m \rangle \langle m | \phi_{1,i}, \ldots, \phi_{N,i}, 0 \rangle \\
&= \sum_{n,m} e^{-(E_n \tau_f - E_m \tau_i)} \langle \phi_{1,f}, \ldots, \phi_{N,f}, 0 | n \rangle \langle n | T \left\{ \hat{\phi}^{k_1}_{a_1} \cdots \hat{\phi}^{k_m}_{a_m} \right\} | m \rangle \langle m | \phi_{1,i}, \ldots, \phi_{N,i}, 0 \rangle \ ,
\end{aligned} \tag{2.59}
$$

where we performed an analytic continuation to imaginary times $t \to -i\tau$ in the last line. This mathematical trick is often called Wick rotation and we will study it in more detail in Chapter [4.1]. The necessity of this approach becomes evident, when we try to study the behavior of Eq. (2.59) for large time arguments, i.e., for $t_f \to \infty$ and $t_i \to -\infty$. Without the Wick rotation, we would obtain a strongly oscillating exponential factor and it would not be clear, how to interpret this result. The Wick rotation transforms the oscillating exponential into an exponentially damped factor. It is obvious that, in the limits $t_f \to \infty$ and $t_i \to -\infty$, the largest contribution to Eq. (2.59) comes from the term $n = m = 0$, i.e., from the ground state $|0\rangle \equiv |\Omega\rangle$. Therefore, we find

$$
\begin{aligned}
&\lim_{\substack{\tau_i \to -\infty, \\ \tau_f \to \infty}} \langle \phi_{1,f}, \ldots, \phi_{N,f}, -i\tau_f | T \left\{ \hat{\phi}^{k_1}_{a_1} \cdots \hat{\phi}^{k_m}_{a_m} \right\} | \phi_{1,i}, \ldots, \phi_{N,i}, -i\tau_i \rangle \\
&= e^{-E_\Omega(\tau_f - \tau_i)} \langle \phi_{1,f}, \ldots, \phi_{N,f}, 0 | \Omega \rangle \langle \Omega | T \left\{ \hat{\phi}^{k_1}_{a_1} \cdots \hat{\phi}^{k_m}_{a_m} \right\} | \Omega \rangle \langle \Omega | \phi_{1,i}, \ldots, \phi_{N,i}, 0 \rangle \ .
\end{aligned} \tag{2.60}
$$

Obviously, we found an expression which contains the vacuum expectation value of a time-ordered product of field operators. In order to find an expression for the n-point functions of the theory, we have to isolate this factor. To this end, we return to the transition amplitude (2.46) and rewrite it in the same way as Eq. (2.56)

$$
\begin{aligned}
&\langle \phi_{1,f}, \ldots, \phi_{N,f}, t_f | \phi_{1,i}, \ldots, \phi_{N,i}, t_i \rangle \\
&= \sum_n \langle \phi_{1,f}, \ldots, \phi_{N,f}, t_f | n \rangle \langle n | \phi_{1,i}, \ldots, \phi_{N,i}, t_i \rangle \\
&= \sum_n e^{-iE_n(t_f - t_i)} \langle \phi_{1,f}, \ldots, \phi_{N,f}, 0 | n \rangle \langle n | \phi_{1,i}, \ldots, \phi_{N,i}, 0 \rangle \\
&= \sum_n e^{-E_n(\tau_f - \tau_i)} \langle \phi_{1,f}, \ldots, \phi_{N,f}, 0 | n \rangle \langle n | \phi_{1,i}, \ldots, \phi_{N,i}, 0 \rangle \\
&\overset{\substack{\tau_i \to -\infty, \\ \tau_f \to \infty}}{=} e^{-E_\Omega(\tau_f - \tau_i)} \langle \phi_{1,f}, \ldots, \phi_{N,f}, 0 | \Omega \rangle \langle \Omega | \phi_{1,i}, \ldots, \phi_{N,i}, 0 \rangle \ .
\end{aligned} \tag{2.61}
$$

Comparing the results (2.60) and (2.61), we observe that the n-point functions can be written as

$$
\begin{aligned}
G^{(n)}_{a_1 \ldots a_n}(x_1, \ldots, x_n) &\equiv \langle \Omega | T \left\{ \phi(x_1) \cdots \phi(x_n) \right\} | \Omega \rangle \\
&= \lim_{\substack{t_i \to -\infty, \\ t_f \to \infty}} \frac{\langle \phi_{1,f}, \ldots, \phi_{N,f}, t_f | T \left\{ \hat{\phi}_{a_1}(x_1) \cdots \hat{\phi}_{a_n}(x_n) \right\} | \phi_{1,i}, \ldots, \phi_{N,i}, t_i \rangle}{\langle \phi_{1,f}, \ldots, \phi_{N,f}, t_f | \phi_{1,i}, \ldots, \phi_{N,i}, t_i \rangle} \\
&= \lim_{\substack{t_i \to -\infty, \\ t_f \to \infty}} \frac{\int \mathscr{D}\vec{\phi}(x) \ \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}^4 x \ \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)) \right\}}{\int \mathscr{D}\vec{\phi}(x) \ \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}^4 x \ \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)) \right\}} \\
&= \frac{\int \mathscr{D}\vec{\phi}(x) \ \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \exp\left\{ i \int \mathrm{d}^4 x \ \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)) \right\}}{\int \mathscr{D}\vec{\phi}(x) \ \exp\left\{ i \int \mathrm{d}^4 x \ \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)) \right\}} \ ,
\end{aligned} \tag{2.62}
$$

where we relabeled the field-space and time indices, $m \to n$. At this point it is common to introduce a so-called generating functional for all n-point functions of the theory. The basic idea of introducing a generating functional arises from the observation that the product of the classical fields in the numerator of Eq. (2.62) can simply be generated by functional derivatives. In order to see this, we define the vacuum-to-vacuum transition amplitude in the presence of external sources

$$
Z[\vec{J}] = \langle \Omega | \Omega \rangle_{\vec{J}} = \mathcal{N} \int \mathscr{D}\vec{\phi}(x) \ \exp\left\{ i \int \mathrm{d}^4 x \ \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)) + J_a(x) \phi^a(x) \right\} , \tag{2.63}
$$

where we introduced the N-dimensional vector $\vec{J}(x) = (J_1(x), \ldots, J_N(x))^T$. The components of this vector are classical sources which describe perturbations of the quantum system. Furthermore, the normalization constant $\mathcal{N}$ can be fixed by the normalization requirement $Z[\vec{J} = \vec{0}] = 1$. It is now easy to see that the functional derivatives of $Z[\vec{J}]$ with respect to the classical sources at $\vec{J} = \vec{0}$ yield Eq. (2.62). Therefore, we conclude that all n-point functions of the theory can be obtained from the vacuum-to-vacuum amplitude (2.63) by taking functional derivatives with respect to the external sources

$$G^{(n)}_{a_1\ldots a_n}(x_1,\ldots,x_n) = (-i)^n \left.\frac{\delta^n Z[\vec{J}]}{\delta J^{a_1}(x)\ldots\delta J^{a_n}(x)}\right|_{\vec{J}=\vec{0}} . \tag{2.64}$$

Due to the property that Eq. (2.63) includes all n-point functions of the theory, i.e., it is the sum of all vacuum-to-vacuum amplitudes in the presence of the classical sources $J_a(x)$, and since it describes a functional of the external sources, it is often called generating functional of all n-point functions or simply generating functional. It is also possible to define another generating functional, denoted as $W[\vec{J}]$, which is the sum of all connected diagrams of the theory. The relation between $W[\vec{J}]$ and $Z[\vec{J}]$ is given by

$$iW[\vec{J}] = \ln(Z[\vec{J}]) . \tag{2.65}$$

This relation can be understood as follows. As mentioned above, the generating functional $Z[\vec{J}]$ is the sum of all diagrams. Now, a general diagram can be written as $diagram = \prod_j (1/n_j!)(connected)_j^{n_j}$, where the variable n_j counts how often a connected subdiagram $(connected)_j$ appears in $diagram$. The factor $1/n_j!$ is the so-called symmetry factor, associated with the subdiagram $(connected)_j$. If we insert this expression in $Z[\vec{J}] \propto \sum_{\{n_j\}} diagram$ and use the definition $W[\vec{J}] = \sum_j (connected)_j$, we immediately arrive at Eq. (2.65). This functional generates the so-called connected n-point functions by taking the functional derivatives of Eq. (2.65) with respect to the classical sources

$$G^{(n),connected}_{a_1\ldots a_n}(x_1,\ldots,x_n) = (-i)^{n-1} \left.\frac{\delta^n W[\vec{J}]}{\delta J^{a_1}(x)\ldots\delta J^{a_n}(x)}\right|_{\vec{J}=\vec{0}} . \tag{2.66}$$

There also exists another special type of diagrams, the so-called one-particle irreducible (1PI) diagrams. Under this term one understands diagrams which are connected in a non-trivial way. In plain language, this means that the diagram is still connected, even if we truncate an arbitrary inner line. Figures [2.1] and [2.2] show simple examples for 1PI diagrams and diagrams which are not one-particle irreducible. As mentioned

(a) (b)

Figure 2.1: The diagrams (a) and (b) show 1PI vacuum corrections to the free 4-point function in φ^4-theory. Obviously, it is possible to truncate an arbitrary inner line without decomposing the diagrams (a) and (b) into disjoint subdiagrams.

(a) (b)

Figure 2.2: The diagrams (a) and (b) show vacuum corrections to the free 2-point function in φ^4-theory, which are not one-particle irreducible. It is evident that we are able to truncate an inner line, so that the diagrams (a) and (b) decompose into disjoint subdiagrams.

before, the 1PI diagrams are special cases of connected diagrams. Therefore, we expect that the generating functional for those diagrams can be obtained from $W[\vec{J}]$ or is at least proportional to $W[\vec{J}]$. In order to construct the generating functional for 1PI diagrams, we start with the definition

$$\varphi_a(x) = \frac{\delta W[\vec{J}]}{\delta J^a(x)} \tag{2.67}$$

of the so-called classical field. The field (2.67) is given by the vacuum expectation value of the field operator $\phi_a(x)$ in the presence of the classical source $J_a(x)$. The designation as classical field can be motivated by an explicit example. To this end, we consider the free Klein-Gordon theory. For this type of theory, the generating functional for connected Green's functions can be explicitly computed as

$$W[J] = -\frac{1}{2}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\ J\left(x_1\right)\Delta_F\left(x_1-x_2\right)J\left(x_2\right)\,. \tag{2.68}$$

Inserting this result into Eq. (2.67) we obtain

$$\begin{aligned}\varphi(x) &= -\frac{1}{2}\frac{\delta}{\delta J(x)}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\ J\left(x_1\right)\Delta_F\left(x_1-x_2\right)J\left(x_2\right)\\ &= -\frac{1}{2}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\ \left\{\frac{\delta J(x_1)}{\delta J(x)}\Delta_F(x_1-x_2)J(x_2) + J(x_1)\Delta_F(x_1-x_2)\frac{\delta J(x_2)}{\delta J(x)}\right\}\\ &= -\int \mathrm{d}^4z\ \Delta_F(x-z)J(z)\,. \end{aligned}\tag{2.69}$$

Next, we insert Eq. (2.69) into the classical equation of motion, i.e., in the Klein-Gordon equation. We find

$$\left(\Box+m^2\right)\varphi(x) = -\int \mathrm{d}^4z\ \left(\Box+m^2\right)\Delta_F(x-z)J(z) = J(x)\,, \tag{2.70}$$

where we used that the Feynman propagator $\Delta_F(x-z)$ is a Green function of the Klein-Gordon operator. Apparently, the classical field $\varphi(x)$ fulfills the classical equation of motion in the presence of the classical source $J(x)$.

Now, we come back to the general case of a theory containing N fields. If we assume the definition (2.67) to be invertible, the classical source $J_a(x)$ can be seen as an independent variable. It is therefore possible to eliminate $J_a(x)$ from the functional by expressing it in terms of the classical field $\varphi_a(x)$. Hence, we define the generating functional for 1PI diagrams as the functional Legendre transform of $W[\vec{J}]$

$$\Gamma[\vec{\varphi}] = W[\vec{J}] - \int \mathrm{d}^4x\ \varphi_a(x)J^a(x)\,. \tag{2.71}$$

By taking the functional derivative of Eq. (2.71) with respect to φ_a, we find another interesting relation

$$\begin{aligned}\frac{\delta\Gamma[\vec{\varphi}]}{\delta\varphi_a(x)} &= \frac{\delta W[\vec{J}]}{\delta\varphi_a(x)} - \int \mathrm{d}^4x'\ \left\{\frac{\delta\varphi_b(x')}{\delta\varphi_a(x)}J^b(x') + \varphi_b(x')\frac{\delta J^b(x')}{\delta\varphi_a(x)}\right\}\\ &= \int \mathrm{d}^4x'\ \left\{\frac{\delta W[\vec{J}]}{\delta J^b(x')}\frac{\delta J^b(x')}{\delta\varphi_a(x)} - \delta^{(4)}(x'-x)\delta^a_bJ^b(x') - \varphi_b(x')\frac{\delta J^b(x')}{\delta\varphi_a(x)}\right\}\\ &= -J^a(x)\,. \end{aligned}\tag{2.72}$$

The above relation shows that for a vanishing classical source $\vec{J}=\vec{0}$ the values for the classical fields are determined by the stationary points of the 1PI generating functional $\Gamma[\vec{\varphi}]$. This fact is comparable to the classical equations of motion which follow from the stationary points of the classical action, compare Eq. (2.7). Due to this similarity, the functional $\Gamma[\vec{\varphi}]$ is often referred to as quantum effective action. Of course, in close analogy to Eqs. (2.64) and (2.66), the 1PI functions can be obtained by taking the functional derivatives of $\Gamma[\vec{\varphi}]$ with respect to the classical fields $\varphi_a(x)$

$$\Gamma^{(n)}_{a_1\ldots a_n}(x_1,\ldots,x_n) = \left.\frac{\delta^n\Gamma[\vec{\varphi}]}{\delta\varphi^{a_1}(x_1)\ldots\delta\varphi^{a_n}(x_n)}\right|_{\vec{\varphi}=\vec{0}}\,. \tag{2.73}$$

Finally, we want to study the symmetries of the quantum effective action. More precisely, we want to find a condition which always holds when the effective action is invariant under a certain transformation of the fields. In the next section, we will use this relation as the starting point for the proof of the Goldstone theorem. To this end, we consider a transformation of the N fields of the form

$$\phi_a(x)\to\phi'_a(x) = \phi_a(x) + \epsilon\Omega_a\left[\phi_a, x\right]\,, \tag{2.74}$$

where Ω_a in general has a functional dependence on ϕ_a. Now, we consider the generating functional of all Green functions $Z[\vec{J}]$ and assume that the classical action as well as the path integral measure are both invariant under the transformation (2.74) of the fields, i.e.,

$$S[\phi'_a] = S[\phi_a], \qquad \mathscr{D}\vec{\phi}'(x) \equiv \prod_x \mathrm{d}\vec{\phi}(x) = \mathscr{D}\vec{\phi}(x)\,. \tag{2.75}$$

Inserting the transformed fields $\phi'_a(x)$ into $Z[\vec{J}]$ and using the assumptions (2.75), one obtains

$$\begin{aligned} Z'[\vec{J}] &= \mathcal{N} \int \mathscr{D}\phi' \ \exp\left\{ iS[\phi'_a] + i \int \mathrm{d}^4x \ J^a(x)\phi'_a(x) \right\} \\ &= \mathcal{N} \int \mathscr{D}\phi \ \exp\left\{ iS[\phi_a] + i \int \mathrm{d}^4x \ J^a(x)\phi_a(x) \right\} \exp\left\{ i\epsilon \int \mathrm{d}^4x \ \Omega_a[\phi_a, x] J^a(x) \right\} \\ &= Z[\vec{J}] + i\epsilon \int \mathscr{D}\phi \ \exp\left\{ iS[\phi_a] + i \int \mathrm{d}^4x \ J_a(x)\phi^a(x) \right\} \int \mathrm{d}^4x \ \Omega_a[\phi_a, x] J^a(x) \\ &= Z[\vec{J}] + \delta Z[\vec{J}] \ , \end{aligned} \tag{2.76}$$

where we expanded the second exponential up to first order in ϵ. In order to be invariant under the transformation (2.74), the second term $\delta Z[\vec{J}]$ has to vanish exactly. Interchanging the space-time and the functional integration and dividing by $Z[\vec{J}]$, this condition can be written as

$$0 = \int \mathrm{d}^4x \ \langle \Omega_a[\phi_a, x] \rangle \, \frac{\delta \Gamma[\vec{\varphi}]}{\delta \varphi^a} \ , \tag{2.77}$$

where we used Eq. (2.72) and defined the average $\langle \Omega_a[\phi_a, x] \rangle$ in presence of the classical source $\vec{J}$ as

$$\langle \Omega_a[\phi_a, x] \rangle = \frac{\int \mathscr{D}\phi \ \exp\left\{ iS[\phi_a] + i \int \mathrm{d}^4x \ J_a(x)\phi^a(x) \right\} \Omega_a[\phi_a, x]}{Z[\vec{J}]} \ . \tag{2.78}$$

The invariance conditions (2.77) are often referred to as Slavnov-Taylor identities. Finally, we can add $\Gamma[\vec{\varphi}]/\epsilon$ on both sides of Eq. (2.77), so that

$$\Gamma[\varphi_a + \epsilon \, \langle \Omega_a[\phi_a, x] \rangle] = \Gamma[\varphi_a] \ , \tag{2.79}$$

where we neclected terms of order $\mathcal{O}(\epsilon^2)$. Obviously, the conditions (2.77) imply the invariance of the quantum effective action under transformations of the type

$$\varphi_a(x) \to \varphi'_a(x) = \varphi_a(x) + \epsilon \, \langle \Omega_a \left[\phi_a, x \right] \rangle \ . \tag{2.80}$$

It has to be taken into account, that the transformations (2.80) are in general not of the same type as the transformations (2.74) which we started with. Fortunately, the equivalence of Eqs. (2.74) and (2.80) holds for an important class of infinitesimal transformations, particularly for infinitesimal transformations which are linear in the fields

$$\Omega_a[\phi_a, x] = \omega_a(x) + \int \mathrm{d}^4z \ T_a{}^b(x, z)\phi_b(z) \ . \tag{2.81}$$

If we combine Eqs. (2.74) and (2.81), the resulting infinitesimal transformation is very similar to an infinitesimal element of a Lie group. For such a group the first term $\omega_a(x)$ vanishes and the matrix $T_a{}^b(x, z)$ is proportional to the product of a generator of the associated Lie group with a delta-distribution $T_a{}^b(x, z) \propto T_a{}^b \delta^{(4)}(x - z)$. For these transformations we are able to find a configuration for the fields $\phi_a(x)$ and the classical sources $J^a(x)$, so that $\langle \phi_a(x) \rangle = \varphi_a(x)$ and therefore $\langle \Omega_a \left[\phi_a, x \right] \rangle = \Omega_a[\varphi_a, x]$.

2.2.2 Spontaneously Broken Symmetries and Degenerate Vacua

In the first section we saw that the invariance of the classical action of a theory under the elements of a continuous group involves a conserved Noether current (2.21) and therefore a conserved Noether charge (2.22). As seen in subsection [2.1.2], this holds for symmetries of space-time as well as for internal symmetries. But it shows that in nature not only the preserved symmetries play an important role, but also the explicitly and spontaneously broken symmetries. In this Subsection we will focus on the systematics of spontaneous symmetry breaking. As already mentioned in the introduction of this section, there are two important prerequisites which must be met. On the one hand, we will see that spontaneous symmetry breaking requires the existence of degenerate vacua. On the other hand, it turns out that an infinite space-time volume will be another important, even if subtle point. In order to simplify matters, we want to study these subjects by using the example of a simple theory containing real scalar fields with a quartic self-interaction. The Lagrangian of this theory is given by

$$\mathscr{L}(\varphi(x), \partial_\mu \varphi(x)) = \frac{1}{2} \left(\partial_\mu \varphi \right)^2 - \mathscr{V}(\varphi) \ , \qquad \mathscr{V}(\varphi) = \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 \ . \tag{2.82}$$

At first sight, Eq. (2.82) is identical to the Lagrangian of the familiar φ^4-theory. But in contrast to this theory, we do not want to fix the sign of the parameter m^2. On the other hand, in accordance with the usual φ^4 Lagrangian, we define $\lambda > 0$ so that the potential density is bounded from below. Obviously, the Lagrangian (2.82) is invariant under Z_2-transformations

$$\varphi(x) \xrightarrow{Z_2} \varphi'(x) = -\varphi(x) \ . \tag{2.83}$$

In general, the cyclic group of rank n, Z_n, is a discrete group describing the rotations of regular polygons with n directed sides. In the case of the field theory (2.82), we have to "rotate" the field $\varphi(x)$ twice, in order to transform it into itself. Therefore, the symmetry group of this theory is given by Z_2. Later in this discussion, we will also associate an operator $\mathcal{C}_\varphi$, acting on the Hilbert space of our system, with this symmetry transformation. But prior to that, we have to clarify the consequences of the sign of the constant m^2. To this end, we start with the case which is realized in ordinary φ^4-theory, i.e., $m^2 > 0$. The potential density for this case is pictured in Fig. [2.3(a)]. From this figure we observe that the choice $m^2 > 0$ leads

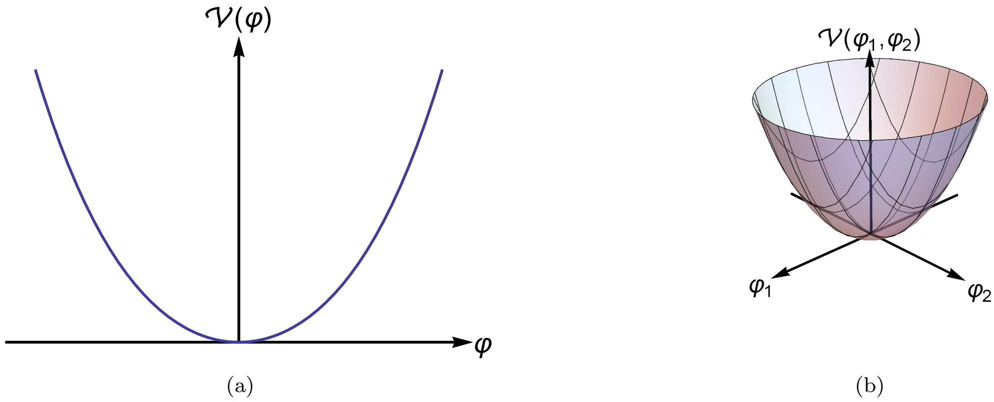


Figure 2.3: Figure [(a)] shows the potential density given in Eq. (2.82) for $m^2 > 0$. Figure [(b)] demonstrates the two dimensional version of Fig. [(a)], i.e., the Wigner-Weyl realization of Eq. (2.82).

to a unique minimum at $\varphi_0 = 0$. Of course, it is also possible to find this minimum by setting the first derivative of $\mathscr{V}(\varphi)$ with respect to φ to zero

$$\left.\frac{\mathrm{d}\mathscr{V}(\varphi)}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0} = m^2\varphi_0 + \frac{\lambda}{3!}\varphi_0^3 \stackrel{!}{=} 0 \ . \tag{2.84}$$

For $m^2 > 0$, the above condition (2.84) immediately leads to the minimum

$$\varphi_0 = 0 \ , \tag{2.85}$$

which underpins the observation from Fig. [2.3(a)]. After quantizing the theory (2.82), this unique minimum of the potential density will lead to a unique ground or vacuum state $|\Omega\rangle$. Figure [2.3(b)] shows the two-dimensional generalization of the potential density (2.82). Also in N-dimensional generalizations of this potential density, we will only obtain a single minimum at $\vec{\varphi} = \vec{0}$, where $\vec{\varphi} = (\varphi_1, \ldots, \varphi_N)^T$ and therefore a unique vacuum state. It has to be taken into account that in the case of $N \geq 2$ the discrete Z_2-symmetry (2.83) of our model becomes a continuous rotational symmetry. This continuous case is often referred to as the Wigner-Weyl realization of a symmetry, compare Fig. [2.3(b)]. As we will see in a moment, the uniqueness of the ground state will not lead to a spontaneous breaking of the Z_2-symmetry of our model. Therefore, we are not interested in this case and turn to the second case $m^2 < 0$. It should be noted that this sign convention of the parameter m^2 is not unphysical, since the Lagrangian (2.82) does not necessarily have to describe particles of mass m. A more detailed discussion of this point is given in Ref. [RQFT]. In the continuous case, this choice of sign is referred to as the Nambu-Goldstone realization of a symmetry. In this case, the condition (2.84) leads to three extreme points

$$\varphi_{0,1/2} = \pm\sqrt{\frac{-6m^2}{\lambda}} \equiv \pm\varphi_0, \qquad \varphi_{0,3} = 0 \ . \tag{2.86}$$

Checking the sufficient condition for the extreme points, it turns out that the point $\varphi_{0,3}$ corresponds to a local maximum of the potential density, while the points $\varphi_{0,1/2}$ describe the two distinct minima of the theory.

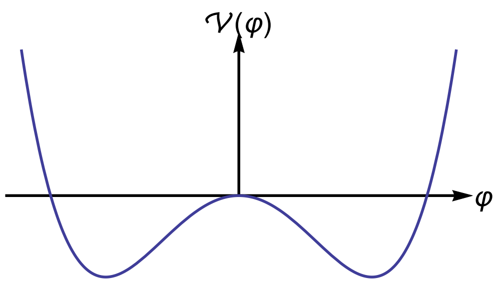
After quantizing the theory, these minima will correspond to two degenerate vacua $|\Omega, +\rangle$ and $|\Omega, -\rangle$, which are obviously distinguishable through the vacuum expectation values $+\varphi_0$ and $-\varphi_0$ of the field operator $\varphi(x)$. This situation and the corresponding two-dimensional case are depicted in Figs. [2.4(a)] and [(b)]. At this point, it is convenient to shift the field by one of its vacuum expectation values and introduce a new dynamical variable

$$\varphi(x) = \sigma(x) \pm \varphi_0 \; . \tag{2.87}$$

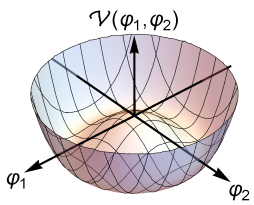
This new field variable describes the field fluctuations around the minima. Therefore, Eq. (2.87) corresponds to an expansion of the initial field variable around the minima of the potential density. The Lagrangian (2.82) can be rewritten in terms of the new fluctuation field $\sigma(x)$

$$\begin{aligned}
\mathscr{L}\left(\sigma(x), \partial_\mu \sigma(x)\right) &= \frac{1}{2}\left(\partial_\mu \sigma\right)^2 - \mathscr{V}(\sigma(x) \pm \varphi_0) \\
&= \frac{1}{2}\left(\partial_\mu \sigma\right)^2 - \frac{m^2}{2}\left(\sigma^2 + \varphi_0^2 \pm 2\sigma\varphi_0\right) - \frac{\lambda}{4!}\left(\sigma^4 \pm 4\sigma^3\varphi_0 + 6\sigma^2\varphi_0^2 \pm 4\sigma\varphi_0^3 + \varphi_0^4\right) \\
&= \frac{1}{2}\left(\partial_\mu \sigma\right)^2 - \frac{1}{2}\left(-2m^2\right)\sigma^2 \pm \frac{\lambda}{3!}\varphi_0\sigma^3 - \frac{\lambda}{4!}\sigma^4 - \frac{m^2}{2}\varphi_0^2 - \frac{\lambda}{4!}\varphi_0^4 \\
&= \frac{1}{2}\left(\partial_\mu \sigma\right)^2 - \frac{1}{2}m_\sigma^2\sigma^2 \pm \frac{\lambda}{3!}\varphi_0\sigma^3 - \frac{\lambda}{4!}\sigma^4 - \mathscr{V}(\varphi_0) \; ,
\end{aligned} \tag{2.88}$$

where we used Eqs. (2.84), (2.86) and defined the mass of the new particle σ as $m_\sigma = \sqrt{-2}m$. The theory in terms of the new dynamical field variable obviously develops not only the quartic self-interaction term, which was already present in Eq. (2.82), but also a new cubic self-interaction term. In addition to that, we obtain a new constant term which is given by the potential density evaluated at its minimum. However, this constant term is irrelevant for the dynamics of the σ-field, since it does not contribute to the equations of motion. Let us come back to the cubic term. This new term ensures that the new Lagrangian (2.88) is



(a)



(b)

Figure 2.4: Figure [(a)] shows the potential density given in Eq. (2.82) for $m^2 < 0$. Figure (b) demonstrates the two dimensional version of figure [(a)], i.e., the Nambu-Goldstone realization of Eq. (2.82).

not invariant under Z_2-transformations of the fluctuation field $\sigma(x) \xrightarrow{Z_2} \sigma'(x) = -\sigma(x)$. This observation could lead one to believe that the initial Z_2-symmetry of the Lagrangian might be broken. But in fact, the initial Z_2-symmetry $\varphi(x) \xrightarrow{Z_2} \varphi'(x) = -\varphi(x)$ is still present in the Lagrangian, even though realized in a different way. This can be seen when we translate the transformation behavior of the initial field $\varphi(x)$ in the language of the new fluctuation field $\sigma(x)$. Under Z_2 the fluctuation field transforms according to $\sigma(x) \xrightarrow{Z_2} -\sigma(x) \mp 2\varphi_0$. This shows us that the term "broken symmetry" is actually not true, since the initial symmetry is just hidden. Nevertheless, we will use this term frequently throughout this work. Furthermore, we recognize that the breakdown of the symmetry only occurs because the theory develops two degenerate vacua for the choice $m^2 < 0$ and selects one of them as the physical vacuum. For the first case $m^2 > 0$ we obtained only a single vacuum state. The vacuum expectation value of the field operator vanishes in this case. Therefore, a shift of the dynamical field would not develop a symmetry breaking term in the Lagrangian. In the next subsection, we will see that the spontaneous breakdown of a global continuous symmetry will have an important consequence, described by the Goldstone theorem. But before we come to this subject, we go back to another subtle point. In the discussion of Eq. (2.86) we naively assumed that the realized vacuum state of the theory is given by either $|\Omega, +\rangle$ or $|\Omega, -\rangle$. But, in a quantized theory it also should be possible to have a vacuum which is described by a superposition of both vacuum states $|\Omega,+\rangle$ and $|\Omega,-\rangle$, e.g.

$$|\tilde{\Omega},+\rangle = \frac{1}{\sqrt{2}}\left(|\Omega,+\rangle + |\Omega,-\rangle\right), \qquad |\tilde{\Omega},-\rangle = \frac{1}{\sqrt{2}}\left(|\Omega,+\rangle - |\Omega,-\rangle\right) \ . \tag{2.89}$$

In the following we want to show that such a vacuum (2.89) is not stable against small external perturbations. To this end, we introduce an operator $\mathcal{C}_\varphi$, associated with the symmetry transformation (2.83). This operator satisfies the following relations

$$\mathcal{C}_\varphi^2 = \mathbb{1}, \qquad \mathcal{C}_\varphi = \mathcal{C}_\varphi^{-1} = \mathcal{C}_\varphi^\dagger \ , \tag{2.90}$$

i.e., it is unitary and also hermitian. The first relation in Eq. (2.90) can be motivated by the fact that the successive action of two operators $\mathcal{C}_\varphi$ on a state vector "reflects" the transformed state onto the initial state. Therefore, the square of the operator must be given by the unit matrix. Multiplying this relation by $\mathcal{C}_\varphi^{-1}$ yields $\mathcal{C}_\varphi = \mathcal{C}_\varphi^{-1}$. Using this relation and considering the matrix element $\langle\varphi'|\mathcal{C}_\varphi|\varphi\rangle$, we are able to show the hermiticity and the unitarity of $\mathcal{C}_\varphi$. The action of this operator also implies that

$$\mathcal{C}_\varphi\,|\Omega,\pm\rangle = |\Omega,\mp\rangle \ , \tag{2.91}$$

which immediately follows from Eq. (2.83). Using this relation, we can show that the linear combinations $|\tilde{\Omega},\pm\rangle$ are invariant or up to a sign invariant under the Z_2-transformation (2.83)

$$\mathcal{C}_\varphi\,|\tilde{\Omega},\pm\rangle = \pm\,|\tilde{\Omega},\pm\rangle \ . \tag{2.92}$$

With these considerations we are able to study an infinitesimal external perturbation which is odd in the field variable $\varphi(x)$. The Hamiltonian describing this perturbation shall be given by $H_I \equiv \delta H_{pert}$, where δ is an infinitesimal coupling constant. The consequences of this perturbation can be studied in the framework of perturbation theory. In our case, the vacuum state $|\Omega\rangle$ is twice degenerate. Therefore, there exists a two-dimensional subspace which is formed by the degenerate vacuum states $|\tilde{\Omega},\pm\rangle$. We also assume that these vacuum states form an orthonormal basis of this subspace, which also implies that $|\Omega,\pm\rangle$ are orthonormal. It turns out, that the proof of this assumption requires an infinitely large spatial volume. For the explicit proof, see for example [Wei2]. Now, the state $|\Omega\rangle$ can be expanded in the orthonormal basis of the subspace

$$|\Omega\rangle = \sum_{j=+,-} c_j\,|\tilde{\Omega},j\rangle \ . \tag{2.93}$$

Up to first order in λ, Schrödinger's equation yields

$$0 = \sum_{j=+,-} c_j\left(H_{pert}^{ij} - E_\Omega^{(1)}\delta^{ij}\right) , \tag{2.94}$$

in which we defined $H_{pert}^{ij} = \langle\tilde{\Omega},i|H_{pert}|\tilde{\Omega},j\rangle$. This equation can be understood as a linear system of equations for the coefficients c_j. The non-trivial solutions of this system require that

$$\det\begin{pmatrix} H_{pert}^{++} - E_\Omega^{(1)} & H_{pert}^{+-} \\ H_{pert}^{-+} & H_{pert}^{++} - E_\Omega^{(1)} \end{pmatrix} = 0 \ . \tag{2.95}$$

In order to calculate the zeros of the characteristic polynomial of the above matrix it will be easier to first consider the matrix elements H_{pert}^{ij}. To this end, we start with the diagonal elements H_{pert}^{++} and H_{pert}^{--}

$$\begin{aligned} H_{pert}^{++} &= \langle\tilde{\Omega},+|H_{pert}|\tilde{\Omega},+\rangle = \langle\tilde{\Omega},+|\mathcal{C}_\varphi^\dagger\mathcal{C}_\varphi H_{pert}\mathcal{C}_\varphi^\dagger\mathcal{C}_\varphi|\tilde{\Omega},+\rangle = -H_{pert}^{++} \\ H_{pert}^{--} &= \langle\tilde{\Omega},-|H_{pert}|\tilde{\Omega},-\rangle = \langle\tilde{\Omega},-|\mathcal{C}_\varphi^\dagger\mathcal{C}_\varphi H_{pert}\mathcal{C}_\varphi^\dagger\mathcal{C}_\varphi|\tilde{\Omega},-\rangle = -H_{pert}^{--} \ , \end{aligned} \tag{2.96}$$

where we have used Eqs. (2.90), (2.92) and the fact that the Hamiltonian H_{pert} is odd in $\varphi(x)$. Obviously, these matrix elements have to vanish. The off-diagonal elements do not have to vanish in general. Defining $H_{pert}^{+-} = H_{pert}^{-+} \equiv \mathcal{E}$, $\mathcal{E} \in \mathbb{R}$ and inserting these results into Eq. (2.95), we obtain $E_\Omega^{(1)} = \pm\mathcal{E}$ and therefore up to first order in δ

$$E_{\tilde{\Omega}} = E_\Omega \pm \delta\mathcal{E} \pm \dots \ . \tag{2.97}$$

This calculation shows that the degenerate energy levels of the vacua are shifted by $\mathcal{E}$ in the presence of an infinitesimal perturbation which is odd in the field. To zeroth order in δ, we obtain the vacuum energies of $|\Omega,\pm\rangle$. Inserting the solutions of Eq. (2.95) into the linear system of equations, we are able to solve for the coefficients c_j. Using the normalization of the vacuum states, we obtain $c_+ = c_- = 1/\sqrt{2}$ for the eigenvalue $\mathcal{E}$ and $c_+ = -1/\sqrt{2}, c_- = 1/\sqrt{2}$ for $-\mathcal{E}$. In order to obtain the vacuum states which diagonalize the perturbed system up to zeroth order in δ, we have to insert these coefficients into Eq. (2.93). It turns out that these states are given by $|\Omega,\pm\rangle$. This calculation shows us that an external perturbation drives the vacuum state of the system into the vicinity of either $|\Omega,+\rangle$ or $|\Omega,-\rangle$. Which of these two vacua is chosen to be the "real" vacuum depends on the form of the perturbation H_{pert}.

2.2.3 The Goldstone Theorem

In the last Subsection we saw that the existence of degenerate vacua is a necessary condition for spontaneous symmetry breaking. In addition to that, we saw that the realized vacuum state of the physical system is not a superposition of all degenerate vacua, because such linear combinations are not stable against infinitesimal external perturbations. It was shown, that such perturbations always drive the system in the vicinity of a vacuum state which is not a superposition of the two vacua. Therefore, only one of the degenerate vacua becomes the physical vacuum state. We studied this phenomenon at the level of a simple bosonic quantum field theory (2.82) with a discrete internal Z_2-symmetry (2.83). Now, we want to extend this analysis and study a system with a continuous internal symmetry. Therefore, we define the vector

$$\vec{\varphi}(x) = (\varphi_1(x), \varphi_2(x))^T \ . \tag{2.98}$$

With this vector, we are able to construct a two-dimensional extension of the model (2.82). The Lagrangian of this model is then given by

$$\mathscr{L}(\varphi_1, \varphi_2, \partial_\mu\varphi_1, \partial_\mu\varphi_2) = \frac{1}{2}(\partial_\mu\varphi_i)(\partial^\mu\varphi^i) - \mathscr{V}(\varphi_1, \varphi_2) \ , \qquad \mathscr{V}(\varphi_1, \varphi_2) = \frac{m^2}{2}\varphi_i\varphi^i + \frac{\lambda}{4}(\varphi_i\varphi^i)^2 \ . \tag{2.99}$$

In contrast to the example of the last subsection, we immediately choose the Nambu-Goldstone realization of the potential density, i.e., $m^2 < 0$. The coupling constant λ is again chosen to be larger than zero, so that the theory is bounded from below. The potential density (2.99) is depicted in Fig. [2.4(b)]. Obviously, this model is invariant under $O(2)$-rotations in the internal φ-space. This transformation is given by

$$\varphi_i(x) \xrightarrow{O(2)} \varphi_i'(x) = O_{ij}\varphi^j(x) \ , \tag{2.100}$$

where $(O_{ij}) \in O(2)$. The group $O(2)$ describes the set of orthogonal (2×2)-matrices, which contains the special orthogonal group $SO(2)$ as a continuous subgroup. The specification "special" arises from the fact that the elements of $SO(2)$ satisfy $\det(O) = 1$, while in general an orthogonal matrix has a determinant of ± 1. The orthogonal group $O(2)$ also contains a Z_2 subgroup. This subgroup is important to realize space reflections. In general, for the set of orthogonal $(N \times N)$-matrices, we have $O(N) = SO(N) \times Z_2$. It is now easy to show that the Lagrangian (2.99) is invariant under the transformation (2.100)

$$\begin{aligned}
\mathcal{L}' &= \frac{1}{2}(\partial_\mu O^{ij}\varphi_j)(\partial^\mu O_{ik}\varphi^k) - \frac{m^2}{2}(O^{ij}\varphi_j)(O_{ik}\varphi^k) - \frac{\lambda}{4}\left[(O^{ij}\varphi_j)(O_{ik}\varphi^k)\right]^2 \\
&= \frac{1}{2}(\partial_\mu\varphi_j)O^{ji}O_{ik}(\partial^\mu\varphi^k) - \frac{m^2}{2}\varphi_j O^{ji}O_{ik}\varphi^k - \frac{\lambda}{4}\left[\varphi_j O^{ji}O_{ik}\varphi^k\right]^2 \\
&= \frac{1}{2}(\partial_\mu\varphi_i)(\partial^\mu\varphi^i) - \frac{m^2}{2}\varphi_i\varphi^i - \frac{\lambda}{4}(\varphi_i\varphi^i)^2 \ ,
\end{aligned} \tag{2.101}$$

where we used the fact that we consider global transformations, the relation $O^{ji}O_{ik} = \delta^j_{\ k}$, and finally renamed the summation index $k \to i$. Now, we have to determine the minima of the potential density. The necessary condition for an extremum gives the following conditions

$$\begin{aligned}
\frac{\partial\mathscr{V}}{\partial\varphi_1} &= \lambda\varphi_1\left[\frac{m^2}{\lambda} + \left(\varphi_1^2 + \varphi_2^2\right)\right] \stackrel{!}{=} 0 \ , \\
\frac{\partial\mathscr{V}}{\partial\varphi_2} &= \lambda\varphi_2\left[\frac{m^2}{\lambda} + \left(\varphi_1^2 + \varphi_2^2\right)\right] \stackrel{!}{=} 0 \ .
\end{aligned} \tag{2.102}$$

The above relations are obviously satisfied for

$$\vec{\varphi}_0 = \vec{0}, \qquad\qquad |\boldsymbol{\varphi}_0|^2 \equiv \varphi_0^2 = \varphi_{1,0}^2 + \varphi_{2,0}^2 = -\frac{m^2}{\lambda} \ . \tag{2.103}$$

In contrast to the last Subsection, we now obtain an infinite number of extrema, because the second condition in Eq. (2.103) defines a 1-sphere containing an infinite number of extrema. This situation is depicted in Fig. [2.5]. As a sufficient condition, we have to check the eigenvalues of the Hessian matrix $H\mathscr{V}(\varphi_1, \varphi_2)$ in the points (2.103). For the first extremum, given by $\vec{\varphi} = \vec{0}$, we obtain

$$H\mathscr{V}(0,0) = \begin{pmatrix} m^2 & 0 \\ 0 & m^2 \end{pmatrix} \ . \tag{2.104}$$

(a) (b)

Figure 2.5: The plots [(a)] and [(b)] show the 1-sphere of minima defined by Eq. (2.103).

Thus, the eigenvalues of this matrix are given by $\zeta_{1/2} = m^2 < 0$, so that the vector $(0,0)^T$ points at a maximum of the potential density. For the second condition of Eq. (2.103) we find

$$H\mathscr{V}(\varphi_{1,0},\varphi_{2,0}) = 2\lambda \begin{pmatrix} \varphi_{1,0}^2 & \varphi_{1,0}\varphi_{2,0} \\ \varphi_{1,0}\varphi_{2,0} & \varphi_{2,0}^2 \end{pmatrix} \ . \tag{2.105}$$

Using $\varphi_{1,0}^2 + \varphi_{2,0}^2 = -\frac{m^2}{\lambda}$, we obtain the following characteristic polynomial

$$p(\zeta) = \zeta\left(\zeta + \frac{m^2}{\lambda}\right) \ . \tag{2.106}$$

The zeros of this polynomial and therefore the eigenvalues of Eq. (2.105) are given by $\zeta_1 = 0$, $\zeta_2 = -m^2/\lambda > 0$. Apparently, the Hessian matrix is indefinite, so that we are not able to give a statement about the extrema. In order to study the properties of the extrema, we note that the potential density (2.99) is only a function of $\varphi_1^2 + \varphi_2^2$. This is of course not surprising, because the theory is invariant under $SO(2)$ rotations[2] in the (φ_1,φ_2)-plane, compare (2.101). Now, we are able to investigate the vicinity of the extrema, by adding an infinitesimal radius δr

$$\begin{aligned} \mathscr{V}(\varphi_{1,0}^2 + \varphi_{2,0}^2 + \delta r) &= \frac{m^2}{2}\left(-\frac{m^2}{\lambda} + \delta r\right) + \frac{\lambda}{4}\left(-\frac{m^2}{\lambda} + \delta r\right)^2 \\ &= -\frac{(m^2)^2}{4\lambda} + \frac{\lambda}{4}(\delta r)^2 \\ &> -\frac{(m^2)^2}{4\lambda} \\ &= \mathscr{V}(\varphi_{1,0}^2 + \varphi_{2,0}^2) \ . \end{aligned} \tag{2.107}$$

The above calculation shows that the points in the vicinity of the extrema in the radial direction lie above them. Obviously, the potential density (2.99) features an infinite number of minima. The condition (2.103) only fixes the radius on which the minima lie. An infinitesimal external perturbation will now drive the system into one of the minima, i.e., the perturbation selects the angle on the 1-sphere, describing the minima of the theory. It is now possible to choose the coordinate system of the internal space in a way that the selected vacuum is given by the vector

$$\vec{\varphi}_0 = (0,\varphi_0)^T \ , \tag{2.108}$$

where $\varphi_0 = \sqrt{-m^2/\lambda}$. In analogy to the discussion of Eq. (2.87), we are able to expand our initial fields around the minimum (2.108) and describe the system in terms of the new dynamical variables

$$\vec{\varphi}(x) = (\pi(x), \sigma(x) + \varphi_0)^T \ . \tag{2.109}$$

[2] As mentioned in the discussion of Eq. (2.101), the whole symmetry group of the theory is given by $O(2)$, but in the discussion of this subsection, it is sufficient to only deal with the $SO(2)$ symmetry.

The meaning of the new dynamical variables $\pi(x)$ and $\sigma(x)$ will become apparent later. At this point, we are in the position to rewrite the Lagrangian (2.99) in terms of the new physical fields $\pi(x)$ and $\sigma(x)$

$$\begin{aligned}\mathscr{L}(\pi(x),\sigma(x),\partial_\mu\pi,\partial_\mu\sigma) &= \frac{1}{2}(\partial_\mu\pi)^2+\frac{1}{2}(\partial_\mu\sigma)^2-\frac{m^2}{2}\left[\pi^2+(\sigma+\varphi_0)^2\right]-\frac{\lambda}{4}\left[\pi^2+(\sigma+\varphi_0)^2\right]^2\\ &= \frac{1}{2}(\partial_\mu\pi)^2+\frac{1}{2}(\partial_\mu\sigma)^2-\frac{1}{2}(-2m^2)\sigma^2-\frac{\lambda}{4}(\pi^2+\sigma^2)^2+\lambda\varphi_0\sigma^3+\lambda\varphi_0\sigma\pi^2-\mathscr{V}(0,\varphi_0)\\ &= \frac{1}{2}(\partial_\mu\pi)^2+\frac{1}{2}(\partial_\mu\sigma)^2-\frac{1}{2}m_\sigma^2\sigma^2-\frac{\lambda}{4}(\pi^2+\sigma^2)^2+\lambda\varphi_0\sigma^3+\lambda\varphi_0\sigma\pi^2\ , \end{aligned}\tag{2.110}$$

where we used the relation (2.103), defined the σ-mass $m_\sigma = \sqrt{-2}m$ and dropped the constant terms $\mathscr{V}(0,\varphi_0) = (m^2/2)\varphi_0^2 + (\lambda/4)\varphi_0^4$, since they do not contribute to the dynamics of the theory. Regarding the above result, we first notice that the $SO(2)$ symmetry is spontaneously broken. But this observation was already made in the discussion of the previous Subsection. A new phenomenon is now that only the radial excitation, i.e., the σ-field becomes massive, while the tangential excitation, i.e., the π-field becomes massless. The appearance of such massless tangential excitations, or strictly spoken, massless particles is a general result. Those particles will always appear, if a global continuous symmetry is spontaneously broken. This result is known as the Goldstone theorem which predicts the occurrence of a massless spin-0 boson with negative parity, denoted as Nambu-Goldstone boson, for each broken generator of the global symmetry group. It is important to stress that the broken symmetry is a global one. In the case of spontaneously broken local symmetries, the conserved currents of the internal symmetry group are coupled to gauge fields. This will result in the so-called Higgs mechanism which will not be discussed in this work. Let us come back to the discussion of the above calculation and the Goldstone theorem. We saw that the breaking of $SO(2)$ results in the existence of one Nambu-Goldstone boson. As mentioned above, the number of massless particles is connected to the amount of broken global symmetries. In order to find a mathematical condition, which indicates the number of Nambu-Goldstone boson, we will prove the Goldstone theorem. To this end, we consider a set of real scalar fields $\phi_a(x)$, $a = 1,\ldots,N$, which infinitesimally transform as

$$\phi_a(x) \xrightarrow{G} \phi'_a(x) = \phi_a(x) - i\alpha_j(T^j)_{ac}\phi^c(x) \tag{2.111}$$

under a certain Lie group G. Here the α_j are infinitesimal group parameters. The T^j are $(N\times N)$-matrix representations of the N_G generators of the Lie algebra of G. We now assume that the classical action of the theory and the functional integral measure are both invariant with respect to the infinitesimal transformations (2.111). According to the discussion of Subsection [2.2.1], the quantum effective action is also invariant under those transformations, if

$$\int \mathrm{d}^4x\,\frac{\delta\Gamma[\vec{\varphi}]}{\delta\varphi_a(x)}(T^j)_{ac}\varphi^c(x) = 0\ . \tag{2.112}$$

Now, from Eq. (2.72) we conclude that the quantum effective action fulfills

$$\frac{\delta\Gamma[\vec{\varphi}]}{\delta\varphi_a(x)} = 0 \tag{2.113}$$

in the absence of classical sources. This shows that for translation-invariant theories, the solutions of Eq. (2.113) are constant with respect to the space-time variables. In such cases, the quantum effective action can be written as

$$\Gamma[\vec{\varphi}] = -\Omega\mathscr{V}_{eff}(\vec{\varphi})\ , \tag{2.114}$$

where Ω denotes the space-time volume. The function $\mathcal{V}_{eff}(\vec{\varphi})$ is the so-called effective potential. This effective potential has the same symmetries as the potential density at the Lagrangian level, but also contains the quantum corrections to the classical potential. At this point, it is clear that this proof is also valid at quantum level. Now, we can insert Eq. (2.114) into Eq. (2.112) and take the derivative with respect to φ_b. We find

$$\frac{\partial^2\mathscr{V}}{\partial\varphi_b\partial\varphi_a}(T^j)_{ac}\varphi^c + \frac{\partial\mathscr{V}}{\partial\varphi_a}(T^j)_a^{\ b} = 0\ , \tag{2.115}$$

where we used $\partial\varphi^c/\partial\varphi_b = \delta^{cb}$ in the last term. If we now evaluate Eq. (2.115) at the vacuum expectation value $\vec{\phi}_0$ of the theory, the second term in the above relation vanishes, because the first derivative of the effective potential with respect to the fields is equal to zero at the minimum. Therefore, we are left with

$$\left.\frac{\partial^2\mathscr{V}}{\partial\varphi_b\partial\varphi_a}\right|_{\vec{\varphi}=\vec{\varphi}_0}(T^j)_{ac}\varphi_0^c = 0\ . \tag{2.116}$$

This equation has two types of solutions. On the one hand, it is possible that the vector $T^j\vec{\phi}_0$ vanishes. This happens exactly , when there is a number of generators T^j, $j = 1,\ldots,N_H$, belonging to a subgroup $H \subset G$ of the initial Lie group G, which annihilate the ground state, i.e., leave the vacuum state invariant. On the other hand, if the vector $T^j\vec{\phi}_0$ does not vanish, it is possible that this vector is an eigenvector to the matrix of the second derivatives of the effective potential with the eigenvalue zero. In this case, there is a set of generators T^j, $j = N_H + 1,\ldots,N_G$, which do not leave the vacuum state invariant. These generators do not belong to the subgroup H, because otherwise we would obtain the first case. Each of those broken generators is associated with a Nambu-Goldstone boson, because the $T^j\vec{\phi}_0$ must be linearly independent. In other words, the number of Nambu-Goldstone bosons arising from spontaneous symmetry breaking of a Lie group G down to a subgroup H is given by the difference $N_G - N_H$, or strictly spoken, by the dimension of the coset $G\backslash H$.
We are now able to check the Goldstone theorem at hand of the example of the $SO(2)$-symmetric model (2.99). In general, there are $N(N-1)/2$ generators, belonging to the Lie algebra $\mathfrak{so}(N)$ of $SO(N)$. This can be seen by considering the properties of the elements of $SO(N)$. The condition $O^{ij}O_{jk} = \delta^i_{\ k}$ gives N equations for $i = k$. In the case $i > k$, we additionally obtain $\sum_{l=1}^{N-1} l = N(N-1)/2$ equations, according to the upper triangle in the $(N\times N)$-matrix. Therefore, we finally obtain $N^2 - N - N(N-1)/2 = N(N-1)/2$ independent parameters, each of them associated with a generator of $SO(N)$. Using this relation, we find only one generator for $SO(2)$. In our model the $SO(2)$ symmetry is completely broken, therefore we have $N_G = 1$ and $N_H = 0$, so that we expect $1 - 0 = 1$ Nambu-Goldstone boson. In the above discussion, we emphasized that the broken generators do not leave the vacuum invariant. We are now able to check this statement in our example. The generator of $SO(2)$ is given by

$$T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \tag{2.117}$$

Using Eq. (2.108), we then obtain

$$T\vec{\varphi}_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix} = -i \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix} \neq \vec{0} , \tag{2.118}$$

so that we have to require that $T\vec{\varphi}_0$ is indeed an eigenvector of the mass matrix with eigenvalue zero.

In order to complete this Subsection and therefore the discussion of the Goldstone theorem, we want to come back to Eq. (2.116). Due to the Eqs. (2.73) and (2.114), the matrix of the second derivatives of the effective potential must be related to the proper two-point function $\Gamma^{(2)}_{ab}$. In addition to that, we can use Eqs. (2.67) and (2.72) to find

$$\delta_a^{\ b}\delta^{(4)}(x-z) = \frac{\delta\varphi_a(x)}{\delta\varphi_b(z)} = \int \mathrm{d}^4y \, \frac{\delta\varphi_a(x)}{\delta J^c(y)} \frac{\delta J^c(y)}{\delta\varphi_b(z)} = -\int \mathrm{d}^4y \, \frac{\delta^2 W[\vec{J}]}{\delta J^a(x)\delta J^c(y)} \frac{\delta^2\Gamma[\vec{\varphi}]}{\delta\varphi_c(y)\delta\varphi_b(z)} . \tag{2.119}$$

When we now study this relation at $\vec{J} = \vec{\varphi} = \vec{0}$, we obtain

$$-i \int \mathrm{d}^4y \, G^{(2),connected}_{ac}(x,y)\Gamma^{(2),cb}(y,z) = \delta_a^{\ b}\delta^{(4)}(x-z) . \tag{2.120}$$

This relation tells us that the proper two-point function is just the inverse of the connected two-point function. Then, the matrix of the second derivatives of the effective potential with respect to the constant fields is, up to a Fourier transform, proportional to the inverse momentum-space Feynman propagator with vanishing momentum $(\Delta^{ab})^{-1}(p^2 = 0)$. According to the discussion of Eq. (2.116) the vector $T^j\vec{\phi}_0$ must be an eigenvector with eigenvalue zero for each broken generator. From the existence of such an eigenvector, it follows that the momentum-space propagator $\Delta^{ab}(p)$ has a pole of rank r in $p^2 = 0$. Now, the rank of this pole is given by the dimension of the space containing the $T^j\vec{\phi}_0$, which is just the dimension of the coset $G\backslash H$.

2.3 Quantum Chromodynamics

In the middle of the sixties about 100 elementary particles were found in various experiments. Due to this large amount of different particles, physicists were sure that those particles cannot be fundamental. In 1964 Gell-Mann, Ne'eman, and Zweig independently introduced a classification scheme for those so-called hadrons. They suggested that the substructure of the hadrons is given by three fundamental particles, called quarks. In the mathematical framework of this classification scheme, the up, the down, and the strange quark form an irreducible representation of the so-called $SU(N_f = 3)_V$ flavor symmetry.[3] In the basis of strong isospin T_3 and hypercharge Y, the Hilbert-space vectors of those three quarks are given by

$$|u\rangle = |T_3 Y\rangle = |\frac{1}{2}\frac{1}{3}\rangle\,, \quad |d\rangle = |-\frac{1}{2}\frac{1}{3}\rangle\,, \quad |s\rangle = |0 - \frac{2}{3}\rangle\ . \tag{2.121}$$

With this fundamental triplet and the anti-triplet, formed by the corresponding anti-quarks, it is possible to classify the known hadrons into multiplets of $SU(N_f = 3)_V$. The physical mesons, regarded as $q\bar{q}$-states in this picture, form a singlet and an octet under $SU(N_f = 3)_V$. The qqq-states, denoted as baryons, form a singlet, two octets, and a decuplet under $SU(N_f = 3)_V$. The underlying interaction of quarks and therefore of all mesons and baryons, i.e, of all hadrons is the strong interaction. Today we know that in addition to the three light quarks there exist three heavy quarks. These quarks are called charm, bottom, and top quark. A summary of all six quark flavors is given in Tab. [2.1]. We also know that quarks have an additional

Generation	Flavor	Mass $[GeV]$	$Q[e]$	Y	J	B	S	C	B'	T
I	u^r, u^g, u^b	$(2.3^{+0.7}_{-0.5}) \cdot 10^{-3}$	$2/3$	$1/3$	$1/2$	$1/3$	0	0	0	0
	d^r, d^g, d^b	$(4.8^{+0.5}_{-0.3}) \cdot 10^{-3}$	$-1/3$	$1/3$	$1/2$	$1/3$	0	0	0	0
II	c^r, c^g, c^b	1.275 ± 0.025	$2/3$	$4/3$	$1/2$	$1/3$	0	1	0	0
	s^r, s^g, s^b	$(95 \pm 5) \cdot 10^{-3}$	$-1/3$	$-2/3$	$1/2$	$1/3$	-1	0	0	0
III	t^r, t^g, t^b	$173.07 \pm 0.52 \pm 0.72$	$2/3$	$4/3$	$1/2$	$1/3$	0	0	0	1
	b^r, b^g, b^b	4.66 ± 0.03	$-1/3$	$-2/3$	$1/2$	$1/3$	0	0	-1	0

Table 2.1: The quantum numbers listed are: $Q \,\hat{=}$ electric charge, $Y \,\hat{=}$ hypercharge, $J \,\hat{=}$ total angular momentum, $B \,\hat{=}$ baryon number, $S \,\hat{=}$ strangeness, $C \,\hat{=}$ charm, $B' \,\hat{=}$ bottomness, $T \,\hat{=}$ topness. The quark masses and the quantum numbers are obtained from Refs. [PDG], [RQM2].

degree of freedom, called color. The introduction of this further quantum number became necessary, when experimentalists discovered the Δ^{++} baryon. This baryon has quantum numbers $I(J^P) = \frac{3}{2}(\frac{3}{2}^+)$ and hypercharge $Y = 1$. Using Eq. ([2.121]) it is evident that the Δ^{++} baryon consists of three up quarks, which leads to a symmetric wave function in flavor space. In addition to that, all spins of the three up quarks are added up to $S = \frac{3}{2}$, which forms a symmetric wave function in spinor space. Finally, from $S = J = \frac{3}{2}$ follows that the angular momentum of Δ^{++} is given by $L = 0$, which results in a symmetric spatial wave function, so that the total wave function $\Psi_{\Delta^{++}} = \psi_{space} \otimes \psi_{spin} \otimes \psi_{flavor}$ is also symmetric. But this result is in contrast to the spin-statistics theorem which tells us that the wave function of a fermion must be antisymmetric. In order to resolve this contradiction Greenberg, Han, and Nambu introduced color as an additional degree of freedom for quarks. The three fundamental colors[4] red, green, and blue form a triplet under the $SU(N_C = 3)_C$ color group. With this additional degree of freedom, the wave function of the hadrons must be modified by a wave function in color space, so that

$$\Psi_{hadron} = \psi_{space} \otimes \psi_{spin} \otimes \psi_{flavor} \otimes \psi_{color}\ . \tag{2.122}$$

Fortunately, it turns out that the wave function associated with the singlet in color space is antisymmetric. Therefore, the total wave function of Δ^{++} becomes antisymmetric and the contradiction with the spin-statistics theorem is resolved.

In the early seventies, the theory of strong interactions, Quantum Chromodynamics (QCD), was formulated in a similar way as Quantum Electrodynamics (QED). But, instead of using an abelian gauge group, as $U(1)_{em}$ in QED, one used the non-abelian gauge group $SU(N_C = 3)_C$. In the upcoming subsections, we want to study the basic properties of Quantum Chromodynamics. Therefore, we start with the construction of the QCD Lagrangian by using the gauge principle. Then, we introduce the so-called chiral symmetry of QCD and the related conserved currents. Finally, we will see that the observed hadron spectrum is not realized as irreducible representations of the chiral group, but of its flavor subgroup $SU(N_f)_V$. From this observation we shall conclude that the chiral symmetry of QCD must be broken spontaneously down to its flavor subgroup.

2.3.1 Quantum Chromodynamics as $SU(N_C)_C$ Gauge Theory

In order to construct the Lagrangian of Quantum Chromodynamics, we start with the Dirac Lagrangian, describing the dynamics of a free fermionic field. This Lagrangian is given by

$$\mathscr{L}_{Dirac} = \bar{\psi}(x)\,(i\gamma^\mu \partial_\mu - m)\,\psi(x)\ , \tag{2.123}$$

[3] Later we will see that $SU(N_f = 3)_V$ is a subgroup of the so-called chiral symmetry.

[4] The existence of three colors is experimentally verified, e.g. by measuring the $R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$ ratio of the cross sections in e^+e^- annihilation processes.

where the γ^μ denotes the usual Dirac matrices in spinor space, compare App. [6.1.3]. The fermionic field $\psi(x)$ is given by a Dirac spinor, i.e., a four-component object, transforming under Lorentz transformations as a $(1/2, 0) \otimes (0, 1/2)$ representation of the Lorentz group. In the introductory discussion of this section, we saw that the quark fields have additional degrees of freedom, given by flavor and color. Therefore, the fermionic field $\psi(x)$ must be modified in a way that it represents a Dirac spinor in spinor space, a $(N_f = 6)$-vector in flavor space, and a $(N_C = 3)$-vector in color space. In the following this quark field will be denoted as $\Psi(x)$. Then, the Lagrangian (2.123) becomes

$$\mathscr{L}_{Quark} = \bar{\Psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\Psi(x)\;. \tag{2.124}$$

At this point we have to notice that the (4×4) mass matrix m in Eq. (2.123) becomes a $(4N_fN_C \times 4N_fN_C)$ matrix in the above equation. Also the first term in the Dirac operator is modified by unit matrices acting in flavor and color space. As mentioned before, the quark and anti-quark fields $\Psi_f(x)$ and $\bar{\Psi}_f(x)$ correspond to a triplet $[3]_C$ and an anti-triplet $[\bar{3}]_C$ in color space. Therefore, the complex quark vector for an arbitrary flavor can be written as

$$\Psi_f(x) = \begin{pmatrix} \Psi_{f,r}(x) \\ \Psi_{f,g}(x) \\ \Psi_{f,b}(x) \end{pmatrix}\;. \tag{2.125}$$

The quark fields $\Psi(x)$ and $\bar{\Psi}(x)$ transform in the fundamental representation of $SU(N_C)_C$, i.e., they transform as

$$\begin{aligned} \Psi(x) &\overset{SU(N_C)_C}{\longrightarrow} \Psi'(x) = U_C\Psi(x) = e^{-i\Lambda_a T^a}\Psi(x), \\ \bar{\Psi}(x) &\overset{SU(N_C)_C}{\longrightarrow} \bar{\Psi}'(x) = \bar{\Psi}(x)U_C^\dagger = \bar{\Psi}(x)e^{i\Lambda_a T^a}\;, \end{aligned} \tag{2.126}$$

where $\Lambda_a,\ a = 1,\ldots,N_C^2 - 1$, are the group parameters and the T^a are the generators of $SU(N_C)_C$. These generators are defined as $T^a = \lambda^a/2$, where λ^a denote the usual Gell-Mann matrices. Furthermore, they are orthogonal in the sense that $\mathrm{Tr}(T^aT^b) = \delta^{ab}/2$ and fulfill the Lie algebra $\mathfrak{su}(N_C)_C$ of $SU(N_C)_C$, $\left[T^a, T^b\right]_- = i f^{ab}{}_c T^c$. Now, the Lagrangian (2.124) is invariant under the global rotations (2.126) in color space

$$\begin{aligned} \mathscr{L}'_{Quark} &= \bar{\Psi}'(x)\,(i\gamma^\mu\partial_\mu - m)\,\Psi'(x) \\ &= \bar{\Psi}(x)e^{i\Lambda_a T^a}\,(i\gamma^\mu\partial_\mu - m)\,e^{-i\Lambda_a T^a}\Psi(x) \\ &= \bar{\Psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\Psi(x) = \mathscr{L}_{Quark}\;, \end{aligned} \tag{2.127}$$

where we used $U_C^\dagger U_C = \mathbb{1}_{N_C\times N_C}$ and the fact that all objects in the Dirac operator, which act on color space, are proportional to the unit matrix $\mathbb{1}_{N_C\times N_C}$ and therefore commute with U_C. In the next step, we require the Lagrangian $\mathscr{L}_{Quark}$ to be invariant under local rotations in color space. To this end, we modify the group elements of $SU(N_C)_C$ by a space-time dependence, such that $\Lambda_a \to \Lambda_a(x)\ \forall\ a$. Now, the invariance of $\mathscr{L}_{Quark}$ under local $SU(N_C)_C$ rotations, i.e., $SU(N_C)_C$ transformations with space-time dependent group parameters, would allow us to pick different group parameters at different space-time points in Minkowski space without changing the physics which is described by $\mathscr{L}_{Quark}$. It is quite obvious that this requirement is much stricter than the invariance under global $SU(N_C)_C$ transformations (2.127). The transformation behaviour of the quark fields (2.126) is then given by

$$\begin{aligned} \Psi(x) &\overset{SU(N_C)_C}{\longrightarrow} \Psi'(x) = U_C(x)\Psi(x) = e^{-i\Lambda_a(x) T^a}\Psi(x), \\ \bar{\Psi}(x) &\overset{SU(N_C)_C}{\longrightarrow} \bar{\Psi}'(x) = \bar{\Psi}(x)U_C^\dagger(x) = \bar{\Psi}(x)e^{i\Lambda_a(x) T^a}\;, \end{aligned} \tag{2.128}$$

so that the Lagrangian (2.124) transforms as

$$\begin{aligned} \mathscr{L}' &= \bar{\Psi}'(x)\,(i\gamma^\mu\partial_\mu - m)\,\Psi'(x) \\ &= \bar{\Psi}(x)e^{i\Lambda_a(x)T^a}\left[i\gamma^\mu\left(\partial_\mu e^{-i\Lambda_a(x)T^a}\right) + e^{-i\Lambda_a(x)T^a}i\gamma^\mu\partial_\mu - e^{-i\Lambda_a(x)T^a}m\right]\Psi(x) \\ &= \bar{\Psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\Psi(x) + \bar{\Psi}(x)\gamma^\mu\left[\partial_\mu\Lambda_a(x)\right]T^a\Psi(x) \\ &= \mathscr{L}_{Quark} + \delta\mathscr{L}_{Quark} \end{aligned} \tag{2.129}$$

under local $SU(N_C)_C$ transformations. Obviously, the Lagrangian is not invariant, because the derivative in the Dirac operator causes an additional term $\delta\mathscr{L}_{Quark}$. In order to compensate this term, we introduce an $SU(N_C)_C$ gauge field $\mathscr{A}^\mu \equiv A^\mu_a T^a$, which transforms under $SU(N_C)_C$ rotations in a way that the additional term in Eq. (2.129) is compensated. The actual transformation behaviour of $\mathscr{A}^\mu$ is given by

$$\mathscr{A}_\mu \overset{SU(N_C)_C}{\longrightarrow} \mathscr{A}'_\mu = U_C(x)\mathscr{A}_\mu U_C^\dagger(x) - \frac{i}{g}\left[\partial_\mu U_C(x)\right]U_C^\dagger(x)\;. \tag{2.130}$$

This gauge field is now introduced into the Lagrangian (2.124) by minimal coupling, i.e., we introduce a covariant derivative

$$D_\mu = \partial_\mu - ig\mathscr{A}_\mu \ , \tag{2.131}$$

which transforms in the adjoint representation of $SU(N_C)_C$

$$\begin{aligned}
D_\mu \to D'_\mu &= \partial_\mu - ig\left\{U_C(x)\mathscr{A}_\mu U_C^\dagger(x) - \frac{i}{g}\left[\partial_\mu U_C(x)\right]U_C^\dagger(x)\right\} \\
&= \partial_\mu + \left[\partial_\mu U_C(x)\right]U_C^\dagger(x) + U_C(x)\partial_\mu U_C^\dagger(x) - igU_C(x)\mathscr{A}_\mu U_C^\dagger(x) - \left[\partial_\mu U_C(x)\right]U_C^\dagger(x) \\
&= U_C(x)(\partial_\mu - ig\mathscr{A}_\mu)U_C^\dagger(x) \\
&= U_C(x)D_\mu U_C^\dagger(x)
\end{aligned} \tag{2.132}$$

and replaces the ordinary 4-gradient. Note that we used $0 = \partial_\mu\left[U_C(x)U_C^\dagger(x)\right] = \left[\partial_\mu U_C(x)\right]U_C^\dagger(x) + U_C(x)\partial_\mu U_C^\dagger(x)$ in the second line of Eq. (2.132). With this covariant derivative, the new minimally coupled Lagrangian is given by

$$\mathscr{L}_{Quark} = \bar{\Psi}(x)\left(i\gamma^\mu D_\mu - m\right)\Psi(x) \ . \tag{2.133}$$

Equation (2.133) is now invariant under local $SU(N_C)_C$ transformations

$$\begin{aligned}
\mathscr{L}'_{Quark} &= \bar{\Psi}'(x)\left(i\gamma^\mu D'_\mu - m\right)\Psi'(x) \\
&= \bar{\Psi}(x)U_C^\dagger(x)\left(i\gamma^\mu U_C(x)D_\mu U_C^\dagger(x) - m\right)U_C(x)\Psi(x) \\
&= \bar{\Psi}(x)\left(i\gamma^\mu D_\mu - m\right)\Psi(x) \\
&= \mathscr{L}_{Quark} \ ,
\end{aligned} \tag{2.134}$$

where we used the same properties as in Eq. (2.127). The new gauge fields $\mathscr{A}^\mu \equiv A_a^\mu T^a$, introduced in Eq. (2.131), are associated with the gluons. According to the definition of $\mathscr{A}^\mu$, the Lagrangian (2.133) contains $a = N_C^2 - 1 = 8$ gauge boson fields A_a^μ and therefore eight gluons. This number of gauge bosons is not surprising, because the gluonic fields correspond to the adjoint representation of $SU(N_C)_C$ and builds therefore an eight-dimensional irreducible representation $[8]_C$ of the color group. Finally, we have to add a kinetic term for the gluon fields to Eq. (2.133). Analogously to Quantum Electrodynamics, this term is given by the square of the field-strength tensor associated with the gauge fields. This field-strength tensor is in general defined as the commutator of the covariant derivatives (2.131). One finds

$$\begin{aligned}
\mathscr{F}_{\mu\nu} &= \frac{i}{g}\left[D_\mu, D_\nu\right]_- \\
&= \frac{i}{g}\left(\partial_\mu - ig\mathscr{A}_\mu\right)\left(\partial_\nu - ig\mathscr{A}_\nu\right) - \left(\partial_\nu - ig\mathscr{A}_\nu\right)\left(\partial_\mu - ig\mathscr{A}_\mu\right) \\
&= \frac{i}{g}\left\{\partial_\mu\partial_\nu - ig(\partial_\mu\mathscr{A}_\nu) - ig\mathscr{A}_\nu\partial_\mu - ig\mathscr{A}_\mu\partial_\nu - g^2\mathscr{A}_\mu\mathscr{A}_\nu - \left[\partial_\nu\partial_\mu - ig(\partial_\nu\mathscr{A}_\mu) - ig\mathscr{A}_\mu\partial_\nu - ig\mathscr{A}_\nu\partial_\mu - g^2\mathscr{A}_\nu\mathscr{A}_\mu\right]\right\} \\
&= \partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu - ig\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]_- \ .
\end{aligned} \tag{2.135}$$

The above equation gives the matrix form of the Yang-Mills field-strength tensor. Using the Lie algebra of $SU(N_C)_C$ and the definition of the gauge fields, the Yang-Mills field-strength tensor can be written as

$$\begin{aligned}
\mathscr{F}_{\mu\nu} &= \partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu - ig\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]_- \\
&= \partial_\mu A_\nu^a T_a - \partial_\nu A_\mu^a T_a - igA_\mu^b A_\nu^c\left[T_b, T_c\right]_- \\
&= \partial_\mu A_\nu^a T_a - \partial_\nu A_\mu^a T_a + gf_{bc}{}^a A_\mu^b A_\nu^c T^a \\
&= \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^a_{\ bc}A_\mu^b A_\nu^c\right)T_a \\
&= F_{\mu\nu}^a T_a \ ,
\end{aligned} \tag{2.136}$$

where we used that the $SU(3)$ structure constants are totally antisymmetric, i.e., $f_{bc}{}^a = -f_b{}^a{}_c = f^a_{\ bc}$. A gauge-invariant kinetic term for the gluonic fields is then given by

$$\mathscr{L}_{Gluon} = -\frac{1}{2}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}\right) \ . \tag{2.137}$$

Using Eqs. (2.132) and (2.135) it is easy to show that (2.137) is indeed gauge invariant

$$\begin{aligned}\mathscr{L}'_{Gluon} &= -\frac{1}{2}\mathrm{Tr}\left(\mathscr{F}'_{\mu\nu}\mathscr{F}'^{\mu\nu}\right)\\ &= -\frac{1}{2}\mathrm{Tr}\left(\frac{i}{g}\left[D'_\mu, D'_\nu\right]_- \frac{i}{g}\left[D'^\mu, D'^\nu\right]_-\right)\\ &= -\frac{1}{2}\mathrm{Tr}\left(\frac{i}{g}U_C(x)\left[D_\mu, D_\nu\right]_- U_C^\dagger(x)\frac{i}{g}U_C(x)\left[D^\mu, D^\nu\right]_- U_C^\dagger(x)\right)\\ &= -\frac{1}{2}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}\right)\\ &= \mathscr{L}_{Gluon}\ ,\end{aligned} \tag{2.138}$$

where we used the cyclic property of the trace. Finally, the Lagrangian of Quantum Chromodynamics is given by the sum of Eqs. (2.133) and (2.137), so that

$$\mathscr{L}_{QCD} = -\frac{1}{2}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}\right) + \bar{\Psi}(x)\left(i\gamma^\mu D_\mu - m\right)\Psi(x)\ . \tag{2.139}$$

When we look at Eq. (2.136), we see that the Yang-Mills field-strength tensor has an additional term which is proportional to two gluonic fields of the colors b and c. This term is of course a consequence of the non-abelian nature of the gluon fields and therefore of the non-abelian gauge symmetry of QCD. This additional term ensures that the gluonic term (2.137) does not only consist of a kinetic part for the gluons, but also includes gluon self-interaction terms. Starting from Eq. (2.137) and using Eq. (2.136), we obtain

$$\begin{aligned}-\frac{1}{2}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}\right) &= -\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu}_b\mathrm{Tr}\left(T_aT^b\right) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\\ &= -\frac{1}{4}\left\{\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) + \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)g f_a^{\ de}A^\mu_d A^\nu_e\right.\\ &\quad\left. + g f^a_{\ bc}A^b_\mu A^c_\nu\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) + g^2 f^a_{\ bc}f_a^{\ de}A^b_\mu A^c_\nu A^\mu_d A^\nu_c\right\}\\ &= -\frac{1}{2}\partial_\mu A^a_\nu\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) - \frac{g}{4}\left\{f_a^{\ de}\left[\partial_\mu A^a_\nu A^\mu_d A^\nu_e - \partial_\nu A^a_\mu A^\mu_d A^\nu_e\right]\right.\\ &\quad\left. + f^a_{\ bc}\left[A^b_\mu A^c_\nu\partial^\mu A^\nu_a - A^b_\mu A^c_\nu\partial^\nu A^\mu_a\right]\right\} - \frac{g^2}{4}f^a_{\ bc}f_a^{\ de}A^b_\mu A^c_\nu A^\mu_d A^\nu_e\ .\end{aligned} \tag{2.140}$$

Now, all terms proportional to the gauge coupling constant g can be summarized by renaming the Lorentz and the color indices and using the fact that the $SU(3)$ structure constants are totally antisymmetric. Finally, we find

$$-\frac{1}{2}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}\right) = -\frac{1}{2}\partial_\mu A^a_\nu\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) - g f_a^{\ bc}\left(\partial_\mu A^a_\nu\right)A^\mu_b A^\nu_c - \frac{g^2}{4}f^a_{\ bc}f_a^{\ de}A^b_\mu A^c_\nu A^\mu_d A^\nu_e\ . \tag{2.141}$$

The first term in the above equation corresponds to the free inverse gluon propagator, the second and the third term correspond to three- and a four-gluon self-interaction vertices. These terms are a consequence of the non-abelian gauge group $SU(N_C)_C$ and therefore a feature of all Yang-Mills theories. It turns out that these gluonic self-interaction terms are the main reason for a phenomenon called asymptotic freedom. This means that the interaction of quarks and gluons at high energies or small distances becomes weak. Figure [2.6] shows all interaction vertices resulting from Eq. (2.139). Finally, it should be taken into account that

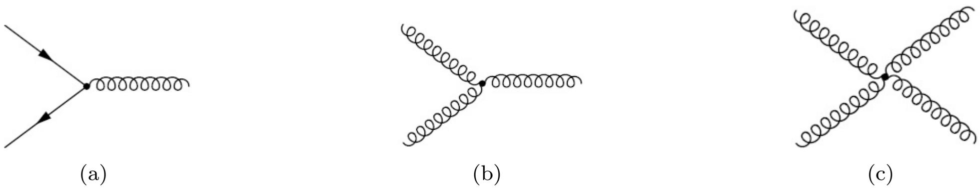


Figure 2.6: Figure [(a)] shows the interaction vertex between quarks and gluons. This vertex results from the covariant derivative in Eq. (2.139). Figures [(b)] and [(c)] show the three- and four-gluon interactions which follow from the Yang-Mills part of the QCD Lagrangian (2.141). Note that the derivative coupling of the gluonic field in the three-gluon vertex will cause a momentum dependence of this vertex. Furthermore we recognize that the four-gluon vertex is of order $\mathcal{O}(g^2)$ in the gauge coupling constant.

the second quantization of the Yang-Mills fields requires the appearance of another term $\mathscr{L}_{gauge}$ in the QCD

Lagrangian. The necessity of this term comes from the fact that the gluonic fields have an additional degree of freedom which is given by the gauge transformations. This additional degree of freedom can be eliminated by adding a term which includes a constraint on the gauge fields. In the functional integral formulation of QCD, this gauge fixing is obtained by the method of Faddeev and Popov. This method introduces a new artificial field whose particle excitations are called ghosts or Faddeev-Popov-ghosts. The name "ghost" arises from the fact that the ghost fields are anticommuting Grassmannian fields, but transform in the $(0,0)$-representation of the Lorentz group, i.e., they transform as scalar particles, but obey Fermi-Dirac statistics. Further, it may happen[5], that the ghosts couple to gluons, resulting in a ghost-gluon vertex which also has to be taken into account in calculations. This ghost-gluon vertex is depicted in Fig. [2.7].

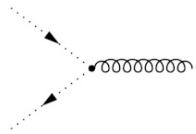


Figure 2.7: Ghost-gluon interaction vertex.

2.3.2 The Chiral Symmetry of Quantum Chromodynamics

In the last subsection, we saw that the Lagrangian of QCD can be obtained by using the gauge principle, i.e., by promoting a global $SU(N_C)_C$ symmetry to a local one. This gauge symmetry is of course the most important symmetry of QCD, because it determines the structure of the Lagrangian and therefore the types of possible interactions. However, the Lagrangian (2.139) also contains important global symmetries. The most important one is the so-called chiral symmetry. In order to introduce this symmetry, we consider the following operators

$$\mathcal{P}_{L/R} = \frac{\mathbb{1} \mp \gamma_5}{2} \ , \tag{2.142}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, compare App. [6.1.3]. These operators are referred to as left- and right-handed projection operators. The designations left- and right-handed come from the fact that these operators project the ultrarelativistic positive and negative energy solutions of the free Dirac equation onto the positive and negative helicity eigenstates. Furthermore, in this zero-mass limit, helicity is equal to chirality. In addition to that, the relations

$$\mathcal{P}_R\mathcal{P}_L = \mathcal{P}_L\mathcal{P}_R = 0, \quad \mathcal{P}_R^2 = \mathcal{P}_R, \quad \mathcal{P}_L^2 = \mathcal{P}_L, \quad \mathcal{P}_R + \mathcal{P}_L = \mathbb{1}_{4\times 4} \tag{2.143}$$

show that these operators are indeed projection operators. Using Eq. (2.142), it is possible to decompose the quark fields $\Psi(x)$ and $\bar{\Psi}(x)$ into their left- and right-handed components

$$\Psi(x) = (\mathcal{P}_L + \mathcal{P}_R)\Psi(x) = \Psi_L(x) + \Psi_R(x), \quad \bar{\Psi}(x) = \bar{\Psi}(x)(\mathcal{P}_R + \mathcal{P}_L) = \bar{\Psi}_L(x) + \bar{\Psi}_R(x) \ . \tag{2.144}$$

This approach can be applied to the fermionic part of the QCD Lagrangian (2.139). One finds

$$\begin{aligned}
\mathscr{L}_{Quark} &= \bar{\Psi}(x)\,(i\gamma^\mu D_\mu - m)\,\Psi(x) \\
&= \bar{\Psi}(x)(\mathcal{P}_L + \mathcal{P}_R)\,(i\gamma^\mu D_\mu - m)\,(\mathcal{P}_L + \mathcal{P}_R)\Psi(x) \\
&= \bar{\Psi}_L(x)\,(i\gamma^\mu D_\mu - m)\,\Psi_L(x) + \bar{\Psi}_R(x)\,(i\gamma^\mu D_\mu - m)\,\Psi_R(x) + \bar{\Psi}_L(x)\,(i\gamma^\mu D_\mu - m)\,\Psi_R(x) \\
&\quad + \bar{\Psi}_R(x)\,(i\gamma^\mu D_\mu - m)\,\Psi_L(x) \\
&= \bar{\Psi}_L(x) i\gamma^\mu D_\mu \Psi_L(x) + \bar{\Psi}_R(x) i\gamma^\mu D_\mu \Psi_R(x) - \bar{\Psi}_L(x) m \Psi_R(x) - \bar{\Psi}_R(x) m \Psi_L(x) \ ,
\end{aligned} \tag{2.145}$$

where we used in the last line that the other four terms vanish because of Eq. (2.143). In detail, we used that

$$\begin{aligned}
&\bar{\Psi}_{L/R}(x) i\gamma^\mu D_\mu \Psi_{R/L}(x) = \bar{\Psi}(x)\mathcal{P}_{R/L} i\gamma^\mu D_\mu \mathcal{P}_{R/L}\Psi(x) = \bar{\Psi}(x)\mathcal{P}_{R/L}\mathcal{P}_{L/R} i\gamma^\mu D_\mu \Psi(x) = 0 \ , \\
&\bar{\Psi}_{L/R}(x) m \Psi_{L/R}(x) = \bar{\Psi}(x)\mathcal{P}_{R/L} m \mathcal{P}_{L/R}\Psi(x) = 0 \ ,
\end{aligned} \tag{2.146}$$

where we used the definitions (2.142), (6.29), and the fact that the projection operators project onto orthogonal subspaces, i.e., the first relation in Eq. (2.143). It is obvious, that the mass term in the QCD Lagrangian mixes the left- and right-handed quark fields, while the derivative term separates the quark fields

[5] The coupling of the ghosts to physical particles depends on the chosen gauge.

of opposite chirality. In the following discussion, we start with the case of vanishing quark masses, $m = 0$. Then, the quark part of the QCD Lagrangian becomes

$$\mathscr{L}_{Quark} = \bar{\Psi}_L(x) i\gamma^\mu D_\mu \Psi_L(x) + \bar{\Psi}_R(x) i\gamma^\mu D_\mu \Psi_R(x) \ . \tag{2.147}$$

In the above Lagrangian, it is possible to rotate the left- and right-handed quark fields separately in flavor space. Therefore, the symmetry group of the Lagrangian (2.147) is given by $U(N_f)_L \times U(N_f)_R$. This global symmetry of the QCD Lagrangian in the chiral limit, i.e., the limit $m = 0$, is called chiral symmetry. In general, the group of unitary $(N \times N)$-matrices $U(N)$ can be decomposed into the direct product of two of its subgroups. One of these subgroups is given by the unitary group of $(N \times N)$-matrices with unit determinant, $SU(N)$. The other subgroup is given by the group of complex phase transformations $U(1)$. Therefore the group of unitary $(N \times N)$-matrices can be written as $U(N) = SU(N) \times U(1)$. Now, it is possible to write the chiral symmetry group as $U(N_f)_L \times U(N_f)_R = SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R$. The elements of $U(N_f)_L$ and $U(N_f)_R$ are given by

$$U_{L/R} = e^{-i\alpha^a_{L/R} T_a} \ , \ a = 0, \ldots, N_f^2 - 1 \ . \tag{2.148}$$

In the above equation, we used that the generator T_0 of $U(1)_{L/R}$ is proportional to the $(N_f \times N_f)$ unit matrix $\mathbb{1}_{N_f \times N_f}$. The possible proportionality constant of this generator can be absorbed in the group parameter $\alpha^0_{L/R}$. For $a = 1, \ldots, N_f^2 - 1$, we have the usual $SU(N_f)$ generators $T_a = \lambda_a/2$, where λ_a denote the $(N_f \times N_f)$ generalizations of the Gell-Mann matrices. These generators are orthogonal in the sense that $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$ and also satisfy the $SU(N_f)$ Lie algebra, $\left[T^a, T^b\right]_- = i f^{ab}{}_c T^c$. Note that the previously mentioned proportionality constant in T_0 is chosen in way that the orthogonality relation is also fulfilled for the whole set of $U(N_f)$ generators. Now, the left- and right-handed quark fields transform as

$$\Psi_{L/R}(x) \overset{U(N_f)_{L/R}}{\longrightarrow} \Psi'_{L/R}(x) = U_{L/R} \Psi_{L/R}(x) = e^{-i\alpha^a_{L/R} T_a} \Psi_{L/R}(x) \ , \tag{2.149}$$

while the Dirac-adjoint fields transform as

$$\bar{\Psi}_{L/R}(x) \overset{U(N_f)_{L/R}}{\longrightarrow} \bar{\Psi}'_{L/R}(x) = \bar{\Psi}_{L/R}(x) U^\dagger_{L/R} = \bar{\Psi}_{L/R}(x) e^{i\alpha^a_{L/R} T_a} \ , \tag{2.150}$$

where we used that the generators of unitary groups are hermitian and that the group parameters are real. According to Noether's theorem Sec. [2.1.2], the invariance of Eq. (2.147) under the chiral rotations (2.149) and (2.150) implies the appearance of $2N_f^2$ conserved Noether currents. Two of these currents are associated with the left- and right-handed phase transformations $U(1)_L \times U(1)_R$, the remaining $2(N_f^2 - 1)$ arise from the $SU(N_f)_L \times SU(N_f)_R$ symmetry. For the calculation of the conserved Noether currents (2.21) we need the infinitesimal forms of Eqs. (2.149) and (2.150). These infinitesimal forms can be obtained by expanding the left- and right-handed rotations (2.149) and (2.150) up to first order into a Taylor polynomial

$$\begin{aligned} U_{L/R} \Psi_{L/R}(x) &\approx \Psi_{L/R}(x) - i\alpha^a_{L/R} T_a \Psi_{L/R}(x) = \Psi_{L/R}(x) + \delta\Psi_{L/R}(x) \ , \\ \bar{\Psi}_{L/R}(x) U^\dagger_{L/R} &\approx \bar{\Psi}_{L/R}(x) + \bar{\Psi}_{L/R}(x) i\alpha^a_{L/R} T_a = \bar{\Psi}_{L/R}(x) + \delta\bar{\Psi}_{L/R}(x) \ . \end{aligned} \tag{2.151}$$

Using the above formulas, Noether's theorem (2.21), and the Lagrangian (2.147), the conserved left-handed current becomes

$$\begin{aligned} J^\mu_L &= \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \Psi_L)} \delta\Psi_L + \delta\bar{\Psi}_L \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \bar{\Psi}_L)} \\ &= \bar{\Psi}_L i\gamma^\lambda \frac{\partial(\partial_\lambda \Psi_L)}{\partial(\partial_\mu \Psi_L)} \left(-i\alpha^a_L T_a \Psi_L\right) \\ &= \bar{\Psi}_L \gamma^\mu \alpha^a_L T_a \Psi_L \\ &\equiv \alpha^a_L J^\mu_{a,L} \ , \end{aligned} \tag{2.152}$$

where we used that the partial derivative of the Lagrangian with respect to $\partial_\mu \bar{\Psi}_L$ vanishes. The right-handed current can be obtained by a similar calculation

$$\begin{aligned} J^\mu_R &= \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \Psi_R)} \delta\Psi_R + \delta\bar{\Psi}_R \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \bar{\Psi}_R)} \\ &= \bar{\Psi}_R i\gamma^\lambda \frac{\partial(\partial_\lambda \Psi_R)}{\partial(\partial_\mu \Psi_R)} \left(-i\alpha^a_R T_a \Psi_R\right) \\ &= \bar{\Psi}_R \gamma^\mu \alpha^a_R T_a \Psi_R \\ &\equiv \alpha^a_R J^\mu_{a,R} \ . \end{aligned} \tag{2.153}$$

For $a = 1, \ldots, N_f^2 - 1$, the $N_f^2 - 1$ left-handed currents transform as a $(N_f^2 - 1, 1)$-multiplet under $SU(N_f)_L \times SU(N_f)_R$ transformations, while the $N_f^2 - 1$ right-handed currents transform as a $(1, N_f^2 - 1)$-multiplet. Of course, the above currents are conserved, because they follow from the exact symmetry of Eq. (2.147) under the chiral transformations (2.149) and (2.150). The conservation of the left- and right-handed currents (2.152) and (2.153) can also be checked explicitly

$$\begin{aligned}
\partial_\mu J^\mu_{L/R} &= \partial_\mu \left\{ \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \Psi_{L/R})} \delta\Psi_{L/R} + \delta\bar{\Psi}_{L/R} \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \bar{\Psi}_{L/R})} \right\} \\
&= \frac{\partial \mathscr{L}_{Quark}}{\partial \Psi_{L/R}} \delta\Psi_{L/R} + \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \Psi_{L/R})} (\partial_\mu \delta\Psi_{L/R}) + (\partial_\mu \delta\bar{\Psi}_{L/R}) \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \bar{\Psi}_{L/R})} + \delta\bar{\Psi}_{L/R} \frac{\partial \mathscr{L}_{Quark}}{\partial \bar{\Psi}_{L/R}} \\
&= -ig\bar{\Psi}_{L/R}\gamma^\lambda \mathscr{A}_\lambda \alpha^a_{L/R} T_a \Psi_{L/R} + \bar{\Psi}_{L/R}\gamma^\mu \left(\partial_\mu \alpha^a_{L/R} T_a \Psi_{L/R} \right) - \bar{\Psi}_{L/R} \alpha^a_{L/R} T_a \gamma^\lambda (\partial_\lambda - ig\mathscr{A}_\lambda) \Psi_{L/R} \\
&= 0 \ ,
\end{aligned} \tag{2.154}$$

where we used the Euler-Lagrange equations (2.8) for the left- and right-handed quark fields. It is convenient not to consider the left- and right-handed conserved currents $J^\mu_{a,L/R}$, but linear combinations of them. The so-called vector current is defined by the sum of Eqs. (2.152) and (2.153)

$$\begin{aligned}
J^\mu_{a,V} &= J^\mu_{a,L} + J^\mu_{a,R} = \bar{\Psi}_L \gamma^\mu T_a \Psi_L + \bar{\Psi}_R \gamma^\mu T_a \Psi_R \\
&= \bar{\Psi} \mathcal{P}_R \gamma^\mu T_a \mathcal{P}_L \Psi + \bar{\Psi} \mathcal{P}_L \gamma^\mu T_a \mathcal{P}_R \Psi \\
&= \bar{\Psi} \frac{\mathbb{1} + \gamma_5}{2} \gamma^\mu T_a \Psi + \bar{\Psi} \frac{\mathbb{1} - \gamma_5}{2} \gamma^\mu T_a \Psi \\
&\equiv \bar{\Psi} \gamma^\mu T_a \Psi \ ,
\end{aligned} \tag{2.155}$$

where we used Eq. (2.143) and the anticommutator relation (6.29). This current is called vector current, because it transforms as a vector under Lorentz transformations. We shall come back to this point in a moment. By taking the difference of Eqs. (2.152) and (2.153) we obtain another structure with distinct transformation behavior under Lorentz transformations

$$\begin{aligned}
J^\mu_{a,A} &= J^\mu_{a,R} - J^\mu_{a,L} = \bar{\Psi}_R \gamma^\mu T_a \Psi_R - \bar{\Psi}_L \gamma^\mu T_a \Psi_L \\
&= \bar{\Psi} \mathcal{P}_L \gamma^\mu T_a \mathcal{P}_R \Psi - \bar{\Psi} \mathcal{P}_R \gamma^\mu T_a \mathcal{P}_L \Psi \\
&= \bar{\Psi} \frac{\mathbb{1} - \gamma_5}{2} \gamma^\mu T_a \Psi - \bar{\Psi} \frac{\mathbb{1} + \gamma_5}{2} \gamma^\mu T_a \Psi \\
&\equiv \bar{\Psi} \gamma^\mu \gamma_5 T_a \Psi \ ,
\end{aligned} \tag{2.156}$$

where we used the same methods as before. This current transforms as an axial-vector under Lorentz transformations, so that it is referred to as axial-vector current. Now, we can check the transformation behavior of Eqs. (2.155) and (2.156) explicitly. Under spatial reflections, the quark and anti-quark fields transform as

$$\Psi(t, \mathbf{r}) \xrightarrow{P} \gamma^0 \Psi(t, -\mathbf{r}), \qquad \bar{\Psi}(t, \mathbf{r}) \xrightarrow{P} \bar{\Psi}(t, -\mathbf{r}) \gamma^0 \ . \tag{2.157}$$

Then, the vector current transforms as

$$\begin{aligned}
J^\mu_{a,V}(t, \mathbf{r}) \xrightarrow{P} J^{\mu\prime}_{a,V}(t, \mathbf{r}) &= \bar{\Psi}(t, -\mathbf{r}) \gamma^0 \gamma^\mu T_a \gamma^0 \Psi(t, -\mathbf{r}) \\
&= (-1)^{(\mu)} \bar{\Psi}(t, -\mathbf{r}) \gamma^\mu T_a \Psi(t, -\mathbf{r}) \\
&= (-1)^{(\mu)} J^\mu_{a,V}(t, -\mathbf{r}) \ ,
\end{aligned} \tag{2.158}$$

where we used the anticommutation (6.28) satisfied by the Dirac matrices. We have to note that the factor $(-1)^{(\mu)}$ is not a tensorial structure, so that (μ) does not describe a Lorentz index. This factor is defined to be equal to 1, if $(\mu) = 0$ and to be equal to -1, if $(\mu) = 1, 2, 3$. Obviously, the vector current (2.155) actually transforms as a vector under parity transformations. Now, we consider the transformation behavior of the axial-vector current

$$\begin{aligned}
J^\mu_{a,A}(t, \mathbf{r}) \xrightarrow{P} J^{\mu\prime}_{a,A}(t, \mathbf{r}) &= \bar{\Psi}(t, -\mathbf{r}) \gamma^0 \gamma^\mu \gamma_5 T_a \gamma^0 \Psi(t, -\mathbf{r}) \\
&= -(-1)^{(\mu)} \bar{\Psi}(t, -\mathbf{r}) \gamma^\mu \gamma_5 T_a \Psi(t, -\mathbf{r}) \\
&= -(-1)^{(\mu)} J^\mu_{a,A}(t, -\mathbf{r}) \ ,
\end{aligned} \tag{2.159}$$

where we again used the anticommutation relations $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$ and $[\gamma^\mu, \gamma_5]_+ = 0$. We recognize that this current truly transforms as an axial-vector under parity transformations. These $2N_f^2$ currents correspond to the symmetry group $U(N_f)_V \times U(N_f)_A = SU(N_f)_V \times SU(N_f)_A \times U(1)_V \times U(1)_A$, which is isomorphic to the chiral group $U(N_f)_R \times U(N_f)_L = SU(N_f)_R \times SU(N_f)_L \times U(1)_R \times U(1)_L$. The vector and axial-vector transformations are given by

$$U_{V/A} = e^{-i\alpha^a_{V/A} T_a} \ , a = 0, \ldots, N_f^2 - 1 \ , \tag{2.160}$$

where $\alpha^a_V = \alpha^a_R + \alpha^a_L$ and $\alpha^a_A = \alpha^a_R - \alpha^a_L$. It turns out that the axial-vector symmetry is explicitly broken by the quark mass term. To study the reason for this phenomenon, we will now consider the full QCD Lagrangian (2.145). It is obvious that the left- and right-handed currents will not change by the presence of the mass terms. Therefore, the vector and axial-vector currents will also stay the same. But, not all of these currents will be conserved anymore, because the mass terms will now add a contribution. In order to see this, we start with the divergence of the left- and right-handed currents. For the left-handed currents, we obtain

$$\begin{aligned}
\partial_\mu J^\mu_L &= \frac{\partial \mathscr{L}_{Quark}}{\partial \Psi_L} \delta\Psi_L + \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \Psi_L)} (\partial_\mu \delta \Psi_L) + (\partial_\mu \delta\bar{\Psi}_L) \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \bar{\Psi}_L)} + \delta\bar{\Psi}_L \frac{\partial \mathscr{L}_{Quark}}{\partial \bar{\Psi}_L} \\
&= \left[g\bar{\Psi}_L \gamma^\lambda \mathscr{A}_\lambda - \bar{\Psi}_R m\right](-i)\alpha^a_L T_a \Psi_L + \bar{\Psi}_L i\gamma^\lambda g_\lambda^{\ \mu}(-i)\alpha^a_L T_a \partial_\mu \Psi_L + i\bar{\Psi}_L \alpha^a_L T_a \left[i\gamma^\lambda D_\lambda \Psi_L - m\Psi_R\right] \\
&= i\bar{\Psi}_R m \alpha^a_L T_a \Psi_L - i\bar{\Psi}_L \alpha^a_L T_a m \Psi_R \\
&= \alpha^a_L i \left(\bar{\Psi}_R m T_a \Psi_L - \bar{\Psi}_L T_a m \Psi_R\right) \\
&\equiv \alpha^a_L \partial_\mu J^\mu_{a,L} \ ,
\end{aligned} \tag{2.161}$$

where we again used the Euler-Lagrange equations (2.8). Similarly, we obtain the 4-divergence of the right-handed currents

$$\begin{aligned}
\partial_\mu J^\mu_R &= \frac{\partial \mathscr{L}_{Quark}}{\partial \Psi_R} \delta\Psi_R + \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \Psi_R)} (\partial \delta \Psi_R) + (\partial \delta\bar{\Psi}_R) \frac{\partial \mathscr{L}_{Quark}}{\partial(\partial_\mu \bar{\Psi}_R)} + \delta\bar{\Psi}_R \frac{\partial \mathscr{L}_{Quark}}{\partial \bar{\Psi}_R} \\
&= \left[g\bar{\Psi}_R \gamma^\lambda \mathscr{A}_\lambda - \bar{\Psi}_L m\right](-i)\alpha^a_R T_a \Psi_R + \bar{\Psi}_R i\gamma^\lambda g_\lambda^{\ \mu}(-i)\alpha^a_R T_a \partial_\mu \Psi_R + i\bar{\Psi}_R \alpha^a_R T_a \left[i\gamma^\lambda D_\lambda \Psi_R - m\Psi_L\right] \\
&= i\bar{\Psi}_L m \alpha^a_R T_a \Psi_R - i\bar{\Psi}_R \alpha^a_R T_a m \Psi_L \\
&= \alpha^a_R i \left(\bar{\Psi}_L m T_a \Psi_R - \bar{\Psi}_R T_a m \Psi_L\right) \\
&\equiv \alpha^a_R \partial_\mu J^\mu_{a,R} \ .
\end{aligned} \tag{2.162}$$

Now, using the results (2.161) and (2.162), we are able to calculate the 4-divergences of the vector currents

$$\begin{aligned}
\partial_\mu J^\mu_{a,V} &= \partial_\mu \left(J^\mu_{a,L} + J^\mu_{a,R}\right) \\
&= i\left(\bar{\Psi}_R m T_a \Psi_L - \bar{\Psi}_L T_a m \Psi_R\right) + i\left(\bar{\Psi}_L m T_a \Psi_R - \bar{\Psi}_R T_a m \Psi_L\right) \\
&= i\bar{\Psi}\frac{\mathbb{1} - \gamma_5}{2}[m, T_a]_- \Psi + i\bar{\Psi}\frac{\mathbb{1} + \gamma_5}{2}[m, T_a]_- \Psi \\
&= i\bar{\Psi}[m, T_a]_- \Psi \ ,
\end{aligned} \tag{2.163}$$

and the 4-divergences of the axial-vector currents

$$\begin{aligned}
\partial_\mu J^\mu_{a,A} &= \partial_\mu \left(J^\mu_{a,R} - J^\mu_{a,L}\right) \\
&= i\left(\bar{\Psi}_L m T_a \Psi_R - \bar{\Psi}_R T_a m \Psi_L\right) - i\left(\bar{\Psi}_R m T_a \Psi_L - \bar{\Psi}_L T_a m \Psi_R\right) \\
&= i\bar{\Psi}\frac{\mathbb{1} + \gamma_5}{2}[m, T_a]_+ \Psi - i\bar{\Psi}\frac{\mathbb{1} - \gamma_5}{2}[m, T_a]_+ \Psi \\
&= i\bar{\Psi}[m, T_a]_+ \gamma_5 \Psi \ ,
\end{aligned} \tag{2.164}$$

where we used the relations (2.143) and the definitions of the left- and right-handed fields (2.144). In the above equations, we only used the currents $J^\mu_{a,L}$ and $J^\mu_{a,R}$without group parameters. These equations require that $\alpha^a_R = \alpha^a_L$. But, if we take equation (2.160) into account, we see that this requirement leads to vanishing group parameters for the axial-vector transformations. This fact shows that the symmetry of Eq. (2.145) under vector transformations is explicitly broken by the quark mass terms.

Now, we are able to discuss the results (2.155), (2.156), (2.163), and (2.164). To this end, we start with the vector currents (2.155), which follow from the invariance of Eq. (2.145) under $SU(N_f)_V \times U(1)_V$ transformations. It is obvious that those currents are only conserved, if the commutator $[m, T_a]_-$ in Eq. (2.163) vanishes. In general, there are only three cases in which this commutator can be equal to zero:

(i) The quark mass matrix vanishes,

(ii) The generators are proportional to the $(N_f \times N_f)$ unit matrix,

(iii) The quark mass matrix is proportional to the $(N_f \times N_f)$ unit matrix.

Case (i) corresponds to the chiral limit of the QCD Lagrangian. We already studied this case in Eq. (2.147) and the following discussion. We showed that this case leads to conserved currents, because in this limit the chiral symmetry becomes an exact symmetry of the QCD Lagrangian, compare Eq. (2.154).

Let us come to the second case (ii). This scenario is only satisfied for the generator T_0 of the $U(1)_V$ transformations. As mentioned in the discussion of Eq. (2.148), the remaining $N_f^2 - 1$ generators are proportional to $(N_f \times N_f)$ generalizations of the Gell-Mann matrices and are therefore not proportional to the identity in flavor space. Obviously, the vector current

$$J^{\mu}_{0,V} = \bar{\Psi}(x)\gamma^{\mu}\Psi(x) \; . \tag{2.165}$$

originating from the $U(1)_V$ transformations, is always conserved. A conserved charge of this form is well known from the Dirac theory. The corresponding conserved charge $Q_{0,V}$ is given by the baryon number.

Now, we turn to the last case (iii). This case is obtained by the assumption that all quark masses are equal. Then, the mass matrix becomes proportional to the unit matrix in flavor space, i.e., $(m_{ij}) \to m(\delta_{ij})$. The $U(N_f)$ generators will, of course, commute with the unit matrix and we obtain N_f^2 conserved currents. In nature, the assumption of equal quark masses is only legitimate in the light quark sector, see Tab. [2.1], where the differences of the quark masses are negligible in contrast to a typical hadronic mass scale. If we just consider the two lightest quark flavors, up and down, the assumption of equal quark masses is very well satisfied, because the difference $\delta m = m_d - m_u$ is small compared to a mass scale of $m_H \sim 1$ GeV. This leads to the so-called $SU(2)_V$ isospin symmetry of strong interactions. If we also include the strange quark, the mass differences $\delta m_{su} = m_s - m_u$ and $\delta m_{sd} = m_s - m_d$ become larger than in the isospin case. But in comparison with the 1 GeV scale, these differences are of order 10%, therefore this $SU(3)_V$ symmetry is also approximately fulfilled. This leads to the so-called flavor symmetry introduced by Gell-Mann, Ne'eman, and Zweig. We will come back to this scenario later.

For the moment, we turn back to the axial-vector currents. According to Eq. (2.164), the divergences of the axial-vector currents are proportional to the anticommutator of the quark mass matrix and the $U(N_f)_A$ generators. In order to obtain conserved axial-vector currents, this anticommutator has to vanish. In general, this is only possible, if the quark mass matrix is equal to zero. Therefore, we conclude that the axial-vector currents are only conserved, if we consider the QCD Lagrangian in the chiral limit, $m = 0$. But this statement is only valid, when we consider the QCD Lagrangian (2.145) at the classical level. At quantum level, the $U(1)_A$ symmetry is never conserved as a consequence of non-perturbative quantum effects. Such a symmetry which is only present at the classical level and disturbed at the quantum level is called anomaly. Therefore, at quantum level, the full symmetry group of the QCD Lagrangian in the chiral limit is given by $SU(N_f)_V \times SU(N_f)_A \times U(1)_V$.

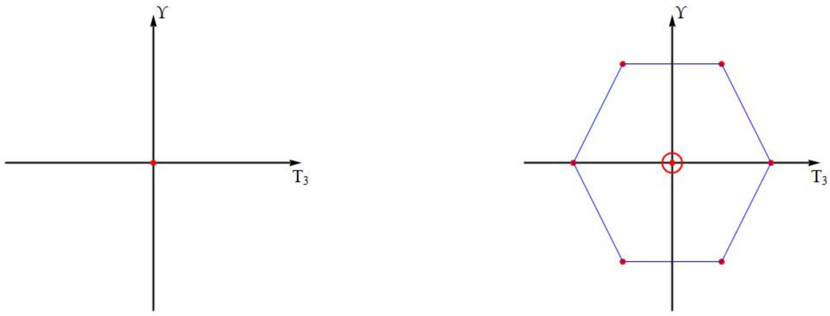
At the end of this section, we want to discuss the $SU(3)_V$ flavor symmetry of QCD. To this end, we only consider three flavors QCD. As mentioned above, the masses of up, down, and strange quark are small compared to a hadronic mass scale of ~ 1 GeV, so that the QCD Lagrangian exhibits an approximate $SU(3)_V \times SU(3)_A \times U(1)_V$ symmetry. But then, this symmetry is also present in the QCD Hamiltonian, so that H_{QCD} commutes with the charge operators $Q_{a,V}, Q_{a,A}$ and $Q_{0,V}$, $a = 1, \ldots, 8$, acting on the Hilbert space of the system. Now, the vector and axial-vector charges have opposite parity, compare Eqs. (2.158) and (2.159). Thus, the commutation relations of $Q_{a,V}$ and $Q_{a,A}$ with H_{QCD} imply the existence of degenerate states with positive and negative parity, but with equal baryon number, strangeness, and spin. This phenomenon of parity doubling is not observed in the hadron spectrum.

A second interesting point is that, due to the symmetry of $\mathscr{L}_{QCD}$, we would expect that the light hadrons can be arranged into irreducible representations of $SU(3)_V \times SU(3)_A \times U(1)_V$. As stated previously, the $U(1)_V$ symmetry corresponds to the conservation of baryon number. This conservation law implies that the obtained multiplets can be divided into $SU(3)_V \times SU(3)_A$ multiplets of different baryon number. For $B = 0$ we would obtain the multiplets of mesons and for $B = 1$ we would obtain those for baryons. But in nature we do not observe the multiplets of $SU(3)_V \times SU(3)_A$, but only those of $SU(3)_V$. These two points indicate a spontaneous breakdown of the chiral $SU(3)_V \times SU(3)_A$ symmetry down to its $SU(3)_V$ subgroup.

The spontaneous breakdown of chiral symmetry is exhibited by the octet of pseudoscalar mesons. According to the discussion of Sec. [2.2.3], we expect the appearance of eight Nambu-Goldstone bosons corresponding to the eight broken generators of $SU(3)_A$. Now, the pseudoscalar octet contains the pion isotriplet

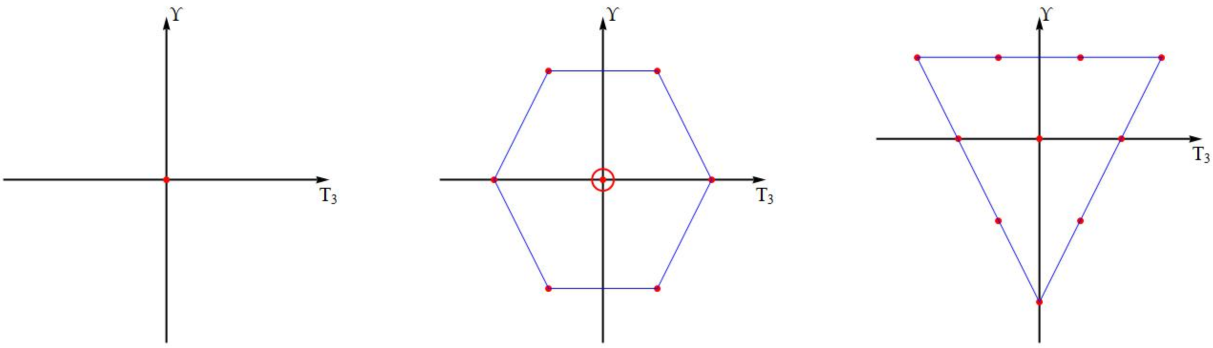
$\{\pi^-, \pi^0, \pi^+\}$, two kaon isodoublets $\{K^+, K^0\}$ and $\{K^-, \bar{K}^0\}$, and the isosinglet $\{\eta\}$. These eight mesons are much lighter than all other mesons and are considered as the Nambu-Goldstone bosons of spontaneous chiral symmetry breaking. Strictly spoken, these mesons are pseudo-Nambu-Goldstone bosons, because they obtain nonzero masses. Those masses arises from the fact that the chiral symmetry is not an exact, but explicitly broken global symmetry of the QCD Lagrangian.

The symmetry under the $SU(3)_V$ subgroup of the chiral group is called flavor symmetry. The quark and antiquark fields build the fundamental triplet $[3]_f$ and antitriplet $[\bar{3}]_f$ of $SU(3)_V$. For mesons, i.e., $q\bar{q}$-states, we can couple the antitriplet with the triplet to obtain a singlet and an octet, $[3]_f \otimes [\bar{3}]_f = [1]_f \oplus [8]_f$. In the case of baryons, we couple three quark triplets to obtain a singlet, two octets, and one decuplet, $[3]_f \otimes [3]_f \otimes [3]_f = [1]_f \oplus [8]_f \oplus [8]_f \oplus [10]_f$. The general forms of those multiplets, represented in the strong isospin-hypercharge plane, are depicted in Figs. [2.8] and [2.9].



(a) (b)

Figure 2.8: Flavor-multiplets for mesons in the (T_3, Y)-plane. In this figure, we neglect the quark content of the different meson resonances.



(a) (b) (c)

Figure 2.9: Flavor-multiplets for baryons in the (T_3, Y)-plane. Again, we neglect the quark content of the different baryon resonances.

Chapter 3

Chiral Perturbation Theory

The last Section [2.3] of the past chapter was dedicated to the theory of strong interactions, Quantum Chromodynamics. We constructed the QCD Lagrangian by using the gauge principle and investigated the so-called chiral symmetry of QCD. In addition to that, we argued that this symmetry must be spontaneously broken. This spontaneous chiral symmetry breaking will be a central topic of the upcoming chapter.

At this point, we could start to investigate strong processes in the framework of QCD by using a perturbative expansion in powers of the strong coupling $\alpha_S \equiv g^2/4\pi$. This approach is valid as long as the coupling α_S is small, so that the interaction can be treated as a perturbation. In the high-energy end of the QCD spectrum, where the interactions are mostly dominated by quarks and gluons, this method works very well - while in the low-energy regime the situation changes. For small energies, the strong coupling becomes very large, so that the interaction cannot be treated as a perturbation. Since we are interested in an analysis of the low-energy properties of QCD, we need to find a technique, which does not rely on a perturbative expansion in powers of the QCD coupling α_S.

In fact, there exist two non-perturbative methods. The first is given by numerical simulations on a lattice, where the whole theory is discretized on a four-dimensional hypercube. The second approach is given by the framework of Effective Field Theory (EFT), which simplifies the initial theory by considering only those degrees of freedom that are important at the energy scale one is interested in. This second approach will be the focus of this work.

The main objective of this chapter is to introduce Chiral Perturbation Theory (ChPT) as a powerful tool to investigate the low-energy regime of strong interactions. Therefore, we start with the basic ideas of ChPT in Sec. [3.1]. This Section includes a brief overview of the basic ingredients of ChPT: the hadronic n-point functions and the Ward-Fradkin-Takahashi identities. It turns out, that the low-energy dynamics of QCD is fully determined by the interaction of the Nambu-Goldstone bosons of chiral symmetry breaking among themselves. The aim of Sec. [3.2] is to introduce the basic mathematical formalism which will be needed to build an effective Lagrangian of a theory with spontaneous symmetry breaking. Finally, in Sec. [3.3], we will combine the results of Sec. [3.1] and [3.2] to construct the next-to-leading order (NLO) Lagrangian of ChPT.

3.1 The Basic Concepts of Chiral Perturbation Theory

In order to outline the basic concepts of ChPT, we start with a short summary of some important results of Sec. [2.3]. To this end, we start with the classical QCD Lagrangian and switch quark masses off. We saw that in this case, the fermionic part of the QCD Lagrangian possesses a global $SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R$ symmetry which originated from the fact that the left- and the right-handed parts of the quark spinors can be rotated independently of each other. As already mentioned, this symmetry is called chiral symmetry. However, this symmetry group is isomorphic to the $SU(N_f)_V \times SU(N_f)_A \times U(1)_V \times U(1)_A$ group of vector and axial-vector transformations. The $2N_f^2$ conserved currents which are associated with these symmetry transformations are given by Eqs. (2.155) and (2.156). If we now consider the QCD Lagrangian at the quantum level, it turns out that the $U(1)_A$ symmetry does not survive quantization. This anomaly results from non-perturbative quantum effects which are associated with instantons. Therefore, at the quantum level, the full symmetry group of the QCD Lagrangian is given by $SU(N_f)_V \times SU(N_f)_A \times U(1)_V$. In the following step, we allow the quarks to have non-vanishing masses and consider the full QCD Lagrangian. As already seen in Eqs. (2.163) and (2.164), the vector and axial-vector currents are not conserved anymore. In particular, when we consider Eq. (2.164), we find that for nonvanishing quark masses, there is no chance that the remaining $N_f^2 - 1$ axial-vector currents are conserved. From Eq. (2.163), we obtain that the $N_f^2 - 1$ currents of the vector transformations are only conserved, if the different quark flavors have identical masses. In addition to that, we realize that the current associated with the $U(1)_V$ transformations is always conserved, as it must be, since this current expresses the conservation of the baryon number. However, this symmetry is not important for the following discussion, so that we restrict ourselves to the remaining $SU(N_f)_V \times SU(N_f)_A$ group. Now, let us turn to the light quark sector, in particular to that part of the QCD Lagrangian, which includes the two lightest quark flavors up and down. Compared to a typical hadronic mass scale $\Lambda_{Hadron} = 1$ GeV, the masses of the up and the down quark can be assumed to be identical, since the difference $\delta m_{ud} = m_d - m_u \approx 2.5$ MeV of their masses is vanishingly small compared to Λ_{Hadron}. Therefore, in the case of the two lightest quark flavors, we obtain an approximate $SU(N_f = 2)_V$ symmetry. This symmetry is well known as the isotopic spin symmetry or isospin symmetry of strong interactions. At this point, it is important to recognize that not only the mass difference δm_{ud} is very small compared to Λ_{Hadron}, but also the masses m_u and m_d itself. Therefore, we extend our assumption to the case in which up and down quark not only have identical, but vanishing masses. This assumption gives rise to an approximate $SU(2)_V \times SU(2)_A$ symmetry, since the three axial-vector currents are also conserved in the case of vanishing quark masses. In the discussion of Sec. [2.3], we already mentioned that there are strong arguments that this chiral $SU(2)_V \times SU(2)_A$ symmetry must be spontaneously broken down to its isospin subgroup $SU(2)_V$. An important object in this context is the vacuum expectation value of the singlet scalar quark condensate

$$\langle\Omega|\bar{\Psi}\Psi|\Omega\rangle \ . \tag{3.1}$$

In fact, it can be shown that a non-vanishing singlet scalar quark condensate is a sufficient condition for spontaneous chiral symmetry breaking. We will return to this important point in Sec. [3.1.2]. For the moment, we interrupt this discussion and turn to other important quantities in Quantum Field Theory, which also play a crucial role in the framework of ChPT, the n-point Green's functions.

3.1.1 n-Point Green's Functions and Pion Pole Dominance

The n-point Green's functions[1] are defined as the time-ordered vacuum expectation values of Heisenberg operators $O_i(x_i)$. The great importance of these n-point functions arises from the fact that the so-called Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism relates them to scattering matrix elements and therefore to physical processes. In our case, we are interested in hadronic processes, in particular in low-energy hadronic processes. The n-point functions corresponding to hadronic processes cannot be formed with the field operators which are included in the QCD Lagrangian, since Eq. (2.139) does not contain field operators describing hadrons, but only field operators describing the fundamental particles of strong interactions, namely quarks and gluons. The relevant n-point functions for our purpose are those which are formed with the time-ordered products of color neutral and hermitian bilinear forms. To be more precise, we are interested in n-point functions formed with the following objects

$$J_V^{\mu,a}(x) = \bar{\Psi}(x)\gamma^\mu T^a \Psi(x) \ , \tag{3.2}$$
$$J_A^{\mu,a}(x) = \bar{\Psi}(x)\gamma^\mu \gamma_5 T^a \Psi(x) \ , \tag{3.3}$$
$$J_S^{a}(x) = \bar{\Psi}(x) T^a \Psi(x) \ , \tag{3.4}$$
$$J_P^{a}(x) = i\bar{\Psi}(x)\gamma_5 T^a \Psi(x) \ , \tag{3.5}$$

where $a = 0, 1, 2, 3$ and $\Psi(x) = (u(x), d(x))^T$. The generators are defined as $T^0 = \mathbb{1}/2, T^i = \tau^i/2$, where τ^i, $i = 1, 2, 3$, denote the usual Pauli matrices, compare [6.1.1] and [6.1.2]. The assignment of the scalar (3.4), the pseudoscalar (3.5), the vector (3.2), and the axial-vector (3.3) bilinear forms to physical mesons which we want to describe, depends on the quantum numbers of the physical states. In order to investigate the low-energy behavior of these n-point functions, we consider a general momentum-space n-point function which involves the time-ordered product of arbitrary operators $O_i(x_i)$

$$G^{(n)}(p_1, \ldots, p_n) = \int \prod_{j=1}^{n} \mathrm{d}^4 x_j \ e^{ip_j x_j} \ \langle\Omega|T\left\{O_1(x_1)\cdots O_n(x_n)\right\}|\Omega\rangle \ . \tag{3.6}$$

[1] We will often call them n-point functions, instead of n-point Green functions.

The vacuum expectation value in the above expression can be rewritten as

$$
\begin{aligned}
&\langle\Omega|T\left\{O_1(x_1)\cdots O_n(x_n)\right\}|\Omega\rangle\\
&=\Theta\left(\min(x_1^0,\ldots,x_m^0)-\max(x_{m+1}^0,\ldots,x_n^0)\right)\langle\Omega|T\left\{O_1(x_1)\cdots O_m(x_m)\right\}T\left\{O_{m+1}(x_{m+1})\cdots O_n(x_n)\right\}|\Omega\rangle\\
&\quad+\ldots\\
&=\sum_s\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\Theta\left(\min(x_1^0,\ldots,x_m^0)-\max(x_{m+1}^0,\ldots,x_n^0)\right)\\
&\quad\times\langle\Omega|T\left\{O_1(x_1)\cdots O_m(x_m)\right\}|\psi_{s,\mathbf{k}}\rangle\langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(x_{m+1})\cdots O_n(x_n)\right\}|\Omega\rangle+\ldots\ ,
\end{aligned}
\tag{3.7}
$$

where we inserted an on-shell single particle state of mass M and spin s. It is clear that the inserted complete set of states involves all multi-particle states, not just only the one-particle state. Therefore, the "..." of the second line stand for all possible multi-particle states including all possible time-orderings, while the "..." of the first line only denote the other possible time-orderings. In addition to that, we assume that the matrix elements

$$\langle\Omega|T\left\{O_1(x_1)\cdots O_m(x_m)\right\}|\psi_{s,\mathbf{k}}\rangle \text{ and } \langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(x_{m+1})\cdots O_n(x_n)\right\}|\Omega\rangle \tag{3.8}$$

are different from zero. Now, exploiting the space-time translational invariance[2] of the Heisenberg operators $O_i(x_i)$, i.e.,

$$e^{ia_\mu P^\mu}O_i(x_i)e^{-ia_\mu P^\mu}=O_i(x_i+a)\ , \tag{3.9}$$

the first matrix element in Eq. (3.8) can be rewritten as

$$
\begin{aligned}
&\langle\Omega|T\left\{O_1(x_1)\cdots O_m(x_m)\right\}|\psi_{s,\mathbf{k}}\rangle\\
&=\langle\Omega|T\left\{e^{ix_1P}e^{-ix_1P}O_1(x_1)e^{ix_1P}\cdots e^{-ix_1P}O_m(x_m)e^{ix_1P}e^{-ix_1P}\right\}|\psi_{s,\mathbf{k}}\rangle\\
&=e^{-ix_1k}\langle\Omega|T\left\{O_1(0)O_2(x_2-x_1)\cdots O_m(x_m-x_1)\right\}|\psi_{s,\mathbf{k}}\rangle\ ,
\end{aligned}
\tag{3.10}
$$

where we suppressed the space-time indices in the exponentials and used that the vacuum and the one-particle state are eigenstates of the four-momentum operator. In an analogous way, the second matrix element in Eq. (3.8) can be written as

$$
\begin{aligned}
&\langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(x_{m+1})\cdots O_n(x_n)\right\}|\Omega\rangle\\
&=\langle\psi_{s,\mathbf{k}}|T\left\{e^{ix_{m+1}P}e^{-ix_{m+1}P}O_{m+1}(x_{m+1})e^{ix_{m+1}P}\cdots e^{-ix_{m+1}P}O_n(x_n)e^{ix_{m+1}P}e^{-ix_{m+1}P}\right\}|\Omega\rangle\\
&=e^{ix_{m+1}k}\langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(0)O_{m+2}(x_{m+2}-x_{m+1})\cdots O_n(x_n-x_{m+1})\right\}|\Omega\rangle\ .
\end{aligned}
\tag{3.11}
$$

Combining Eqs. (3.10) and (3.11) with Eq. (3.7) and inserting this result into Eq. (3.6), we obtain

$$
\begin{aligned}
&G^{(n)}(p_1,\ldots,p_n)\\
&=\int\prod_{j=1}^{n}\mathrm{d}^4x_j\ e^{ip_jx_j}\sum_s\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\Theta\left(\min(x_1^0,\ldots,x_m^0)-\max(x_{m+1}^0,\ldots,x_n^0)\right)e^{-ix_1k}e^{ix_{m+1}k}\\
&\quad\times\langle\Omega|T\left\{O_1(0)O_2(x_2-x_1)\ldots O_m(x_m-x_1)\right\}|\psi_{s,\mathbf{k}}\rangle\\
&\quad\otimes\langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(0)O_{m+2}(x_{m+2}-x_{m+1})\ldots O_n(x_n-x_{m+1})\right\}|\Omega\rangle+\ldots\ .
\end{aligned}
\tag{3.12}
$$

The above expression can be simplified by introducing new variables

$$\xi_j=\begin{cases}x_j-x_1 & ,\ j=2,\ldots,m\ ,\\ x_j-x_{m+1} & ,\ j=m+2,\ldots,n\ ,\end{cases} \tag{3.13}$$

while x_1 and x_{m+1} remain unchanged. In order to see how Eq. (3.12) changes under the coordinate transformation (3.13), we consider each part of the n-point function separately. First of all, we note that the Jacobian associated with the above change of variables is equal to one. Then, the exponentials in Eq. (3.12) become

$$
\begin{aligned}
\prod_{j=1}^{n}e^{ip_jx_j}e^{-ix_1k}e^{ix_{m+1}k}&=e^{i(p_1-k)x_1}e^{i(p_{m+1}+k)x_{m+1}}\prod_{j=2}^{m}e^{ip_j(\xi_j+x_1)}\prod_{j=m+2}^{n}e^{ip_j(\xi_j+x_{m+1})}\\
&=e^{i(p_1+\ldots+p_m-k)x_1}e^{i(p_{m+1}+\ldots+p_n+k)x_{m+1}}\prod_{\substack{j=2,\\ j\neq m+1}}^{n}e^{ip_j\xi_j}\ .
\end{aligned}
\tag{3.14}
$$

[2]Translational invariance with respect to space-time translations implies that the Heisenberg operators fulfill $[P_\mu,O_i(x_i)]_-=-i\partial_\mu^{x_i}O_i(x_i)$. Using $e^XYe^{-X}=Y+[X,Y]_-+\ldots$ in combination with the commutation relations, satisfied by the Heisenberg operators, leads to Eq. (3.9).

The next object in Eq. (3.12) we turn our attention to is the Θ-function. Using Eq. (3.13), we obtain

$$\begin{aligned}&\Theta\left(\min(x_1^0,\ldots,x_m^0)-\max(x_{m+1}^0,\ldots,x_n^0)\right)\\&=\Theta\left(\min(x_1^0,\xi_2^0+x_1^0,\ldots,\xi_m^0+x_1^0)-\max(x_{m+1}^0,\xi_{m+2}^0+x_{m+1}^0,\ldots,\xi_n^0+x_{m+1}^0)\right)\\&=\Theta\left(x_1^0-x_{m+1}^0+\min(0,\xi_2^0,\ldots,\xi_m^0)-\max(0,\xi_{m+2}^0,\ldots,\xi_n^0)\right)\\&=\int\frac{\mathrm{d}\omega}{2\pi}\frac{i}{\omega+i\epsilon}e^{-i\omega\left(x_1^0-x_{m+1}^0+\min(0,\xi_2^0,\ldots,\xi_m^0)-\max(0,\xi_{m+2}^0,\ldots,\xi_n^0)\right)}\ ,\end{aligned}\tag{3.15}$$

where we used the Fourier representation of the Θ-function

$$\Theta(\tau)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{i}{\omega+i\epsilon}e^{-i\omega\tau}\ .\tag{3.16}$$

Finally, the matrix elements (3.8) simplify to

$$\langle\Omega|T\left\{O_1(0)O_2(\xi_2)\ldots O_m(\xi_m)\right\}|\psi_{s,\mathbf{k}}\rangle\ \text{ and }\ \langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(0)O_{m+2}(\xi_{m+2})\ldots O_n(\xi_n)\right\}|\Omega\rangle\ .\tag{3.17}$$

Combining the results (3.14), (3.15), and (3.17), we obtain

$$\begin{aligned}&G^{(n)}(p_1,\ldots,p_n)\\&=\sum_s\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\int\frac{\mathrm{d}\omega}{2\pi}\int\prod_{\substack{j=2,\\ j\neq m+1}}^{n}\mathrm{d}^4\xi_j\ e^{ip_j\xi_j}\int\mathrm{d}^4x_1\mathrm{d}^4x_{m+1}\ e^{i(p_1+\ldots+p_m-k)x_1}e^{i(p_{m+1}+\ldots+p_n+k)x_{m+1}}\\&\quad\times e^{-i\omega\left(x_1^0-x_{m+1}^0+\min(0,\xi_2^0,\ldots,\xi_m^0)-\max(0,\xi_{m+2}^0,\ldots,\xi_n^0)\right)}\frac{i}{\omega+i\epsilon}\\&\quad\times\langle\Omega|T\left\{O_1(0)O_2(\xi_2)\ldots O_m(\xi_m)\right\}|\psi_{s,\mathbf{k}}\rangle\langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(0)O_{m+2}(\xi_{m+2})\ldots O_n(\xi_n)\right\}|\Omega\rangle+\ldots\\&=\sum_s\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\int\frac{\mathrm{d}\omega}{2\pi}\int\prod_{\substack{j=2,\\ j\neq m+1}}^{n}\mathrm{d}^4\xi_j\ e^{ip_j\xi_j}\int\mathrm{d}^4x_1\mathrm{d}^4x_{m+1}\ e^{-i(\mathbf{p}_1+\ldots+\mathbf{p}_m-\mathbf{k})\cdot\mathbf{x}_1}e^{-i(\mathbf{p}_{m+1}+\ldots+\mathbf{p}_n+\mathbf{k})\cdot\mathbf{x}_{m+1}}\\&\quad\times e^{i(p_1^0+\ldots+p_m^0-\omega_{\mathbf{k}}-\omega)x_1^0}e^{i(p_{m+1}^0+\ldots+p_n^0+\omega_{\mathbf{k}}+\omega)x_{m+1}^0}\frac{ie^{-i\omega\left(\min(0,\xi_2^0,\ldots,\xi_m^0)-\max(0,\xi_{m+2}^0,\ldots,\xi_n^0)\right)}}{\omega+i\epsilon}\\&\quad\times\langle\Omega|T\left\{O_1(0)O_2(\xi_2)\ldots O_m(\xi_m)\right\}|\psi_{s,\mathbf{k}}\rangle\langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(0)O_{m+2}(\xi_{m+2})\ldots O_n(\xi_n)\right\}|\Omega\rangle+\ldots\\&=(2\pi)^4\sum_s\int\mathrm{d}^3\mathbf{k}\frac{1}{2\omega_{\mathbf{k}}}\int\mathrm{d}\omega\int\prod_{\substack{j=2,\\ j\neq m+1}}^{n}\mathrm{d}^4\xi_j\ e^{ip_j\xi_j}\ \delta\left(p_1^0+\ldots+p_m^0-\omega_{\mathbf{k}}-\omega\right)\delta\left(p_{m+1}^0+\ldots+p_n^0+\omega_{\mathbf{k}}+\omega\right)\\&\quad\times\delta^{(3)}\left(\mathbf{p}_1+\ldots+\mathbf{p}_m-\mathbf{k}\right)\delta^{(3)}\left(\mathbf{p}_{m+1}+\ldots+\mathbf{p}_n+\mathbf{k}\right)\frac{ie^{-i\omega\left(\min(0,\xi_2^0,\ldots,\xi_m^0)-\max(0,\xi_{m+2}^0,\ldots,\xi_n^0)\right)}}{\omega+i\epsilon}\\&\quad\times\langle\Omega|T\left\{O_1(0)O_2(\xi_2)\ldots O_m(\xi_m)\right\}|\psi_{s,\mathbf{k}}\rangle\langle\psi_{s,\mathbf{k}}|T\left\{O_{m+1}(0)O_{m+2}(\xi_{m+2})\ldots O_n(\xi_n)\right\}|\Omega\rangle+\ldots\ .\end{aligned}\tag{3.18}$$

It is obvious that an integration over $\mathrm{d}^3\mathbf{k}$ gives a three-dimensional delta-distribution which expresses the conservation of the 3-momentum. In addition to that, the energy $\omega_{\mathbf{k}}$ of the on-shell intermediate state becomes $\omega_{\mathbf{p}_1+\ldots+\mathbf{p}_m}=\sqrt{(\mathbf{p}_1+\ldots+\mathbf{p}_m)^2+M^2}$. Defining $p_\mu=p_{1,\mu}+\ldots+p_{m,\mu}$, this energy can be written as $\omega_{\mathbf{p}}=\sqrt{\mathbf{p}^2+M^2}$, so that

$$\begin{aligned}&G^{(n)}(p_1,\ldots,p_n)\\&=\frac{(2\pi)^4}{2\omega_{\mathbf{p}}}\sum_s\int\mathrm{d}\omega\int\prod_{\substack{j=2,\\ j\neq m+1}}^{n}\mathrm{d}^4\xi_j\ e^{ip_j\xi_j}\ \delta\left(p_1^0+\ldots+p_m^0-\omega_{\mathbf{p}}-\omega\right)\delta\left(p_{m+1}^0+\ldots+p_n^0+\omega_{\mathbf{p}}+\omega\right)\\&\quad\times\delta^{(3)}\left(\mathbf{p}_1+\ldots+\mathbf{p}_n\right)\frac{ie^{-i\omega\left(\min(0,\xi_2^0,\ldots,\xi_m^0)-\max(0,\xi_{m+2}^0,\ldots,\xi_n^0)\right)}}{\omega+i\epsilon}\\&\quad\times\langle\Omega|T\left\{O_1(0)O_2(\xi_2)\ldots O_m(\xi_m)\right\}|\psi_{s,\mathbf{p}}\rangle\langle\psi_{s,\mathbf{p}}|T\left\{O_{m+1}(0)O_{m+2}(\xi_{m+2})\ldots O_n(\xi_n)\right\}|\Omega\rangle+\ldots\ .\end{aligned}\tag{3.19}$$

Apparently, the above expression of the n-point function has a pole at $\omega=0$. Near this pole, the contribution of the exponential in the numerator is one, so that we are able to drop it. This allows us to solve the ω

integration in order to obtain

$$
\begin{aligned}
& G^{(n)}(p_1,\ldots,p_n) \\
&= \frac{(2\pi)^4}{2\omega_{\mathbf{p}}} \sum_s \int \prod_{\substack{j=2,\\ j\neq m+1}}^{n} \mathrm{d}^4\xi_j \; e^{ip_j\xi_j} \; \delta^{(4)}\left(p_1+\ldots+p_n\right) \frac{i}{p_1^0+\ldots+p_m^0-\omega_{\mathbf{p}}+i\epsilon} \\
&\quad \times \langle\Omega| T\left\{O_1(0)O_2(\xi_2)\ldots O_m(\xi_m)\right\}|\psi_{s,\mathbf{p}}\rangle \langle\psi_{s,\mathbf{p}}| T\left\{O_{m+1}(0)O_{m+2}(\xi_{m+2})\ldots O_n(\xi_n)\right\}|\Omega\rangle + \ldots \; . \qquad (3.20)
\end{aligned}
$$

The fraction in Eq. (3.20) can be rewritten as follows

$$
\begin{aligned}
\frac{i}{p_1^0+\ldots+p_m^0-\omega_{\mathbf{p}}+i\epsilon} &= \frac{i}{p_1^0+\ldots+p_m^0-\sqrt{\left(\mathbf{p}_1+\ldots+\mathbf{p}_m\right)^2+M^2}+i\epsilon} \\
&= \frac{i\left(p^0+\sqrt{\mathbf{p}^2+M^2}\right)}{\left(p^0\right)^2-\left(\mathbf{p}^2+M^2\right)+i\epsilon} \\
&= \frac{2\omega_{\mathbf{p}} i}{p^2-M^2+i\epsilon} \; , \qquad (3.21)
\end{aligned}
$$

where we frequently redefined ϵ, which is valid, since ϵ is an arbitrary positive infinitesimal real number. In addition to that, we used in the last line that in the vicinity of the pole, p^0 is equal to $\omega_{\mathbf{p}}$. Inserting (3.21) into (3.20), we finally obtain

$$
\begin{aligned}
& G^{(n)}(p_1,\ldots,p_n) \\
&= \frac{(2\pi)^4}{2\omega_{\mathbf{p}}} \sum_s \int \prod_{\substack{j=2,\\ j\neq m+1}}^{n} \mathrm{d}^4\xi_j \; e^{ip_j\xi_j} \; \delta^{(4)}\left(p_1+\ldots+p_n\right) \frac{2\omega_{\mathbf{p}} i}{p^2-M^2+i\epsilon} \\
&\quad \times \langle\Omega| T\left\{O_1(0)O_2(\xi_2)\ldots O_m(\xi_m)\right\}|\psi_{s,\mathbf{p}}\rangle \langle\psi_{s,\mathbf{p}}| T\left\{O_{m+1}(0)O_{m+2}(\xi_{m+2})\ldots O_n(\xi_n)\right\}|\Omega\rangle + \ldots \\
&= (2\pi)^4\delta^{(4)}\left(p_1+\ldots+p_n\right) \frac{i}{p^2-M^2+i\epsilon} \sum_s \int \prod_{\substack{j=2,\\ j\neq m+1}}^{n} \mathrm{d}^4\xi_j \; e^{ip_j\xi_j} \\
&\quad \times \langle\Omega| T\left\{O_1(0)O_2(\xi_2)\ldots O_m(\xi_m)\right\}|\psi_{s,\mathbf{p}}\rangle \langle\psi_{s,\mathbf{p}}| T\left\{O_{m+1}(0)O_{m+2}(\xi_{m+2})\ldots O_n(\xi_n)\right\}|\Omega\rangle + \ldots \; . \qquad (3.22)
\end{aligned}
$$

First of all, it should be emphasized that the equal signs in Eqs. (3.20) - (3.22) are only valid in the vicinity of the pole, so that Eq. (3.22) determines the residue of the pole of the initial n-point function. Considering (3.22), we conclude that, for $p_\mu = p_{1,\mu}+\ldots+p_{m,\mu} = -p_{m+1,\mu}-\ldots-p_{n,\mu}$ and non-vanishing matrix elements (3.17), n-point functions always have poles when on-shell intermediate particles can be created. In order to understand the great importance of this result, we have to turn back to the previous discussion, namely to the spontaneous chiral symmetry breaking in QCD: According to the Goldstone theorem, Sec. [2.2.3], this spontaneously broken symmetry gives rise to the occurrence of $\dim\left(SU(2)_V \times SU(2)_A\right) - \dim\left(SU(2)_V\right) = 3$ massless pseudoscalar bosons. But, since the $SU(2)_V \times SU(2)_A$ chiral symmetry is only an approximate symmetry, these bosons acquire small but nonzero masses, so that they are often referred to as pseudo-Nambu-Goldstone bosons. These pseudo-Nambu-Goldstone bosons are identified as the three pions, since they are significantly lighter than all other particles of the hadronic spectrum and have these quantum numbers we are looking for, i.e., they are spin-0 bosons with negative parity and transform as a triplet under the unbroken isospin subgroup. Especially the fact that the pions are the lightest particles of the QCD spectrum is important for us, since they are the first degrees of freedom which we observe at low energies. Therefore, at low energies, the hadronic n-point functions formed with Eqs. (3.2) - (3.5) are dominated by a pole, Eq. (3.22), which is generated by the pions. Or formulated differently: While the high-energy end of the QCD spectrum is dominated by quarks and gluons, the low-energy dynamics of strong interactions is governed by the exchange of the pseudo-Nambu-Goldstone bosons of chiral symmetry breaking, i.e., by the exchange of pions.

3.1.2 The Singlet Scalar Quark Condensate

In the introduction of Sec. [3.1], we stated that a non-vanishing vacuum expectation value of the singlet scalar quark condensate is a sufficient condition for the spontaneous breakdown of the chiral symmetry of QCD. In order to prove this statement, we start with the zero-component of the axial-vector current (2.156)

$$
Q_A^a(t,\mathbf{r}) = \int \mathrm{d}^3\mathbf{r} \; \bar{\Psi}(t,\mathbf{r})\gamma_0\gamma_5 T^a \Psi(t,\mathbf{r}) \; , \qquad (3.23)
$$

the definitions of the scalar and pseudoscalar bilinear forms (3.4) and (3.5) and consider the chiral limit of QCD. In addition to that, we assume the vacuum expectation value of the singlet scalar quark bilinear to be different from zero, i.e.,

$$\langle\Omega|J_S^0(x)|\Omega\rangle = \langle\Omega|\bar{\Psi}(x)T^0\Psi(x)|\Omega\rangle = \frac{1}{2}\langle\Omega|\bar{\Psi}(0)\Psi(0)|\Omega\rangle \equiv \frac{1}{2}\langle\Omega|\bar{\Psi}\Psi|\Omega\rangle \neq 0\ , \tag{3.24}$$

where we used the translational invariance of $J_S^0(x)$. Now, we evaluate the equal-time commutation relation of Eq. (3.23) and the pseudoscalar quark bilinear form (3.5)

$$\begin{aligned}
& i\left[Q_A^a(t,\mathbf{r}),P^a(t,\mathbf{r}')\right]_- \\
&= i\int \mathrm{d}^3\mathbf{r}\ \left[\bar{\Psi}(t,\mathbf{r})\gamma_0\gamma_5T^a\Psi(t,\mathbf{r}), i\bar{\Psi}(t,\mathbf{r}')\gamma_5T^a\Psi(t,\mathbf{r}')\right]_- \\
&= i\int \mathrm{d}^3\mathbf{r}\ \left[\Psi^\dagger(t,\mathbf{r})_{\alpha,i}\,(\gamma_5)^{\alpha\beta}\,(T^a)^{ij}\,\Psi(t,\mathbf{r})_{\beta,j}, i\Psi^\dagger(t,\mathbf{r}')_{\alpha',i'}\,(\gamma_0)^{\alpha'\beta'}\,(\gamma_5)_{\beta'\gamma'}\,(T^a)^{i'j'}\,\Psi(t,\mathbf{r}')^{\gamma'}_{\ j'}\right]_- \\
&= i^2\,(\gamma_5)^{\alpha\beta}\,(T^a)^{ij}\,(\gamma_0)^{\alpha'\beta'}\,(\gamma_5)_{\beta'\gamma'}\,(T^a)^{i'j'}\int \mathrm{d}^3\mathbf{r}\left[\Psi^\dagger(t,\mathbf{r})_{\alpha,i}\Psi(t,\mathbf{r})_{\beta,j},\Psi^\dagger(t,\mathbf{r}')_{\alpha',i'}\Psi(t,\mathbf{r}')^{\gamma'}_{\ j'}\right]_-\ ,
\end{aligned}\tag{3.25}$$

where we explicitly denote the spinor and isospin indices. In addition to that, it should be taken into account that no summation over a is implied in the above equation. Furthermore, the index a runs only from one to three, since the singlet axial-vector current carries an anomaly. The commutator in Eq. (3.25) can be evaluated by using

$$[AB,CD]_- = A\,[B,C]_+\,D - AC\,[B,D]_+ + [A,C]_+\,DB - C\,[A,D]_+\,B \tag{3.26}$$

and the usual equal-time anticommutation relations[3] of the quark operators

$$\left[\Psi(t,\mathbf{r})_{\alpha,i},\Psi^\dagger(t,\mathbf{r}')_{\beta,j}\right]_+ = \delta_{\alpha\beta}\delta_{ij}\delta^{(3)}(\mathbf{r}-\mathbf{r}')\ ,\quad [\Psi(t,\mathbf{r})_{\alpha,i},\Psi(t,\mathbf{r}')_{\beta,j}]_+ = \left[\Psi^\dagger(t,\mathbf{r})_{\alpha,i},\Psi^\dagger(t,\mathbf{r}')_{\beta,j}\right]_+ = 0\ . \tag{3.27}$$

Then, Eq. (3.25) becomes

$$\begin{aligned}
i\left[Q_A^a(t,\mathbf{r}),P^a(t,\mathbf{r}')\right]_- &= i^2\int \mathrm{d}^3\mathbf{r}\ \Big[\Psi^\dagger(t,\mathbf{r})_{\alpha,i}\,(T^a)^i_{\ i'}\,(\gamma_5)^\alpha_{\ \alpha'}\,(\gamma_0)^{\alpha'\beta'}\,(\gamma_5)_{\beta'\gamma'}\,(T^a)^{i'j'}\,\Psi(t,\mathbf{r}')^{\gamma'}_{\ j'} \\
&\qquad - \Psi^\dagger(t,\mathbf{r}')_{\alpha',i'}\,(T^a)^{i'}_{\ i}\,(\gamma_0)^{\alpha'\beta'}\,(\gamma_5)_{\beta'\alpha}\,(\gamma_5)^{\alpha\beta}\,(T^a)^{ij}\,\Psi(t,\mathbf{r})_{\beta,j}\Big]\,\delta^{(3)}(\mathbf{r}-\mathbf{r}') \\
&= i^2\Psi^\dagger(t,\mathbf{r})\,[\gamma_5T^a,\gamma_0\gamma_5T^a]_-\,\Psi(t,\mathbf{r}) \\
&= \frac{1}{2}\bar{\Psi}(t,\mathbf{r})\Psi(t,\mathbf{r}) \\
&= J_S^0(t,\mathbf{r})\ ,\ \text{for } a=1,2,3\ ,
\end{aligned}\tag{3.28}$$

where we used Eq. (6.29) and $(T^a)^2 = \mathbb{1}_{2\times2}/4$, for $a=1,2,3$. The above equation obviously relates the singlet scalar quark condensate to the triplet axial-vector charge operator Q_A^a. Now, we can evaluate the vacuum expectation value of the commutator (3.28) and obtain

$$\langle\Omega|i\left[Q_A^a(t,\mathbf{r}),P^a(t,\mathbf{r}')\right]_-|\Omega\rangle = \langle\Omega|J_S^0(t,\mathbf{r})|\Omega\rangle \propto \langle\Omega|\bar{\Psi}\Psi|\Omega\rangle\ . \tag{3.29}$$

Since we started with the assumption (3.24), it follows that the triplet axial-vector charge operator does not annihilate the vacuum. Therefore, we conclude that a non-vanishing vacuum expectation value of the singlet scalar quark condensate is a sufficient condition for the spontaneous breakdown of chiral symmetry.

3.1.3 Ward-Fradkin-Takahashi Identities, Gauge Invariance, and the Generating Functional

In Sec. [3.1.1], we saw that at low energies the strong processes are dominated by the exchange of pions. This behavior arises from the fact that the n-point functions formed with (3.2)-(3.5) exhibit a pole which is generated by the pions. Now, the physical amplitudes and therefore also the n-point functions are affected by the symmetries of the physical system in question. In the case of QCD, these constraints originate from chiral symmetry. And in fact, it turns out that chiral symmetry does not only determine the transformation behavior of the n-point functions, but also relates the n-point functions among themselves. These

[3] In general, the quark operators also carry color-space indices, but since the bilinear forms (3.2) - (3.5) describe color neutral objects, it is possible to omit those indices.

symmetry relations are the so-called Ward-Fradkin-Takahashi (WFT) identities. These identities relate the 4-divergence of a symmetry current to a linear combination of other n-point functions.

In the case of QCD, this means that the 4-divergence of an arbitrary n-point function formed with a least one factor of $J_V^{\mu,a}$ or $J_A^{\mu,a}$ can be related to a sum of other n-point functions of the theory. In general, this sum involves one n-point function which will contain the 4-divergence of the symmetry current. For an exact symmetry, this term obviously vanishes. The other terms of the sum each contain an equal-time commutator of the charge density of the symmetry current with one of the remaining operators. These equal-time commutation relations result from the derivatives of the Θ-functions which are contained in the time-ordered product of the n-point functions. In principle, these equal-time commutators are determined by the algebra which is formed by the symmetry currents and are therefore fully governed by the underlying symmetries of the theory. Generally speaking, this algebra relates the equal-time commutator to another current[4]. Therefore, we conclude that the 4-divergence of the initial n-point function, in general, is related to another n-point function which expresses the current conservation and a linear combination of $(n-1)$-point functions.

Before we illustrate these considerations by means of an example, we come to another important object which is closely connected to the n-point functions, the so-called generating functional. At the beginning of Sec. [2.2.1], using the example of a scalar theory containing N fields $\phi_a(x)$, $a = 1,\ldots,N$, we already saw that all n-point functions of the theory are included in the generating functional $Z[\vec{J}]$, with $\vec{J}(x) = (J_1(x),\ldots,J_N(x))^T$. In addition to that, the n-point functions of the theory can be obtained from the generating functional by taking the functional derivatives of $Z[\vec{J}]$ with respect to the classical sources $J_a(x)$, $a = 1,\ldots,N$, see Eq. (2.64). The connection of the generating functional and the WFT identities is given by the fact that the complete set of all WFT identities is contained as an invariance property of the generating functional. To be particular: In the absence of anomalies, the WFT identities Eare equivalent to the invariance of the generating functional under local transformations of the symmetry group in question. This invariance property can be understood as the manifestation of the underlying symmetry of the theory in question at the level of the n-point functions.

In order to illustrate these general meditations, we consider a complex scalar theory which is described by the following classical Lagrangian

$$\mathscr{L}\left(\varphi^\dagger(x),\varphi(x),\partial_\mu\varphi^\dagger(x),\partial_\mu\varphi(x)\right) = \left(\partial_\mu\varphi\right)^\dagger\left(\partial^\mu\varphi\right) - \mathscr{V}(\varphi^\dagger,\varphi) \ , \qquad \mathscr{V}(\varphi^\dagger,\varphi) = m^2\varphi^\dagger\varphi + \lambda\left(\varphi^\dagger\varphi\right)^2 \ . \quad (3.30)$$

The constants m^2 and λ are chosen in way, so that the potential density is bounded from below and possesses only a single minimum, i.e., $\lambda > 0$ and $m^2 > 0$. The importance of the second requirement will become clear in a moment. At this point, it is appropriate to calculate the canonical momenta associated with the fields $\varphi(x)$ and $\varphi^\dagger(x)$

$$\pi(x) = \frac{\partial\mathscr{L}}{\partial\left(\partial_0\varphi(x)\right)} = \partial^0\varphi^\dagger(x) \ , \qquad \pi^\dagger(x) = \frac{\partial\mathscr{L}}{\partial\left(\partial_0\varphi^\dagger(x)\right)} = \partial^0\varphi(x) \ . \quad (3.31)$$

Now, it is obvious that the Lagrangian (3.30) is invariant under global $U(1)$ transformations[5] of the fields

$$\varphi(x) \xrightarrow{U(1)} \varphi'(x) = e^{-i\alpha}\varphi(x) \ , \quad (3.32)$$

$$\varphi^\dagger(x) \xrightarrow{U(1)} \varphi'^\dagger(x) = \varphi^\dagger(x)e^{i\alpha} \ , \quad (3.33)$$

where $\alpha \in \mathbb{R}$ is the group parameter of $U(1)$. Using the transformation behavior (3.32) of the field $\varphi(x)$ and its complex conjugate $\varphi^\dagger(x)$, Eq. (3.33), the $U(1)$ invariance of Eq. (3.30) is easy to prove

$$\begin{aligned}\mathscr{L} \xrightarrow{U(1)} \mathscr{L}' &= \left(\partial_\mu\varphi'\right)^\dagger\left(\partial^\mu\varphi'\right) - \mathscr{V}(\varphi'^\dagger,\varphi') \\ &= e^{i\alpha}e^{-i\alpha}\left(\partial_\mu\varphi\right)^\dagger\left(\partial^\mu\varphi\right) - \mathscr{V}(\varphi^\dagger,\varphi) \\ &= \mathscr{L} \ , \end{aligned} \quad (3.34)$$

where we used that the $U(1)$ transformation is a global transformation and that the potential density only depends on $\varphi^\dagger(x)\varphi(x) = |\varphi(x)|^2$. Now, it becomes clear that the requirement $m^2 > 0$ forbids a spontaneous

[4] At this point, one has to be careful, because these equal-time commutators are only determined up to so-called Schwinger-terms.

[5] If we decompose the fields $\varphi(x)$ and $\varphi^\dagger(x)$ into their real and imaginary parts, we observe that the global $U(1)$ transformations corresponds to global $SO(2)$ rotations in the field space of $\mathrm{Re}(\varphi)$ and $\mathrm{Im}(\varphi)$, compare Sec. [2.2.3].

breakdown of the $U(1)$ symmetry. According to Noether's theorem, the invariance of Eq. (3.30) under the continuous transformations (3.32) and (3.33) gives rise to a conserved current. In order to derive this conserved Noether current, we need the infinitesimal forms of Eqs. (3.32) and (3.33)

$$\varphi(x) \xrightarrow{U(1)} \varphi'(x) \approx (1 - i\alpha)\varphi(x) = \varphi(x) + \delta\varphi(x) \ , \tag{3.35}$$

$$\varphi^\dagger(x) \xrightarrow{U(1)} \varphi'^\dagger(x) \approx \varphi^\dagger(x)(1 + i\alpha) = \varphi^\dagger(x) + \delta\varphi^\dagger(x) \ . \tag{3.36}$$

Using the above transformations and Eq. (2.21), the Noether current can be calculated as follows

$$\begin{aligned} J^\mu_\alpha &= \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu\varphi\right)}\delta\varphi + \delta\varphi^\dagger \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu\varphi^\dagger\right)} \\ &= -i\alpha \frac{\partial\left[\left(\partial_\lambda\varphi^\dagger\right)\left(\partial^\lambda\varphi\right)\right]}{\partial\left(\partial_\mu\varphi\right)}\varphi + i\alpha\varphi^\dagger \frac{\partial\left[\left(\partial_\lambda\varphi^\dagger\right)\left(\partial^\lambda\varphi\right)\right]}{\partial\left(\partial_\mu\varphi^\dagger\right)} \\ &= i\alpha\left[\varphi^\dagger g^\mu{}_\lambda \partial^\lambda\varphi - g^{\mu\lambda}\left(\partial_\lambda\varphi^\dagger\right)\varphi\right] \\ &\equiv \alpha J^\mu \ , \end{aligned} \tag{3.37}$$

where the Noether current J^μ is given by

$$J^\mu(x) = i\left[\varphi^\dagger(x)\partial^\mu\varphi(x) - \left(\partial^\mu\varphi^\dagger(x)\right)\varphi(x)\right] \ . \tag{3.38}$$

According to Eq. (2.22), this Noether current is associated with a conserved charge

$$Q = \int_V \mathrm{d}^3\mathbf{r}\ J^0 = i\int_V \mathrm{d}^3\mathbf{r}\ \left[\varphi^\dagger(x)\partial^0\varphi(x) - \left(\partial^0\varphi^\dagger(x)\right)\varphi(x)\right] = i\int_V \mathrm{d}^3\mathbf{r}\ \left[\varphi^\dagger(x)\pi^\dagger(x) - \pi(x)\varphi(x)\right] \ , \tag{3.39}$$

where we used the definitions of the conjugate momenta, Eq. (3.31). Now, we can quantize the theory and consider Eq. (3.30) at quantum level. To this end, we require the usual equal-time commutation relations between the fields and their conjugate momenta

$$\left[\varphi(t,\mathbf{r}), \pi(t,\mathbf{r}')\right]_- = \left[\varphi^\dagger(t,\mathbf{r}), \pi^\dagger(t,\mathbf{r}')\right]_- = i\delta^{(3)}(\mathbf{r}-\mathbf{r}') \ , \tag{3.40}$$

$$\left[\varphi(t,\mathbf{r}), \varphi^\dagger(t,\mathbf{r}')\right]_- = \left[\pi(t,\mathbf{r}), \pi^\dagger(t,\mathbf{r}')\right]_- = \left[\varphi(t,\mathbf{r}), \pi^\dagger(t,\mathbf{r}')\right]_- = \left[\varphi^\dagger(t,\mathbf{r}), \pi(t,\mathbf{r}')\right]_- = 0 \ . \tag{3.41}$$

It should be noted that the equal-time commutators of the fields and momenta with itself, of course, also vanish. Using the above relations, it is easy to derive the equal-time commutation relations of $\varphi(x), \varphi^\dagger(x), \pi(x)$ $, \pi^\dagger(x)$, and the charge density J^0

$$\begin{aligned} \left[J^0(t,\mathbf{r}), \varphi(t,\mathbf{r}')\right]_- &= i\left[\varphi^\dagger(t,\mathbf{r})\pi^\dagger(t,\mathbf{r}) - \pi(t,\mathbf{r})\varphi(t,\mathbf{r}), \varphi(t,\mathbf{r}')\right]_- \\ &= i\Big\{\varphi^\dagger(t,\mathbf{r})\left[\pi^\dagger(t,\mathbf{r}), \varphi(t,\mathbf{r}')\right]_- + \left[\varphi^\dagger(t,\mathbf{r}), \varphi(t,\mathbf{r}')\right]_- \pi^\dagger(t,\mathbf{r}) - \pi(t,\mathbf{r})\left[\varphi(t,\mathbf{r}), \varphi(t,\mathbf{r}')\right]_- \\ &\quad - \left[\pi(t,\mathbf{r}), \varphi(t,\mathbf{r}')\right]_- \varphi(t,\mathbf{r})\Big\} \\ &= -\delta^{(3)}(\mathbf{r}-\mathbf{r}')\varphi(t,\mathbf{r}) \ , \end{aligned} \tag{3.42}$$

$$\begin{aligned} \left[J^0(t,\mathbf{r}), \varphi^\dagger(t,\mathbf{r}')\right]_- &= i\left[\varphi^\dagger(t,\mathbf{r})\pi^\dagger(t,\mathbf{r}) - \pi(t,\mathbf{r})\varphi(t,\mathbf{r}), \varphi^\dagger(t,\mathbf{r}')\right]_- \\ &= i\Big\{\varphi^\dagger(t,\mathbf{r})\left[\pi^\dagger(t,\mathbf{r}), \varphi^\dagger(t,\mathbf{r}')\right]_- + \left[\varphi^\dagger(t,\mathbf{r}), \varphi^\dagger(t,\mathbf{r}')\right]_- \pi^\dagger(t,\mathbf{r}) - \pi(t,\mathbf{r})\left[\varphi(t,\mathbf{r}), \varphi^\dagger(t,\mathbf{r}')\right]_- \\ &\quad - \left[\pi(t,\mathbf{r}), \varphi^\dagger(t,\mathbf{r}')\right]_- \varphi(t,\mathbf{r})\Big\} \\ &= \delta^{(3)}(\mathbf{r}-\mathbf{r}')\varphi^\dagger(t,\mathbf{r}) \ , \end{aligned} \tag{3.43}$$

$$\begin{aligned} \left[J^0(t,\mathbf{r}), \pi(t,\mathbf{r}')\right]_- &= i\left[\varphi^\dagger(t,\mathbf{r})\pi^\dagger(t,\mathbf{r}) - \pi(t,\mathbf{r})\varphi(t,\mathbf{r}), \pi(t,\mathbf{r}')\right]_- \\ &= i\Big\{\varphi^\dagger(t,\mathbf{r})\left[\pi^\dagger(t,\mathbf{r}), \pi(t,\mathbf{r}')\right]_- + \left[\varphi^\dagger(t,\mathbf{r}), \pi(t,\mathbf{r}')\right]_- \pi^\dagger(t,\mathbf{r}) - \pi(t,\mathbf{r})\left[\varphi(t,\mathbf{r}), \pi(t,\mathbf{r}')\right]_- \\ &\quad - \left[\pi(t,\mathbf{r}), \pi(t,\mathbf{r}')\right]_- \varphi(t,\mathbf{r})\Big\} \\ &= \delta^{(3)}(\mathbf{r}-\mathbf{r}')\pi(t,\mathbf{r}) \ , \end{aligned} \tag{3.44}$$

$$\begin{aligned}
\left[J^0(t,\mathbf{r}),\pi^\dagger(t,\mathbf{r}')\right]_- &= i\left[\varphi^\dagger(t,\mathbf{r})\pi^\dagger(t,\mathbf{r})-\pi(t,\mathbf{r})\varphi(t,\mathbf{r}),\pi^\dagger(t,\mathbf{r}')\right]_- \\
&= i\Big\{\varphi^\dagger(t,\mathbf{r})\left[\pi^\dagger(t,\mathbf{r}),\pi^\dagger(t,\mathbf{r}')\right]_- + \left[\varphi^\dagger(t,\mathbf{r}),\pi^\dagger(t,\mathbf{r}')\right]_-\pi^\dagger(t,\mathbf{r}) - \pi(t,\mathbf{r})\left[\varphi(t,\mathbf{r}),\pi^\dagger(t,\mathbf{r}')\right]_- \\
&\quad - \left[\pi(t,\mathbf{r}),\pi^\dagger(t,\mathbf{r}')\right]_-\varphi(t,\mathbf{r})\Big\} \\
&= -\delta^{(3)}(\mathbf{r}-\mathbf{r}')\pi^\dagger(t,\mathbf{r})\ ,
\end{aligned} \tag{3.45}$$

where we frequently used Eqs. (3.40), (3.41), the linearity of the commutator and $[AB,C]_- = A\,[B,C]_- + [A,C]_-\,B$. At this point, we are in the position to consider the 4-divergence of n-point functions which are formed with the Noether current (3.38) and the field operators $\varphi(x)$ and $\varphi^\dagger(x)$. As an explicit example, we consider the following 3-point function

$$G^{(3),\mu}(x_1,x_2,x_3) = \langle\Omega|T\left\{\varphi(x_1)J^\mu(x_2)\varphi^\dagger(x_3)\right\}|\Omega\rangle\ . \tag{3.46}$$

Before we calculate the 4-divergence of Eq. (3.46), we note that the above 3-point function is invariant under the global $U(1)$ transformations (3.32) and (3.33), since the symmetry current J^μ is invariant itself. This invariance of the Noether current tells us that it transforms as a singlet under the $U(1)$ phase transformation, i.e., it transforms as a neutral object with respect to the $U(1)$ charge. This is not a general result, since the transformation behavior of the fields under the group in question determines in which way the symmetry current transforms. An example for that are the left- and right-handed currents (2.152) and (2.153). The left-handed Noether current transforms as a $(3,1)$ multiplet under $SU(2)_L\times SU(2)_R$, while the right-handed current transforms as a $(1,3)$ multiplet. This beautifully demonstrates that the transformation behavior of the n-point functions is determined by the underlying symmetry. Now, we are able to calculate the 4-divergence of Eq. (3.46) with respect to x_2. Using the short-hand notations $\varphi(x_i)\equiv\varphi_i$, $\varphi^\dagger(x_i)\equiv\varphi_i^\dagger$, $J^\mu(x_i)\equiv J_i^\mu$ as well as $\Theta(x_i^0-x_j^0)\equiv\Theta_{ij}$ and $\delta^{(n)}(x_i-x_j)\equiv\delta_{ij}^n=\delta_{ji}^n$, we find

$$\begin{aligned}
\partial_\mu^{x_2}G^{(3),\mu}(x_1,x_2,x_3) &= \partial_\mu^{x_2}\langle\Omega|T\left\{\varphi(x_1)J^\mu(x_2)\varphi^\dagger(x_3)\right\}|\Omega\rangle \\
&= \partial_\mu^{x_2}\Big[\Theta_{12}\Theta_{23}\langle\Omega|\varphi_1J_2^\mu\varphi_3^\dagger|\Omega\rangle + \Theta_{13}\Theta_{32}\langle\Omega|\varphi_1\varphi_3^\dagger J_2^\mu|\Omega\rangle + \Theta_{21}\Theta_{13}\langle\Omega|J_2^\mu\varphi_1\varphi_3^\dagger|\Omega\rangle \\
&\qquad + \Theta_{23}\Theta_{31}\langle\Omega|J_2^\mu\varphi_3^\dagger\varphi_1|\Omega\rangle + \Theta_{31}\Theta_{12}\langle\Omega|\varphi_3^\dagger\varphi_1J_2^\mu|\Omega\rangle + \Theta_{32}\Theta_{21}\langle\Omega|\varphi_3^\dagger J_2^\mu\varphi_1|\Omega\rangle\Big] \\
&= \delta_{12}\Big[\Theta_{13}\langle\Omega|J_2^0\varphi_1\varphi_3^\dagger|\Omega\rangle - \Theta_{31}\langle\Omega|\varphi_3^\dagger\varphi_1J_2^0|\Omega\rangle\Big] \\
&\quad + \delta_{12}\Big[\Theta_{32}\langle\Omega|\varphi_3^\dagger J_2^0\varphi_1|\Omega\rangle - \Theta_{23}\langle\Omega|\varphi_1J_2^0\varphi_3^\dagger|\Omega\rangle\Big] \\
&\quad + \delta_{23}\Big[\Theta_{12}\langle\Omega|\varphi_1J_2^0\varphi_3^\dagger|\Omega\rangle - \Theta_{21}\langle\Omega|\varphi_3^\dagger J_2^0\varphi_1|\Omega\rangle\Big] \\
&\quad + \delta_{23}\Big[\Theta_{31}\langle\Omega|J_2^0\varphi_3^\dagger\varphi_1|\Omega\rangle - \Theta_{13}\langle\Omega|\varphi_1\varphi_3^\dagger J_2^0|\Omega\rangle\Big]\ ,
\end{aligned} \tag{3.47}$$

where we used $\partial_\mu^{x_i}\Theta_{ij} = -\partial_\mu^{x_i}\Theta_{ji} = g_{\mu0}\delta_{ij}$ and the fact that the symmetry current (3.38) is conserved, i.e., $\partial_\mu J^\mu = 0$. Now, the delta-distributions $\delta_{12}\equiv\delta(x_1^0-x_2^0)$ and $\delta_{23}\equiv\delta(x_2^0-x_3^0)$ allow us to use the equal-time commutation relations (3.42) and (3.43). Using these relations and inserting suitable Θ-functions, we observe that two terms of each square bracket in Eq. (3.47) vanish. Finally, we obtain

$$\partial_\mu^{x_2}G^{(3),\mu}(x_1,x_2,x_3) = \left[\delta^{(4)}(x_2-x_3)-\delta^{(4)}(x_1-x_2)\right]\langle\Omega|T\left\{\varphi(x_1)\varphi^\dagger(x_3)\right\}|\Omega\rangle\ . \tag{3.48}$$

Using the same manipulations, it is now possible to prepare the complete set of WFT identities of all n-point functions of the theory. As already mentioned at the beginning of this discussion, we find that the WFT identities relate the 4-divergence of an n-point function with a linear combination of $(n-1)$-point functions. In the case of the above example, the additional n-point function expressing the current conservation vanishes, since the local $U(1)$ symmetry of Eq. (3.30) is exact. In order to complete this example, we want to study the WFT identities for all n-point functions instead of only considering the special case of Eq. (3.48). This will also show us that the symmetry properties at the level of the n-point functions manifest themselves as an invariance property of the generating functional. Therefore, we start with the path integral expression of the generating functional of all n-point functions of the theory

$$Z[j^\dagger,j,j_\mu] = \int\mathscr{D}\varphi^\dagger(x)\mathscr{D}\varphi(x)\ e^{iS\left[\varphi^\dagger,\varphi,j^\dagger,j,j_\mu\right]}\ , \tag{3.49}$$

where $\mathscr{D}\varphi(x) \equiv \prod_x d\varphi(x), \mathscr{D}\varphi^\dagger(x) \equiv \prod_x d\varphi^\dagger(x)$ and

$$\begin{aligned} S\left[\varphi^\dagger,\varphi,j^\dagger,j,j_\mu\right] &= S[\varphi^\dagger,\varphi] + \int \mathrm{d}^4x\, j^\dagger(x)\varphi(x) + \varphi^\dagger(x)j(x) + j_\mu(x)J^\mu(x) \\ &= \int \mathrm{d}^4x\, \left\{[\partial_\mu\varphi(x)]^\dagger\,\partial^\mu\varphi(x) - \mathscr{V}\left(\varphi^\dagger(x),\varphi(x)\right)\right\} + \int \mathrm{d}^4x\, \left[j^\dagger(x)\varphi(x) + \varphi^\dagger(x)j(x)\right. \\ &\quad \left. + j_\mu(x)J^\mu(x)\right] \ . \end{aligned} \tag{3.50}$$

Now, we want to promote the global $U(1)$ invariance of Eq. (3.30) to a local one of Eq. (3.50). As already seen in Sec. [2.3.1], the group elements of $U(1)$ may now depend on the space-time, i.e., they correspond to a mapping from Minkowski space to the group $U(1)$, $x \mapsto U(x)$. The transformation properties of the fields (3.32) and (3.33) then become

$$\varphi(x) \xrightarrow{U(1)} \varphi'(x) = e^{-i\alpha(x)}\varphi(x) \ , \tag{3.51}$$

$$\varphi^\dagger(x) \xrightarrow{U(1)} \varphi'^\dagger(x) = \varphi^\dagger(x)e^{i\alpha(x)} \ . \tag{3.52}$$

In addition to that, we require that the classical sources $j^\dagger(x)$ and $j(x)$ corresponding to the fields transform in the same way as the fields, respectively,

$$j(x) \xrightarrow{U(1)} j'(x) = e^{-i\alpha(x)}j(x) \ , \tag{3.53}$$

$$j^\dagger(x) \xrightarrow{U(1)} j'^\dagger(x) = j^\dagger(x)e^{i\alpha(x)} \ . \tag{3.54}$$

The transformation behavior of the external field $j_\mu(x)$ corresponding to the symmetry current $J^\mu(x)$ should stay indefinite for the moment. Then, using Eqs. (3.51)-(3.54), the action including the source terms (3.50) transforms under local $U(1)$ transformations according to

$$\begin{aligned} S\left[\varphi^\dagger,\varphi,j^\dagger,j,j_\mu\right] \xrightarrow{U(1)} S\left[\varphi'^\dagger,\varphi',j'^\dagger,j',j'_\mu\right] &= S[\varphi'^\dagger,\varphi'] + \int \mathrm{d}^4x\, \left[j'^\dagger(x)\varphi'(x) + \varphi'^\dagger(x)j'(x) + j'_\mu(x)J^{\mu\prime}(x)\right] \\ &= S[\varphi'^\dagger,\varphi'] + \int \mathrm{d}^4x\, \left[j^\dagger(x)\varphi(x) + \varphi^\dagger(x)j(x) + j'_\mu(x)J^{\mu\prime}(x)\right] \ , \end{aligned} \tag{3.55}$$

where we used that the terms $j^\dagger(x)\varphi(x)$ and $\varphi^\dagger(x)j(x)$ are trivially invariant due to their transformation behavior. Now, the transformation behavior of the action $S[\varphi^\dagger,\varphi]$ is given by

$$\begin{aligned} S[\varphi^\dagger,\varphi] \xrightarrow{U(1)} S[\varphi^{\dagger\prime},\varphi'] &= \int \mathrm{d}^4x\, \left\{[\partial_\mu\varphi'(x)]^\dagger\,\partial^\mu\varphi'(x) - \mathscr{V}(\varphi^{\dagger\prime},\varphi')\right\} \\ &= \int \mathrm{d}^4x\, \Big(\left\{\left[\partial_\mu\varphi^\dagger(x)\right]e^{i\alpha(x)} + i\left[\partial_\mu\alpha(x)\right]e^{i\alpha(x)}\varphi^\dagger(x)\right\} \\ &\quad \times \left\{-i\left[\partial^\mu\alpha(x)\right]e^{-i\alpha(x)}\varphi(x) + \left[\partial^\mu\varphi(x)\right]e^{-i\alpha(x)}\right\} - \mathscr{V}(\varphi^\dagger,\varphi)\Big) \\ &= S[\varphi^\dagger,\varphi] + \int \mathrm{d}^4x\, \left\{[\partial_\mu\alpha(x)]\,J^\mu(x) + [\partial_\mu\alpha(x)]^2\,\varphi^\dagger(x)\varphi(x)\right\} \ , \end{aligned} \tag{3.56}$$

where we used the definition of the Noether current (3.38) and the fact that the potential density only depends on $|\varphi(x)|^2$. In close analogy to the construction of the QCD Lagrangian in Sec. [2.3.1], the kinetic term produces an additional term

$$\delta S\left[\varphi^\dagger,\varphi\right] = \int \mathrm{d}^4x\, \left\{[\partial_\mu\alpha(x)]\,J^\mu(x) + [\partial_\mu\alpha(x)]^2\,\varphi^\dagger(x)\varphi(x)\right\} \ , \tag{3.57}$$

which has to compensated by the transformation behavior of other objects in Eq. (3.50) to obtain a local $U(1)$ invariance. Another object which we expect to transform in a nontrivial manner is the Noether current J^μ since it also contains 4-divergences of the fields. We find

$$\begin{aligned} J^\mu(x) \xrightarrow{U(1)} J^{\mu\prime}(x) &= i\left\{\varphi^{\dagger\prime}(x)\partial^\mu\varphi'(x) - \left[\partial^\mu\varphi^{\dagger\prime}(x)\right]\varphi'(x)\right\} \\ &= i\Big(\varphi^\dagger(x)e^{i\alpha(x)}\left\{-i\left[\partial^\mu\alpha(x)\right]e^{-i\alpha(x)}\varphi(x) + e^{-i\alpha(x)}\partial^\mu\varphi(x)\right\} \\ &\quad - \left\{\left[\partial^\mu\varphi^\dagger(x)\right]e^{i\alpha(x)} + i\left[\partial^\mu\alpha(x)\right]e^{i\alpha(x)}\varphi^\dagger(x)\right\}e^{-i\alpha(x)}\varphi(x)\Big) \\ &= J^\mu(x) + 2\left[\partial^\mu\alpha(x)\right]\varphi^\dagger(x)\varphi(x) \ . \end{aligned} \tag{3.58}$$

Inserting the results (3.56) and (3.58) into Eq. (3.55), we obtain

$$S\left[\varphi'^{\dagger},\varphi',j'^{\dagger},j',j'_{\mu}\right]=S[\varphi^{\dagger},\varphi]+\int \mathrm{d}^4x\ \Big(j^{\dagger}(x)\varphi(x)+\varphi^{\dagger}(x)j(x)+\left[\partial_{\mu}\alpha(x)\right]J^{\mu}(x)+\left[\partial_{\mu}\alpha(x)\right]^2\varphi^{\dagger}(x)\varphi(x)$$
$$+j'_{\mu}(x)\left\{J^{\mu}(x)+2\left[\partial^{\mu}\alpha(x)\right]\varphi^{\dagger}(x)\varphi(x)\right\}\Big)\ . \tag{3.59}$$

In order to eliminate the additional terms in the above equation, we use the same approach as in Sec. [2.3.1]. To this end, we require that the external field j_{μ} transforms as a $U(1)$ gauge field

$$\begin{aligned} j_{\mu}(x)\xrightarrow{U(1)} j'_{\mu}(x) &= U(x)j_{\mu}(x)U^{\dagger}(x)-i\left[\partial_{\mu}U(x)\right]U^{\dagger}(x)\\ &= j_{\mu}(x)-\partial_{\mu}\alpha(x)\ , \end{aligned} \tag{3.60}$$

where we used $U(x)=e^{-i\alpha(x)}$. Then, Eq. (3.59) becomes

$$\begin{aligned} S\left[\varphi'^{\dagger},\varphi',j^{\dagger\prime},j',j'_{\mu}\right] &= S[\varphi^{\dagger},\varphi]+\int \mathrm{d}^4x\ \Big(j^{\dagger}(x)\varphi(x)+\varphi^{\dagger}(x)j(x)+\left[\partial_{\mu}\alpha(x)\right]J^{\mu}(x)+\left[\partial_{\mu}\alpha(x)\right]^2\varphi^{\dagger}(x)\varphi(x)\\ &\quad+\left[j_{\mu}(x)-\partial_{\mu}\alpha(x)\right]\left\{J^{\mu}(x)+2\left[\partial^{\mu}\alpha(x)\right]\varphi^{\dagger}(x)\varphi(x)\right\}\Big)\\ &= S[\varphi^{\dagger},\varphi]+\int \mathrm{d}^4x\ \Big\{j^{\dagger}(x)\varphi(x)+\varphi^{\dagger}(x)j(x)+j_{\mu}(x)J^{\mu}(x)\\ &\quad+2\left[\partial_{\mu}\alpha(x)\right]j^{\mu}(x)\varphi^{\dagger}(x)\varphi(x)-\left[\partial_{\mu}\alpha(x)\right]^2\varphi^{\dagger}(x)\varphi(x)\Big\}\\ &= S\left[\varphi^{\dagger},\varphi,j^{\dagger},j,j_{\mu}\right]+\delta S\left[\varphi^{\dagger},\varphi,j^{\dagger},j,j_{\mu}\right]\ , \end{aligned} \tag{3.61}$$

where we used Eq. (3.50) and defined

$$\delta S\left[\varphi^{\dagger},\varphi,j^{\dagger},j,j_{\mu}\right]=\int \mathrm{d}^4x\ \Big\{2\left[\partial_{\mu}\alpha(x)\right]j^{\mu}(x)\varphi^{\dagger}(x)\varphi(x)-\left[\partial_{\mu}\alpha(x)\right]^2\varphi^{\dagger}(x)\varphi(x)\Big\}\ . \tag{3.62}$$

In order to get rid of the above two terms, we have to introduce a new term into the action (3.50). This term has to be chosen in a way, so that the additional terms arising from its transformation behavior under local $U(1)$ transformations compensate the two terms in Eq. (3.62). Obviously, this term is given by

$$S_1[\varphi^{\dagger},\varphi,j_{\mu}]=\int \mathrm{d}^4x\ \varphi^{\dagger}(x)\varphi(x)j_{\mu}(x)j^{\mu}(x)\ , \tag{3.63}$$

since it transforms under local $U(1)$ transformations according to

$$\begin{aligned} \varphi^{\dagger}(x)\varphi(x)j_{\mu}(x)j^{\mu}(x)\xrightarrow{U(1)}\varphi'^{\dagger}(x)\varphi'(x)j'_{\mu}(x)j^{\mu\prime}(x) &= \varphi^{\dagger}(x)\varphi(x)\left[j_{\mu}(x)-\partial_{\mu}\alpha(x)\right]\left[j^{\mu}(x)-\partial^{\mu}\alpha(x)\right]\\ &= \varphi^{\dagger}(x)\varphi(x)j_{\mu}(x)j^{\mu}(x)-2\left[\partial_{\mu}\alpha(x)\right]j^{\mu}(x)\varphi^{\dagger}(x)\varphi(x)\\ &\quad+\left[\partial_{\mu}\alpha(x)\right]^2\varphi^{\dagger}(x)\varphi(x)\ . \end{aligned} \tag{3.64}$$

Finally, combining Eqs. (3.50) and (3.61), we obtain the action of a theory with local $U(1)$ symmetry

$$\begin{aligned} \tilde{S}\left[\varphi^{\dagger},\varphi,j^{\dagger},j,j_{\mu}\right] &= S[\varphi^{\dagger},\varphi]+S_1[\varphi^{\dagger},\varphi,j_{\mu}]+\int \mathrm{d}^4x\ \left[j^{\dagger}(x)\varphi(x)+\varphi^{\dagger}(x)j(x)+j_{\mu}(x)J^{\mu}(x)\right]\\ &= \int \mathrm{d}^4x\ \Big\{\left[D_{\mu}\varphi(x)\right]^{\dagger}D^{\mu}\varphi(x)-\mathscr{V}(\varphi^{\dagger},\varphi)+j^{\dagger}(x)\varphi(x)+\varphi^{\dagger}(x)j(x)\Big\}\ , \end{aligned} \tag{3.65}$$

where, in close analogy to Sec. [2.3.1], we defined a covariant derivative of the form

$$D_{\mu}=\partial_{\mu}-ij_{\mu}(x)\ . \tag{3.66}$$

The first term in Eq. (3.65) contains the source term $j_{\mu}(x)J^{\mu}(x)$ as well as the additionally inserted term (3.63). Since $D_{\mu}\varphi(x)$ and its complex conjugate transform in the same way as the fields (3.51) and (3.52), the invariance of Eq.(3.65) can be seen easily. Now, we are in the position to prove that also the generating functional (3.49) is invariant under local $U(1)$ transformations. We obtain

$$\begin{aligned} Z[j^{\dagger},j,j_{\mu}]\xrightarrow{U(1)} Z[j'^{\dagger},j',j'_{\mu}] &= \int \mathscr{D}\varphi^{\dagger}\mathscr{D}\varphi\ e^{i\tilde{S}\left[\varphi^{\dagger},\varphi,j'^{\dagger},j',j'_{\mu}\right]}\\ &= \int \mathscr{D}\varphi^{\dagger\prime}\mathscr{D}\varphi'\ e^{i\tilde{S}\left[\varphi'^{\dagger},\varphi',j'^{\dagger},j',j'_{\mu}\right]}\\ &= \int \mathscr{D}\varphi^{\dagger}\mathscr{D}\varphi\ e^{i\tilde{S}\left[\varphi'^{\dagger},\varphi',j'^{\dagger},j',j'_{\mu}\right]} \end{aligned}$$

$$
\begin{aligned}
&= \int \mathscr{D}\varphi^\dagger \mathscr{D}\varphi \; e^{i\tilde{S}[\varphi^\dagger,\varphi,j^\dagger,j,j_\mu]} \\
&= Z[j^\dagger,j,j_\mu] \;,
\end{aligned} \tag{3.67}
$$

where we renamed the fields $\varphi(x) \to \varphi'(x)$ and $\varphi^\dagger(x) \to \varphi'^\dagger(x)$ in the first line. In addition to that, we used the invariance of the action and of the functional measures under local $U(1)$ transformations. In the following, we want to use the invariance of the generating functional, Eq. (3.67), to derive a "master equation" which contains the complete set of WFT identities of the theory (3.30). However, it will be useful to first recall another form of Eq. (3.49), which is given by

$$
\begin{aligned}
Z[j^\dagger,j,j_\mu] &= \lim_{\substack{t_i\to-\infty \\ t_f\to\infty}} \langle \Omega, t_f | \Omega, t_i\rangle_{j^\dagger,j,j_\mu} \\
&= \langle \Omega | T e^{i\int \mathrm{d}^4x\, \mathscr{L}_{ext}} |\Omega\rangle \\
&= \langle \Omega | T e^{i\int \mathrm{d}^4x\, \left[j^\dagger(x)\varphi(x)+\varphi^\dagger(x)j(x)+j_\mu(x)J^\mu(x)\right]} |\Omega\rangle \;.
\end{aligned} \tag{3.68}
$$

Keeping the above definition in mind, we use Eq. (3.67) to obtain

$$
\begin{aligned}
0 &= Z[j'^\dagger,j',j'_\mu] - Z[j^\dagger,j,j_\mu] \\
&= \int \mathscr{D}\varphi^\dagger(x)\mathscr{D}\varphi(x)\; \Big\{ e^{i\int \mathrm{d}^4x\, \left[j'^\dagger(x)\varphi(x)+\varphi^\dagger(x)j'(x)+j'_\mu(x)J^\mu(x)+j'_\mu(x)j^{\mu,\prime}(x)\varphi^\dagger(x)\varphi(x)\right]} \\
&\qquad - e^{i\int \mathrm{d}^4x\, \left[j^\dagger(x)\varphi(x)+\varphi^\dagger(x)j(x)+j_\mu(x)J^\mu(x)+j_\mu(x)j^\mu(x)\varphi^\dagger(x)\varphi(x)\right]} \Big\}\, e^{iS[\varphi^\dagger,\varphi]} \;.
\end{aligned} \tag{3.69}
$$

For infinitesimal $U(1)$ transformations, the first exponential in the above equation can be expanded up to first order in the group parameter $\alpha(x)$

$$
\begin{aligned}
& e^{i\int \mathrm{d}^4x\, \left[j'^\dagger(x)\varphi(x)+\varphi^\dagger(x)j'(x)+j'_\mu(x)J^\mu(x)+j'_\mu(x)j^{\mu,\prime}(x)\varphi^\dagger(x)\varphi(x)\right]} \\
&= e^{i\int \mathrm{d}^4x\, \left\{j^\dagger(x)[1+i\alpha(x)]\varphi(x)+\varphi^\dagger(x)[1-i\alpha(x)]j(x)+[j_\mu(x)-\partial_\mu\alpha(x)]J^\mu(x)+[j_\mu(x)-\partial_\mu\alpha(x)][j^\mu(x)-\partial^\mu\alpha(x)]\varphi^\dagger(x)\varphi(x)\right\}} \\
&= e^{i\int \mathrm{d}^4x\, \left[j^\dagger(x)\varphi(x)+\varphi^\dagger(x)j(x)+j_\mu(x)J^\mu(x)+j_\mu(x)j^\mu(x)\varphi^\dagger(x)\varphi(x)\right]} \\
&\quad \times \left\{ 1 + \int \mathrm{d}^4x\; \alpha(x)\left[\varphi^\dagger(x)j(x) - j^\dagger(x)\varphi(x)\right] - i[\partial_\mu\alpha(x)]\left[J^\mu(x) + 2j^\mu(x)\varphi^\dagger(x)\varphi(x)\right] + \mathcal{O}(\alpha^2) \right\} \;.
\end{aligned} \tag{3.70}
$$

Inserting Eq. (3.70) into Eq. (3.69), we find

$$
\begin{aligned}
0 &= \int \mathscr{D}\varphi^\dagger(x)\mathscr{D}\varphi(x)\; \left\{ \int \mathrm{d}^4x\; \alpha(x)\left[\varphi^\dagger(x)j(x) - j^\dagger(x)\varphi(x)\right] - i[\partial_\mu\alpha(x)]\left[J^\mu(x)+2j^\mu(x)\varphi^\dagger(x)\varphi(x)\right] \right\} e^{iS[\varphi^\dagger,\varphi,j^\dagger,j,j_\mu]} \\
&= \int \mathscr{D}\varphi^\dagger(x)\mathscr{D}\varphi(x)\; \left\{ \int \mathrm{d}^4x\; \alpha(x)\left[\frac{1}{i}j(x)\frac{\delta}{\delta j(x)} - \frac{1}{i}j^\dagger(x)\frac{\delta}{\delta j^\dagger(x)}\right] - i[\partial_\mu\alpha(x)]\frac{1}{i}\frac{\delta}{\delta j_\mu(x)} \right\} e^{iS[\varphi^\dagger,\varphi,j^\dagger,j,j_\mu]} \\
&= \int \mathrm{d}^4x\; \alpha(x)\left[\frac{1}{i}j(x)\frac{\delta}{\delta j(x)} - \frac{1}{i}j^\dagger(x)\frac{\delta}{\delta j^\dagger(x)} + \partial^x_\mu\frac{\delta}{\delta j_\mu(x)}\right] Z[j^\dagger,j,j_\mu] \;,
\end{aligned} \tag{3.71}
$$

where we replaced the fields by appropriate functional derivatives in order to interchange the space-time and the functional integral and to identify the generating functional (3.49). In addition to that, we used an integration by parts. We obtain

$$
\left[\frac{1}{i}j(x)\frac{\delta}{\delta j(x)} - \frac{1}{i}j^\dagger(x)\frac{\delta}{\delta j^\dagger(x)} + \partial^x_\mu\frac{\delta}{\delta j_\mu(x)}\right] Z[j^\dagger,j,j_\mu] = 0 \;, \tag{3.72}
$$

since $\alpha(x)$ can be arbitrarily chosen and Eq. (3.71) must hold for all group parameters. In order to show that Eq. (3.72) is the desired "master equation" for deriving the complete set of WFT identities, we use it to replicate the WFT identity of the 3-point function (3.48)

$$
\begin{aligned}
\partial^{x_2}_\mu G^{(3),\mu}(x_1,x_2,x_3) &= \left(\frac{1}{i}\right)^3 \partial^{x_2}_\mu \left.\frac{\delta^3 Z[j^\dagger,j,j_\mu]}{\delta j^\dagger(x_1)\delta j_\mu(x_2)\delta j(x_3)}\right|_{j^\dagger=j=j_\mu=0} \\
&= \left(\frac{1}{i}\right)^3 \left\{ \frac{\delta^2}{\delta j^\dagger(x_1)\delta j(x_3)}\left[\partial^{x_2}_\mu \frac{\delta Z[j^\dagger,j,j_\mu]}{\delta j_\mu(x_2)}\right]\right\}_{j^\dagger=j=j_\mu=0} \\
&= \left(\frac{1}{i}\right)^4 \left\{ \frac{\delta^2}{\delta j^\dagger(x_1)\delta j(x_3)}\left[j^\dagger(x_2)\frac{\delta}{\delta j^\dagger(x_2)} - j(x_2)\frac{\delta}{\delta j(x_2)}\right] Z[j^\dagger,j,j_\mu]\right\}_{j^\dagger=j=j_\mu=0}
\end{aligned}
$$

$$
\begin{aligned}
&= \delta^{(4)}(x_1-x_2)\left.\frac{\delta^2 Z[j^\dagger,j,j_\mu]}{\delta j(x_3)\delta j^\dagger(x_2)}\right|_{j^\dagger=j=j_\mu=0} - \delta^{(4)}(x_2-x_3)\left.\frac{\delta^2 Z[j^\dagger,j,j_\mu]}{\delta j^\dagger(x_1)\delta j(x_2)}\right|_{j^\dagger=j=j_\mu=0}\\
&= \left[\delta^{(4)}(x_2-x_3)-\delta^{(4)}(x_1-x_2)\right]\langle\Omega|T\left\{\varphi(x_1)\varphi^\dagger(x_3)\right\}|\Omega\rangle\ ,
\end{aligned}
\tag{3.73}
$$

where we used Eqs. (3.72), (3.68), and the fact that the functional derivatives of the generating functional in the second to last line correspond, up to a factor of i^2, to the respective 2-point functions. It is obvious that in principle each WFT identity of the theory can easily be obtained by using Eq. (3.72). This example beautifully demonstrates that the local invariance of the generating functional is equivalent to the WFT identities of the theory.

At the end of this subsection, we apply this external field method to two-flavor QCD. Since the gauge part of Eq. (2.139) is not important for the following discussion, we will only consider the quark part (2.133) of the QCD Lagrangian. In order to generate the hadronic n-point functions, formed with Eqs. (3.2)-(3.5), we have to modify the quark part of the QCD Lagrangian by an additional term which contains a coupling of these quadratic forms to external fields. Then, Eq. (2.133) becomes

$$
\begin{aligned}
\mathscr{L}_{Quark} &\longrightarrow \mathscr{L}_{0,Quark}+\mathscr{L}_{ext}\\
&= \bar\Psi(x)i\gamma^\mu D_\mu\Psi(x)+\bar\Psi\gamma^\mu\big[v_\mu(x)+a_\mu(x)\gamma_5\big]\Psi(x)-\bar\Psi(x)\big[s(x)-i\gamma_5 p(x)\big]\Psi(x)\ ,
\end{aligned}
\tag{3.74}
$$

where we introduced the external fields $v_\mu(x), a_\mu(x), s(x)$, and $p(x)$ as hermitian and color neutral matrices acting on flavor space

$$
v_\mu(x)=v_{\mu,i}(x)T^i\ ,\quad a_\mu(x)=a_{\mu,i}(x)T^i\ ,\quad s(x)=s_a(x)T^a\ ,\quad p(x)=p_a(x)T^a\ ,
\tag{3.75}
$$

where $i=1,2,3$ and $a=0,\ldots,3$. At this point, two important things have to be taken into account: First of all, in Eq. (3.74), we omitted a coupling to the vector and axial-vector singlet currents in order to avoid the occurrence of anomalies. Then, secondly, we absorbed the mass matrix in the definition of the external field

$$
s(x)=m+s'(x)\ ,\qquad m=\begin{pmatrix} m_u & 0\\ 0 & m_d\end{pmatrix}\ ,
\tag{3.76}
$$

so that the usual quark part of the QCD Lagrangian can be restored by setting $v_\mu(x)=a_\mu(x)=p(x)=s'(x)=0$. If, in analogy to Eq. (3.68), we use the definition of the generating functional as vacuum-to-vacuum transition amplitude in the presence of external fields, we are able to generate the complete set of hadronic n-point functions, involving $J_S^a, J_P^a, J_V^{\mu,i}$, and $J_A^{\mu,i}$, by taking the functional derivatives of

$$
Z[v_\mu,a_\mu,s,p]=\langle\Omega|Te^{i\int \mathrm{d}^4x\,\mathscr{L}_{ext}}|\Omega\rangle
\tag{3.77}
$$

with respect to the respective external fields. In order to ensure the validity of the WFT identities, we promote the global $SU(2)_L\times SU(2)_R$ chiral symmetry of QCD to a local one and require the generating functional to stay invariant under this modification. From this requirement, we are able to extract the transformation properties of the external fields (3.75) under local $SU(2)_L\times SU(2)_R$ transformations. To this end, we rewrite the Lagrangian (3.74) in terms of left- and right-handed quark fields. According to the discussion of Sec.[2.3.2], the first term $\mathscr{L}_{0,Quark}$ can be rewritten as in Eq. (2.147). Omitting the space-time dependence of the various fields, the second term, $\mathscr{L}_{ext}$, can be expressed as

$$
\begin{aligned}
\mathscr{L}_{ext} &= \bar\Psi\gamma^\mu\big(v_\mu+a_\mu\gamma_5\big)\Psi-\bar\Psi\big(s-i\gamma_5p\big)\Psi\\
&= \bar\Psi\left(\mathcal{P}_R+\mathcal{P}_L\right)\gamma^\mu\big(v_\mu+a_\mu\gamma_5\big)\left(\mathcal{P}_L+\mathcal{P}_R\right)\Psi-\bar\Psi\left(\mathcal{P}_R+\mathcal{P}_L\right)\big(s-i\gamma_5p\big)\left(\mathcal{P}_L+\mathcal{P}_R\right)\Psi\\
&= \bar\Psi_L\gamma^\mu\left[\frac{1}{2}(l_\mu+r_\mu)+\frac{1}{2}(r_\mu-l_\mu)\gamma_5\right]\Psi_L+\bar\Psi_L\gamma^\mu\left[\frac{1}{2}(l_\mu+r_\mu)+\frac{1}{2}(r_\mu-l_\mu)\gamma_5\right]\Psi_R\\
&\quad+\bar\Psi_R\gamma^\mu\left[\frac{1}{2}(l_\mu+r_\mu)+\frac{1}{2}(r_\mu-l_\mu)\gamma_5\right]\Psi_L+\bar\Psi_R\gamma^\mu\left[\frac{1}{2}(l_\mu+r_\mu)+\frac{1}{2}(r_\mu-l_\mu)\gamma_5\right]\Psi_R\\
&\quad-\bar\Psi_L\left(s-i\gamma_5p\right)\Psi_L-\bar\Psi_L\left(s-i\gamma_5p\right)\Psi_R-\bar\Psi_R\left(s-i\gamma_5p\right)\Psi_L-\bar\Psi_R\left(s-i\gamma_5p\right)\Psi_R\\
&= \bar\Psi_L\gamma^\mu l_\mu\Psi_L+\bar\Psi_R\gamma^\mu r_\mu\Psi_R-\bar\Psi_L\left(s-ip\right)\Psi_R-\bar\Psi_R(s+ip)\Psi_L\ ,
\end{aligned}
\tag{3.78}
$$

where we defined the left- and right-handed external fields as

$$
l_\mu(x)=v_\mu(x)-a_\mu(x)\ ,\qquad r_\mu(x)=v_\mu(x)+a_\mu(x)
\tag{3.79}
$$

and frequently used the properties (2.143) of the left- and right-handed projection operators. Now, combining the results (2.147) and (3.78), the Lagrangian in terms of the left- and right-handed quark fields is given by

$$\mathscr{L}_{0,Quark}+\mathscr{L}_{ext}=\bar{\Psi}_L i\gamma^\mu D_\mu\Psi_L+\bar{\Psi}_R i\gamma^\mu D_\mu\Psi_R+\bar{\Psi}_L\gamma^\mu l_\mu\Psi_L+\bar{\Psi}_R\gamma^{\ \mu} r_\mu\Psi_R-\bar{\Psi}_L\left(s-ip\right)\Psi_R-\bar{\Psi}_R(s+ip)\Psi_L\ . \tag{3.80}$$

We observe that the scalar and pseudoscalar external fields mix the left- and right-handed quark fields. On the other hand, this is not a surprising result, since the quark mass matrix which already induced this kind of mixing in Sec. [2.3.2], is contained in the definition of the scalar external field $s(x)$, Eq. (3.76). Now, transforming this Lagrangian under local $SU(2)_L\times SU(2)_R$, we obtain

$$\begin{aligned}
\mathscr{L}_{0,Quark}+\mathscr{L}_{ext}\overset{SU(2)_L\times SU(2)_R}{\longrightarrow}\ &\mathscr{L}'_{0,Quark}+\mathscr{L}'_{ext}\\
&=\bar{\Psi}_L U_L^\dagger(x)i\gamma^\mu D_\mu U_L(x)\Psi_L+\bar{\Psi}_R U_R^\dagger(x)i\gamma^\mu D_\mu U_R(x)\Psi_R+\bar{\Psi}_L U_L^\dagger(x)\gamma^\mu l'_\mu U_L(x)\Psi_L\\
&\quad+\bar{\Psi}_R U_R^\dagger(x)\gamma^\mu r'_\mu U_R(x)\Psi_R-\bar{\Psi}_L U_L^\dagger(x)\left(s'-ip'\right)U_R(x)\Psi_R-\bar{\Psi}_R U_R^\dagger(x)\left(s'+ip'\right)U_L(x)\Psi_L\\
&=\mathscr{L}_{0,Quark}+\bar{\Psi}_L i\gamma^\mu U_L^\dagger(x)\left[\partial_\mu U_L(x)\right]\Psi_L+\bar{\Psi}_R i\gamma^\mu U_R^\dagger(x)\left[\partial_\mu U_R(x)\right]\Psi_R+\bar{\Psi}_L U_L^\dagger(x)\gamma^\mu l'_\mu U_L(x)\Psi_L\\
&\quad+\bar{\Psi}_R U_R^\dagger(x)\gamma^\mu r'_\mu U_R(x)\Psi_R-\bar{\Psi}_L U_L^\dagger(x)\left(s'-ip'\right)U_R(x)\Psi_R-\bar{\Psi}_R U_R^\dagger(x)\left(s'+ip'\right)U_L(x)\Psi_L\ .
\end{aligned}\tag{3.81}$$

Using $\left[\partial_\mu U_{L/R}^\dagger(x)\right]U_{L/R}(x)=-U_{L/R}^\dagger(x)\partial_\mu U_{L/R}(x)$, the invariance of Eq. (3.81) under local $SU(2)_L\times SU(2)_R$ chiral transformations is guaranteed, if the external fields transform according to

$$l_\mu(x)\overset{SU(2)_L\times SU(2)_R}{\longrightarrow} l'_\mu(x)=U_L(x)l_\mu(x)U_L^\dagger(x)+iU_L(x)\partial_\mu U_L^\dagger(x)\ , \tag{3.82}$$

$$r_\mu(x)\overset{SU(2)_L\times SU(2)_R}{\longrightarrow} r'_\mu(x)=U_R(x)r_\mu(x)U_R^\dagger(x)+iU_R(x)\partial_\mu U_R^\dagger(x)\ , \tag{3.83}$$

$$s(x)+ip(x)\overset{SU(2)_L\times SU(2)_R}{\longrightarrow} s'(x)+ip'(x)=U_R(x)\big[s(x)+ip(x)\big]U_L^\dagger(x)\ , \tag{3.84}$$

$$s(x)-ip(x)\overset{SU(2)_L\times SU(2)_R}{\longrightarrow} s'(x)-ip'(x)=U_L(x)\big[s(x)-ip(x)\big]U_R^\dagger(x)\ . \tag{3.85}$$

Obviously, the left- and right-handed external fields have to transform as gauge fields under the local chiral rotations. This transformation behavior of external fields which are associated with symmetry currents is not new. In the previous example of a complex scalar theory, Eq. (3.30), it was already shown that the invariance of the generating functional under local symmetry transformations is connected to a gauge field-like transformation behavior of the external field j_μ. Now, the external fields (3.75) are also subject to other symmetries, since the complete QCD Lagrangian is also invariant under Lorentz and CPT transformations. The $SU(3)_C$ color symmetry is trivially satisfied, because the external fields are introduced as color neutral matrices, i.e., they transform as a singlet under $SU(3)_C$. In order to determine the transformation behavior of the external fields under Lorentz and CPT transformations, we proceed in the same manner as for the chiral rotations. Under proper orthochronous Lorentz transformations[6], the quark fields transform in the $(1/2,0)\oplus(0,1/2)$ representation of the Lorentz group, i.e., they transform according to

$$\Psi(x)\overset{SO^+(1,3)}{\longrightarrow}\Psi'(x')=S(\Lambda)\Psi(\Lambda^{-1}x)\ , \tag{3.86}$$

$$\bar{\Psi}(x)\overset{SO^+(1,3)}{\longrightarrow}\bar{\Psi}'(x')=\bar{\Psi}(\Lambda^{-1}x)S^{-1}(\Lambda)\ , \tag{3.87}$$

where

$$S(\Lambda)=e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\ ,\qquad \sigma^{\mu\nu}=\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]_-\ , \tag{3.88}$$

and $\omega_{\mu\nu}$ contains the group parameters of $SO^+(1,3)$. Again omitting the space-time dependence of the fields and using Eqs. (3.86) and (3.87), the Lagrangian of the external fields transforms according to

$$\begin{aligned}
\mathscr{L}_{ext}\overset{SO^+(1,3)}{\longrightarrow}\mathscr{L}'_{ext}&=\bar{\Psi}S^{-1}(\Lambda)\gamma^\mu\left(v'_\mu+\gamma_5 a'_\mu\right)S(\Lambda)\Psi-\bar{\Psi}S^{-1}(\Lambda)\left(s'-i\gamma_5 p'\right)S(\Lambda)\Psi\\
&=\bar{\Psi}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\left(v'_\mu+\gamma_5 a'_\mu\right)\Psi-\bar{\Psi}S^{-1}(\Lambda)S(\Lambda)\left(s'-i\gamma_5 p'\right)\Psi\\
&=\bar{\Psi}\Lambda^\mu_{\ \nu}\gamma^\nu\left(v'_\mu+\gamma_5 a'_\mu\right)\Psi-\bar{\Psi}\left(s'-i\gamma_5 p'\right)\Psi\ ,
\end{aligned}\tag{3.89}$$

[6] The proper orthochronous Lorentz group $SO^+(1,3)=\left\{\Lambda\in O(1,3)|\Lambda^0_{\ 0}\geq 1,\det(\Lambda)=1\right\}$ forms a subgroup of the Lorentz group $O(1,3)$.

where we used $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$ and the fact that $[\gamma_5, S(\Lambda)]_- = 0$. Requiring the $SO^+(1,3)$ invariance of Eq. (3.89), we obtain

$$v_\mu(x) \stackrel{SO^+(1,3)}{\longrightarrow} v'_\mu(x') = \Lambda_\mu{}^\nu v_\nu(\Lambda^{-1}x) \ , \tag{3.90}$$

$$a_\mu(x) \stackrel{SO^+(1,3)}{\longrightarrow} a'_\mu(x') = \Lambda_\mu{}^\nu a_\nu(\Lambda^{-1}x) \ , \tag{3.91}$$

$$s(x) \stackrel{SO^+(1,3)}{\longrightarrow} s'(x') = s(\Lambda^{-1}x) \ , \tag{3.92}$$

$$p(x) \stackrel{SO^+(1,3)}{\longrightarrow} p'(x') = p(\Lambda^{-1}x) \ . \tag{3.93}$$

Finally, we have to determine the transformation behavior of the external fields under C-, P- and T-transformations. Due to the CPT-Theorem, we only have to check parity and charge conjugation, because the symmetry under time-reversal is then automatically satisfied. Under parity, the quark fields transform as

$$\Psi(x) \xrightarrow{P} \Psi'(x') = \gamma^0\Psi(t, -\mathbf{r}) \ , \tag{3.94}$$

$$\bar\Psi(x) \xrightarrow{P} \bar\Psi'(x') = \bar\Psi(t, -\mathbf{r})\gamma^0 \ . \tag{3.95}$$

Then, using Eqs. (3.94) and (3.95), the Lagrangian involving the external fields transforms according to

$$\begin{aligned}
\mathscr{L}_{ext} \xrightarrow{P} \mathscr{L}'_{ext} &= \bar\Psi\gamma^0\gamma^\mu\left(v'_\mu + a'_\mu\gamma_5\right)\gamma^0\Psi - \bar\Psi\gamma^0\left(s' - i\gamma_5 p'\right)\gamma^0\Psi \\
&= (-1)^{(\mu)}\bar\Psi\gamma^\mu\left(v'_\mu\gamma^0 + a'_\mu\gamma^0\gamma_5\right)\gamma^0\Psi - \bar\Psi\left(s'\gamma^0 - i\gamma^0\gamma_5 p'\right)\gamma^0\Psi \\
&= (-1)^{(\mu)}\bar\Psi\gamma^\mu\left(v'_\mu - a'_\mu\gamma_5\right)\Psi - \bar\Psi\left(s' + i\gamma_5 p'\right)\Psi \ ,
\end{aligned} \tag{3.96}$$

where we again omitted the arguments of the fields. Furthermore, we used the anticommutation relations $[\gamma^\mu, \gamma_5]_+ = 0$ and $\left[\gamma^0, \gamma^i\right]_+ = 0$, for $i = 1, 2, 3$ and introduced the factor $(-1)^{(\mu)}$. This factor is defined to be equal to 1 for $\mu = 0$ and -1 for $\mu = 1, 2, 3$ and originates from the permutation of γ^0 and γ^μ in the first term of the second line of Eq. (3.96). Now, requiring the invariance of the Lagrangian under parity transformations, we find

$$v_\mu(x) \xrightarrow{P} v'_\mu(x') = (-1)^{(\mu)}v_\mu(t, -\mathbf{r}) = v^\mu(t, -\mathbf{r}) \ , \tag{3.97}$$

$$a_\mu(x) \xrightarrow{P} a'_\mu(x') = -(-1)^{(\mu)}a_\mu(t, -\mathbf{r}) = -a^\mu(t, -\mathbf{r}) \ , \tag{3.98}$$

$$s(x) \xrightarrow{P} s'(x') = s(t, -\mathbf{r}) \ , \tag{3.99}$$

$$p(x) \xrightarrow{P} p'(x') = -p(t, -\mathbf{r}) \ . \tag{3.100}$$

In order to complete the discussion of the transformation behavior of the external fields, we come to charge conjugation. To this end, it will be useful to consider the transformation behavior of the i-th component of the quark fields in isospin space. We have

$$\Psi_i(x) \xrightarrow{C} C\bar\Psi_i^t(x) \ , \tag{3.101}$$

$$\bar\Psi_i(x) \xrightarrow{C} \Psi_i^t(x)C \ , \tag{3.102}$$

where the transposition ^{t} acts in spinor space and the charge conjugation operator is defined as $C = i\gamma^2\gamma^0$. Then, omitting the space-time dependence of the fields, the Lagrangian transforms as

$$\begin{aligned}
\mathscr{L}_{ext} \xrightarrow{C} \mathscr{L}'_{ext} &= \Psi_i^t C\gamma^\mu\left(v_\mu^{ij\prime} + a_\mu^{ij\prime}\gamma_5\right)C\bar\Psi_j^t - \Psi_i^t C\left(s^{ij\prime} - i\gamma_5 p^{ij\prime}\right)C\bar\Psi_j^t \\
&= \Psi_i^t C\gamma^\mu C\left(v_\mu^{ij\prime} + a_\mu^{ij\prime}\gamma_5^t\right)\bar\Psi_j^t - \Psi_i^t CC\left(s^{\prime,i'j'} - i\gamma_5^t p^{ij\prime}\right)\bar\Psi_j^t \\
&= \left\{-\bar\Psi_j\left(a_\mu^{ij\prime}\gamma_5 + v_\mu^{ij\prime}\right)\gamma^\mu\Psi_i\right\}^t + \left\{-\bar\Psi_j\left(-i\gamma_5 p^{ij\prime} + s^{ij\prime}\right)\Psi_i\right\}^t \\
&= \bar\Psi_j\gamma^\mu\left(-v_\mu^{ij\prime} + a_\mu^{ij\prime}\gamma_5\right)\Psi_i - \bar\Psi_j\left(s^{ij\prime} - i\gamma_5 p^{ij\prime}\right)\Psi_i \ ,
\end{aligned} \tag{3.103}$$

where we used $[\gamma_5, C]_- = 0, [\gamma^\mu, \gamma_5]_+ = 0$ as well as $\gamma_5 = \gamma_5^t, C\gamma^\mu C = (\gamma^\mu)^t$ and $CC = -\mathbb{1}_{4\times4}$. The additional minus sign in the third line arises from the fact that we interchanged the quark fields which obey Fermi-Dirac statistics. Furthermore, we are able to drop the ^{t}, since the quadratic form is contracted to a scalar in spinor space. Finally, using Eq. (3.103), we are able to read off the transformation behavior of the external fields

$$v_\mu(x) \xrightarrow{C} v'_\mu(x) = -v^T_\mu(x) \ , \tag{3.104}$$

$$a_\mu(x) \xrightarrow{C} a'_\mu(x) = a^T_\mu(x) \ , \tag{3.105}$$

$$s(x) \xrightarrow{C} s'(x) = s^T(x) \ , \tag{3.106}$$

$$p(x) \xrightarrow{C} p'(x) = p^T(x) \ , \tag{3.107}$$

where the ^{T} refers to transposition in isospin space.

3.2 The Nonlinear Realization of a Symmetry

In Sec. [2.2.2] we introduced the concept of a spontaneously broken symmetry at hand of the example of a simple toy model with a discrete internal symmetry, Eq. (2.82). In the Nambu-Goldstone realization, the potential density developed two distinct minima which correspond to two degenerate vacuum states in the quantized theory. Then, we decomposed the initial field variable $\varphi(x)$ into a constant part, corresponding to one of the minima of the potential density and a dynamic part, describing the fluctuations of the field around the minima, Eq. (2.87). In terms of the fluctuation field $\sigma(x)$, the potential density developed a new cubic term. Due to this new term, it seemed that the Z_2-symmetry was not present anymore, since the Lagrangian (2.88) is not invariant under a Z_2-transformation of the fluctuation field $\sigma(x) \xrightarrow{Z_2} -\sigma(x)$. Actually, the Lagrangian (2.88) is still Z_2-symmetric, because Eq. (2.87) requires that the fluctuation field transforms according to $\sigma(x) \xrightarrow{Z_2} -\sigma(x) \mp 2\varphi_0$. Based on this observation, we concluded that the initial symmetry is not broken but hidden, since the symmetry is now realized in a nontrivial way.

It is possible to extend these considerations to models with continuous symmetries and in general, it turns out that the new realization of the symmetry is a nonlinear one. The aim of this section will be a brief, but also quite general introduction to the concept of nonlinearly realized symmetries. Since this framework is closely connected to the spontaneous breakdown of a continuous symmetry, it is not surprising that the Nambu-Goldstone bosons will play a crucial role in the following discussion. In fact, this approach allows a systematic expansion of the Lagrangian in powers of Nambu-Goldstone boson momenta, which makes the concept of nonlinearly realized symmetries a powerful tool in the construction of phenomenological Lagrangians.

In order to keep the discussion general, we consider a theory whose Lagrangian $\mathscr{L}(\phi_a(x))$ is invariant under the space-time independent transformations of an arbitrary compact, connected, semisimple Lie group G, acting linearly on the fields

$$\phi_a(x) \xrightarrow{G} \phi'_a(x) = g_{ab}\phi^b(x) \ , \text{ for } g \in G \ . \tag{3.108}$$

Now, we expect that this symmetry group G is spontaneously broken down to a continuous subgroup $H \subset G$. In addition to that, we require the symmetry transformations $h \in H$ to leave the vacuum expectation values $\phi_{0,a}$ invariant, i.e.,

$$\phi_{0,a} \xrightarrow{H} \phi'_{0,a} = h_{ab}\phi_0^b = \phi_{0,a} \ \forall \ h \in H \ . \tag{3.109}$$

Finally, we denote the generators of the subgroup H as $T_i^{(H)}, i = 1, \ldots, \dim(H)$. The remaining generators of G shall be denoted as $T_{i'}^{(G\backslash H)}, i' = 1, \ldots, \dim(G) - \dim(H)$, and should be chosen in a way that the $\left\{T_i^{(H)}, T_{i'}^{(G\backslash H)}\right\}$ form a complete set of all generators of G. Since G is a compact Lie group, it is possible to choose a basis for the generators, in which the structure constants f_{lmn} are totally antisymmetric. Without the loss of generality, it is possible to choose this basis in a way, so that we are working with a real representation of G. This implies that the generators are purely imaginary and antisymmetric. Additionally, the generators fulfill the following commutation relations[7]

$$\left[T_i^{(H)}, T_j^{(H)}\right]_- = i f_{ij}{}^k T_k^{(H)} \ , \tag{3.110}$$

$$\left[T_i^{(H)}, T_{j'}^{(G\backslash H)}\right]_- = i f_{ij'}{}^{k'} T_{k'}^{(G\backslash H)} \ , \tag{3.111}$$

$$\left[T_{i'}^{(G\backslash H)}, T_{j'}^{(G\backslash H)}\right]_- = i f_{i'j'}{}^{k'} T_{k'}^{(G\backslash H)} + i f_{i'j'}{}^{k} T_k^{(H)} \ . \tag{3.112}$$

[7] From now on, we use the indices $i, j, k, \ldots$ to label objects connected to the subgroup H, while the primed indices $i', j', k', \ldots$ label objects which are connected to the coset space $G\backslash H$.

Relation (3.110) originates from the fact that H is a subgroup of G, so that the generators $\left\{T_i^{(H)}\right\}$ of H form a closed subalgebra. From this commutation relation also follows that the $f_{ijk'}$ are equal to zero, as they would otherwise appear on the right-hand side of Eq. (3.110). The antisymmetry of the structure constants requires that the $f_{ik'j}$ also vanish, so that we obtain Eq. (3.111). The last commutation relation (3.112) is a general expression, since the $\left\{T_{i'}^{(G\backslash H)}\right\}$ in general do not need to form a closed algebra.

Following Ref. [Wei2], we express an arbitrary field configuration $\vec{\phi}(x)$ as a transformation $\Sigma(x) \in G$ acting on a field configuration $\vec{\bar{\phi}}(x)$ from which the Nambu-Goldstone modes have been eliminated

$$\phi_a(x) = \Sigma(x)_{ab}\bar{\phi}^b(x) \ . \tag{3.113}$$

The condition that $\vec{\bar{\phi}}(x)$ does not contain Nambu-Goldstone boson fields can be formulated as follows. First of all, we have to remember the discussion of the Goldstone theorem in Sec. [2.2.3]. In this context, we concluded that the massless excitations, i.e., the Nambu-Goldstone bosons, are contained in the linearly independent eigenvectors $T^{(G\backslash H)i'}\vec{\phi}_0$ of the mass matrix with vanishing eigenvalues. Then, if the $\vec{\bar{\phi}}(x)$ do not contain the Nambu-Goldstone "directions" in field space, they should be orthogonal to the massless eigenvectors. Thus, this condition may be written as

$$\bar{\phi}_a(x)(T^{(G\backslash H)i'})^{ab}\phi_{0,b} = 0 \ , \tag{3.114}$$

where the index i' labels the respective generator which does not annihilate the vacuum. Therefore, the number of independent conditions (3.114) is given by the dimension of the coset space $G\backslash H$. The validity of Eq. (3.114) can be illustrated by considering an explicit example: At the beginning of the discussion of Goldstone's theorem, we started with a scalar field theory (2.99), whose initial $SO(2)$ symmetry was spontaneously broken by picking a non-vanishing vacuum expectation value (2.108). This vacuum expectation value can be interpreted as $\vec{\bar{\phi}}$. The spontaneous breakdown of the $SO(2)$ symmetry generated one Nambu-Goldstone boson associated with one broken generator (2.117). The massless eigenvector corresponding to this Nambu-Goldstone mode was given by Eq. (2.118). It is obvious that the condition (3.114) is satisfied by Eqs. (2.108) and (2.118). It should be mentioned that it is always possible to find a suitable $\Sigma(x) \in G$ such that Eq. (3.114) is satisfied.[8] Another important observation is that Eq. (3.114) is not only satisfied by $\vec{\bar{\phi}}$, since

$$\bar{\phi}_a(x) \to \bar{\phi}'_a(x) = \left[\delta_{ab} + i\Lambda_i \left(T^{(H)i}\right)_{ab}\right]\bar{\phi}^b(x) \tag{3.115}$$

is also a solution. This statement can be proven by inserting Eq. (3.115) into Eq. (3.114). Omitting the matrix indices in field space, we obtain[9]

$$\begin{aligned}
\vec{\bar{\phi}}'^T(x)T^{(G\backslash H)j'}\vec{\phi}_0 &= \left[\vec{\bar{\phi}}^T(x) - i\Lambda_i\vec{\bar{\phi}}^T(x)T^{(H)i}\right]T^{(G\backslash H)j'}\vec{\phi}_0 \\
&= \vec{\bar{\phi}}^T(x)T^{(G\backslash H)j'}\vec{\phi}_0 - i\Lambda_i\vec{\bar{\phi}}^T(x)T^{(H)i}T^{(G\backslash H)j'}\vec{\phi}_0 \\
&= -i\Lambda_i\vec{\bar{\phi}}^T(x)\left[T^{(H)i}, T^{(G\backslash H)j'}\right]_-\vec{\phi}_0 \\
&= 0 \ ,
\end{aligned} \tag{3.116}$$

where we used the antisymmetry of the generators in the first line and eliminated the first term of the second line by exploiting Eq. (3.114). The expression in the third line can be obtained by inserting $i\Lambda_i\vec{\bar{\phi}}^T(x)T^{(G\backslash H)j'}T^{(H)i}\vec{\phi}_0$, which is obviously equal to zero, because the generators of H annihilate the vacuum expectation value. Finally, we can use the commutation relation (3.111), which yields an expression proportional to Eq. (3.114). The importance of this result is based on the fact that Eq. (3.115) is the infinitesimal form of an H transformation of $\vec{\bar{\phi}}$. Therefore, using Eqs. (3.113) and (3.115), we conclude that $\Sigma(x)$ is only determined up to a right multiplication by an element $h(x) \in H$.

This allows us to arrange the group elements of G into so-called equivalence classes[10]. Each of those equivalence classes contains those elements of G, which only differ by a right multiplication of an $h(x) \in H$. This is exactly the definition of the right cosets of H in G. Therefore, in order to find a parametrization of $\Sigma(x)$, we have to find a parametrization for the coset space $G\backslash H$. The discussion of the Goldstone

[8] A proof of this statement can be found in Ref. [Wei2].

[9] Here the ^{T} stands for the transposition of the respective vector in field space.

[10] This is possible, because the Σ's fulfill equivalence relations, i.e., $\Sigma_i(x)$ and $\Sigma_j(x)$ are equivalent, if $\Sigma_i(x) = \Sigma_j(x)h(x)$. It is obvious that this relation is reflexive, transitive, and symmetric.

theorem showed that the number of the Nambu-Goldstone bosons is given by the dimension of the coset space $G\backslash H$. Therefore, we are able to use the fields, describing the Nambu-Goldstone bosons, as the coordinates of the coset space. In fact, it is not surprising that the Nambu-Goldstone bosons arise in the parametrization of $\Sigma(x)$. This information was already contained in Eq. (3.113), because we expressed a general field configuration as a transformation of $\Sigma(x)$ acting on a field, from which the Nambu-Goldstone modes have been eliminated. Since the field $\vec{\bar{\phi}}(x)$ does not contain the information about the massless excitations, these degrees of freedom must be contained in the transformation $\Sigma(x)$. In order to obtain a possible parametrization of $\Sigma(x)$, we can use the fact that in the vicinity of the identity an arbitrary element $g(x) \in G$ can be decomposed as

$$g(x) = e^{i\pi(x)_{l'}T^{(G\backslash H)l'}} e^{i\zeta(x)_i T^{(H)i}} , \tag{3.117}$$

where the $\pi(x)_{l'}$ and the $\zeta(x)_i$ are a set of group parameters. Since we know that $\Sigma(x)$ is only determined up to a right multiplication of a $h(x) \in H$, a possible parametrization of $\Sigma(x)$ is given by

$$\Sigma(x) = e^{i\pi(x)_{l'}T^{(G\backslash H)l'}} \equiv \Sigma(\pi_{l'}(x)) . \tag{3.118}$$

Up to normalization factors, the group parameters $\pi(x)_{l'}$ can be identified as the Nambu-Goldstone boson fields. The definition $\Sigma(x) \equiv \Sigma(\pi_{l'}(x))$ shall remind us that the transformation Σ is in principle a continuous function of the Nambu-Goldstone fields. It should be mentioned that Eq. (3.118) provides only one possible parametrization of the coset space $G\backslash H$. In order to investigate the transformation properties of $\Sigma(\pi_{l'}(x))$ under G and H transformations, we have to come back to the linear transformation properties of the fields $\phi_a(x)$ under the group G. Inserting Eq. (3.113) into Eq. (3.108), we obtain

$$\phi_a(x) \xrightarrow{G} \phi'_a(x) = g_{ab}\phi^b(x) = g_{ad}\Sigma^{dc}(\pi_{l'}(x))\bar{\phi}_c(x) . \tag{3.119}$$

It is obvious that the transformation $g_{ad}\Sigma^{dc}(\pi_{l'}(x))$ is an element of G. Therefore, this transformation must be in the same right coset as $\Sigma'_{ab} \equiv \Sigma_{ab}(\pi'_{l'}(x))$, so that

$$g_{ad}\Sigma^{dc}(\pi_{l'}(x)) = \Sigma_{ae}(\pi'_{l'}(x))h^{ec}(\pi_{l'}(x), g) . \tag{3.120}$$

This transformation can be rewritten by multiplying Eq. (3.120) with the inverse H transformation from the right. The transformation behavior of $\Sigma(\pi_{l'}(x))$ can therefore be written as

$$\Sigma_{ab}(\pi_{l'}(x)) \xrightarrow{G} \Sigma_{ab}(\pi'_{l'}(x)) = g_{ac}\Sigma^{cd}(\pi_{l'}(x))h^{-1}_{db}(\pi_{l'}(x), g) . \tag{3.121}$$

Before we illustrate Eq. (3.121) by means of an explicit example, we use the above results to show that the transformation behavior of the massive fields $\bar{\phi}_a(x)$ under G is fully determined by their transformation properties under the unbroken subgroup H. To this end, we start again at the left-hand side of Eq. (3.119) and write the transformation of a general field configuration $\phi_a(x)$ as

$$\phi_a(x) \xrightarrow{G} \phi'_a(x) = \Sigma_{ab}(\pi'_{l'}(x))\bar{\phi}'^{,b}(x) , \tag{3.122}$$

where we used Eq. (3.113). On the other hand, we can use Eq. (3.120) in Eq. (3.119) to obtain

$$\phi_a(x) \xrightarrow{G} \phi'_a(x) = \Sigma_{ae}(\pi'_{l'}(x))h^{ec}(\pi_{l'}(x), g)\bar{\phi}_c(x) . \tag{3.123}$$

Since the above equations should describe the same transformation, we are able to identify the transformation properties of the massive fields by comparing Eqs. (3.122) and (3.123)

$$\bar{\phi}_a(x) \xrightarrow{G} \bar{\phi}'_a(x) = h_{ab}(\pi_{l'}(x), g)\bar{\phi}^b(x) . \tag{3.124}$$

Obviously, under global G transformations, the massive field configuration $\bar{\phi}_a(x)$ transforms linearly under a local transformation of the unbroken subgroup H.

As mentioned before, we now consider an explicit example to illustrate the abstract transformation behavior (3.120). To this end, we consider the rotation of a three-dimensional real vector in space and set $G = SO(3)$ and $H = SO(2)$. In this case, the coset space $G\backslash H = SO(3)\backslash SO(2)$ is isomorphic to the 2-sphere S^2 which is a curved manifold. Now, we consider a vector $\vec{\varphi}_M$ of unit length, pointing onto the point M. This vector can be interpreted as the vector $\vec{\bar{\phi}}(x)$. In order to obtain an arbitrary 3-vector of unit length $\vec{\varphi}_A$, we can rotate the vector $\vec{\varphi}_M$ by applying an $SO(3)\backslash SO(2)$ transformation. It is obvious that the vector $\vec{\varphi}_A$ is associated with the arbitrary field configuration $\vec{\phi}(x)$, while the $SO(3)\backslash SO(2)$ rotation plays the part of the transformation $\Sigma(\pi_{l'}(x))$. Finally, we can apply a second transformation and rotate the vector from A to another point T, which is described by the unit vector $\vec{\varphi}_T$. In the context of the above discussion, this second rotation is given by the space-time independent G transformation acting on $\vec{\phi}(x)$, Eq. (3.119).

But instead of rotating the vector from M to A and then to T, we are able to find a rotation that directly rotates the vector from M to T. These two cases are depicted in Fig. [3.1]. The direct rotation can be understood as the transformation $\Sigma(\pi'_{l'}(x))$. In general, the transformations, describing the two different ways of rotating the vector from M to T are different. However, it is possible to express the rotation from M to A to T as an $SO(2)$ transformation which rotates the vector $\vec{\varphi}_M$ about itself, followed by the direct rotation from M to T. This compensating $H = SO(2)$ transformation ensures, that the two transformations become the same, as already indicated by Eq. (3.120).

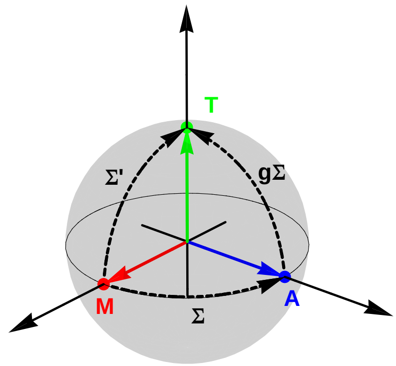


Figure 3.1: Two ways of rotating the vector $\vec{\varphi}_M$ from M to T.

Up to now, we investigated the transformation behavior of $\Sigma(\pi_{l'}(x))$ under general G transformations. In the following step, we can turn to the case where the G transformation is also an element of H, i.e., $g = h \in H \subset G$. To this end, we consider the transformation of the broken generators under global H transformations

$$\begin{aligned} hT^{(G\setminus H)}_{j'}h^{-1} &= \left(\mathbb{1} + i\alpha_i T^{(H)i}\right) T^{(G\setminus H)}_{j'} \left(\mathbb{1} - i\alpha_k T^{(H)k}\right) \\ &= T^{(G\setminus H)}_{j'} + i\alpha_i \left[T^{(H)i}, T^{(G\setminus H)}_{j'}\right]_- \\ &= \mathscr{R}(h)_{j'k'} T^{(G\setminus H)k'} \ , \end{aligned} \tag{3.125}$$

where we used Eq. (3.111) and neglected terms of order $\mathcal{O}(\alpha_i^2)$. Obviously, the broken generators transform according to a linear representation $\mathscr{R}(h)_{j'k'}$ of H. It is also possible to choose the $\pi_{l'}(x)$, describing the Nambu-Goldstone boson fields, in a way that they transform according to a linear representation of H under global H transformations. Using the parametrization (3.118) of the coset space and exploiting the total antisymmetry of the structure constants, we find

$$\begin{aligned} h\Sigma(\pi_{l'}(x))h^{-1} &= \left(\mathbb{1} + i\alpha_j T^{(H)j}\right)\left(\mathbb{1} + i\pi_{k'}(x)T^{(G\setminus H)k'}\right)\left(\mathbb{1} - i\alpha_k T^{(H)k}\right) \\ &= \mathbb{1} - \alpha_j \pi_{k'}(x)\left[T^{(G\setminus H)k'}, T^{(H)j}\right]_- + i\pi_{k'}(x)T^{(G\setminus H)k'} \\ &= \mathbb{1} + i\mathscr{R}(h)_{j'k'}\pi^{k'}(x)T^{(G\setminus H),j'} \\ &= \Sigma(\mathscr{R}(h)_{j'k'}\pi^{k'}(x)) \ , \end{aligned} \tag{3.126}$$

where we again used Eq. (3.111) and neglected terms of order $\mathcal{O}(\alpha_i^2)$. Comparing this result with Eq. (3.121) and remembering the derivation of Eq. (3.124), we obtain

$$\pi_{l'}(x) \xrightarrow{H} \pi'_{l'}(x) = \mathscr{R}(h)_{l'k'}\pi^{k'}(x) \ , \qquad \bar{\phi}_a(x) \xrightarrow{H} \bar{\phi}'_a(x) = h_{ab}\bar{\phi}^b(x) \ . \tag{3.127}$$

It is quite obvious that the transformation behavior of $\pi_{l'}(x)$ and $\bar{\phi}_a(x)$ simplifies enormously, when we consider the case $g = h \in H \subset G$. As mentioned above, we see that the Nambu-Goldstone boson fields transform linearly under a representation $\mathscr{R}(h)$ of the unbroken subgroup H. In addition to that, Eq. (3.127) shows that the massive field configuration transforms according to $h \in H$ itself.

Up to now we investigated the transformation behavior of $\Sigma(\pi_{l'}(x))$ and $\vec{\bar{\phi}}(x)$ under general G transformations and under transformations of the unbroken subgroup H. Another important point is the question how the Nambu-Goldstone bosons enter the Lagrangian. At the beginning of this section, we started with a Lagrangian $\mathscr{L}(\phi_a(x))$ which is invariant under global G transformations. In general, such a Lagrangian contains terms involving derivatives of the fields $\phi_a(x)$ and terms without derivatives. Using Eq. (3.113), we are able to replace the general field configurations $\phi_a(x)$ by the local G transformation $\Sigma_{ab}(\pi_{l'}(x))$ acting on the matter fields $\bar{\phi}^b(x)$. Since the initial Lagrangian is globally G invariant, the transformation $\Sigma_{ab}(\pi_{l'}(x))$ and therefore the Nambu-Goldstone bosons will not enter the Lagrangian through terms without derivatives. Obviously, the terms we are interested in are those involving derivatives of the field configurations $\phi_a(x)$. With Eq. (3.113), the 4-gradient of the fields $\phi_a(x)$ can be written as

$$\begin{aligned}
\partial_\mu\phi_a(x) &= \partial_\mu\left(\Sigma_{ab}(\pi_{l'}(x))\bar{\phi}^b(x)\right) \\
&= \left[\partial_\mu\Sigma_{ab}(\pi_{l'}(x))\right]\bar{\phi}^b(x) + \Sigma_{ab}(\pi_{l'}(x))\partial_\mu\bar{\phi}^b(x) \\
&= \Sigma_{ab}(\pi_{l'}(x))\left[\partial_\mu\bar{\phi}^b(x) + \left(\Sigma^{-1}\right)^{bc}(\pi_{l'}(x))\left[\partial_\mu\Sigma_{cd}(\pi_{l'}(x))\right]\bar{\phi}^d(x)\right] \ .
\end{aligned} \tag{3.128}$$

It is obvious that the Lagrangian becomes a function of the matter fields $\bar{\phi}_a(x)$ and their derivatives $\partial_\mu\bar{\phi}_a(x)$. In addition to that, the matrix $\Sigma^{-1}(\pi_{l'}(x))\left[\partial_\mu\Sigma(\pi_{l'}(x))\right]$ enters the Lagrangian. Since this matrix is an object defined in the algebra of G, it can be decomposed as

$$\Sigma^{-1}(\pi_{l'}(x))\partial_\mu\Sigma(\pi_{l'}(x)) = iT_{i'}^{(G\setminus H)}C_\mu^{(G\setminus H)i'} + iT_i^{(H)}C_\mu^{(H)i} \ , \tag{3.129}$$

where the coefficients of the above expansion are given by

$$C_\mu^{(G\setminus H)i'} = C^{(G\setminus H)i'j'}(\pi_{l'}(x))\partial_\mu\pi_{j'}(x) \ , \tag{3.130}$$

$$C_\mu^{(H)i} = C^{(H)ij'}(\pi_{l'}(x))\partial_\mu\pi_{j'}(x) \ . \tag{3.131}$$

This means that the Nambu-Goldstone boson fields $\pi_{l'}(x)$ enter the Lagrangian through the appearance of the matrix $\Sigma^{-1}(\pi_{l'}(x))\partial_\mu\Sigma(\pi_{l'}(x))$. More precisely, the coefficients (3.130), (3.131) are responsible for the occurrence of the Nambu-Goldstone boson fields.

An important point is that for exact broken symmetries each term of the Lagrangian containing Nambu-Goldstone bosons involves at least one derivative of the respective fields. The physical meaning of this fact can be understood, if we translate the derivative terms of the Lagrangian into momentum space. There, the derivatives become momenta and we are able to conclude that the interaction of Nambu-Goldstone bosons among themselves and among matter fields becomes arbitrarily weak for small momenta and vanishes for zero momentum.

Another important consequence is that mass terms for the Nambu-Goldstone bosons are forbidden by the symmetry of the Lagrangian, since each term must involve at least one derivative. At this point, it should be taken into account that this situation changes, when the spontaneously broken symmetry was not exact. In this case, the Nambu-Goldstone bosons are able to generate terms without derivatives, so that mass terms and interaction terms without derivatives are possible. It will also turn out that the masses, generated by the Nambu-Goldstone bosons, are small.

An example for this phenomenon is given by Quantum Chromodynamics: If we consider the QCD Lagrangian for the two lightest quark flavors up and down, the symmetry group is given by $G = SU(2)_V \times SU(2)_A$.[11] However, this symmetry is not an exact symmetry, since the masses of the up and down quark are different. Therefore, an explicit symmetry breaking is introduced quite naturally into the QCD Lagrangian.

[11] Note that there is also a $U(1)_V$ symmetry which is associated with the baryon number conservation, compare the discussion at the end of Sec. [2.3.2].

As mentioned in Sec. [2.3.2], this approximate symmetry is spontaneously broken down to its diagonal subgroup $H = SU(2)_V$. The Nambu-Goldstone bosons, associated with this symmetry breaking pattern, are the three pions $\left\{\pi^{\pm}, \pi^0\right\}$. Since the initial symmetry group $G = SU(2)_V \times SU(2)_A$ was not exact, the pions are not massless, but significantly lighter than all other hadrons of the QCD spectrum. Later in Chapter [4] we will see other examples for this phenomenon.

Now, let us turn back to the situation where the initial symmetry under G is exact. We saw that the Nambu-Goldstone boson fields and their derivatives enter the Lagrangian through the expansion coefficients (3.130) and (3.131). Therefore, the transformation properties of these objects under global transformations of the group G become interesting. In order to investigate their transformation behavior, we transform the matrix $\Sigma^{-1}(\pi_{l'}(x))\partial_\mu\Sigma(\pi_{l'}(x))$ under global G transformations. Using Eq. (3.121) and omitting the space-time dependence of the Nambu-Goldstone boson fields, we find

$$
\begin{aligned}
\Sigma^{-1}(\pi_{l'})\partial_\mu\Sigma(\pi_{l'}) &\xrightarrow{G} \Sigma^{-1}(\pi'_{l'})\partial_\mu\Sigma(\pi'_{l'}) \\
&= \left[g\Sigma(\pi_{l'})h^{-1}(\pi_{l'},g)\right]^{-1}\partial_\mu\left[g\Sigma(\pi_{l'})h^{-1}(\pi_{l'},g)\right] \\
&= h(\pi_{l'},g)\Sigma^{-1}(\pi_{l'})g^{-1}g\left\{\left[\partial_\mu\Sigma(\pi_{l'})\right]h^{-1}(\pi_{l'},g) + \Sigma(\pi_{l'})\partial_\mu h^{-1}(\pi_{l'},g)\right\} \\
&= h(\pi_{l'},g)\left[\Sigma^{-1}(\pi_{l'})\partial_\mu\Sigma(\pi_{l'})\right]h^{-1}(\pi_{l'},g) + h(\pi_{l'},g)\partial_\mu h^{-1}(\pi_{l'},g) \\
&= h(\pi_{l'},g)\left[\Sigma^{-1}(\pi_{l'})\partial_\mu\Sigma(\pi_{l'})\right]h^{-1}(\pi_{l'},g) - \left[\partial_\mu h(\pi_{l'},g)\right]h^{-1}(\pi_{l'},g)\ ,
\end{aligned}
\tag{3.132}
$$

where we used $0 = \partial_\mu\left[h(\pi_{l'},g)h^{-1}(\pi_{l'},g)\right] = h(\pi_{l'},g)\partial_\mu h^{-1}(\pi_{l'},g) + \left[\partial_\mu h(\pi_{l'},g)\right]h^{-1}(\pi_{l'},g)$ in the last line. Now, we can insert the decomposition (3.129) into the left- and the right-hand sides of Eq. (3.132)

$$
\begin{aligned}
iT^{(G\backslash H)}_{i'}C'^{(G\backslash H)i'}_\mu + iT^{(H)}_iC'^{(H)i}_\mu &= h(\pi_{l'},g)\left[iT^{(G\backslash H)}_{i'}C^{(G\backslash H)i'}_\mu + iT^{(H)}_iC^{(H)i}_\mu\right]h^{-1}(\pi_{l'},g) \\
&\quad - \left[\partial_\mu h(\pi_{l'},g)\right]h^{-1}(\pi_{l'},g) \\
&= ih(\pi_{l'},g)T^{(G\backslash H)}_{i'}C^{(G\backslash H)i'}_\mu h^{-1}(\pi_{l'},g) + ih(\pi_{l'},g)T^{(H)}_iC^{(H)i}_\mu h^{-1}(\pi_{l'},g) \\
&\quad - \left[\partial_\mu h(\pi_{l'},g)\right]h^{-1}(\pi_{l'},g)\ .
\end{aligned}
\tag{3.133}
$$

In order to obtain the transformation behavior of $C^{(G\backslash H)i'}_\mu$ and $C^{(H)i}_\mu$, we have to compare the coefficients of the generators $T^{(G\backslash H)}_{i'}$ and $T^{(H)}_i$. This procedure is valid since these two sets of generators are independent of each other. We find

$$
T^{(G\backslash H)}_{i'}C^{(G\backslash H)i'}_\mu \xrightarrow{G} T^{(G\backslash H)}_{i'}C'^{(G\backslash H)i'}_\mu = h(\pi_{l'},g)T^{(G\backslash H)}_{i'}C^{(G\backslash H)i'}_\mu h^{-1}(\pi_{l'},g)\ , \tag{3.134}
$$

$$
T^{(H)}_iC^{(H)i}_\mu \xrightarrow{G} T^{(H)}_iC'^{(H)i}_\mu = h(\pi_{l'},g)T^{(H)}_iC^{(H)i}_\mu h^{-1}(\pi_{l'},g) + i\left[\partial_\mu h(\pi_{l'},g)\right]h^{-1}(\pi_{l'},g)\ . \tag{3.135}
$$

The above transformation properties can be studied in more detail. Using the commutation relation (3.111), Eq. (3.134) can be written as

$$
\begin{aligned}
T^{(G\backslash H)}_{i'}C'^{(G\backslash H)i'}_\mu &= h(\pi_{l'},g)T^{(G\backslash H)}_{i'}h^{-1}(\pi_{l'},g)C^{(G\backslash H),i'}_\mu \\
&= \left[\mathbb{1} + i\alpha_i(\pi_{l'},g)T^{(H)i}\right]T^{(G\backslash H)}_{i'}\left[\mathbb{1} - i\alpha_j(\pi_{l'},g)T^{(H)j}\right]C^{(G\backslash H)i'}_\mu \\
&= \left\{T^{(G\backslash H)}_{i'} + i\alpha_i(\pi_{l'},g)\left[T^{(H)i}, T^{(G\backslash H)}_{i'}\right]_-\right\}C^{(G\backslash H)i'}_\mu \\
&= \mathscr{R}_{i'j'}\left(h(\pi_{l'},g)\right)C^{(G\backslash H)i'}_\mu T^{(G\backslash H)j'}\ ,
\end{aligned}
\tag{3.136}
$$

where we neglected terms of order $\mathcal{O}(\alpha_i^2)$. It is also possible to rewrite Eq. (3.135). Using the commutation relation (3.110), one finds

$$
\begin{aligned}
T^{(H)}_iC'^{(H)i}_\mu &= h(\pi_{l'},g)T^{(H)}_iC^{(H)i}_\mu h^{-1}(\pi_{l'},g) + i\left[\partial_\mu h(\pi_{l'},g)\right]h^{-1}(\pi_{l'},g) \\
&= h(\pi_{l'},g)T^{(H)}_ih^{-1}(\pi_{l'},g)C^{(H)i}_\mu - T^{(H)}_l\mathscr{C}^l_\mu \\
&= \left[\mathbb{1} + i\alpha_j(\pi_{l'},g)T^{(H)j}\right]T^{(H)}_i\left[\mathbb{1} - i\alpha_k(\pi_{l'},g)T^{(H)k}\right]C^{(H)i}_\mu - T^{(H)}_l\mathscr{C}^l_\mu \\
&= \left\{T^{(H)}_i + i\alpha_k(\pi_{l'},g)\left[T^{(H)k}, T^{(H)}_i\right]_-\right\}C^{(H)i}_\mu - T^{(H)}_l\mathscr{C}^l_\mu \\
&= \tilde{\mathscr{R}}_{ij}\left(h(\pi_{l'},g)\right)C^{(H)i}_\mu T^{(H)j} - T^{(H)}_l\mathscr{C}^l_\mu\ .
\end{aligned}
\tag{3.137}
$$

In the second line of the above calculation, we decomposed the object $[\partial_\mu h(\pi_{l'}, g)]\, h^{-1}(\pi_{l'}, g)$ similarly to Eq. (3.129)

$$[\partial_\mu h(\pi_{l'}, g)]\, h^{-1}(\pi_{l'}, g) = iT_i^{(H)}\mathscr{C}_\mu^i = iT_i^{(H)}\mathscr{C}^{ij}(\pi_{l'}(x))\partial_\mu \pi_j(x) \ . \tag{3.138}$$

Obviously, the quantity (3.130) transforms in a similar way as the massive field configuration $\vec{\vec{\phi}}(x)$. More precisely, the transformation behavior under global G transformations is determined by a local H transformation $h(\pi_{l'}, g)$, even though in a different representation of this transformation. In the transformation behavior of Eq. (3.131) arises an additional term which is proportional to the 4-gradient of the local H transformation. Finally, we can study the transformation properties of the 4-gradient of the massive field configuration $\vec{\vec{\phi}}(x)$. Using Eq. (3.124), we find

$$\begin{aligned}\partial_\mu \vec{\vec{\phi}}(x) &\xrightarrow{G} \partial_\mu \vec{\vec{\phi}}'(x) \\ &= [\partial_\mu h(\pi_{l'}, g)]\, \vec{\vec{\phi}}(x) + h(\pi_{l'}, g)\partial_\mu \vec{\vec{\phi}}(x) \\ &= h(\pi_{l'}, g)\left\{\partial_\mu \vec{\vec{\phi}}(x) + h^{-1}(\pi_{l'}, g)\, [\partial_\mu h(\pi_{l'}, g)]\, \vec{\vec{\phi}}(x)\right\} \ .\end{aligned} \tag{3.139}$$

The last term of the above equation is of the same form as the additional term in Eq. (3.135). Therefore, this term can be compensated by defining an appropriate covariant derivative of the matter fields $\vec{\vec{\phi}}(x)$. We choose

$$D_\mu \vec{\vec{\phi}}(x) = \partial_\mu \vec{\vec{\phi}}(x) + iT_i^{(H)} C_\mu^{(H)i} \vec{\vec{\phi}}(x) \ . \tag{3.140}$$

It is obvious that this object transforms in an analogous manner as Eq. (3.124) under global G transformations

$$\begin{aligned}D_\mu \vec{\vec{\phi}}(x) &\xrightarrow{G} \left(D_\mu \vec{\vec{\phi}}(x)\right)' \\ &= \partial_\mu \vec{\vec{\phi}}'(x) + iT_i^{(H)} C_\mu^{\prime(H)i} \vec{\vec{\phi}}'(x) \\ &= h(\pi_{l'}, g)\left\{\partial_\mu \vec{\vec{\phi}}(x) + h^{-1}(\pi_{l'}, g)\, [\partial_\mu h(\pi_{l'}, g)]\, \vec{\vec{\phi}}(x)\right\} \\ &\quad + i\left\{h(\pi_{l'}, g) T_i^{(H)} C_\mu^{(H)i} h^{-1}(\pi_{l'}, g) + i\, [\partial_\mu h(\pi_{l'}, g)]\, h^{-1}(\pi_{l'}, g)\right\} h(\pi_{l'}, g)\vec{\vec{\phi}}(x) \\ &= h(\pi_{l'}, g)\left[\partial_\mu \vec{\vec{\phi}}(x) + iT_i^{(H)} C_\mu^{(H)i} \vec{\vec{\phi}}(x)\right] \\ &= h(\pi_{l'}, g) D_\mu \vec{\vec{\phi}}(x) \ ,\end{aligned} \tag{3.141}$$

where we used the results (3.135) and (3.139). It should be mentioned that it is possible to construct higher covariant derivatives which transform in the same manner, see Ref. [Wei2].

The important point is now that an H invariant theory which is constructed from the massive field vector $\vec{\vec{\phi}}(x)$, its covariant derivative $D_\mu \vec{\vec{\phi}}(x)$, and the coefficients $C_\mu^{(G\backslash H)i'}$, is automatically invariant under the whole symmetry group G. This means that the theory "remembers" its initial spontaneously broken symmetry group, so that the only difference between the broken and the unbroken symmetry group is given by the realization of the symmetry. For the unbroken subgroup H this realization is linear, while on the Nambu-Goldstone boson manifold, i.e., the coset space $G\backslash H$, the realization is nonlinear.

At the end of this section, we apply this formalism to the case of two-flavor QCD. In Sec. [2.3.2], we already introduced the chiral symmetry of Quantum Chromodynamics. We saw that the fermionic part of the QCD Lagrangian with N_f quark-flavors has a global $SU(N_f)_V \times SU(N_f)_A \times U(1)_V$ symmetry for vanishing quark masses. Since the $U(1)_V$ symmetry is related to the conservation of the baryon number and not important for the current discussion, we restrict ourselves to the remaining $SU(N_f)_V \times SU(N_f)_A$ symmetry. Now, in the real world, the quarks are not massless. But in case of two-flavor QCD the masses of the up and down quark are vanishingly small compared to a typical energy scale $\Lambda_{QCD} \approx 1$ GeV. Therefore, we are dealing with an approximate $SU(2)_V \times SU(2)_A$ chiral symmetry of the quark part of the QCD Lagrangian (2.133). In the previous discussion of Sec. [2.3.2] of the chiral symmetry, we started by projecting the quark field onto its left- and right-handed components. We saw that these left- and right-handed quark fields can be rotated independently of each other, resulting in an $SU(2)_L \times SU(2)_R$[12] symmetry which is isomorphic to $SU(2)_V \times SU(2)_A$.

[12] From now on, we will take the general results of Sec. [2.3.2] for the special case $N_f = 2$.

In this section, we will take a slightly different way in order to obtain the desired results. First of all, we recognize, that in the chiral limit $m = 0$, the fermionic part (2.133) is invariant under the following transformation of the quark field

$$\Psi(x) \overset{SU(2)_V\times SU(2)_A}{\longrightarrow} \Psi'(x) = e^{-i\alpha_{V,a}T_V^a - i\alpha_{A,b}T_A^b}\Psi(x) \ , \ a,b = 1,2,3 \ , \tag{3.142}$$

where $\Psi(x) = (u(x), d(x))^T$. In addition to that, the vector and axial-vector generators are defined as

$$T_V^a = \tau^a/2 \ , \tag{3.143}$$
$$T_A^a = \gamma_5 T_V^a \ , \tag{3.144}$$

where τ^a are the usual Pauli matrices. The numerical factor 1/2 is chosen, so that $\mathrm{Tr}\left(T_{V/A}^a T_{V/A}^b\right) = \delta^{ab}/2$. Using Noether's theorem (2.21), we obtain the same conserved currents (2.155) and (2.156) as in the previous discussion in Sec. [2.3.2]. Now, we define new generators which project only on the left- and right-handed parts of the quark field $\Psi(x)$

$$T_L^a = \mathcal{P}_L T_V^a \ , \tag{3.145}$$
$$T_R^a = \mathcal{P}_R T_V^a \ , \tag{3.146}$$

where $\mathcal{P}_L$ and $\mathcal{P}_R$ are the left- and right-handed projection operators, cf. Eq. (2.142). These generators fulfill the $SU(2)_L \times SU(2)_R$ Lie algebra, since

$$\left[T_L^a, T_L^b\right]_- = i\epsilon^{abc}T_{L,c} \ , \tag{3.147}$$
$$\left[T_R^a, T_R^b\right]_- = i\epsilon^{abc}T_{R,c} \ , \tag{3.148}$$
$$\left[T_L^a, T_R^b\right]_- = 0 \ . \tag{3.149}$$

The relations (3.147) and (3.148) follow from Eq. (2.143) and the $SU(2)$ Lie algebra which is satisfied by the Pauli matrices, compare App. [6.1.2]. The last commutation relation follows from the fact that the projection operators project onto orthogonal subspaces, i.e., $\mathcal{P}_L\mathcal{P}_R = \mathcal{P}_R\mathcal{P}_L = 0$. It is obvious that the vector and axial-vector generators (3.143), (3.144) are connected to the left- and right-handed generators (3.145), (3.146)

$$T_V^a = T_L^a + T_R^a \ , \qquad T_A^a = T_R^a - T_L^a \ . \tag{3.150}$$

Using the commutation relations (3.147) - (3.149), we can show that

$$\left[T_V^a, T_V^b\right]_- = i\epsilon^{abc}T_{V,c} \ , \tag{3.151}$$
$$\left[T_V^a, T_A^b\right]_- = i\epsilon^{abc}T_{A,c} \ , \tag{3.152}$$
$$\left[T_A^a, T_A^b\right]_- = i\epsilon^{abc}T_{V,c} \ . \tag{3.153}$$

The above commutation relations form the Lie algebra of $SU(2)_V \times SU(2)_A$. These relations are of the same form as those we started with in our general discussion at the beginning of this section. Comparing Eqs. (3.151) - (3.153) with Eqs. (3.110) - (3.112), we see that the $f_{i'j'k'}$ vanish in the case of chiral $SU(2)_V \times SU(2)_A$. Therefore, the coset space $SU(2)_V \times SU(2)_A \backslash SU(2)_V$ is called a symmetric space. Following the same steps as before in the general case, we write a general $SU(2)_V \times SU(2)_A$ transformation $g(x)$ as

$$g(x) = e^{-i\pi_a(x)T_A^a}e^{-i\alpha_{V,a}T_V^a} \ . \tag{3.154}$$

Since the second exponential in Eq. (3.154) is a transformation of the unbroken $SU(2)_V$ subgroup, we identify

$$\Sigma(x) = e^{-i\pi_a(x)T_A^a} \tag{3.155}$$

as the transformation which represents each right coset of $SU(2)_V \times SU(2)_A \backslash SU(2)_V$. Again, the group parameters $\pi_a(x)$ are, up to a normalization factor, identified with the Nambu-Goldstone bosons fields, i.e., the fields describing the pion triplet $\left\{\pi^\pm, \pi^0\right\}$. The transformation behavior of Eq. (3.155) is fully determined by Eq. (3.120), so that

$$e^{-i\alpha_{V,a}T_V^a - i\alpha_{A,b}T_A^b}e^{-i\pi_a(x)T_A^a} = e^{-i\pi'_a(x)T_A^a}e^{-i\alpha_{V,a}T_V^a} \ . \tag{3.156}$$

Using Eq. (3.150), we are able to rewrite the above equation as follows

$$e^{-i\alpha_{L,a}T_L^a - i\alpha_{R,a}T_R^a}e^{-i\pi_a(x)(T_R^a - T_L^a)} = e^{-i\pi'_a(x)(T_R^a - T_L^a)}e^{-i\alpha_{V,a}(x)(T_L^a + T_R^a)} \ . \tag{3.157}$$

According to Eq. (3.149), the left- and right-handed generators commute, so that the four exponentials in Eq. (3.157), involving linear combinations of those generators, can be factorized. Then, in general, each of the eight exponentials is of the following form

$$\tilde{U}_{L/R}(x) = e^{-i\tilde{\alpha}_a(x)T^a_{L/R}} \, , \tag{3.158}$$

where the $\tilde{\alpha}_a$ denote the different group parameters. Using Eq. (2.143), the above exponential can be written as

$$\begin{aligned}\tilde{U}_{L/R}(x) &= (\mathcal{P}_L + \mathcal{P}_R)\,\tilde{U}_{L/R}(x) \\ &= (\mathcal{P}_L + \mathcal{P}_R)\left(\mathbb{1} - i\tilde{\alpha}_a \mathcal{P}_{L/R} T^a_V - \ldots\right) \\ &= \mathcal{P}_{L/R} e^{-i\tilde{\alpha}_a T^a_V} + \mathcal{P}_{R/L} \, ,\end{aligned} \tag{3.159}$$

where we used Eqs. (3.145), (3.146), and the fact that the $\mathcal{P}_{L/R}$ project onto orthogonal subspaces. Applying this general consideration to Eq. (3.157), we obtain

$$\begin{aligned}&\left[\mathcal{P}_L e^{-i\alpha_{L,a}T^a_V} + \mathcal{P}_R e^{-i\alpha_{R,a}T^a_V}\right]\left[\mathcal{P}_L e^{i\pi_a(x)T^a_V} + \mathcal{P}_R e^{-i\pi_a(x)T^a_V}\right] \\ &= \left[\mathcal{P}_L e^{i\pi'_a(x)T^a_V} + \mathcal{P}_R e^{-i\pi'_a(x)T^a_V}\right]\left[\mathcal{P}_L e^{-i\alpha_{V,a}(x)T^a_V} + \mathcal{P}_R e^{-i\alpha_{V,a}(x)T^a_V}\right] \, .\end{aligned} \tag{3.160}$$

The above equation can be divided into two parts, one part proportional to $\mathcal{P}_L$ and another part proportional to $\mathcal{P}_R$. The left-handed part of Eq. (3.160) is given by

$$e^{-i\alpha_{L,a}T^a_V} e^{i\pi_a(x)T^a_V} = e^{i\pi'_a(x)T^a_V} e^{-i\alpha_{V,a}(x)T^a_V} \, . \tag{3.161}$$

For the right-handed part, we obtain a similar expression

$$e^{-i\alpha_{R,a}T^a_V} e^{-i\pi_a(x)T^a_V} = e^{-i\pi'_a(x)T^a_V} e^{-i\alpha_{V,a}(x)T^a_V} \, . \tag{3.162}$$

In order to solve Eq. (3.162) for the exponential including the transformed Nambu-Goldstone boson fields, we multiply Eq. (3.162) from the right by the inverse of Eq. (3.161). Defining

$$\mathcal{U}(x) = e^{-2i\pi_a(x)T^a_V} \, , \tag{3.163}$$

we obtain the following transformation behaviour of $\mathcal{U}(x)$

$$\mathcal{U}(x) \overset{SU(2)_V \times SU(2)_A}{\longrightarrow} \mathcal{U}'(x) = U_R \mathcal{U}(x) U_L^\dagger \, , \tag{3.164}$$

where we used Eq. (2.148) to identify the left- and right-handed transformations. It should be taken into account that the canonical parametrization (3.163) of the coset space is only one possible parametrization. In the special case of $N_f = 2$, the coset space can also be parametrized by

$$\mathcal{U}(x) = \frac{1}{f_\pi}\left[\sigma(x) + i\pi_i(x)\tau^i\right] \, , \tag{3.165}$$

where $\sigma(x) = f_\pi\sqrt{1 - \pi_i^2(x)/f_\pi^2}$ and $f_\pi \simeq 93$ MeV is the so-called pion decay constant. This parametrization originates from the fact that the unit sphere embedded in 4-dimensional Euclidean space, $S^3 = \{\mathbf{x} \in \mathbb{R}^4 | \mathbf{x}^T\mathbf{x} = 1\}$, is diffeomorphic[13] to the group space of $SU(2)$. Then, we can use that every unitary, unimodular (2×2)-matrix $\mathcal{U}$ can be decomposed as $\mathcal{U} = \mathscr{U}_0 \mathbb{1}_{2\times 2} + i\mathscr{U}_i\tau^i$, in which $\vec{\mathscr{U}} = (\mathscr{U}_0, \mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3)^T \in \mathbb{R}^4$ defines a unit vector. Finally, identifying $(\sigma, \pi_1, \pi_2, \pi_3)^T = f_\pi\,(\mathscr{U}_0, \mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3)^T$ and solving $\vec{\mathscr{U}}^T\vec{\mathscr{U}} = 1$ for σ, we are left with Eq. (3.165). A very important point in this context is that the physical quantities, i.e., the scattering matrix elements as well as the n-point functions, are not affected by the choice of the coset parametrization, compare Ref. [ScSc].

3.3 The Chiral Lagrangian up to $\mathcal{O}(p^4)$

To conclude this chapter about ChPT, we finally have to use the results of the previous sections in order to illustrate the construction of the chiral Lagrangian. To this end, we begin to introduce the main building blocks of the chiral Lagrangian. In general, it will be possible to construct an infinite sequence of terms which satisfy the required symmetries of QCD. Therefore, we have to introduce a classification scheme for these terms, which will lead us to the so-called chiral expansion. Then, using this approach, we construct the leading and next-to-leading order terms of the chiral Lagrangian.

[13] A mapping $f : \mathcal{M} \mapsto \mathcal{N}$ between two manifolds $\mathcal{M}$ and $\mathcal{N}$ is said to be a diffeomorphism, if f is a bijection with the property that f and its inverse f^{-1} are differentiable. If such a diffeomorphism exists, then the manifolds $\mathcal{M}$ and $\mathcal{N}$ are said to be diffeomorphic.

3.3.1 Basic Objects and Chiral Expansion

As a starting point for the motivation of the main building blocks of the chiral Lagrangian, we have to go back to the hadronic n-point functions. In the first section of this chapter, we introduced two important properties of these n-point functions, which will help us to understand the emergence of the basic objects of the chiral Lagrangian. In Sec. [3.1.1], we studied the pole structure of an arbitrary n-point function. It was shown that these functions will always develop a pole, whenever it is possible to create an on-shell intermediate particle. Furthermore, this pole can be interpreted as a propagator of the intermediate particle, compare Eq. (3.22). In the language of QCD this means that in the low-energy regime, all hadronic n-point functions are dominated by the pole which arises from the existence of the pion. The physical interpretation of this fact is that, at low energies, all hadronic processes are determined by the exchange of pions. Therefore, it seems to be quite intuitive to describe the low-energy regime of QCD by an effective field theory with pionic fields as dynamical variables.

The implementation of the pion fields is realized by using the formalism of Sec. [3.2], namely by using the nonlinear realization of $SU(2)_V \times SU(2)_A$. At this point, we do not want to restrict ourselves to a special parametrization of the coset space $SU(2)_V \times SU(2)_A \backslash SU(2)_V$, so that the $SU(2)$ matrix, containing the pion fields will be simply denoted as $\mathcal{U}(x)$. Since the chiral Lagrangian should fulfill the same symmetries as the original QCD Lagrangian, it will be useful to study the transformation behavior of $\mathcal{U}(x)$ under local[14] $SU(2)_L \times SU(2)_R$ transformations, under proper orthochronous Lorentz transformations, and under CP-transformations[15]. In addition to that, there is the local $SU(3)_C$ color symmetry of the QCD Lagrangian. But this symmetry is trivially fulfilled, since the dynamical variables of ChPT are mesonic fields, which transform as color singlets under the QCD gauge group.
The transformation behavior of $\mathcal{U}(x)$ under local chiral rotations is given by

$$\mathcal{U}(x) \overset{SU(2)_L \times SU(2)_R}{\longrightarrow} \mathcal{U}'(x) = U_R(x)\mathcal{U}(x)U_L^\dagger(x) \ , \tag{3.166}$$

which simply describes a generalization of Eq. (3.164) to local left- and right-handed rotations. For proper orthochronous Lorentz transformations, the transformation behavior becomes quite simple, since the matrix $\mathcal{U}(x)$ describes pseudoscalar fields. We find, cf. Eq. (3.93),

$$\mathcal{U}(x) \overset{SO^+(1,3)}{\longrightarrow} \mathcal{U}'(x') = \mathcal{U}(\Lambda^{-1}x) \ . \tag{3.167}$$

Finally, the transformation properties under the discrete symmetries are given by, cf. Eqs. (3.100), (3.101),

$$\mathcal{U}(x) \overset{C}{\longrightarrow} \mathcal{U}'(x) = \mathcal{U}^T(x) \ , \tag{3.168}$$

$$\mathcal{U}(t,\mathbf{r}) \overset{P}{\longrightarrow} \mathcal{U}'(t,\mathbf{r}') = \mathcal{U}^\dagger(t,-\mathbf{r}) \ . \tag{3.169}$$

In order to illustrate the above relations, we consider the canonical parametrization (3.163) of the coset space $SU(2)_V \times SU(2)_A \backslash SU(2)_V$ and define $\Phi(x) \equiv -2\pi_a(x)T_V^a$. Since the pionic fields are pseudoscalar fields, the $SU(2)$ matrix $\Phi(x)$ should have the same transformation properties like the pseudoscalar current

$$\Phi_{ij}(x) \mathrel{\hat{=}} \bar{\Psi}_j i\gamma_5 \Psi_i \ . \tag{3.170}$$

Under charge conjugation and parity transformations, this object transforms as $\Phi \overset{C}{\longrightarrow} \Phi^T$ and $\Phi \overset{P}{\longrightarrow} -\Phi$. Inserting these relations back into the exponential, we finally end up with Eqs. (3.168) and (3.169).

At the beginning of the discussion concerning the transformation behavior of the matrix $\mathcal{U}(x)$, we stated that the chiral Lagrangian should be invariant under local chiral rotations. This statement requires some explanations which automatically will lead us to further main building blocks of the chiral Lagrangian. To this end, we remember that the hadronic n-point functions not only have an important pole structure, but also fulfill symmetry relations among themselves, the so-called WFT identities. In Sec. [3.1.3], we used an explicit example to show that these symmetry relations manifest themselves as an invariance property of the generating functional, i.e., they hold if and only if the generating functional remains invariant under local transformations of the symmetry group in question.

[14] The necessity of local chiral transformations originates from the WFT identities which have to be fulfilled by the hadronic n-point functions. This important point will be discussed in more detail in a moment.
[15] Of course, the QCD Lagrangian is also invariant under time-reversal transformations, but due to the CPT-theorem, we will only focus on CP-transformations.

At the end of this section, we saw that the application of this approach requires the appearance of additional terms in the QCD Lagrangian, see Eq. (3.74). These additional terms contain quark/anti-quark bilinear forms which are coupled to external fields (3.75) and enable us to derive the hadronic n-point functions directly from the new generating functional of QCD (3.77). Finally, it was shown that the resulting Lagrangian admits a local chiral symmetry, so that we are able to analyze the chiral WFT identities. It is therefore not surprising that these external fields should also enter the chiral Lagrangian. And in fact, besides the matrix $\mathcal{U}(x)$, they form the main building blocks of the chiral Lagrangian. To this end, we define the field-strength tensors for the left- and right-handed external fields

$$f^{(L)}_{\mu\nu}(x) = \partial_\mu l_\nu(x) - \partial_\nu l_\mu(x) - i\left[l_\mu(x), l_\nu(x)\right]_- \ , \tag{3.171}$$

$$f^{(R)}_{\mu\nu}(x) = \partial_\mu r_\nu(x) - \partial_\nu r_\mu(x) - i\left[r_\mu(x), r_\nu(x)\right]_- \ . \tag{3.172}$$

The scalar and pseudoscalar external fields will be summarized as

$$\chi(x) = 2B_0\left[s(x) + ip(x)\right] \ , \tag{3.173}$$

where the constant B_0 is proportional to the scalar quark condensate, compare Ref. [ScSc]. The transformation properties of Eqs. (3.171)-(3.173) under the various symmetry transformations follow immediately from those of the external fields. Using Eqs. (3.82) and (3.83), we find

$$\begin{aligned} f^{(L)}_{\mu\nu}(x) \stackrel{SU(2)_L\times SU(2)_R}{\longrightarrow} f^{(L)\prime}_{\mu\nu}(x) &= \partial_\mu\left[U_L(x)l_\nu(x)U_L^\dagger(x) + iU_L(x)\partial_\nu U_L^\dagger(x)\right] - \partial_\nu\left[U_L(x)l_\mu(x)U_L^\dagger(x) + iU_L(x)\partial_\mu U_L^\dagger(x)\right] \\ &\quad - i\left[U_L(x)l_\mu(x)U_L^\dagger(x) + iU_L(x)\partial_\mu U_L^\dagger(x), U_L(x)l_\nu(x)U_L^\dagger(x) + iU_L(x)\partial_\nu U_L^\dagger(x)\right]_- \\ &= U_L(x)\left[\partial_\mu l_\nu(x) - \partial_\nu l_\mu(x) - i\left[l_\mu(x), l_\nu(x)\right]_-\right]U_L^\dagger(x) \\ &= U_L(x) f^{(L)}_{\mu\nu}(x) U_L^\dagger(x) \end{aligned} \tag{3.174}$$

and similarly

$$\begin{aligned} f^{(R)}_{\mu\nu}(x) \stackrel{SU(2)_L\times SU(2)_R}{\longrightarrow} f^{(R)\prime}_{\mu\nu}(x) &= \partial_\mu\left[U_R(x)r_\nu(x)U_R^\dagger(x) + iU_R(x)\partial_\nu U_R^\dagger(x)\right] - \partial_\nu\left[U_R(x)r_\mu(x)U_R^\dagger(x) + iU_R(x)\partial_\mu U_R^\dagger(x)\right] \\ &\quad - i\left[U_R(x)r_\mu(x)U_R^\dagger(x) + iU_R(x)\partial_\mu U_R^\dagger(x), U_R(x)r_\nu(x)U_R^\dagger(x) + iU_R(x)\partial_\nu U_R^\dagger(x)\right]_- \\ &= U_R(x)\left[\partial_\mu r_\nu(x) - \partial_\nu r_\mu(x) - i\left[r_\mu(x), r_\nu(x)\right]_-\right]U_R^\dagger(x) \\ &= U_R(x) f^{(R)}_{\mu\nu}(x) U_R^\dagger(x) \ , \end{aligned} \tag{3.175}$$

where we frequently used that $(\partial_\mu U^\dagger_{L/R}(x))U_{L/R}(x) = -U^\dagger_{L/R}(x)\partial_\mu U_{L/R}(x)$. Under proper orthochronous Lorentz transformations, the left- and right-handed field-strength tensors transform as a Lorentz tensor, i.e.,

$$f^{(L/R)}_{\mu\nu} \stackrel{SO^+(1,3)}{\longrightarrow} f^{(L/R)\prime}_{\mu\nu}(x') = \Lambda_\mu^{\ \alpha}\Lambda_\nu^{\ \beta} f^{(L/R)}_{\alpha\beta}(\Lambda^{-1}x) \ , \tag{3.176}$$

where we used the definitions (3.79) in combination with Eqs. (3.90) and (3.91). In order to derive the transformation behavior under charge conjugation and parity transformations, we have to start from the left- and right-handed external fields. From Eqs. (3.79) and (3.104), (3.105), respectively (3.97), (3.98) we find

$$l_\mu(x) \stackrel{C}{\longrightarrow} l'_\mu(x) = -r^T_\mu(x) \ , \tag{3.177}$$

$$r_\mu(x) \stackrel{C}{\longrightarrow} r'_\mu(x) = -l^T_\mu(x) \tag{3.178}$$

and

$$l_\mu(t,\mathbf{r}) \stackrel{P}{\longrightarrow} l'_\mu(t,\mathbf{r}') = (-1)^{(\mu)} r_\mu(t,-\mathbf{r}) = r^\mu(t,-\mathbf{r}) \ , \tag{3.179}$$

$$r_\mu(t,\mathbf{r}) \stackrel{P}{\longrightarrow} r'_\mu(t,\mathbf{r}') = (-1)^{(\mu)} l_\mu(t,-\mathbf{r}) = l^\mu(t,-\mathbf{r}) \ . \tag{3.180}$$

Combining the results (3.177)-(3.180), we finally obtain

$$f^{(L/R)}_{\mu\nu}(x) \stackrel{C}{\longrightarrow} f^{(L/R)\prime}_{\mu\nu}(x) = -f^{(R/L)T}_{\mu\nu}(x) \ , \tag{3.181}$$

$$f^{(L/R)}_{\mu\nu}(t,\mathbf{r}) \stackrel{P}{\longrightarrow} f^{(L/R)\prime}_{\mu\nu}(t,\mathbf{r}') = f^{(R/L)\mu\nu}(t,-\mathbf{r}) \tag{3.182}$$

as transformation properties of the left- and right-handed field-strength tensors. Finally, we are left with the transformation behavior of the scalar/pseudoscalar matrix $\chi(x)$. According to Eq. (3.84), this object transforms in the same manner as the matrix $\mathcal{U}(x)$ under local chiral rotations, i.e.,

$$\chi(x) \overset{SU(2)_L\times SU(2)_R}{\longrightarrow} \chi'(x) = U_R(x)\chi(x)U_L^\dagger(x) \ . \tag{3.183}$$

The transformation behavior under proper orthochronous Lorentz transformations can be obtained from Eqs. (3.92) and (3.93). We find

$$\chi(x) \overset{SO^+(1,3)}{\longrightarrow} \chi'(x') = \chi(\Lambda^{-1}x) \ . \tag{3.184}$$

Finally, under the discrete symmetry operations, the scalar/pseudoscalar matrix transforms according to

$$\chi(x) \overset{C}{\longrightarrow} \chi'(x) = \chi^T(x) \ , \tag{3.185}$$

$$\chi(t,\mathbf{r}) \overset{P}{\longrightarrow} \chi'(t,\mathbf{r}') = \chi^\dagger(t,-\mathbf{r}) \ , \tag{3.186}$$

where we used Eqs. (3.106), (3.107) and (3.99), (3.100). Now, the space-time dependence of the left- and right-handed rotations requires the definition of a covariant derivative in order to construct chirally invariant terms involving derivatives. This covariant derivative should be defined in a way that it transforms in the same way as the object it acts on. In the case of the pion matrix $\mathcal{U}(x)$ and the scalar/pseudoscalar matrix $\chi(x)$, this covariant derivative should transform from the left with a right-handed chiral rotation and from the right with an adjoint left-handed transformation matrix. To this end, we define

$$D_\mu\mathscr{O}(x) = \partial_\mu\mathscr{O}(x) - ir_\mu(x)\mathscr{O}(x) + i\mathscr{O}(x)l_\mu(x) \tag{3.187}$$

as the covariant derivative for an object $\mathscr{O}(x)$ transforming as $\mathscr{O}(x) \overset{SU(2)_L\times SU(2)_R}{\longrightarrow} U_R(x)\mathscr{O}(x)U_L^\dagger(x)$ under local chiral rotations. It is now easy to verify that Eq. (3.187) transforms in the desired way. Using Eqs. (3.82) and (3.83), we obtain

$$\begin{aligned} D_\mu\mathscr{O}(x) \overset{SU(2)_L\times SU(2)_R}{\longrightarrow} (D_\mu\mathscr{O}(x))' &= \partial_\mu\left[U_R(x)\mathscr{O}(x)U_L^\dagger\right] - i\left[U_R(x)r_\mu(x)U_R^\dagger(x) + iU_R(x)\partial_\mu U_R^\dagger(x)\right]U_R(x)\mathscr{O}(x)U_L^\dagger \\ &\quad + iU_R(x)\mathscr{O}(x)U_L^\dagger\left[U_L(x)l_\mu(x)U_L^\dagger(x) + iU_L(x)\partial_\mu U_L^\dagger(x)\right] \\ &= U_R(x)\left[\partial_\mu\mathscr{O}(x) - ir_\mu(x)\mathscr{O}(x) + i\mathscr{O}(x)l_\mu(x)\right]U_L^\dagger(x) \\ &= U_R(x)D_\mu\mathscr{O}(x)U_L^\dagger(x) \ . \end{aligned} \tag{3.188}$$

If we now identify the object $\mathscr{O}(x)$ with $\mathcal{U}(x)$ or $\chi(x)$, the transformation properties of the covariant derivative under proper orthochronous Lorentz transformations and CP-transformations immediately follow from the previous results. We obtain

$$D_\mu\mathscr{O}(x) \overset{SO^+(1,3)}{\longrightarrow} (D_\mu\mathscr{O}(x))' = \Lambda_\mu^{\ \alpha}D_\alpha\mathscr{O}(\Lambda^{-1}x) \ , \tag{3.189}$$

$$D_\mu\mathscr{O}(x) \overset{C}{\longrightarrow} (D_\mu\mathscr{O}(x))' = (D_\mu\mathscr{O}(x))^T \ , \tag{3.190}$$

$$D_\mu\mathscr{O}(t,\mathbf{r}) \overset{P}{\longrightarrow} (D_\mu\mathscr{O}(t,\mathbf{r}))' = (D^\mu\mathscr{O}(t,-\mathbf{r}))^\dagger \ . \tag{3.191}$$

At this point, it has to be taken into account that we are also able to define covariant derivatives for objects that do not transform according to $\mathscr{O}(x) \overset{SU(2)_L\times SU(2)_R}{\longrightarrow} U_R(x)\mathscr{O}(x)U_L^\dagger(x)$ under local $SU(2)_L \times SU(2)_R$ transformations. Therefore, we consider an object $\tilde{\mathscr{O}}(x)$ which transforms according to $\tilde{\mathscr{O}}(x) \overset{SU(2)_L\times SU(2)_R}{\longrightarrow} U_L(x)\tilde{\mathscr{O}}(x)U_R^\dagger(x)$. Using the same manipulations as in Eq. (3.188), we are able to show that the covariant derivative

$$D_\mu\tilde{\mathscr{O}}(x) = \partial_\mu\tilde{\mathscr{O}}(x) + i\tilde{\mathscr{O}}(x)r_\mu(x) - il_\mu(x)\tilde{\mathscr{O}}(x) \tag{3.192}$$

has the same transformation properties as the object $\tilde{\mathscr{O}}(x)$ itself. If we identify this object with $\mathcal{U}^\dagger(x)$ or $\chi^\dagger(x)$, the transformation properties under the proper orthochronous Lorentz group and under CP-transformations follow immediately from Eqs. (3.189)-(3.191). In addition to that, we are able to define objects $\mathscr{O}_L(x)$ and $\mathscr{O}_R(x)$, that transform like $\mathscr{O}_L(x) \overset{SU(2)_L\times SU(2)_R}{\longrightarrow} U_L(x)\mathscr{O}_L(x)U_L^\dagger(x)$ or $\mathscr{O}_R(x) \overset{SU(2)_L\times SU(2)_R}{\longrightarrow} U_R(x)\mathscr{O}_R(x)U_R^\dagger(x)$, respectively. An example for objects of this type is given by the left- and right-handed field-strength tensors. The covariant derivatives of these objects read

$$D_\mu\mathscr{O}_L(x) = \partial_\mu\mathscr{O}_L(x) - il_\mu(x)\mathscr{O}_L(x) + i\mathscr{O}_L(x)l_\mu(x) \ , \tag{3.193}$$

$$D_\mu\mathscr{O}_R(x) = \partial_\mu\mathscr{O}_R(x) - ir_\mu(x)\mathscr{O}_R(x) + i\mathscr{O}_R(x)r_\mu(x) \ . \tag{3.194}$$

It turns out that the above covariant derivatives will only play a minor role for our purposes, therefore we do not give their transformation behavior under proper orthochronous Lorentz transformations and under CP-transformations.

Before we start with the construction of invariant terms, we have to find a systematic way which will help us to organize the terms of the chiral Lagrangian. To this end, we have to go back to the level of the hadronic n-point functions. The complete set of these n-point functions as well as their symmetry relations are collected in the generating functional of QCD (3.77). Now, the low-energy analysis of QCD in the framework of ChPT relies on a simultaneous expansion of the "true" generating functional in powers of Nambu-Goldstone boson momenta, i.e., pion momenta and quark masses. In this so-called chiral expansion, the pion matrix corresponds to an object of order one, i.e.,

$$\mathcal{U}(x) \sim \mathcal{O}(1) \ , \tag{3.195}$$

while the covariant derivative of $\mathcal{U}(x)$ as well as the left- and right-handed external fields are counted as objects of $\mathcal{O}(p)$

$$D_\mu \mathcal{U}(x) \sim \mathcal{O}(p) \ , \tag{3.196}$$

$$l_\mu(x), r_\mu(x) \sim \mathcal{O}(p) \ . \tag{3.197}$$

This assignment seems to be quite intuitive, since the derivative acting on $\mathcal{U}(x)$ can be translated into pion momenta. Furthermore, the left- and right-handed external fields will only enter the chiral Lagrangian in combination with a space-time derivative. At order $\mathcal{O}(p^2)$, there are three fundamental objects which may enter the chiral Lagrangian, namely the scalar/pseudoscalar matrix $\chi(x)$ as well as the left- and right-handed field-strength tensors

$$\chi(x) \sim \mathcal{O}(p^2) \ , \tag{3.198}$$

$$f_{\mu\nu}^{(L/R)}(x) \sim \mathcal{O}(p^2) \ . \tag{3.199}$$

3.3.2 Constructing the Chiral Lagrangian up to $\mathcal{O}(p^4)$

After introducing the basic objects of the chiral Lagrangian, we are now in the position to construct the leading order and next-to-leading order (NLO) terms of ChPT. To this end, denoting the powers of pion momenta and quark masses as n, we recognize that the full chiral Lagrangian must be of the form

$$\mathscr{L}_{\chi PT} = \sum_{n=0} \mathscr{L}_{2n} \ , \tag{3.200}$$

since parity conservation requires an even number of derivatives. For $n = 0$, we are only able to construct one term

$$\mathrm{Tr}\left\{\mathcal{U}^\dagger \mathcal{U}\right\} \ , \tag{3.201}$$

which is obviously a trivial term, since the pion matrix $\mathcal{U}(x)$ is unitary, so that the trace is equal to two[16] and therefore a constant. For $n \geq 1$, it is possible to construct nontrivial interactions. The number of all possible terms increases rapidly for large n. For example, in the case of $N_f = 2$, there are only two possible terms at leading order, i.e., for $n = 1$. At NLO ($n = 2$) the number of possible terms increases to ten, while at NNLO $n = 3$, there already exist 56 independent interaction terms[17]. Before, we start with the construction of the leading-order term, we want to illustrate the general way of constructing invariant terms. To this end, we recognize that two objects $\mathscr{O}_1(x)$ and $\mathscr{O}_2(x)$ which transform according to

$$\mathscr{O}_i(x) \overset{SU(2)_L \times SU(2)_R}{\longrightarrow} \mathscr{O}_i'(x) = U_R(x)\mathscr{O}_i(x)U_L^\dagger(x) \ , \text{for } i = 1,2 \ , \tag{3.202}$$

under local chiral transformations can easily be combined to form an invariant, because

$$\begin{aligned}\mathrm{Tr}\left\{\mathscr{O}_1(x)\mathscr{O}_2^\dagger(x)\right\} \overset{SU(2)_L \times SU(2)_R}{\longrightarrow} \mathrm{Tr}\left\{\mathscr{O}_1(x)\mathscr{O}_2^\dagger(x)\right\}' &= \mathrm{Tr}\left\{U_R(x)\mathscr{O}_1(x)U_L^\dagger(x)\left[U_R(x)\mathscr{O}_2(x)U_L^\dagger(x)\right]^\dagger\right\} \\ &= \mathrm{Tr}\left\{\mathscr{O}_1(x)\mathscr{O}_2^\dagger(x)\right\} \ ,\end{aligned} \tag{3.203}$$

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[16] In the case of two quark flavors.

[17] In the case of $N_f = 3$, the number of possible terms is again different. At NLO, there are twelve possible terms, while at NNLO, the number increases to 98, compare Ref. [BiEc] and refs. therein. The number of possible terms varies for different numbers of quark flavors, because there exist different techniques in order to eliminate redundant terms.

where we used the cyclic property of the trace. This procedure can be expanded to arbitrarily high powers of $\mathscr{O}_i(x)\mathscr{O}_j^\dagger(x)$ operators under the trace, since

$$\begin{aligned}
&\mathrm{Tr}\left\{\mathscr{O}_1(x)\mathscr{O}_2^\dagger(x)\cdots\mathscr{O}_{n-1}(x)\mathscr{O}_n^\dagger(x)\right\} \overset{SU(2)_L\times SU(2)_R}{\longrightarrow} \mathrm{Tr}\left\{\mathscr{O}_1(x)\mathscr{O}_2^\dagger(x)\cdots\mathscr{O}_{n-1}(x)\mathscr{O}_n^\dagger(x)\right\}' \\
&= \mathrm{Tr}\left\{U_R(x)\mathscr{O}_1(x)U_L^\dagger(x)\left[U_R(x)\mathscr{O}_2(x)U_L^\dagger(x)\right]^\dagger\cdots U_R(x)\mathscr{O}_{n-1}(x)U_L^\dagger(x)\left[U_R(x)\mathscr{O}_n(x)U_L^\dagger(x)\right]^\dagger\right\} \\
&= \mathrm{Tr}\left\{U_R(x)\mathscr{O}_1(x)\mathscr{O}_2^\dagger(x)U_R^\dagger(x)\cdots U_R(x)\mathscr{O}_{n-1}(x)\mathscr{O}_n^\dagger(x)U_R^\dagger(x)\right\} \\
&= \mathrm{Tr}\left\{\mathscr{O}_1(x)\mathscr{O}_2^\dagger(x)\cdots\mathscr{O}_{n-1}(x)\mathscr{O}_n^\dagger(x)\right\}\ ,
\end{aligned} \tag{3.204}$$

where we again used the cyclic property of the trace. In order to construct further terms, we can use two or more traces which are constructed in the way described above and build products of them. In general, these terms are of the following form

$$\mathrm{Tr}\left\{\mathscr{O}_{i_1}(x)\mathscr{O}_{i_2}^\dagger(x)\cdots\mathscr{O}_{i_{n-1}}(x)\mathscr{O}_{i_n}^\dagger(x)\right\}\cdots\mathrm{Tr}\left\{\mathscr{O}_{j_1}(x)\mathscr{O}_{j_2}^\dagger(x)\cdots\mathscr{O}_{j_{m-1}}(x)\mathscr{O}_{j_m}^\dagger(x)\right\}\ . \tag{3.205}$$

Now, we can use this approach for the construction of the leading-order term of the chiral Lagrangian. The first step consists of combining the fundamental building blocks (3.166), (3.171)-(3.173), and (3.187) to objects which transform according to Eq. (3.202) under local $SU(2)_R\times SU(2)_L$ transformations. Up to $\mathcal{O}(p^2)$ in the chiral expansion, the complete list of those objects is given by

$$\mathcal{U}(x), D_\mu\mathcal{U}(x), D_\mu D_\nu\mathcal{U}(x),\mathcal{U}(x)f_{\mu\nu}^{(L)}(x), f_{\mu\nu}^{(R)}(x)\mathcal{U}(x), \text{ and } \chi(x)\ . \tag{3.206}$$

Now, the second step consists of building invariant terms in the fashion of Eqs. (3.204) and (3.205). Due to the latter equation, we have to collect all invariant traces starting from chiral order $\mathcal{O}(1)$ up to $\mathcal{O}(p^2)$. As already mentioned before, the lowest-order term is given by a constant Eq. (3.201). Therefore, we start with the first-order terms. At $\mathcal{O}(p)$, there are only two possible structures which we can construct by using the objects listed in Eq. (3.206)

$$\mathrm{Tr}\left\{\left[D_\mu\mathcal{U}(x)\right]\mathcal{U}^\dagger(x)\right\},\quad \mathrm{Tr}\left\{\mathcal{U}(x)\left[D_\mu\mathcal{U}(x)\right]^\dagger\right\}\ . \tag{3.207}$$

In the following we want to show that (i) both traces are, up to a sign, equivalent and that (ii) both traces vanish identically. In order to show the first claim, we leave the traces out and add up both expressions

$$\begin{aligned}
&\left[D_\mu\mathcal{U}(x)\right]\mathcal{U}^\dagger(x)+\mathcal{U}(x)\left[D_\mu\mathcal{U}(x)\right]^\dagger \\
&= \left[\partial_\mu\mathcal{U}(x)-ir_\mu(x)\mathcal{U}(x)+i\mathcal{U}(x)l_\mu(x)\right]\mathcal{U}^\dagger(x)+\mathcal{U}(x)\left[\partial_\mu\mathcal{U}^\dagger+i\mathcal{U}^\dagger(x)r_\mu(x)-il_\mu(x)\mathcal{U}^\dagger(x)\right] \\
&= \partial_\mu\left[\mathcal{U}(x)\mathcal{U}^\dagger(x)\right]-ir_\mu(x)\left[\mathcal{U}(x)\mathcal{U}^\dagger(x)\right]+i\left[\mathcal{U}(x)\mathcal{U}^\dagger(x)\right]r_\mu(x) \\
&= D_\mu\left[\mathcal{U}(x)\mathcal{U}^\dagger(x)\right] \\
&= 0\ ,
\end{aligned} \tag{3.208}$$

where we used Eq. (3.194) and the fact that $\mathcal{U}(x)$ is unitary. Taking the trace of the above equation completes the proof of (i). According to (i), it will be sufficient to show the second claim for the first trace in Eq. (3.207). Using the canonical parametrization (3.163), we find

$$\begin{aligned}
\mathrm{Tr}\left\{\left[D_\mu\mathcal{U}(x)\right]\mathcal{U}^\dagger(x)\right\} &= \mathrm{Tr}\left\{\left[\partial_\mu\mathcal{U}(x)\right]\mathcal{U}^\dagger(x)\right\} \\
&= \mathrm{Tr}\left\{\left[\partial_\mu\left(\mathbb{1}_{2\times2}-i\pi_i(x)\tau^i+\frac{1}{2}(i\pi_i(x)\tau^i)^2-\dots\right)\right]\mathcal{U}^\dagger(x)\right\} \\
&= -\mathrm{Tr}\left\{i\left(\partial_\mu\pi_i(x)\right)\tau^i\left[\mathbb{1}_{2\times2}-i\pi_j(x)\tau^j+\dots\right]\mathcal{U}^\dagger(x)\right\} \\
&= 0\ ,
\end{aligned} \tag{3.209}$$

where we used that the Pauli matrices and therefore the external fields are traceless, compare Eq. (3.75). Furthermore, we made use of the fact that the matrices $\pi_i\tau^i$ and $\mathcal{U}^\dagger(x)$ commute. The results (3.208) and (3.209) ensure that, at leading order, a product of two traces may not enter the Lagrangian. Therefore, we have to use the objects (3.206) in order to construct terms which contain only one trace. The possible terms with respect to local chiral rotations are given by

(a) $\mathrm{Tr}\left\{\left[D_\mu D_\nu\mathcal{U}(x)\right]\mathcal{U}^\dagger(x)\right\}$, $\mathrm{Tr}\left\{\left[D_\nu\mathcal{U}(x)\right]\left[D_\mu\mathcal{U}(x)\right]^\dagger\right\}$, $\mathrm{Tr}\left\{\mathcal{U}(x)\left[D_\nu D_\mu\mathcal{U}(x)\right]^\dagger\right\}$,

(b) $\mathrm{Tr}\left\{\mathcal{U}(x)f^{(L)}_{\mu\nu}(x)\mathcal{U}^\dagger(x)\right\}$, $\mathrm{Tr}\left\{f^{(L)}_{\mu\nu}(x)\right\}$, $\mathrm{Tr}\left\{\mathcal{U}^\dagger(x)f^{(R)}_{\mu\nu}(x)\mathcal{U}(x)\right\}$, $\mathrm{Tr}\left\{f^{(R)}_{\mu\nu}(x)\right\}$,

(c) $\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x)\right\}$ and $\mathrm{Tr}\left\{\mathcal{U}(x)\chi^\dagger(x)\right\}$.

It is evident that not all of the above terms may enter the chiral Lagrangian at leading order. In order to reduce the amount of possible terms, we start with those listed in (a). First of all, we want to show that all three terms are not independent of each other. We have

$$\begin{aligned}\mathrm{Tr}\left\{[D_\mu D_\nu\mathcal{U}(x)]\,\mathcal{U}^\dagger(x)\right\} &= \mathrm{Tr}\left\{[\partial_\mu\left(D_\nu\mathcal{U}(x)\right) - ir_\mu(x)D_\nu\mathcal{U}(x) + i\left(D_\nu\mathcal{U}(x)\right)l_\mu(x)]\,\mathcal{U}^\dagger(x)\right\} \\ &= \partial_\mu\mathrm{Tr}\left\{[D_\nu\mathcal{U}(x)]\,\mathcal{U}^\dagger(x)\right\} - \mathrm{Tr}\left\{\left(D_\nu\mathcal{U}(x)\right)\left[\partial_\mu\mathcal{U}^\dagger(x) + i\mathcal{U}^\dagger(x)r_\mu(x) - il_\mu(x)\mathcal{U}^\dagger(x)\right]\right\} \\ &= -\mathrm{Tr}\left\{[D_\nu\mathcal{U}(x)]\,[D_\mu\mathcal{U}(x)]^\dagger\right\}\ ,\end{aligned} \tag{3.210}$$

where we used the product rule in combination with Eq. (3.209) and the definition of the covariant derivative (3.192). On the other hand, we obtain

$$\begin{aligned}\mathrm{Tr}\left\{\mathcal{U}(x)\,[D_\nu D_\mu\mathcal{U}(x)]^\dagger\right\} &= \mathrm{Tr}\left\{\mathcal{U}(x)\left[\partial_\nu\left(D_\mu\mathcal{U}(x)\right)^\dagger + i\left(D_\mu\mathcal{U}(x)\right)^\dagger r_\nu(x) - il_\nu(x)\left(D_\mu\mathcal{U}(x)\right)^\dagger\right]\right\} \\ &= \partial_\nu\mathrm{Tr}\left\{\mathcal{U}(x)\left(D_\mu\mathcal{U}(x)\right)^\dagger\right\} - \mathrm{Tr}\left\{\left[\partial_\nu\mathcal{U}(x) - ir_\nu(x)\mathcal{U}(x) + i\mathcal{U}(x)l_\nu(x)\right]\left(D_\mu\mathcal{U}(x)\right)^\dagger\right\} \\ &= -\mathrm{Tr}\left\{[D_\nu\mathcal{U}(x)]\,[D_\mu\mathcal{U}(x)]^\dagger\right\}\ ,\end{aligned} \tag{3.211}$$

where we used similar manipulations as in the previous calculation. In addition to that, all space-time indices have to be contracted, since the Lagrangian has to be Lorentz invariant, so that we are left with

$$\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D^\mu\mathcal{U}(x)]^\dagger\right\}\ . \tag{3.212}$$

All terms in (b) are proportional to the left- or right-handed field-strength tensors. It is quite obvious, using the cyclic property of the trace and the unitarity of $\mathcal{U}(x)$, that the first two terms and the last two terms are identical, respectively. Furthermore, we can use that the field-strength tensors are traceless tensors, i.e.,

$$\mathrm{Tr}\left\{f^{(L)}_{\mu\nu}(x)\right\} = \mathrm{Tr}\left\{\partial_\mu l_\nu(x) - \partial_\nu l_\mu(x) - i\left[l_\mu(x), l_\nu(x)\right]_-\right\} = 0\ , \tag{3.213}$$

$$\mathrm{Tr}\left\{f^{(R)}_{\mu\nu}(x)\right\} = \mathrm{Tr}\left\{\partial_\mu r_\nu(x) - \partial_\nu r_\mu(x) - i\left[r_\mu(x), r_\nu(x)\right]_-\right\} = 0\ , \tag{3.214}$$

where we used that the left- and right-handed external fields are traceless. In addition to that, the trace of the commutator term vanishes, due to the Lie algebra of $SU(2)$. Therefore, all terms in (b) vanish identically and do not have to be considered. The last two terms in (c) are independent of each other, so that we can build two linear combinations

$$\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) \pm \mathcal{U}(x)\chi^\dagger(x)\right\} \tag{3.215}$$

from them. Since both objects in Eq. (3.215) are Lorentz scalars, it is easy to see that both linear combinations are symmetric under proper orthochronous Lorentz transformations. Therefore, we have to check the transformation behavior of Eq. (3.215) under C- and P-transformations. We find

$$\begin{aligned}\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) \pm \mathcal{U}(x)\chi^\dagger(x)\right\} &\xrightarrow{C} \mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) \pm \mathcal{U}(x)\chi^\dagger(x)\right\}' \\ &= \mathrm{Tr}\left\{\chi(x)^T\left(\mathcal{U}^\dagger(x)\right)^T \pm \mathcal{U}^T(x)\left(\chi^\dagger(x)\right)^T\right\} \\ &= \mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) \pm \mathcal{U}(x)\chi^\dagger(x)\right\}\end{aligned} \tag{3.216}$$

and

$$\begin{aligned}\mathrm{Tr}\left\{\chi(t,\mathbf{r})\mathcal{U}^\dagger(t,\mathbf{r}) \pm \mathcal{U}(t,\mathbf{r})\chi^\dagger(t,\mathbf{r})\right\} &\xrightarrow{P} \mathrm{Tr}\left\{\chi(t,\mathbf{r})\mathcal{U}^\dagger(t,\mathbf{r}) \pm \mathcal{U}(t,\mathbf{r})\chi^\dagger(t,\mathbf{r})\right\}' \\ &= \mathrm{Tr}\left\{\chi(t,-\mathbf{r})^\dagger\mathcal{U}(t,-\mathbf{r}) \pm \mathcal{U}^\dagger(t,-\mathbf{r})\chi(t,-\mathbf{r})\right\} \\ &= \mathrm{Tr}\left\{\pm\chi(t,-\mathbf{r})\mathcal{U}^\dagger(t,-\mathbf{r}) + \mathcal{U}(t,-\mathbf{r})\chi^\dagger(t,-\mathbf{r})\right\}\ ,\end{aligned} \tag{3.217}$$

where we used the transformation properties of the previous section and the cyclic property of the trace. Obviously, the negative linear combination breaks the symmetry under parity transformations, because it transforms with the wrong sign. Combining this result with Eq. (3.212), the leading-order chiral Lagrangian is given by

$$\mathscr{L}_2 = \frac{f_\pi^2}{4}\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D^\mu\mathcal{U}(x)]^\dagger\right\} + \frac{f_\pi^2}{4}\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) + \mathcal{U}(x)\chi^\dagger(x)\right\}\ , \tag{3.218}$$

where f_π denotes the pion decay constant. At leading order, the chiral Lagrangian includes only two free parameters, the pion decay constant and the constant B_0 which is contained in the definition of $\chi(x)$. Furthermore, in the case of three-flavor ChPT, we would end up with the same result. At this point, we could use the same techniques that lead to Eq. (3.218) in order to construct the NLO Lagrangian. In this case the whole procedure becomes a bit more involved, since there are much more invariants with respect to local chiral transformations. Furthermore, one has to use more trace identities and the equation of motion to identify terms which are not independent of each other. For the sake of brevity, we skip the technical details[18] and only quote the final results in the case of two and three quark flavors. In the case of $N_f = 2$ ChPT, one obtains

$$\begin{aligned}
\mathscr{L}_4 = {} & \frac{\ell_1}{4}\left(\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D^\mu\mathcal{U}(x)]^\dagger\right\}\right)^2 + \frac{\ell_2}{4}\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D_\nu\mathcal{U}(x)]^\dagger\right\}\mathrm{Tr}\left\{[D^\mu\mathcal{U}(x)]\,[D^\nu\mathcal{U}(x)]^\dagger\right\} \\
& + \frac{h_1 - h_3 + \ell_3}{16}\left(\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) + \mathcal{U}(x)\chi^\dagger(x)\right\}\right)^2 + \frac{\ell_4}{4}\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D^\mu\chi(x)]^\dagger + [D_\mu\chi(x)]\,[D^\mu\mathcal{U}(x)]^\dagger\right\} \\
& + \ell_5\mathrm{Tr}\left\{f^{(R)}_{\mu\nu}(x)\mathcal{U}(x)f^{(L)\mu\nu}(x)\mathcal{U}^\dagger(x)\right\} - \left(\frac{\ell_5}{2} + 2h_2\right)\mathrm{Tr}\left\{f^{(L)}_{\mu\nu}(x)f^{(L)\mu\nu}(x) + f^{(R)}_{\mu\nu}(x)f^{(R)\mu\nu}(x)\right\} \\
& + i\frac{\ell_6}{2}\mathrm{Tr}\left\{f^{(R)}_{\mu\nu}\,[D^\mu\mathcal{U}(x)]\,[D^\nu\mathcal{U}(x)]^\dagger + f^{(L)}_{\mu\nu}\,[D^\mu\mathcal{U}(x)]^\dagger\,[D^\nu\mathcal{U}(x)]\right\} + \frac{h_1 - h_3 - \ell_7}{16}\left(\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) - \mathcal{U}(x)\chi^\dagger(x)\right\}\right)^2 \\
& + \frac{h_1 + h_3}{4}\mathrm{Tr}\left\{\chi(x)\chi^\dagger(x)\right\} - \frac{h_1 - h_3}{8}\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x)\chi(x)\mathcal{U}^\dagger(x) + \mathcal{U}(x)\chi^\dagger(x)\mathcal{U}(x)\chi^\dagger(x)\right\}\ . \qquad (3.219)
\end{aligned}$$

The above NLO term contains seven low-energy constants (LECs), $\ell_i, i = 1, \ldots, 7$, and three constants belonging to contact interactions, $h_j, j = 1, 2, 3$. In the case of three quark flavors, the NLO term of the chiral Lagrangian reads

$$\begin{aligned}
\mathscr{L}_4 = {} & L_1\left(\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D^\mu\mathcal{U}(x)]^\dagger\right\}\right)^2 + L_2\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D_\nu\mathcal{U}(x)]^\dagger\right\}\mathrm{Tr}\left\{[D^\mu\mathcal{U}(x)]\,[D^\nu\mathcal{U}(x)]^\dagger\right\} \\
& + L_3\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D^\mu\mathcal{U}(x)]^\dagger\,[D_\nu\mathcal{U}(x)]\,[D^\nu\mathcal{U}(x)]^\dagger\right\} + L_4\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D^\mu\mathcal{U}(x)]^\dagger\right\}\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) + \mathcal{U}(x)\chi^\dagger(x)\right\} \\
& + L_5\mathrm{Tr}\left\{[D_\mu\mathcal{U}(x)]\,[D^\mu\mathcal{U}(x)]^\dagger\left[\chi(x)\mathcal{U}^\dagger(x) + \mathcal{U}(x)\chi^\dagger(x)\right]\right\} + L_6\left(\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) + \mathcal{U}(x)\chi^\dagger(x)\right\}\right)^2 \\
& + L_7\mathrm{Tr}\left\{\chi(x)\mathcal{U}^\dagger(x) - \mathcal{U}(x)\chi^\dagger(x)\right\}^2 + L_8\mathrm{Tr}\left\{\mathcal{U}(x)\chi^\dagger(x)\mathcal{U}(x)\chi^\dagger(x) + \chi(x)\mathcal{U}^\dagger(x)\chi(x)\mathcal{U}^\dagger(x)\right\} \\
& - iL_9\mathrm{Tr}\left\{f^{(R)}_{\mu\nu}(x)\,[D^\mu\mathcal{U}(x)]\,[D^\nu\mathcal{U}(x)]^\dagger + f^{(L)}_{\mu\nu}(x)\,[D^\mu\mathcal{U}(x)]^\dagger\,[D^\nu\mathcal{U}(x)]\right\} + L_{10}\mathrm{Tr}\left\{\mathcal{U}(x)f^{(L)}_{\mu\nu}(x)\mathcal{U}^\dagger(x)f^{(R)\mu\nu}(x)\right\} \\
& + H_1\mathrm{Tr}\left\{f^{(R)}_{\mu\nu}(x)f^{(R)\mu\nu}(x) + f^{(L)}_{\mu\nu}(x)f^{(L)\mu\nu}(x)\right\} + H_2\mathrm{Tr}\left\{\chi(x)\chi^\dagger(x)\right\}\ . \qquad (3.220)
\end{aligned}$$

As already mentioned in Ref. [17], the number of possible terms and therefore the number of LECs increases, when we include the strange quark. In this case, we obtain ten LECs, $L_i, i = 1, \ldots, 10$, and two constants, H_1, H_2, which couple terms that only involve external fields. In general and not depending on the number of included quark flavors, the LECs can be understood as measure for the inability to solve QCD in the low-energy regime. Nevertheless, it is possible to fix their values, for example with lattice calculations, empirical data, or QCD inspired models. If one is interested in a low-energy analysis beyond tree-level, the LECs also open up a practical use: When considering Eq. (3.218) at one-loop order, the resulting vertices are of order $\mathcal{O}(p^4)$, so that the resulting divergences cannot be absorbed by an appropriate renormalization of the LECs of $\mathscr{L}_2$. At this point, the LECs of $\mathscr{L}_4$ come into play. The NLO term was constructed in a way that it represents the most general chiral Lagrangian at order $\mathcal{O}(p^4)$ in the chiral expansion. Therefore, it is possible to renormalize the LECs as well as the constants from the contact interactions in a way that all divergences of the one-loop graphs can be absorbed.

3.3.3 Four-Pion Interaction Vertices

In the last part of this chapter, we want to use the results (3.218) and (3.219) in order to identify all four-pion interaction terms which arise up to NLO in the chiral Lagrangian. The use of this step will become apparent in Sec. [4.3.1], when we introduce our approach for the determination of the low-energy coupling constants of the extended linear sigma model. The desired four-pion interaction terms can easily be found by expanding the pion matrix $\mathcal{U}(x)$. To this end, we choose from now on

$$\mathcal{U}(x) = \frac{1}{f_\pi}\left[\sigma(x) + i\pi_i(x)\tau^i\right]\ , \qquad \sigma(x) = f_\pi\sqrt{1 - \pi_i^2(x)/f_\pi^2}\ , \qquad (3.221)$$

[18] For more details, see Refs. [ScSc], [FeSc].

as a parametrization of the coset space. Expanding the square root in inverse powers of the pion decay constant, we find

$$\sigma(x) = f_\pi \left\{ 1 - \frac{1}{2}\left(\frac{\vec{\pi}}{f_\pi}\right)^2 - \frac{1}{8}\left(\frac{\vec{\pi}^2}{f_\pi^2}\right)^2 + \mathcal{O}(\pi^6) \right\} . \tag{3.222}$$

Furthermore, it will be useful to determine the expansion of the 4-gradient of $\sigma(x)$, since the chiral Lagrangian contains a lot of derivatively coupled interaction terms. We find

$$\begin{aligned} \partial_\mu \sigma(x) &= -\frac{1}{f_\pi} \frac{\vec{\pi}\cdot\partial_\mu\vec{\pi}}{\sqrt{1-\pi_i^2(x)/f_\pi^2}} \\ &= -\frac{1}{f_\pi}\vec{\pi}\cdot(\partial_\mu\vec{\pi}) \left\{ 1 + \frac{1}{2}\left(\frac{\vec{\pi}}{f_\pi}\right)^2 + \frac{3}{8}\left(\frac{\vec{\pi}^2}{f_\pi^2}\right)^2 + \mathcal{O}(\pi^6) \right\} . \end{aligned} \tag{3.223}$$

In addition to that, we set the left- and right-handed external fields to zero

$$l_\mu(x) = r_\mu(x) = 0 \tag{3.224}$$

and define the scalar/pseudoscalar matrix $\chi(x)$ as

$$\chi(x) = M^2 \mathbb{1}_{2\times 2} , \tag{3.225}$$

where M^2 defines a mass parameter of dimension $\left[\text{Energy}^2\right]$. As a consequence of these definitions, the NLO chiral Lagrangian simplifies to

$$\begin{aligned} \mathscr{L}_{\chi PT} &= \frac{f_\pi^2}{4}\text{Tr}\left\{[\partial_\mu\mathcal{U}(x)]\,[\partial^\mu\mathcal{U}(x)]^\dagger\right\} + \frac{f_\pi^2}{4}M^2\text{Tr}\left\{\mathcal{U}^\dagger(x)+\mathcal{U}(x)\right\} \\ &+ \frac{\ell_1}{4}\left(\text{Tr}\left\{[\partial_\mu\mathcal{U}(x)]\,[\partial^\mu\mathcal{U}(x)]^\dagger\right\}\right)^2 + \frac{\ell_2}{4}\text{Tr}\left\{[\partial_\mu\mathcal{U}(x)]\,[\partial_\nu\mathcal{U}(x)]^\dagger\right\}\text{Tr}\left\{[\partial^\mu\mathcal{U}(x)]\,[\partial^\nu\mathcal{U}(x)]^\dagger\right\} \\ &+ \frac{h_1-h_3+\ell_3}{16}M^4\left(\text{Tr}\left\{\mathcal{U}^\dagger(x)+\mathcal{U}(x)\right\}\right)^2 + \frac{h_1-h_3-\ell_7}{16}M^4\left(\text{Tr}\left\{\mathcal{U}^\dagger(x)-\mathcal{U}(x)\right\}\right)^2 \\ &- \frac{h_1-h_3}{8}M^4\text{Tr}\left\{\mathcal{U}^\dagger(x)\mathcal{U}^\dagger(x)+\mathcal{U}(x)\mathcal{U}(x)\right\} + \frac{h_1+h_3}{2}M^4 + \mathcal{O}(p^6) . \end{aligned} \tag{3.226}$$

Now, using the coset parametrization (3.221) and the expansions (3.222), (3.223), we are able to obtain an explicit expression of the above Lagrangian in terms of the pion fields. In order to keep track of all terms, we now expand all terms of Eq. (3.226) order by order in the chiral expansion. At LO, we are left with only two terms

$$\mathscr{L}_2 = \frac{f_\pi^2}{4}\text{Tr}\left\{[\partial_\mu\mathcal{U}(x)]\,[\partial^\mu\mathcal{U}(x)]^\dagger\right\} + \frac{f_\pi^2}{4}M^2\text{Tr}\left\{\mathcal{U}^\dagger(x)+\mathcal{U}(x)\right\} . \tag{3.227}$$

Dropping the space-time arguments and using Eqs. (3.222), (3.223) and (6.10), we find

$$\begin{aligned} \mathscr{L}_2 &= \frac{1}{4}\text{Tr}\left\{(\partial_\mu\sigma)^2 + (\partial_\mu\pi_i)(\partial^\mu\pi_j)\,\tau^i\tau^j\right\} + \frac{M^2 f_\pi}{4}\text{Tr}\left\{2\sigma\right\} \\ &= \frac{1}{2}(\partial_\mu\vec{\pi})^2 - \frac{1}{2}M^2\vec{\pi}^2 - \frac{M^2}{8f_\pi^2}\left(\vec{\pi}^2\right)^2 + \frac{1}{2f_\pi^2}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + M^2 f_\pi^2 + \mathcal{O}\left(\pi^6\right) . \end{aligned} \tag{3.228}$$

From the above result, we make two observations: On the one hand, the mass parameter M^2 corresponds to the full tree-level mass of the pion at LO in the chiral expansion. In a moment, we will see that at NLO, the tree-level mass of the pion gets another contribution. On the other hand, we find that this mass parameter is also contained in the coefficient of the $\left(\vec{\pi}^2\right)^2$ interaction term. But, in the discussion of Sec. [3.2], we argued that Nambu-Goldstone bosons may only interact through derivatively coupled terms among each other. In order to resolve this problem, we have to remember that the previous statement is only true, if we consider an exact chiral symmetric Lagrangian. In the case of Eq. (3.228), chiral symmetry is explicitly broken by the occurrence of the mass parameter M^2. Therefore, we have to take the chiral limit, $M^2 \longrightarrow 0$, in order to show that the pion-interaction terms always have to contain derivatives. At NLO, we have to deal with five possible terms

$$\begin{aligned} \mathscr{L}_4 &= \frac{\ell_1}{4}\left(\text{Tr}\left\{[\partial_\mu\mathcal{U}(x)]\,[\partial^\mu\mathcal{U}(x)]^\dagger\right\}\right)^2 + \frac{\ell_2}{4}\text{Tr}\left\{[\partial_\mu\mathcal{U}(x)]\,[\partial_\nu\mathcal{U}(x)]^\dagger\right\}\text{Tr}\left\{[\partial^\mu\mathcal{U}(x)]\,[\partial^\nu\mathcal{U}(x)]^\dagger\right\} \\ &+ \frac{h_1-h_3+\ell_3}{16}M^4\left(\text{Tr}\left\{\mathcal{U}^\dagger(x)+\mathcal{U}(x)\right\}\right)^2 + \frac{h_1-h_3-\ell_7}{16}M^4\left(\text{Tr}\left\{\mathcal{U}^\dagger(x)-\mathcal{U}(x)\right\}\right)^2 \\ &- \frac{h_1-h_3}{8}M^4\text{Tr}\left\{\mathcal{U}^\dagger(x)\mathcal{U}^\dagger(x)+\mathcal{U}(x)\mathcal{U}(x)\right\} + \frac{h_1+h_3}{2}M^4 + \mathcal{O}(p^6) . \end{aligned} \tag{3.229}$$

For the sake of simplicity, we consider each NLO term individually. The expansion of the first term in (3.229) yields

$$\begin{aligned}\frac{\ell_1}{4}\left(\mathrm{Tr}\left\{[\partial_\mu\mathcal{U}]\,[\partial^\mu\mathcal{U}]^\dagger\right\}\right)^2 &= \frac{\ell_1}{4f_\pi^4}\left(\mathrm{Tr}\left\{(\partial_\mu\sigma)^2+(\partial_\mu\pi_i)\,(\partial^\mu\pi_j)\,\tau^i\tau^j\right\}\right)^2\\ &= \frac{\ell_1}{f_\pi^4}\,(\partial_\mu\vec{\pi})^2\,(\partial_\nu\vec{\pi})^2+\mathcal{O}\left(\pi^6\right)\ ,\end{aligned} \tag{3.230}$$

where we made use of Eqs. (3.223) and (6.10). Using the same techniques, we find a similar result for the second term

$$\begin{aligned}\frac{\ell_2}{4}\mathrm{Tr}\left\{[\partial_\mu\mathcal{U}]\,[\partial_\nu\mathcal{U}]^\dagger\right\}\mathrm{Tr}\left\{[\partial^\mu\mathcal{U}]\,[\partial^\nu\mathcal{U}]^\dagger\right\} &= \frac{\ell_2}{4f_\pi^4}\mathrm{Tr}\left\{(\partial_\mu\sigma)\,\partial_\nu\sigma+(\partial_\mu\pi_i)\,\partial_\nu\pi_j\tau^i\tau^j\right\}\\ &\quad\times\mathrm{Tr}\left\{(\partial^\mu\sigma)\,\partial^\nu\sigma+(\partial^\mu\pi_i)\,\partial^\nu\pi_j\tau^i\tau^j\right\}\\ &= \frac{\ell_2}{f_\pi^4}\left[(\partial_\mu\vec{\pi})\cdot\partial_\nu\vec{\pi}\right]^2+\mathcal{O}\left(\pi^6\right)\ .\end{aligned} \tag{3.231}$$

As already mentioned before, at NLO, the tree-level mass of the pion gets a second contribution. This new contribution originates from the third and the fifth term in Eq. (3.229)

$$\begin{aligned}&\frac{h_1-h_3+\ell_3}{16}M^4\left(\mathrm{Tr}\left\{\mathcal{U}^\dagger(x)+\mathcal{U}(x)\right\}\right)^2-\frac{h_1-h_3}{8}M^4\mathrm{Tr}\left\{\mathcal{U}^\dagger(x)\mathcal{U}^\dagger(x)+\mathcal{U}(x)\mathcal{U}(x)\right\}\\ &=\frac{h_1-h_3+\ell_3}{16f_\pi^2}M^4\left(\mathrm{Tr}\left\{2\sigma\right\}\right)^2-\frac{h_1-h_3}{4f_\pi^2}M^4\mathrm{Tr}\left\{\sigma^2-\pi_i\pi_j\tau^i\tau^j\right\}\\ &=(h_1-h_3+\ell_3)\,M^4-\frac{(h_1-h_3+\ell_3)M^4}{f_\pi^2}\vec{\pi}^2-\frac{h_1-h_3}{2}M^4+\frac{h_1-h_3}{f_\pi^2}M^4\vec{\pi}^2+\mathcal{O}\left(\pi^6\right)\\ &=-\frac{\ell_3M^4}{f_\pi^2}\vec{\pi}^2+\left(\frac{h_1-h_3}{2}+\ell_3\right)M^4+\mathcal{O}\left(\pi^6\right)\ ,\end{aligned} \tag{3.232}$$

where we used (3.222). The fourth term vanishes exactly, since the trace of a Pauli matrix vanishes. Normally, this term shifts the masses of the charged and the neutral pions. Since the chiral expansion of the "true" generating functional of QCD involves all terms which are allowed by symmetry, the chiral expansion quite naturally introduces the explicit breaking of the isospin symmetry at NLO. In our case, this term has to vanish, since the definition (3.225) corresponds to the isospin symmetric limit where all pions have the same mass. Therefore, the NLO term of the chiral Lagrangian becomes

$$\mathscr{L}_4=\frac{\ell_1}{f_\pi^4}\,(\partial_\mu\vec{\pi})^2\,(\partial_\nu\vec{\pi})^2+\frac{\ell_2}{f_\pi^4}\left[(\partial_\mu\vec{\pi})\cdot(\partial_\nu\vec{\pi})\right]^2-\frac{\ell_3M^4}{f_\pi^2}\vec{\pi}^2+(h_1+\ell_3)\,M^4+\mathcal{O}\left(\pi^6\right)\ . \tag{3.233}$$

Finally, combining the results (3.228) and (3.233), the chiral Lagrangian up to NLO is given by

$$\begin{aligned}\mathscr{L}_{\chi PT}&=\mathscr{L}_2+\mathscr{L}_4\\ &=\frac{1}{2}\,(\partial_\mu\vec{\pi})^2-\frac{1}{2}M_\pi^2\vec{\pi}^2+C_{1,\chi PT}\left(\vec{\pi}^2\right)^2+C_{2,\chi PT}\,(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2+C_{3,\chi PT}\,(\partial_\mu\vec{\pi})^2\,(\partial_\nu\vec{\pi})^2\\ &\quad+C_{4,\chi PT}\left[(\partial_\mu\vec{\pi})\cdot\partial_\nu\vec{\pi}\right]^2+\mathcal{O}\left(\pi^6\right)\ ,\end{aligned} \tag{3.234}$$

where we introduced

$$M_\pi^2=M^2+\frac{2\ell_3}{f_\pi^2}M^4 \tag{3.235}$$

as the tree-level mass of the pion. Furthermore, we defined the following coupling constants of the different four-pion interaction terms

$$C_{1,\chi PT}=-\frac{M^2}{8f_\pi^2}\ , \tag{3.236}$$

$$C_{2,\chi PT}=\frac{1}{2f_\pi^2}\ , \tag{3.237}$$

$$C_{3,\chi PT}=\frac{\ell_1}{f_\pi^4}\ , \tag{3.238}$$

$$C_{4,\chi PT}=\frac{\ell_2}{f_\pi^4}\ . \tag{3.239}$$

These coefficient will be important in the discussion of Chapter [4], when we calculate the LECs of the extended linear sigma model.

Chapter 4

Calculation of the Low Energy Constants

After introducing the technical basics and approaches in the last chapters, we are now in the position to apply these concepts. To this end, we start with a very detailed introduction to the techniques which we want to use later, in order to determine the tree-level LECs of the eLSM at the hand of a simple example, see Sec. [4.1]. Then, in Sec. [4.2], we finally consider the eLSM. During this section, we introduce the physical content which is described by the eLSM and illustrate how it is technically realized. Then, we study all terms of the eLSM in detail and finally derive expressions for the LECs in terms of the various coupling constants of the model.

4.1 A Toy-Model Introduction

The aim of this first section is, on the one hand, to introduce the basic concepts of the upcoming sections, but on the other hand, also to show the technical methods which we will use to determine the LECs of the eLSM. To this end, we consider a simple toy model describing the interaction of a light scalar field $\varphi_l(x)$ with a heavy scalar field $\varphi_H(x)$ and study the important properties of this theory in detail. In Sec. [4.1.1], we therefore focus on the very basic properties of the theory and derive the generating functional of the theory. Based on the explicit form of the generating functional, we also derive the perturbative expansion up to first order in the coupling constants of the theory and introduce the position-space Feynman rules. The necessity of this step will be become apparent in Sec. [4.1.2], when we integrate the heavy field out of the theory in order to find a low-energy description of the theory.

4.1.1 General Considerations

As mentioned above, we want to consider a toy model describing the interaction of two scalar particles. The classical action of this theory shall be given by

$$\begin{aligned} S_{toy}[\varphi_l, \varphi_H] &= \int \mathrm{d}^4x \; \mathscr{L}_{toy}\left(\varphi_l, \varphi_H, \partial_\mu\varphi_l, \partial_\mu\varphi_H\right) \\ &= \int \mathrm{d}^4x \; \left[\mathscr{L}_{0,l}(\varphi_l, \partial_\mu\varphi_l) + \mathscr{L}_{0,H}(\varphi_H, \partial_\mu\varphi_H) + \mathscr{L}_{int}(\varphi_l, \varphi_H)\right] \; , \end{aligned} \tag{4.1}$$

where the free parts of the Lagrangian $\mathscr{L}_{toy}$ are given by the free Klein-Gordon Lagrangians

$$\mathscr{L}_{0,i}(\varphi_i, \partial_\mu\varphi_i) = \frac{1}{2}\left(\partial_\mu\varphi_i\right)\left(\partial^\mu\varphi_i\right) - \frac{m_i^2}{2}\varphi_i^2 \; , \; i = l, H \; . \tag{4.2}$$

The interaction part of the Lagrangian shall be given by

$$\mathscr{L}_{int}(\varphi_l, \varphi_H) = -\frac{g_1}{4}\varphi_H^2\varphi_l^2 - \frac{g_2}{2}\varphi_H\varphi_l^2 \; , \tag{4.3}$$

where we introduced the coupling constants g_1 and g_2. In order to quantize the theory, we remember the discussion of Sec. [2.2.1]. Comparing the above Lagrangian with the general approach in Eq. (2.63), we are immediately able to write down the generating functional of the toy model

$$Z[J_l, J_H] = \mathcal{N} \int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H(x) \; \exp\left\{i\int \mathrm{d}^4x \; \left[\mathscr{L}_{toy}\left(\varphi_l, \varphi_H, \partial_\mu\varphi_l, \partial_\mu\varphi_H\right) + J_l\varphi_l + J_H\varphi_H\right]\right\} \; , \tag{4.4}$$

where we introduced the classical sources $J_l(x)$ and $J_H(x)$. The normalization constant $\mathcal{N}$ is fixed by the requirement $Z[0,0]=1$, so that

$$\mathcal{N}^{-1} = \int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H(x)\ \exp\left\{i\int \mathrm{d}^4x\ \mathscr{L}_{toy}\left(\varphi_l,\varphi_H,\partial_\mu\varphi_l,\partial_\mu\varphi_H\right)\right\}. \tag{4.5}$$

For later purposes, it will be useful to bring Eq. (4.4) into another form which is easier to handle. Therefore, we recognize that the generating functional of the interaction-free theory, i.e., $g_1 = g_2 = 0$, would factorize into two Gaussian functional integrals which can be solved analytically. But this observation would also be useful for nonzero coupling constants, if we were able to eliminate the interaction terms (4.3) from the functional integral. And in fact, by using a simple trick, we are able to rewrite Eq. (4.4) in the desired way. First of all, we factorize the free and the interacting part of the Lagrangian

$$\begin{aligned}
Z[J_l,J_H] &= \mathcal{N}\int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{toy}\left(\varphi_l,\varphi_H,\partial_\mu\varphi_l,\partial_\mu\varphi_H\right)+J_l\varphi_l+J_H\varphi_H\right]\right\}\\
&= \mathcal{N}_{int}\int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H(x)\ \exp\left\{i\int \mathrm{d}^4x\ \mathscr{L}_{int}(\varphi_l,\varphi_H)\right\}\\
&\quad\times \mathcal{N}_0\exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{0,l}(\varphi_l,\partial_\mu\varphi_l)+\mathscr{L}_{0,H}(\varphi_H,\partial_\mu\varphi_H)+J_l\varphi_l+J_H\varphi_H\right]\right\}\\
&= \mathcal{N}_{int}\int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H(x)\ \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\left[\int \mathrm{d}^4x\ \left(\frac{g_1}{4}\varphi_H^2\varphi_l^2+\frac{g_2}{2}\varphi_H\varphi_l^2\right)\right]^n\\
&\quad\times \mathcal{N}_0\exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{0,l}(\varphi_l,\partial_\mu\varphi_l)+\mathscr{L}_{0,H}(\varphi_H,\partial_\mu\varphi_H)+J_l\varphi_l+J_H\varphi_H\right]\right\}\\
&= \mathcal{N}_{int}\int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H(x)\ \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{g_1}{4}\int \mathrm{d}^4x\ \varphi_H^2\varphi_l^2\right)^{n-k}\left(\frac{g_2}{2}\int \mathrm{d}^4x'\ \varphi_H\varphi_l^2\right)^k\\
&\quad\times \mathcal{N}_0\exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{0,l}(\varphi_l,\partial_\mu\varphi_l)+\mathscr{L}_{0,H}(\varphi_H,\partial_\mu\varphi_H)+J_l\varphi_l+J_H\varphi_H\right]\right\},
\end{aligned}\tag{4.6}$$

where we also factorized the normalization constant $\mathcal{N}\equiv\mathcal{N}_{int}\mathcal{N}_0$ and rewrote the first exponential factor into its power-series representation and used the binomial theorem. Now, we can use the same trick as in the discussion of Eq. (3.71), i.e., we replace the fields in the series expansion with functional derivatives with respect to the respective fields. This trick is based on the observation that

$$\begin{aligned}
&\frac{1}{i}\frac{\delta}{\delta J_i(x')}\exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{0,l}(\varphi_l,\partial_\mu\varphi_l)+\mathscr{L}_{0,H}(\varphi_H,\partial_\mu\varphi_H)+J_l\varphi_l+J_H\varphi_H\right]\right\}\\
&=\frac{1}{i}\left\{i\frac{\delta}{\delta J_i(x')}\int \mathrm{d}^4x\ \left[J_l\varphi_l+J_H\varphi_H\right]\right\}\exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{0,l}(\varphi_l,\partial_\mu\varphi_l)+\mathscr{L}_{0,H}(\varphi_H,\partial_\mu\varphi_H)+J_l\varphi_l+J_H\varphi_H\right]\right\}\\
&=\varphi_i(x')\exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{0,l}(\varphi_l,\partial_\mu\varphi_l)+\mathscr{L}_{0,H}(\varphi_H,\partial_\mu\varphi_H)+J_l\varphi_l+J_H\varphi_H\right]\right\},
\end{aligned}\tag{4.7}$$

where we used $\delta J_j(x)/\delta J_i(x') = \delta^i{}_j\delta^{(4)}(x-x')$ for $i,j=l,H$. The importance of Eq. (4.7) arises from the fact that the functional derivatives with respect to the classical sources can be pulled out of the functional integral, since they do not depend on the fields itself. Using Eq. (4.7) in order to substitute each field in the interaction part of the Lagrangian, Eq. (4.6) can be written as

$$\begin{aligned}
Z[J_l,J_H] &= \mathcal{N}_{int}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{g_1}{4}\int \mathrm{d}^4x\ \left(\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)^2\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2\right]^{n-k}\\
&\quad\times\left[\frac{g_2}{2}\int \mathrm{d}^4x'\ \left(\frac{1}{i}\frac{\delta}{\delta J_H(x')}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_l(x')}\right)^2\right]^k\\
&\quad\times \mathcal{N}_0\int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{0,l}(\varphi_l,\partial_\mu\varphi_l)+\mathscr{L}_{0,H}(\varphi_H,\partial_\mu\varphi_H)+J_l\varphi_l+J_H\varphi_H\right]\right\}\\
&= \mathcal{N}_{int}\exp\left\{i\int \mathrm{d}^4x\ \mathscr{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)},\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)\right\}Z_0[J_l,J_H],
\end{aligned}\tag{4.8}$$

where we defined the free generating functional as

$$\begin{aligned} Z_0[J_l, J_H] &= \mathcal{N}_0 \int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H(x)\ \exp\left\{ i \int \mathrm{d}^4x\ [\mathscr{L}_{0,l}(\varphi_l, \partial_\mu\varphi_l) + \mathscr{L}_{0,H}(\varphi_H, \partial_\mu\varphi_H) + J_l\varphi_l + J_H\varphi_H] \right\} \\ &= \mathcal{N}_l \int \mathscr{D}\varphi_l(x)\ \exp\left\{ i \int \mathrm{d}^4x\ [\mathscr{L}_{0,l}(\varphi_l, \partial_\mu\varphi_l) + J_l\varphi_l] \right\} \\ &\quad \times \mathcal{N}_H \int \mathscr{D}\varphi_H(x)\ \exp\left\{ i \int \mathrm{d}^4x\ [\mathscr{L}_{0,H}(\varphi_H, \partial_\mu\varphi_H) + J_H\varphi_H] \right\} \\ &\equiv Z_{0,l}[J_l] Z_{0,H}[J_H]\ , \end{aligned} \tag{4.9}$$

with $\mathcal{N}_0 \equiv \mathcal{N}_l\mathcal{N}_H$. As mentioned before, we obtain a free generating functional which is given by the product of two Gaussian functional integrals. In order to simplify Eq. (4.9), we have to evaluate these Gaussian integrals. Since both integrals are exactly of the same form, it is sufficient to evaluate only one of them. Integrating the kinetic part of the Klein-Gordon Lagrangian by parts, we find for $i = l, H$

$$\begin{aligned} Z_{0,i}[J_i] &= \mathcal{N}_i \int \mathscr{D}\varphi_i(x)\ \exp\left\{ i \int \mathrm{d}^4x\ \left[\frac{1}{2}\left(\partial_\mu\varphi_i\right)\left(\partial^\mu\varphi_i\right) - \frac{m_i^2}{2}\varphi_i^2 + J_i\varphi_i \right] \right\} \\ &= \mathcal{N}_i \int \mathscr{D}\varphi_i(x)\ \exp\left\{ -\frac{i}{2} \int \mathrm{d}^4x\ \left[\varphi_i\left(\Box + m_i^2\right)\varphi_i + J_i\varphi_i \right] \right\} \\ &= \mathcal{N}_i \int \mathscr{D}\varphi_i(x)\ \exp\left\{ -\frac{i}{2} \int \mathrm{d}^4x\mathrm{d}^4y\ \varphi_i(x)\mathscr{O}_i(x,y)\varphi_i(y) + i \int \mathrm{d}^4x\ J_i\varphi_i \right\}\ , \end{aligned} \tag{4.10}$$

where we defined the operator

$$\mathscr{O}_i(x,y) = \left(\Box_x + m_i^2\right)\delta^{(4)}(x-y)\ , \tag{4.11}$$

where the index $_x$ reminds us that the d'Alembertian acts on the space-time variable x. At this point, it is important to stress that the convergence of the Gaussian functional integral (4.10) is not guaranteed, since the integrand is a strongly oscillating function for large arguments. This problem can be resolved, if we perform an analytic continuation to imaginary times, i.e., if we set

$$x^0 \equiv t \to -i\tau\ ,\ \tau \in \mathbb{R}\ . \tag{4.12}$$

Geometrically, the above identification can be understood as a clockwise rotation of the time-integration contour about $\pi/2$ into the complex plane, since $e^{-i(\pi/2)}\tau = -i\tau$. This situation is illustrated in Fig. [4.1]. It should be noted that this so-called Wick rotation is only possible, if there are no singularities in that part

Figure 4.1: The diagram [(a)] shows the initial time-integration contour in Minkowski space-time (red dashed line), which is rotated clockwise into the complex time-plane (red line). The second diagram [(b)] shows the Euclidean time-integration contour, which corresponds to the Minkowski case of [(a)].

of the complex time-plane we are integrating over. The result of the Wick rotation is that the integrand in Eq. (4.10) gets a real part which dampens the oscillation of the exponential, so that convergence is guaranteed. In order to see this, we have to evaluate, which consequences are caused by the Wick rotation. First of all, we define a Euclidean 4-vector

$$x_E^\mu = (\tau, \mathbf{r})\ , \tag{4.13}$$

so that

$$x^\mu x_\mu = t^2 - \mathbf{r}^2 = -\tau^2 - \mathbf{r}^2 = -x_E^\mu x_{\mu,E}\ . \tag{4.14}$$

Obviously, the analytic continuation to imaginary times transforms the Minkowski space-time into a four-dimensional Euclidean space-time. This transformation occurs because of the additional i-factor which compensates the different signs of the time and the spatial variables in Minkowski space-time. Since the metric of this four-dimensional Euclidean space-time is given by the four-dimensional identity matrix, in principle we do not have to distinguish between co- and contravariant indices. But, in order to indicate the summation over the contracted indices, we will keep the co- and contravariant notation. Furthermore, it is also convenient to introduce a Euclidean momentum space by setting

$$k^0 \to i\kappa \ , \ \kappa \in \mathbb{R} \ , \tag{4.15}$$

such that

$$k^\mu k_\mu = \left(k^0\right)^2 - \mathbf{k}^2 = -\kappa^2 - \mathbf{k}^2 = -k_E^\mu k_{\mu,E} \ , \tag{4.16}$$

where we defined the four-dimensional Euclidean momentum-space vector

$$k_E^\mu = (\kappa, \mathbf{k}) \ . \tag{4.17}$$

The transformation (4.15) corresponds to a counterclockwise rotation of the k^0-axis about $\pi/2$ into the complex k^0-plane, compare Fig. [4.2]. It is clear that the variable transformations (4.13) and (4.15) give rise

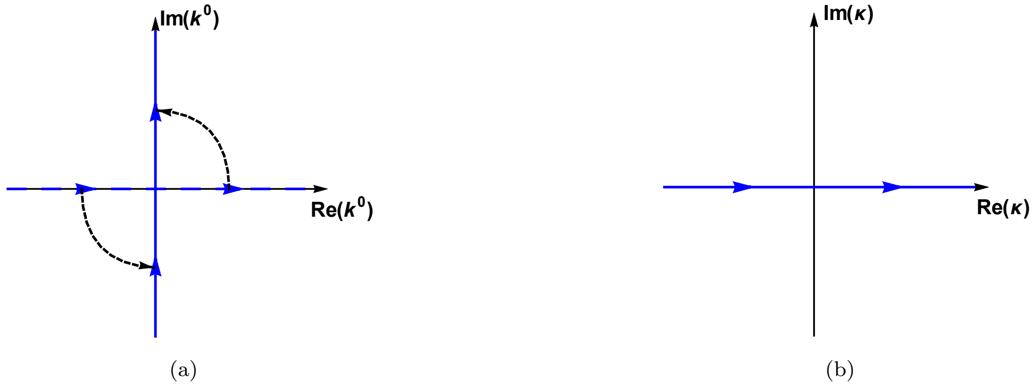


Figure 4.2: The diagram [(a)] shows the initial k^0-integration contour in Minkowski momentum-space (blue dashed line), which is counterclockwise rotated into the complex k^0-plane (blue line). The second diagram [(b)] shows the Euclidean κ-integration contour which corresponds to the Minkowski case of [(a)].

to the existence of a Jacobian,

$$\mathrm{d}^4x = \det\left(\frac{\partial\,(t,\mathbf{r})}{\partial\,(\tau,\mathbf{r})}\right)\mathrm{d}^4x_E = -i\mathrm{d}^4x_E \ , \qquad \mathrm{d}^4k = \det\left(\frac{\partial\,(k^0,\mathbf{k})}{\partial\,(\kappa,\mathbf{k})}\right)\mathrm{d}^4k_E = i\mathrm{d}^4k_E \ . \tag{4.18}$$

Due to Eq. (4.13), it is obvious that also differential operators which involve the time variable will change under this transformation. The Minkowski 4-gradient becomes

$$(\partial_\mu) = (\partial_0, \boldsymbol{\nabla}) = \left(\frac{\partial}{\partial(-i\tau)}, \boldsymbol{\nabla}\right) = (i\partial_\tau, \boldsymbol{\nabla}) = \partial_{\mu,E} \ , \tag{4.19}$$

so that the d'Alembertian can be written as

$$\Box = \partial^\mu\partial_\mu = \partial_t^2 - \Delta = -\partial_\tau^2 - \Delta = -\left(\partial_\tau^2 + \Delta\right) = -\partial_E^\mu \partial_{\mu,E} = -\Box_E \ . \tag{4.20}$$

Finally, we have to find the Euclidean representation of the delta-distribution. Using its Fourier representation, we obtain

$$\begin{aligned}
\delta^{(4)}(x-y) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{-i(x-y)k} \\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i(\mathbf{x}-\mathbf{y})\mathbf{k}} \int \frac{\mathrm{d}k^0}{2\pi}\, e^{-i(x_0-y_0)k^0} \\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i(\mathbf{x}-\mathbf{y})\mathbf{k}}\; i \int \frac{\mathrm{d}\kappa}{2\pi}\, e^{-i(\tau_x-\tau_y)\kappa} \\
&= i \int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\, e^{-i[(\tau_x-\tau_y)\kappa-(\mathbf{x}-\mathbf{y})\mathbf{k}]}
\end{aligned}$$

$$
\begin{aligned}
&= i \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4}\, e^{-i(x_E-y_E)k_E} \\
&= i\delta^{(4)}(x_E - y_E)\ , \qquad (4.21)
\end{aligned}
$$

where we substituted $\mathbf{k} \to -\mathbf{k}$ in the fourth step. At this point, we are able to rewrite the functional integral (4.10)

$$
Z_{0,i}[J_i] = \mathcal{N}_i \int \mathscr{D}\varphi_i(x)\ \exp\left\{-\frac{1}{2}\int \mathrm{d}^4x_E \mathrm{d}^4y_E\ \varphi_i(x_E)\mathscr{O}_E(x_E,y_E)\varphi_i(y_E) + \int \mathrm{d}^4x_E\ J_i(x_E)\varphi_i(x_E)\right\}\ , \tag{4.22}
$$

where we defined the Euclidean operator

$$
\mathscr{O}_{i,E}(x_E,y_E) = \left(-\Box_{x,E} + m_i^2\right)\delta^{(4)}(x_E - y_E)\ . \tag{4.23}
$$

By considering Eq. (4.22), it is now evident that the Wick rotation induces a damped real part in the exponential, so that the convergence of the integral is ensured. Applying Eq. (6.39) to the above functional integral, we obtain

$$
Z_{0,i}[J_i] = \mathcal{N}_i\tilde{\mathcal{N}}_i\Big(\det\mathscr{O}_{i,E}(x_E,y_E)\Big)^{-1/2}\ \exp\left\{\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\ J_i(x_E)\mathscr{O}_{i,E}^{-1}(x_E,y_E)J_i(y_E)\right\}\ . \tag{4.24}
$$

The usual normalization requirement, $Z_{0,i}[0] = 1$, fixes the normalization constant as

$$
\mathcal{N}_i^{-1} = \tilde{\mathcal{N}}_i\Big(\det\mathscr{O}_{i,E}(x_E,y_E)\Big)^{-1/2}\ , \tag{4.25}
$$

so that the factor involving the functional determinant of the operator $\mathscr{O}_{i,E}(x_E,y_E)$, cancels. In the following subsection, we will see that this result is quite advantageous, since the functional determinant in Eq. (4.24) corresponds to an infinite constant. But in general it turns out that these functional determinants have a physical meaning. An example for this will be presented in the upcoming Sec. [4.1.2]. Before we consider the above functional determinant in detail, it will be useful to determine the inverse operator $\mathscr{O}_{i,E}^{-1}(x_E,y_E)$ in Eq. (4.24). To this end, we remember that the inverse of an $(N\times N)$-matrix is defined by $A^{-1}A = \mathbb{1}_{N\times N}$. Since the operator (4.23) can be considered as an infinite-dimensional matrix, we have to consider a generalized version of this definition

$$
\begin{aligned}
\delta^{(4)}(x_E - x'_E) &= \int \mathrm{d}^4y_E\ \mathscr{O}_{i,E}(x_E,y_E)\mathscr{O}_{i,E}^{-1}(y_E,x'_E) \\
&= \int \mathrm{d}^4y_E\ \left(-\Box_{x,E} + m_i^2\right)\delta^{(4)}(x_E - y_E)\mathscr{O}_{i,E}^{-1}(y_E,x'_E) \\
&= \left(-\Box_{x,E} + m_i^2\right)\mathscr{O}_{i,E}^{-1}(x_E,x'_E)\ , \qquad (4.26)
\end{aligned}
$$

which shows that the inverse operator is a Green function of the Euclidean Klein-Gordon operator, since it fulfills the Euclidean Klein-Gordon equation with a delta-distribution as inhomogeneity. In order to determine the explicit form of this Green function, it will be useful to switch to Euclidean momentum space and consider the Fourier transform of Eq. (4.26). We have

$$
\begin{aligned}
\int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\, e^{-i(x_E - x'_E)k_E} &= \int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{\mathrm{d}^4k'_E}{(2\pi)^4}\ \left(-\Box_{x,E} + m_i^2\right)e^{-ix_Ek_E}\tilde{\mathscr{O}}_{i,E}^{-1}(k_E,k'_E)e^{ix'_Ek'_E} \\
&\stackrel{!}{=} \int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{\mathrm{d}^4k'_E}{(2\pi)^4}\ \left(k_E^2 + m_i^2\right)\tilde{\mathscr{O}}_{i,E}^{-1}(k_E,k'_E)e^{-i(k_Ex_E - k'_Ex'_E)}\ , \qquad (4.27)
\end{aligned}
$$

which is obviously fulfilled for

$$
\tilde{\mathscr{O}}_{i,E}^{-1}(k_E,k'_E) = (2\pi)^4\delta^{(4)}(k_E - k'_E)\frac{1}{k_E^2 + m_i^2}\ . \tag{4.28}
$$

Finally, we have to transform this result back to position space. We find

$$
\begin{aligned}
\mathscr{O}_{i,E}^{-1}(x_E,x'_E) &= \int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{\mathrm{d}^4k'_E}{(2\pi)^4}\ e^{-ix_Ek_E}(2\pi)^4\delta^{(4)}(k_E - k'_E)\frac{1}{k_E^2 + m_i^2}e^{ix'_Ek'_E} \\
&= \int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\ \frac{1}{k_E^2 + m_i^2}e^{-i(x_E - x'_E)k_E} \\
&\equiv \Delta_{E,F}(x_E - x'_E)\ , \qquad (4.29)
\end{aligned}
$$

where we defined the Euclidean Feynman propagator. Now, from Eq. (4.29), we observe that the Euclidean Feynman propagator only depends on the difference of the Euclidean space-time points x_E, x'_E. This fact reflects the space-time translation invariance of the theory. Furthermore, it is obvious that the Feynman propagator is a symmetric function under the transposition of its arguments. This function defines only one special example of a whole class of Lorentz-invariant propagation functions which are Green's functions of the Euclidean Klein-Gordon operator. The distinction between these propagators arises from the integration contour in the k^0-plane, which we select to rewrite the four-dimensional Fourier representation of the propagator into a three-dimensional one. In the case of Eq. (4.29) this integration contour is fixed, because the poles of the above propagator are given by

$$\kappa_{1/2} = \pm i\sqrt{\mathbf{k}^2 + m_i^2} \equiv \pm i\omega_{i,\mathbf{k}} \ , \tag{4.30}$$

so that they are not located on the integration contour from $-\infty$ to ∞ along the real κ-axis, compare Fig. [4.3]. At this point, we are able to perform the analytic continuation of Eq. (4.29) back to Minkowski

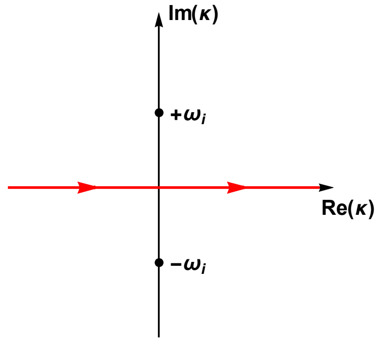


Figure 4.3: Poles and integration contour (red line) of the Euclidean Feynman propagator.

space-time. Using Eqs. (4.13) and (4.15) as well as (4.18), we obtain

$$\begin{aligned}
\Delta_{E,F}(x_E - x'_E) &= \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m_i^2} e^{-i(x_E - x'_E)k_E} \\
&= i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m_i^2} e^{-i\left[(x_0 - x'_0)k^0 - (\mathbf{r}-\mathbf{r}')\cdot\mathbf{k}\right]} \\
&= i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m_i^2} e^{-i(x-x')k} \\
&\equiv i\Delta_F(x - x') \ ,
\end{aligned} \tag{4.31}$$

where we substituted $\mathbf{k} \to -\mathbf{k}$ in the second line. Obviously we obtain, up to a phase-factor, the well-known Feynman propagator $\Delta_F(x-y)$ in Minkowski space-time as analytic continuation of the Euclidean Feynman propagator. In this case, the poles are given by

$$k^0_{1/2} = \pm\sqrt{\mathbf{k}^2 + m_i^2} \equiv \pm\omega_{i,\mathbf{k}} \ , \tag{4.32}$$

so that they lie on the real k^0 axis. Again, we have a fixed integration contour, since the transformation $\kappa \to -ik^0$ in Eq. (4.30) requires an integration along the complex k^0-axis. But due to the residue theorem, we are able to deform this integration contour to the well-known Feynman contour $\mathscr{C}_F$, since there are no further poles on the complex k^0-plane. These two equivalent integration contours are depicted in Fig. [4.4]. Instead of using $\mathscr{C}_F$ as integration contour, it is quite convenient to push the poles (4.32) from the real k^0-axis into the complex k^0-plane. This can be obtained by adding a small imaginary part of the squared mass m_i^2 in the denominator of Eq. (4.31)

$$\Delta_F(x - x') = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m_i^2 + i\epsilon^+} e^{-i(x-x')k} \ . \tag{4.33}$$

The poles of the Feynman propagator are now given by

$$k^0_{1/2} = \pm\sqrt{\omega_{i,\mathbf{k}}^2 - i\epsilon} \approx \pm\omega_{i,\mathbf{k}} \mp i\delta \ , \tag{4.34}$$

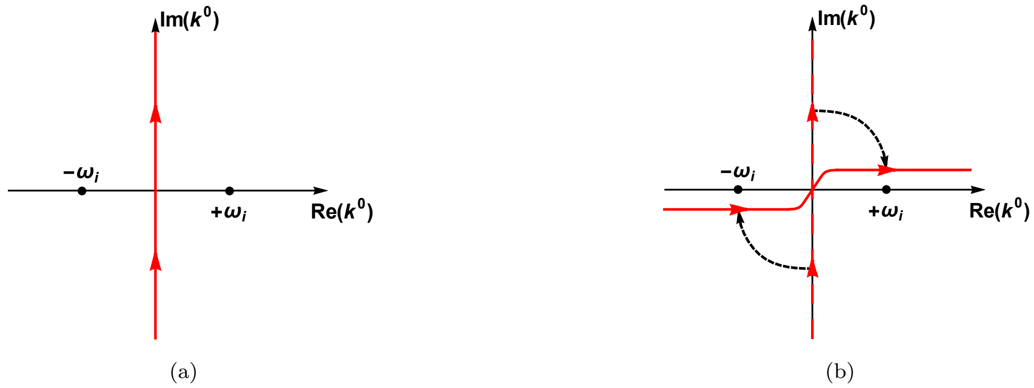


Figure 4.4: The diagram [(a)] shows the fixed integration contour in the complex k^0-plane. The second diagram [(b)] shows the usual Feynman integration contour $\mathscr{C}_F$, which can be obtained by deforming the inital contour of [(a)].

where we expanded the square root up to first order in $\epsilon/\omega_{i,\mathbf{k}}^2$ and defined $\delta \equiv \epsilon/2\omega_{i,\mathbf{k}}$. Since the poles of Eq. (4.33) are in the complex k^0-plane, it is possible to integrate along the real k^0-axis in order to obtain the desired Feynman propagator. The shifted poles and the new integration contour are illustrated in Fig. [4.5]. Using the results (4.31), (4.33) we are in the position to write down the final expression for the Gaussian

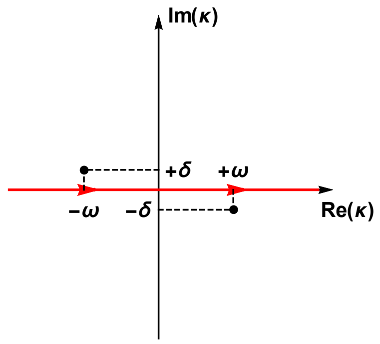


Figure 4.5: Poles and integration contour (red line) of the Feynman-Propagator after shifting this poles (4.32) into the complex plane (4.34).

functional integral (4.24). We find

$$Z_{0,i}[J_i] = \exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\; J_i(x)\Delta_{F,i}(x-y)J_i(y)\right\}\,, \tag{4.35}$$

where we used the normalization (4.25) and also performed an analytic continuation of the space-time integration measures inside the exponential. Then, the generating functional of the interaction-free toy model takes the following form

$$\begin{aligned} Z_0[J_l,J_H] &= Z_{0,l}[J_l]Z_{0,H}[J_H] \\ &= \exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\; J_l(x)\Delta_{F,l}(x-y)J_l(y)\right\}\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\; J_H(x)\Delta_{F,H}(x-y)J_H(y)\right\} \\ &= \exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y \sum_{i=l,H} J_i(x)\Delta_{F,i}(x-y)J_i(y)\right\}\,. \end{aligned} \tag{4.36}$$

Inserting this result into Eq. (4.8), the generating functional of the full theory is given by

$$Z[J_l,J_H] = \mathcal{N}_{int}\exp\left\{i\int \mathrm{d}^4x\; \mathscr{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)},\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)\right\}\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y \sum_{i=l,H} J_i(x)\Delta_{F,i}(x-y)J_i(y)\right\}\,. \tag{4.37}$$

In order to fix the normalization constant $\mathcal{N}_{int}$, we again require that $Z[0,0]=1$, which leads to

$$\mathcal{N}_{int}^{-1} = \exp\left\{i\int \mathrm{d}^4x\; \mathscr{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)},\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)\right\}Z_0[J_l,J_H]\bigg|_{J_l=J_H=0}\,. \tag{4.38}$$

Before we continue with the derivation of the position-space Feynman rules for the toy model (4.1), we interrupt this discussion and turn back to the functional determinant arising from the Gaussian functional integration. In the discussion of Eqs. (4.24) and (4.25), we stated that this functional determinant corresponds to an infinite constant. In order to prove this statement, we start from the definition of the operator (4.11) and use the Fourier representation of the delta-distribution

$$\begin{aligned}\mathscr{O}_i(x,y) &= \left(\Box_x + m_i^2\right)\delta^{(4)}(x-y) = \left(\Box_x + m_i^2\right)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{ik(x-y)} \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \left(-k^2+m_i^2\right) e^{ik(x-y)} \ .\end{aligned} \tag{4.39}$$

Now, we know that the factor involving the functional determinant can be rewritten as

$$\Big(\det\mathscr{O}(x,y)\Big)^{-1/2} = \exp\left\{-\frac{1}{2}\mathrm{Tr}\ln\mathscr{O}(x,y)\right\} , \tag{4.40}$$

where we used that $\ln\det\mathscr{O}(x,y) = \mathrm{Tr}\ln\mathscr{O}(x,y)$. In order to rewrite the logarithm of the Fourier representation (4.39), we remember that the logarithm of a matrix M is, in general, defined by a power series of the form $\ln M = \sum_j c_j M^j$. In principle, we know the exact form of the series representation of the logarithm, but for the following discussion it will be very useful to consider this general form. Then, the logarithm of the operator $\mathscr{O}(x,y)$ is given by

$$\begin{aligned}\ln\mathscr{O}(x,y) &= \ln\int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \left(-k^2+m_i^2\right) e^{ik(x-y)} \\ &= \sum_j c_j\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \left(-k^2+m_i^2\right) e^{ik(x-y)}\right]^j \ .\end{aligned} \tag{4.41}$$

Now, we consider the m-th term of this power series, which can be written as

$$\begin{aligned}&\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \left(-k^2+m_i^2\right) e^{ik(x-y)}\right]^m \\ &= \int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_{m-1}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_m}{(2\pi)^4}\left(-k_1^2+m_i^2\right)\cdots\left(-k_m^2+m_i^2\right)e^{ik_1(x-x_1)}e^{ik_2(x_1-x_2)}\cdots e^{ik_m(x_{m-1}-y)} \\ &= \int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_{m-1}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_m}{(2\pi)^4}\left(-k_1^2+m_i^2\right)\cdots\left(-k_m^2+m_i^2\right)e^{-ik_1x}e^{ik_my}e^{ix_1(k_1-k_2)}e^{ix_2(k_2-k_3)} \\ &\quad\times\cdots e^{ix_{m-1}(k_{m-1}-k_m)} \\ &= \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_m}{(2\pi)^4}\left(-k_1^2+m_i^2\right)\cdots\left(-k_m^2+m_i^2\right)e^{-ik_1x}e^{ik_my}(2\pi)^4\delta^{(4)}(k_1-k_2)(2\pi)^4\delta^{(4)}(k_2-k_3) \\ &\quad\times\cdots(2\pi)^4\delta^{(4)}(k_{m-1}-k_m) \\ &= \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\left(-k_1^2+m_i^2\right)^m e^{-ik_1(x-y)} \ .\end{aligned} \tag{4.42}$$

Applying these steps to each term of the power series, the logarithm of the operator (4.11) is given by

$$\ln\mathscr{O}(x,y) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(-k^2+m_i^2\right)e^{-ik(x-y)} \tag{4.43}$$

Inserting this result into Eq. (4.40), we are left with

$$\Big(\det\mathscr{O}(x,y)\Big)^{-1/2} = \exp\left\{-\frac{1}{2}\int\mathrm{d}^4x\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(-k^2+m_i^2\right)\right\} , \tag{4.44}$$

which tends to zero, since the operator trace of the logarithm diverges. As already mentioned before, this divergent constant factor is not important for us, because the normalization of the functional integral is chosen in a way that this factor cancels out. The reason why we showed the divergence of this functional determinant becomes clear in the following Subsection, when have to consider another functional determinant. Then, it will turn out that functional determinants may also have a physical meaning.

After this short excursion we have to continue with the derivation of the Feynman rules. In order to define these rules for the toy model (4.1), we have to expand the full generating functional (4.37) into a perturbative series of the coupling constants g_1, g_2. In principle, we are able to calculate this perturbative series order by order in the coupling constants by rewriting the exponential in Eq. (4.37) into its series representation. For our purpose, it will be sufficient to consider this perturbative series only up to first order in g_1, g_2. It should be taken into account that the expansion of the full generating functional involves the expansion of the exponential in Eq. (4.37) as well as the expansion of the normalization constant (4.38). In order to simplify the book-keeping of this calculation, we start with the expansion of the exponential, so that

$$\begin{aligned}
\frac{Z[J_l,J_H]}{\mathcal{N}_{int}} &= \exp\left\{i\int \mathrm{d}^4x\, \mathscr{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)},\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)\right\}Z_0[J_l,J_H] \\
&= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{g_1}{4}\int \mathrm{d}^4x\,\left(\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)^2\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2\right]^{n-k}\left[\frac{g_2}{2}\int \mathrm{d}^4x'\,\frac{1}{i}\frac{\delta}{\delta J_H(x')}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x')}\right)^2\right]^{k} \\
&\quad\times Z_0[J_l,J_H] \\
&= \left\{1 - i\frac{g_1}{4}\int \mathrm{d}^4x\,\left(\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)^2\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2 - i\frac{g_2}{2}\int \mathrm{d}^4x\,\frac{1}{i}\frac{\delta}{\delta J_H(x)}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2 + \mathcal{O}(g_1^2,g_2^2,g_1g_2)\right\} \\
&\quad\times Z_0[J_l,J_H]\ . \qquad (4.45)
\end{aligned}$$

The action of the curly bracket on the interaction-free generating functional now produces the different terms of the expansion of the full generating functional. The zeroth-order term just reproduces the free generating functional, which is an expected result, since the zeroth order of a perturbative expansion corresponds to the unperturbed case, i.e., to the interaction-free case. The first-order corrections can be obtained by calculating the action of the terms proportional to g_1 and g_2 on the free generating functional. In order to simplify matters, we consider both terms separately. Using the abbreviation $\Delta_{F,i}(x-y)\equiv\Delta_{F,i}(x,y)$, the first term yields

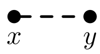
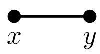
$$\begin{aligned}
&-i\frac{g_1}{4}\int \mathrm{d}^4x\,\left(\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)^2\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2 Z_0[J_l,J_H] \\
&= -i\frac{g_1}{4}\int \mathrm{d}^4x\,\left(\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)^2\frac{1}{i}\frac{\delta}{\delta J_l(x)}\left[-\int \mathrm{d}^4z_1\,\Delta_{F,l}(x,z_1)J_l(z_1)\right]Z_0[J_l,J_H] \\
&= -i\frac{g_1}{4}\int \mathrm{d}^4x\,\left(\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)^2\left[i\Delta_{F,l}(0) + \int \mathrm{d}^4z_1\mathrm{d}^4z_2\,\Delta_{F,l}(x,z_1)\Delta_{F,l}(x,z_2)J_l(z_1)J_l(z_2)\right]Z_0[J_l,J_H] \\
&= -i\frac{g_1}{4}\int \mathrm{d}^4x\,\frac{1}{i}\frac{\delta}{\delta J_H(x)}\left[-i\Delta_{F,l}(0)\int \mathrm{d}^4z_1\,\Delta_{F,H}(x,z_1)J_H(z_1)\right. \\
&\quad\left.-\int \mathrm{d}^4z_1\mathrm{d}^4z_2\mathrm{d}^4z_3\,\Delta_{F,l}(x,z_1)\Delta_{F,l}(x,z_2)\Delta_{F,H}(x,z_3)J_l(z_1)J_l(z_2)J_H(z_3)\right]Z_0[J_l,J_H] \\
&= \left\{-i\frac{g_1}{4}\int \mathrm{d}^4x\, i\Delta_{F,l}(0)i\Delta_{F,H}(0) - i\frac{g_1}{4}\int \mathrm{d}^4x\, i\Delta_{F,l}(0)\int \mathrm{d}^4z_1\mathrm{d}^4z_2\, i\Delta_{F,H}(x,z_1)i\Delta_{F,H}(x,z_2)iJ_H(z_1)iJ_H(z_2)\right. \\
&\quad - i\frac{g_1}{4}\int \mathrm{d}^4x\, i\Delta_{F,H}(0)\int \mathrm{d}^4z_1\mathrm{d}^4z_2\, i\Delta_{F,l}(x,z_1)i\Delta_{F,l}(x,z_2)iJ_l(z_1)iJ_l(z_2) \\
&\quad\left.-i\frac{g_1}{4}\int \mathrm{d}^4x\int \mathrm{d}^4z_1\mathrm{d}^4z_2\mathrm{d}^4z_3\mathrm{d}^4z_4\, i\Delta_{F,l}(x,z_1)i\Delta_{F,l}(x,z_2)i\Delta_{F,H}(x,z_3)i\Delta_{F,H}(x,z_4)iJ_l(z_1)iJ_l(z_2)iJ_H(z_3)iJ_H(z_4)\right\} \\
&\quad\times Z_0[J_l,J_H]\ , \qquad (4.46)
\end{aligned}$$

where we frequently used the symmetry of the Feynman propagator and inserted additional i-factors in the last line. The second term gives

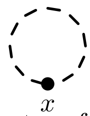
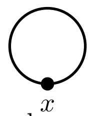
$$\begin{aligned}
&-i\frac{g_2}{2}\int \mathrm{d}^4x\,\frac{1}{i}\frac{\delta}{\delta J_H(x)}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2 Z_0[J_l,J_H] \\
&= -i\frac{g_2}{2}\int \mathrm{d}^4x\,\frac{1}{i}\frac{\delta}{\delta J_H(x)}\frac{1}{i}\frac{\delta}{\delta J_l(x)}\left[-\int \mathrm{d}^4z_1\,\Delta_{F,l}(x,z_1)J_l(z_1)\right]Z_0[J_l,J_H] \\
&= -i\frac{g_2}{2}\int \mathrm{d}^4x\,\frac{1}{i}\frac{\delta}{\delta J_H(x)}\left[i\Delta_{F,l}(0) + \int \mathrm{d}^4z_1\mathrm{d}^4z_2\,\Delta_{F,l}(x,z_1)\Delta_{F,l}(x,z_2)J_l(z_1)J_l(z_2)\right]Z_0[J_l,J_H]
\end{aligned}$$

$$= \left\{ -i\frac{g_2}{2}\int \mathrm{d}^4x\; i\Delta_{F,l}(0)\int \mathrm{d}^4z_1\; i\Delta_{F,H}(x,z_1)iJ_H(z_1) \right.$$
$$\left. -i\frac{g_2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4z_1\mathrm{d}^4z_2\mathrm{d}^4z_3\; i\Delta_{F,l}(x,z_1)i\Delta_{F,l}(x,z_2)i\Delta_{F,H}(x,z_3)iJ_l(z_1)iJ_l(z_2)iJ_H(z_3) \right\} Z_0[J_l,J_H]\;, \tag{4.47}$$

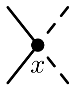
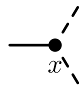
where we again introduced additional i-factors. When we compare both expressions, we are able to introduce a consistent diagrammatic language in order to substitute the cumbersome mathematical expressions. To this end, we consider the different mathematical elements of Eqs. (4.46) and (4.47) and introduce a diagrammatical equivalent for each of those elements. First of all, we observe that the Feynman propagators for both fields are present in all expressions. The physical meaning of these objects is simply the propagation of a particle of the species l or H from one space-time point to another. Therefore, it is quite natural to introduce

$i\Delta_{F,l}(x,y) =$  , $i\Delta_{F,H}(x,y) =$  (4.48)

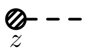
as diagrammatic equivalent of the different propagators. In addition to that, the expressions (4.46) and (4.47) involve Feynman propagators with vanishing argument, which will be identified with

$i\Delta_{F,l}(x,x) \equiv i\Delta_{F,l}(0) =$  , $i\Delta_{F,H}(x,x) \equiv i\Delta_{F,H}(0) =$  . (4.49)

Furthermore, each expression involves a factor of the respective coupling constant and a space-time integration over x. These vertices can be represented as

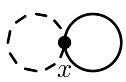
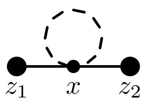
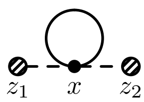
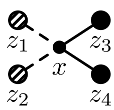
$-ig_1\int \mathrm{d}^4x =$  , $-ig_2\int \mathrm{d}^4x =$  . (4.50)

Finally, we are left with the external sources which are always contracted with a Feynman propagator. In this context, the "contraction" has to be understood with respect to the space-time variable which is integrated over. This has to be understood as an infinite-dimensional generalization of a matrix-vector product. In our diagrammatic language, these external sources shall be illustrated by

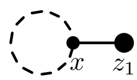
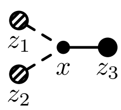
$\int \mathrm{d}^4z\; iJ_l(z) =$  , $\int \mathrm{d}^4z\; iJ_H(z) =$  . (4.51)

Using these Feynman rules, we are able to translate the mathematical expressions (4.46), (4.47) into a diagrammatic language. We find

$$-i\frac{g_1}{4}\int \mathrm{d}^4x \left(\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)^2\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2 Z_0[J_l,J_H]$$

$= \frac{1}{4}\Big\{$  $+$  $+$  $+$  $\Big\} Z_0[J_l,J_H]$ (4.52)

and

$-i\frac{g_2}{2}\int \mathrm{d}^4x\; \frac{1}{i}\frac{\delta}{\delta J_H(x)}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2 Z_0[J_l,J_H] = \frac{1}{2}\Big\{$  $+$  $\Big\} Z_0[J_l,J_H]\;.$ (4.53)

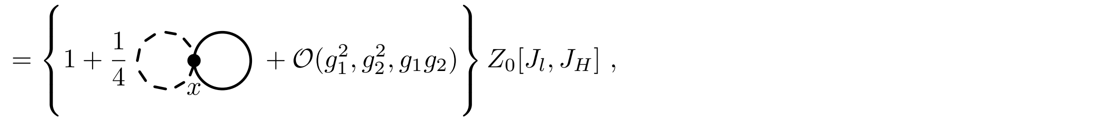
In order to obtain the full expression of the generating functional up to first order in the coupling constants, we also have to evaluate the normalization constant (4.38). It is quite obvious that the normalization constant will give us the same terms as in Eqs. (4.46) and (4.47), since we have to calculate the same functional derivatives of the free generating functional. But in contrast to Eq. (4.39) we have to evaluate these terms for vanishing external sources $J_l = J_H = 0$, so that only the first term of Eq. (4.46) survives. Therefore, the normalization constant can be written as

$$\mathcal{N}_{int}^{-1} = \exp\left\{ i\int \mathrm{d}^4x\; \mathscr{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J_l(x)},\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)\right\} Z_0[J_l,J_H]\Big|_{J_l=J_H=0}$$

$$
= \left\{ 1 - i\frac{g_1}{4} \int \mathrm{d}^4x \left(\frac{1}{i}\frac{\delta}{\delta J_H(x)}\right)^2 \left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2 - i\frac{g_2}{2} \int \mathrm{d}^4x \,\frac{1}{i}\frac{\delta}{\delta J_H(x)} \left(\frac{1}{i}\frac{\delta}{\delta J_l(x)}\right)^2 + \mathcal{O}(g_1^2, g_2^2, g_1 g_2) \right\}
$$

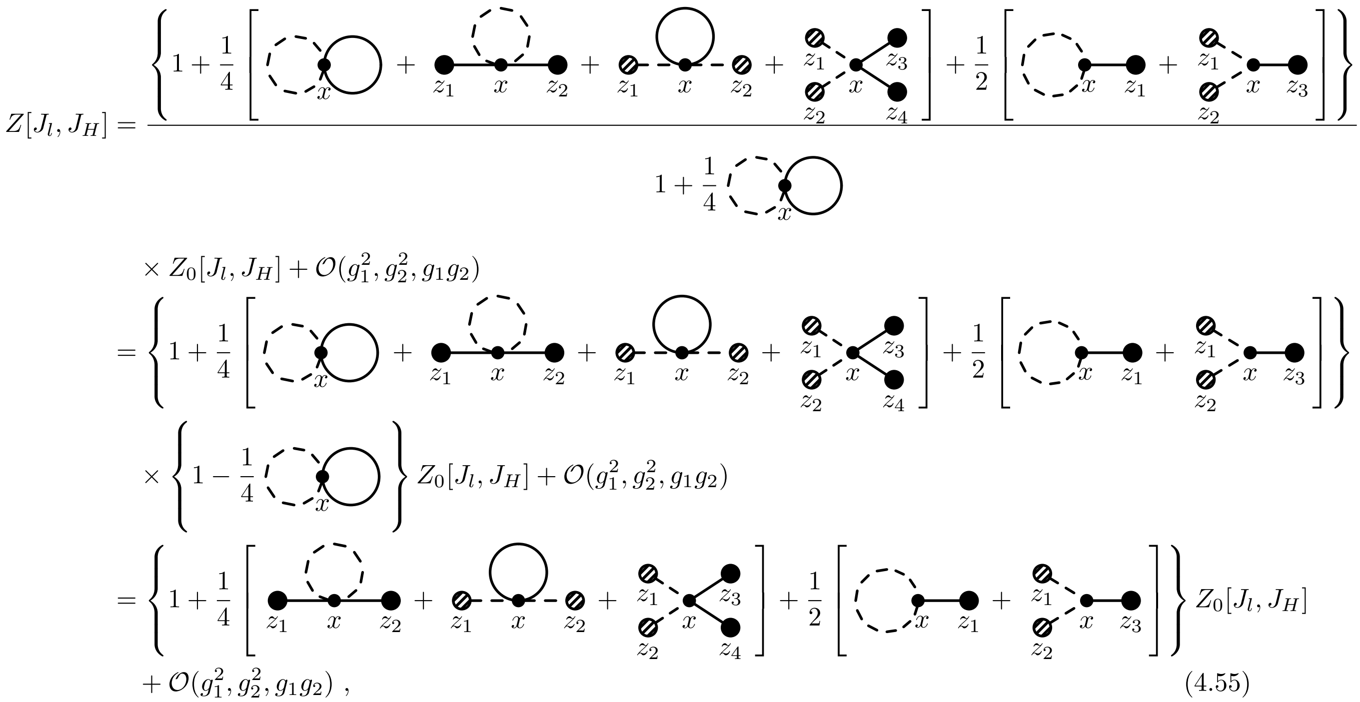
$$
\times Z_0[J_l, J_H] \Big|_{J_l = J_H = 0}
$$

$$
= \left\{ 1 - i\frac{g_1}{4} \int \mathrm{d}^4x \; i\Delta_{F,l}(0) i\Delta_{F,H}(0) + \mathcal{O}(g_1^2, g_2^2, g_1 g_2) \right\} Z_0[J_l, J_H]
$$



(4.54)

where we used the results (4.39), (4.46), and (4.47) and substituted the mathematical expression of the non-vanishing term by its Feynman diagram. The final expression of the full generating functional (4.37), up to first order in g_1, g_2, can now easily be stated by collecting the results (4.39), (4.52), (4.53), and (4.54) and inserting them into Eq. (4.37). Since the mathematical expressions are quite long and cumbersome, it will be easier to perform this calculation in the shape of the corresponding Feynman diagrams. We obtain



(4.55)

where we expanded the denominator according to $(1 + x)^{-1} \approx 1 - x$ in the second step. Obviously, the normalization constant ensures that the two-loop vacuum diagram cancels from the generating functional. This is not a coincidence, since one can show to all orders in perturbation theory that the vacuum diagrams, i.e., those diagrams without external legs, have to cancel, if the generating functional is normalized. A vivid proof of this statement for ordinary φ^4-theory can be found in Ref. [PeSc]. In principle, using Eq. (2.64), it would now be possible to calculate the first-order corrections of the n-point functions of our toy model. Then, the whole procedure, presented in this subsection could be used to calculate higher-order corrections of the full generating functional and of the n-point functions. Of course, these calculations become more and more tedious for higher orders in the perturbative expansion. But since this analysis is not important for the following sections, we stop the perturbative expansion at this point and turn to the low-energy analysis of our scalar field theory.

4.1.2 Low-Energy Effective Model

The basic considerations of the previous subsection were intended as a brief introduction into the functional treatment of a scalar quantum field theory, but also as a starting point of the low-energy analysis of this Subsection. To be particular, starting from the toy model (4.1), we want to find an effective model which describes the dynamics and phenomena of our scalar field theory at low energies. The basic idea of this kind of analysis is the simple observation that each physical process involves many energy scales. The starting point of a low-energy analysis of such a process is now based on the fact that, in general, these energy scales are widely separated. This allows us to find an approximate low energy description of the physical system without including the details of high-energy phenomena. To this end, one has to identify the relevant degrees of freedom at low energies. Other parameters and degrees of freedom which are large compared to the chosen energy scale can be sent to infinity. In order to improve this approximation, it is possible to introduce the interactions of the neglected energy scales as small perturbations of the low-energy description. As already mentioned in the introduction of this section, our toy model describes the interaction of a light scalar field $\varphi_l(x)$ with a heavy scalar field $\varphi_H(x)$. From now on, we assume that the mass of the heavy field excitations m_H is very large compared to the light particle mass m_l, i.e., $m_H \gg m_l$. Now, at energies smaller than the heavy particle mass, we expect that the dynamics of the toy model is fully determined by the self-interaction of light scalar field $\varphi_l(x)$. In order to derive an effective field theory for the low energy regime of the theory, we have to integrate the heavy field out of the action (4.1) of our model. The starting point for this analysis is given by the functional integral

$$\begin{aligned}
\langle\varphi_{l,f},\varphi_{H,f},\infty|\varphi_{l,i},\varphi_{H,i},-\infty\rangle &= \mathcal{N}\int \mathscr{D}\varphi_l(x)\mathscr{D}\varphi_H\ \exp\left\{iS_{toy}[\varphi_l,\varphi_H]\right\} \\
&= \mathcal{N}\int \mathscr{D}\varphi_l(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left[\frac{1}{2}\left(\partial_\mu\varphi_l\right)\partial^\mu\varphi_l - \frac{m_l^2}{2}\varphi_l^2\right]\right\} \\
&\quad\times\int \mathscr{D}\varphi_H(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left[\frac{1}{2}\left(\partial_\mu\varphi_H\right)\partial^\mu\varphi_H - \frac{m_H^2}{2}\varphi_H^2 - \frac{g_1}{4}\varphi_H^2\varphi_l^2 - \frac{g_2}{2}\varphi_H\varphi_l^2\right]\right\} \\
&= \mathcal{N}\int \mathscr{D}\varphi_l(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left[\frac{1}{2}\left(\partial_\mu\varphi_l\right)\partial^\mu\varphi_l - \frac{m_l^2}{2}\varphi_l^2\right]\right\}\times I_H[\varphi_l]\ , \qquad (4.56)
\end{aligned}$$

where we used Eq. (2.52) and defined the functional integral of the heavy field as

$$I_H[\varphi_l] = \int \mathscr{D}\varphi_H(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left[\frac{1}{2}\left(\partial_\mu\varphi_H\right)\partial^\mu\varphi_H - \frac{m_H^2}{2}\varphi_H^2 - \frac{g_1}{4}\varphi_H^2\varphi_l^2 - \frac{g_2}{2}\varphi_H\varphi_l^2\right]\right\}. \qquad (4.57)$$

Since the interactions between both fields only involve at most quadratic powers of the respective fields, the above functional integral can be solved analytically. According to the discussion of the previous subsection, we rewrite this functional integral as

$$\begin{aligned}
I_H[\varphi_l] &= \int \mathscr{D}\varphi_H(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left[-\frac{1}{2}\varphi_H\Box\varphi_H - \frac{m_H^2}{2}\varphi_H^2 - \frac{g_1}{4}\varphi_H^2\varphi_l^2 - \frac{g_2}{2}\varphi_H\varphi_l^2\right]\right\} \\
&= \int \mathscr{D}\varphi_H(x)\ \exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\ \left[\varphi_H\left(\Box + m_H^2 + \frac{g_1}{2}\varphi_l^2\right)\varphi_H + g_2\varphi_H\varphi_l^2\right]\right\} \\
&= \int \mathscr{D}\varphi_H(x)\ \exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ \varphi_H(x)\mathscr{O}_{\varphi_l}(x,y)\varphi_H(y) + i\int \mathrm{d}^4x\ J_{\varphi_l}(x)\varphi_H(x)\right\}, \qquad (4.58)
\end{aligned}$$

where we integrated the kinetic term by parts and defined the operator

$$\mathscr{O}_{\varphi_l}(x,y) = \left[\Box_x + m_H^2 + \frac{g_1}{2}\varphi_l^2(x)\right]\delta^{(4)}(x-y) \qquad (4.59)$$

and the "source"

$$J_{\varphi_l}(x) = -\frac{g_2}{2}\varphi_l^2(x)\ . \qquad (4.60)$$

In order to ensure the convergence of the Gaussian functional integral (4.58), we have to perform the Wick rotation of the integrand. Using Eqs. (4.12)-(4.21) we find

$$I_H[\varphi_l] = \int \mathscr{D}\varphi_H(x_E)\ \exp\left\{-\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\ \varphi_H(x_E)\mathscr{O}_{\varphi_l,E}(x_E,y_E)\varphi_H(y_E) + \int \mathrm{d}^4x_E\ J_{\varphi_l}(x_E)\varphi_H(x_E)\right\}, \qquad (4.61)$$

where we defined the Euclidean operator

$$\mathscr{O}_{\varphi_l,E}(x_E,y_E) = \left[-\Box_{x,E} + m_H^2 + \frac{g_1}{2}\varphi_l^2(x_E)\right]\delta^{(4)}(x_E-y_E)\ . \qquad (4.62)$$

Again, we obtain a damped real part of the integrand, so that the convergence of the functional integral is guaranteed. Finally, using Eq. (6.39), the functional integral is given by

$$I_H[\varphi_l] = \mathcal{N}_H\left(\det\mathscr{O}_{\varphi_l,E}(x_E,y_E)\right)^{-1/2}\ \exp\left\{\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\ J_{\varphi_l}(x_E)\mathscr{O}^{-1}_{\varphi_l,E}(x_E,y_E)J_{\varphi_l}(y_E)\right\}$$

$$
\begin{aligned}
&= \mathcal{N}_H \left(\det \mathscr{O}_{\varphi_l}(x,y)\right)^{-1/2} \exp\left\{\frac{i}{2}\int \mathrm{d}^4x \mathrm{d}^4y \; J_{\varphi_l}(x)\mathscr{O}^{-1}_{\varphi_l}(x,y)J_{\varphi_l}(y)\right\} \\
&= \mathcal{N}_H \left(\det \mathscr{O}_{\varphi_l}(x,y)\right)^{-1/2} \exp\left\{i\int \mathrm{d}^4x \mathrm{d}^4y \; \frac{g_2^2}{8}\varphi_l^2(x)\left[\Box_x + m_H^2 + \frac{g_1}{2}\varphi_l^2(x)\right]^{-1} \delta^{(4)}(x-y)\varphi_l^2(y)\right\} \\
&= \mathcal{N}_H \left(\det \mathscr{O}_{\varphi_l}(x,y)\right)^{-1/2} \exp\left\{i\int \mathrm{d}^4x \; \frac{g_2^2}{8}\varphi_l^2(x)\left[\Box + m_H^2 + \frac{g_1}{2}\varphi_l^2(x)\right]^{-1} \varphi_l^2(x)\right\} , \qquad (4.63)
\end{aligned}
$$

where we performed the analytic continuation back to Minkowski space-time and introduced the inverse operator

$$
\mathscr{O}^{-1}_{\varphi_l}(x,y) = \left[\Box_x + m_H^2 + \frac{g_1}{2}\varphi_l^2(x)\right]^{-1} \delta^{(4)}(x-y) \qquad (4.64)
$$

in the third line. This result can now be inserted into the transition amplitude (4.56), so that

$$
\begin{aligned}
&\langle \varphi_{l,f}, \varphi_{H,f}, \infty | \varphi_{l,i}, \varphi_{H,i}, -\infty\rangle \\
&= \mathcal{N}\int \mathscr{D}\varphi_l(x) \; \exp\left\{i\int \mathrm{d}^4x \; \left[\frac{1}{2}\left(\partial_\mu\varphi_l\right)\partial^\mu\varphi_l - \frac{m_l^2}{2}\varphi_l^2\right]\right\} \\
&\quad \times \mathcal{N}_H \left(\det \mathscr{O}_{\varphi_l}(x,y)\right)^{-1/2} \exp\left\{i\int \mathrm{d}^4x \; \frac{g_2^2}{8}\varphi_l^2\left(\Box + m_H^2 + \frac{g_1}{2}\varphi_l^2\right)^{-1}\varphi_l^2\right\} \\
&= \mathcal{N}_{eff}\int \mathscr{D}\varphi_l(x) \left(\det \mathscr{O}_{\varphi_l}(x,y)\right)^{-1/2} \exp\left\{i\int \mathrm{d}^4x \; \left[\frac{1}{2}\left(\partial_\mu\varphi_l\right)\partial^\mu\varphi_l - \frac{m_l^2}{2}\varphi_l^2 + \frac{g_2^2}{8}\varphi_l^2\left(\Box + m_H^2 + \frac{g_1}{2}\varphi_l^2\right)^{-1}\varphi_l^2\right]\right\} , \\
&\qquad (4.65)
\end{aligned}
$$

with $\mathcal{N}_{eff} = \mathcal{N}\mathcal{N}_H$. The inverse operator in the above exponential can be expanded as follows

$$
\begin{aligned}
\left[\Box + m_H^2 + \frac{g_1}{2}\varphi_l^2(x)\right]^{-1} &= \left\{\left(\Box + m_H^2\right)\left[1 + \frac{g_1}{2}\left(\Box + m_H^2\right)^{-1}\varphi_l^2(x)\right]\right\}^{-1} \\
&= \sum_{m=0}^{\infty}(-1)^m\left[\frac{g_1}{2}\left(\Box + m_H^2\right)^{-1}\varphi_l^2(x)\right]^m \frac{1}{m_H^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box}{m_H^2}\right)^n \\
&= \frac{1}{m_H^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box}{m_H^2}\right)^n + \text{terms involving two or more } \varphi_l \text{ fields} , \qquad (4.66)
\end{aligned}
$$

where we only took the $m = 0$ term of the second sum. The reason for this step will become apparent, when we apply the methods of this section to the extended Linear Sigma Model in the following sections. Since, by assumption, the mass of the heavy-field excitations is very large in comparison with the other parameters of the model, we are able to neglect terms of order $\mathcal{O}(m_H^{-6})$. Then, the action of the low-energy effective theory for our toy model is given by

$$
\begin{aligned}
S^{(1)}_{toy,eff} &= \int \mathrm{d}^4x \; \left[\frac{1}{2}\left(\partial_\mu\varphi_l\right)^2 - \frac{m_l^2}{2}\varphi_l^2 + \frac{g_2^2}{8m_H^2}\varphi_l^2\left(1 - \frac{\Box}{m_H^2}\right)\varphi_l^2\right] \\
&= \int \mathrm{d}^4x \; \left[\frac{1}{2}\left(\partial_\mu\varphi_l\right)^2 - \frac{m_l^2}{2}\varphi_l^2 + \frac{g_2^2}{8m_H^2}\varphi_l^4 + \frac{g_2^2}{2m_H^4}\varphi_l^2\left(\partial_\mu\varphi_l\right)^2\right] , \qquad (4.67)
\end{aligned}
$$

where the superscript (1) refers to the fact that we expanded the inverse operator (4.66) only up to $n = 1$. Furthermore, we integrated the last term of the above equation by parts. Before we want to interpret this result, we turn back to the transition amplitude (4.65). After integrating the heavy field $\varphi_H(x)$ out of the theory, we are left with an exponential including the low-energy effective action of our toy model times the functional determinant of Eq. (4.59). In the following, we want to find the physical interpretation of this functional determinant. To this end, we first have to rewrite the operator (4.59). The basic idea is to represent this operator as a generalized matrix-matrix product of two other operators, i.e.,

$$
\begin{aligned}
\mathscr{O}_{\varphi_l}(x,x') &= \left[\Box_x + m_H^2 + \frac{g_1}{2}\varphi_l^2(x)\right]\delta^{(4)}(x-x') \\
&= \left(\Box_x + m_H^2\right)\delta^{(4)}(x-x') + \frac{g_1}{2}\left(\Box_x + m_H^2\right)\Delta_{F,H}(x,x')\varphi_l^2(x') \\
&= \int \mathrm{d}^4y \; \left[\left(\Box_x + m_H^2\right)\delta^{(4)}(x-y)\delta^{(4)}(y-x') + \frac{g_1}{2}\left(\Box_x + m_H^2\right)\delta^{(4)}(x-y)\Delta_{F,H}(y,x')\varphi_l^2(x')\right] \\
&= \int \mathrm{d}^4y \; \mathscr{O}_0(x,y)\mathscr{O}_{\varphi_l,I}(y,x') , \qquad (4.68)
\end{aligned}
$$

where we used that the heavy-particle Feynman propagator is a Green function of the respective Klein-Gordon operator. Furthermore, we defined

$$\mathscr{O}_0(x,y) = \left(\Box_x + m_H^2\right)\delta^{(4)}(x-y) \ , \tag{4.69}$$

$$\mathscr{O}_{\varphi_l,I}(y,x') = \delta^{(4)}(y-x') + \frac{g_1}{2}\Delta_{F,H}(y,x')\varphi_l^2(x') \ . \tag{4.70}$$

The reason why we wanted to represent the operator (4.59) as a product of two other operators becomes apparent, when we insert Eq. (4.68) into the functional determinant. Since the determinant of a product of matrices is equal to the product of the determinants of the respective matrices, we are able to rewrite the determinant of the expression (4.59) as the product of the determinants of Eqs. (4.69) and (4.70). At this point, we remember the discussion of the functional determinant of the previous subsection. There, we showed that the functional determinant of Eq. (4.11), which is identical to Eq. (4.69), corresponds to an infinite constant. Now, since we are able to factorize this infinite factor, we are can absorb it into the normalization constant. Using these considerations, we find

$$\begin{aligned}\left(\det\mathscr{O}_{\varphi_l}(x,x')\right)^{-1/2} &= \left(\det\mathscr{O}_0(x,y)\det\mathscr{O}_{\varphi_l,I}(y,x')\right)^{-1/2}\\ &= \mathcal{N}_{det}\exp\left\{-\frac{1}{2}\mathrm{Tr}\ln\mathscr{O}_{\varphi_l,I}(y,x')\right\} \ ,\end{aligned} \tag{4.71}$$

where we again used that the logarithm of the determinant of a matrix is equal to the trace over the logarithm of this matrix. As mentioned before, the infinite factor $\mathcal{N}_{det}$ can be absorbed into the overall normalization constant. At this point, we are able to study the physical meaning of the above exponential. To this end, we have to expand the matrix logarithm into its series representation

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n \ . \tag{4.72}$$

In contrast to the discussion of the previous subsection, it will be advantageous to consider the above series representation instead of the general case (4.41), since the operator (4.70) is already in the right form to use Eq. (4.72). We find

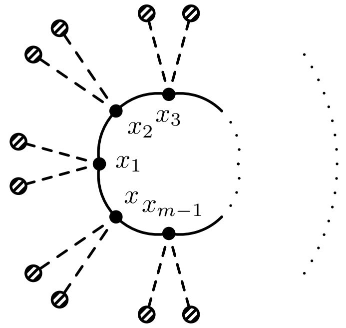
$$\begin{aligned}&\exp\left\{-\frac{1}{2}\mathrm{Tr}\ln\mathscr{O}_{\varphi_l,I}(x,x')\right\}\\ &= \exp\left\{-\frac{1}{2}\mathrm{Tr}\ln\left[\delta^{(4)}(x-x') + \frac{g_1}{2}\Delta_{F,H}(x,x')\varphi_l^2(x')\right]\right\}\\ &= \exp\left\{-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Tr}\left(\frac{g_1}{2}\Delta_{F,H}(x,x')\varphi_l^2(x')\right)^n\right\} \ ,\end{aligned} \tag{4.73}$$

where we renamed the space-time variable $y \to x$. In order to interpret this result, we consider the m-th term of the above series

$$\begin{aligned}&\frac{(-1)^{m+1}}{m}\mathrm{Tr}\left(\frac{g_1}{2}\Delta_{F,H}(x,x')\varphi_l^2(x')\right)^m\\ &= \frac{(-1)^{m+1}}{m}\mathrm{Tr}\left[\left(\frac{g_1}{2}\right)^m\int \mathrm{d}^4x_1\mathrm{d}^4x_2\cdots\mathrm{d}^4x_{m-1}\Delta_{F,H}(x,x_1)\varphi_l^2(x_1)\Delta_{F,H}(x_1,x_2)\varphi_l^2(x_2)\cdots\Delta_{F,H}(x_{m-1},x')\varphi_l^2(x')\right]\\ &= \frac{(-1)^{m+1}}{m}\left(\frac{g_1}{2}\right)^m\int \mathrm{d}^4x\mathrm{d}^4x_1\cdots\mathrm{d}^4x_{m-1}\Delta_{F,H}(x,x_1)\varphi_l^2(x_1)\Delta_{F,H}(x_1,x_2)\varphi_l^2(x_2)\cdots\Delta_{F,H}(x_{m-1},x)\varphi_l^2(x)\\ &= \frac{-1}{m}\left(\frac{-ig_1}{2}\right)^m\int \mathrm{d}^4x\mathrm{d}^4x_1\cdots\mathrm{d}^4x_{m-1}i\Delta_{F,H}(x,x_1)i^2\varphi_l^2(x_1)i\Delta_{F,H}(x_1,x_2)i^2\varphi_l^2(x_2)\cdots i\Delta_{F,H}(x_{m-1},x)i^2\varphi_l^2(x) \ .\end{aligned} \tag{4.74}$$

The reason why we introduced the additional i-factors in the last line is that we are now able to use the Feynman rules of the last section in order to translate the above expression into a Feynman diagram. The external sources of the previous discussion are now given by the light fields $\varphi_l(x)$, because we are able to interpret them as external fields in this context. Translating Eq. (4.74) into a Feynman diagram, we find

$$\frac{(-1)^{m+1}}{m}\mathrm{Tr}\left(\frac{g_1}{2}\Delta_{F,H}(x,x')\varphi_l^2(x')\right)^m = (-1)$$

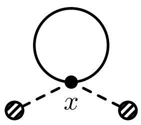


$$. \qquad (4.75)$$

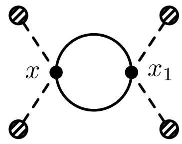
Obviously, this term describes a connected one-loop diagram with m vertices. When we naively compare the left- and the right-hand side of Eq. (4.75), it seems that the constant factor $1/(2^m m)$ is missing. But in fact, this is not the case, since this factor exactly cancels out because of the symmetry properties of the diagram. To be particular, at each of the m vertices, the external light fields can be interchanged, which gives a factor of 2^m. Furthermore, the diagram is symmetric under the cyclic group of rank m, Z_m, i.e., there are m possibilities to rotate a distinct vertex into a neighboring vertex, yielding another factor of m. Combining both factors, we end up with a symmetry factor of the diagram, which is equivalent to the constant factor of the logarithm series. Finally, translating each term of the logarithmic series into a Feynman diagram, Eq. (4.73) becomes

$$\begin{aligned}
&\exp\left\{-\frac{1}{2}\mathrm{Tr}\ln \mathscr{O}_{\varphi_l,I}(x,x')\right\}\\
&= \exp\left\{-\frac{1}{2}\left[\frac{g_1}{2}\mathrm{Tr}\left(\Delta_{F,H}(x,x')\varphi_l^2(x')\right) + \frac{(-1)}{2}\left(\frac{g_1}{2}\right)^2\mathrm{Tr}\left[\left(\Delta_{F,H}(x,x')\varphi_l^2(x')\right)^2\right] + \ldots\right.\right.\\
&\qquad\left.\left. + \frac{(-1)^{m+1}}{m}\left(\frac{g_1}{2}\right)^m \mathrm{Tr}\left[\left(\Delta_{F,H}(x,x')\varphi_l^2(x')\right)^m\right] + \ldots\right]\right\}
\end{aligned}$$

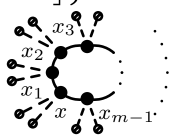
$$= \exp\left\{\frac{1}{2}\left[\right.\right.$$



$$+$$



$$+\ldots+$$



$$\left.\left. + \ldots\right]\right\}, \qquad (4.76)$$

where we used that each diagram contains one factor of (-1), so that the exponential gets a positive sign. Obviously, the factor containing the functional determinant, Eq. (4.71), corresponds to the one-loop corrections of the $2n$-point functions of the light scalar field $\varphi_l(x)$.

Before we discuss the consequences of this result, we turn back to Eq. (4.65). After integrating the heavy field out of the theory, the low-energy effective theory contains an infinite series of interaction terms for the light field. Each of these terms is coupled with a so-called Low Energy Constant (LEC), which in principle contains all the information about the heavy particles which have been removed from the theory. An explicit example for these LECs is given by Eq. (4.67), where we neglected all interaction terms involving more than four light fields and all terms of order $\mathcal{O}(m_H^{-6})$. The remaining interaction terms in $S^{(1)}_{toy,eff}$ are coupled by the LECs $g_2^2/8m_H^2$ and $g_2^2/2m_H^4$. These constants can also be obtained in another way. In order to see this, we observe that the action (4.1) is symmetric under the Z_2 transformation $\varphi_l \to -\varphi_l$. At low energies, this symmetry must be conserved, so that it also must be present in the corresponding effective theory. Therefore, a possible ansatz for the low-energy effective theory would be given by the most general Z_2 symmetric Lagrangian

$$S_{toy,eff}[\varphi_l] = \int \mathrm{d}^4x\ \left[\mathscr{L}_{0,l}(\varphi_l,\partial_\mu\varphi_l) + \ell_1\varphi_l^4 + \ell_2\varphi_l^2\Box\varphi_l^2 + \ldots\right] , \qquad (4.77)$$

with coefficients ℓ_i to be determined. These LECs can be obtained by matching the effective theory to the initial theory, since both theories must be identical, i.e., must yield the same scattering matrix elements, at high energy scales. The matching condition for ℓ_1 and ℓ_2 is depicted in Fig. [4.6] and shows how, at tree-level, the effective theory must be matched to the full theory. To be particular, one has to calculate the scattering matrix element of the s-, t-, and u-channel diagrams of the full theory and expand the resulting expression for large m_H. This is possible, since the heavy particle mass is very large compared to the energy scale at which our effective theory should be valid. Then, we calculate the same expressions for the effective field theory and compare both expressions in order to determine the LECs ℓ_1 and ℓ_2. This matching

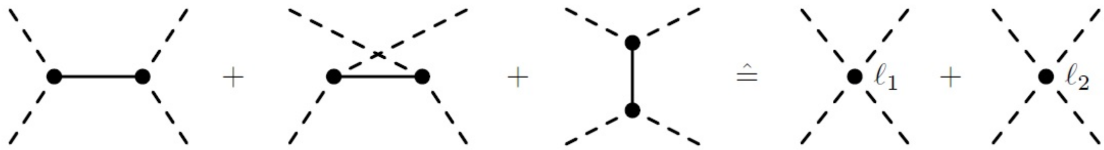


Figure 4.6: Tree-level contributions to $\varphi_l\varphi_l \to \varphi_l\varphi_l$ of the full and the effective field theory.

procedure can also be extended to loop diagrams, which will result in loop corrections of the LECs of the theory. At this point, the importance of the functional determinant (4.76) comes into play, since it includes the one-loop contributions to the $2n$-point functions of the light fields. But in this work, we will only focus on the determination of tree-level LECs, so that we do not have to consider the functional determinants of the theory.

4.2 The Extended Linear Sigma Model (eLSM)

As already mentioned in the introduction of this chapter, we now turn to the eLSM. In the following Subsections, we present the eLSM in detail. In Sec. [4.2.1], we introduce the physical content of the model. Then, in Sec. [4.2.2], we focus on the mathematical implementation of this physical content and introduce the main building blocks of the eLSM. Furthermore, we will use these basic objects to construct the relevant interaction terms of the model.

4.2.1 The Physical Content of the eLSM

In Sec. [2.3], we briefly introduced Quantum Chromodynamics by deriving the QCD Lagrangian. It was shown that the quark part of this Lagrangian possesses an approximate $SU(N_f)_V \times SU(N_f)_A$ symmetry, the so-called chiral symmetry. In the third Chapter, it turned out that chiral symmetry plays an essential role in the non-perturbative treatment of the low-energy regime of strong interactions. The framework of Chiral Perturbation Theory basically relies on the analysis of the hadronic n-point functions and their symmetry relations. Furthermore, it can be shown that ChPT yields the most general solution of the chiral WFT identities, see Ref. [Leut]. In this approach, the dynamical fields enter the Lagrangian through a nonlinear realization of the underlying chiral symmetry, compare Sec. [3.2]. An important characteristic of this nonlinear realization is that the scalar hadronic degrees of freedom are already integrated out, so that only the Nambu-Goldstone bosons, i.e., pseudoscalar objects, enter the Lagrangian. This is beautifully demonstrated by Eq. (3.165), where the zeroth component of the Euclidean unit vector $\vec{\mathscr{U}}$ is eliminated by solving the normalization constraint. In Sec. [3.2], we also saw that, in the case of an exact underlying symmetry, the interaction terms of these fields always have to involve space-time derivatives. The physical meaning of this circumstance can be understood, when we translate the space-time derivatives into momenta of the Nambu-Goldstone bosons. Then, we recognize that their interaction becomes arbitrarily weak at small momenta, which is a basic feature of Nambu-Goldstone bosons. In addition to that, the derivative couplings offer a way for a systematic expansion of the scattering-matrix elements in powers of Nambu-Goldstone boson momenta. Another possibility to realize chiral symmetry in an effective field theory containing hadronic degrees of freedom is given by a linear realization. In these so-called linear sigma models, scalar and pseudoscalar fields enter the Lagrangian. In addition to that, it is possible to extend this linear sigma model in a way that also vector and axial-vector degress of freedom can be included, see also Ref. [Par2]. In the following, we want to introduce these degrees of freedom for the case of two quark flavors.

4.2.1.1 Scalar Mesons

Mesons with spin quantum number $S = 1$ and angular momentum $L = 1$, coupled to a total angular momentum of $J = 0$, are called scalar mesons. The term "scalar" derives from two important properties of the corresponding field. On the one hand, the vanishing total angular momentum requires that the corresponding field transforms as a singlet (or a scalar) under proper orthochronous Lorentz transformations. On the other hand, using

$$P = (-1)^{L+1} \,, \tag{4.78}$$

we obtain a positive parity, which ensures that the scalar field does not change its sign under spatial reflections. Furthermore, using

$$C = (-1)^{L+S} \,, \tag{4.79}$$

we obtain that scalar mesons also have a positive sign under charge conjugation transformations. It is conventional to summarize the above properties by classifying

$$I(J^{PC}) \ , \tag{4.80}$$

where I denotes the isospin quantum number. In this convention, the scalar mesons are classified as $I(0^{++})$ mesons. The possible values of the isospin quantum number can be derived as follows: In the case of $N_f = 2$, the QCD Lagrangian possesses an approximate global $SU(2)_V$ symmetry, compare Sec. [2.3.2]. The quark and anti-quark fields arise as irreducible representations of the isospin group. In particular, the quark field $\Psi(x)$ defines an isodoublet, denoted as $[2]_f$, while the anti-quark field emerges as an anti-doublet. Now, the quark/anti-quark nature of a meson requires the coupling of the doublet with the anti-doublet. According to the usual coupling rules, one finds

$$[2]_f \otimes [\bar{2}]_f = [1]_f \oplus [3]_f \ , \tag{4.81}$$

i.e., a singlet, $I = 0$, and a triplet, $I = 1$. The eLSM contains both, a singlet scalar meson, denoted as σ_N, and a triplet of scalar mesons, which will be described by the isospin vector $\vec{a}_0 = (a_{0,1}, a_{0,2}, a_{0,3})^T$. Especially the singlet σ_N will be of great importance in the analysis of Sec. [4.3]. This significance originates from the fact that a singlet scalar meson has the same quantum numbers as the vacuum, so that the σ_N-field will be allowed to acquire a non-vanishing vacuum expectation value. This fact will be used in Sec. [4.3] to model the spontaneous breakdown of chiral symmetry in the framework of the eLSM. Now, we have to identify these fields with physical particles[1]. This allows a certain scope, since there are various possibilities. For example in the isotriplet sector, there is the $a_0(980)$ meson with a mass of $m_{a_0(980)} = (980 \pm 20)$ MeV and in addition to that, there is the $a_0(1450)$ meson with a mass of $m_{a_0(1450)} = (1474 \pm 19)$ MeV. In the singlet sector, the situation is more involved, since many $0(0^{++})$ states were found by the experimentalists: The $f_0(500)$ with a mass of $m_{f_0(500)} = (400 - 550)$ MeV; the $f_0(980)$ with a mass of $m_{f_0(980)} = (990 \pm 20)$ MeV; the $f_0(1370)$ with a mass of $m_{f_0(1370)} = (1200 - 1500)$ MeV; the $f_0(1500)$ with a mass of $m_{f_0(1500)} = (1508 \pm 6)$ MeV and the $f_0(1710)$ with a mass of $m_{f_0(1710)} = 1722^{+6}_{-5}$ MeV. In addition to that, there is another ambiguity concerning the scalars, which arises from the inner structure of these resonances. To be particular, some of these resonances, like the $f_0(980)$, are sometimes considered as exotic objects like multiquark states, glueballs, or meson-meson bound states. But the discussion of these possibilities lies beyond the aim of this work, so that we, for example, refer to Ref. [Par2] for a more detailed summary of the singlet scalar mesons and to the literature for more information concerning the exotic objects. Throughout this work, we will interpret the σ_N-field as well as the $\vec{a}_0$-fields as quark/anti-quark states, so that all four fields can be combined to a four-vector in isospin space, which can be assigned to the following quadratic form

$$S^a = \sqrt{2}\bar{\Psi}(x) T_V^a \Psi(x) \ , \ a = 0, \ldots, 3 \ , \tag{4.82}$$

where the generators are given by Eqs. (6.13) and (6.15). In the following, we want to study the transformation behavior of Eqs. (4.82) with respect to infinitesimal vector- and axial-vector transformations and show that S^a really transforms like a scalar meson. To this end, we also check the transformation properties of Eq. (4.82) with respect to proper orthochronous Lorentz transformations, C-transformations, and P-transformations.

Before we are able to determine the transformation behavior of Eq. (4.82) under infinitesimal $SU(2)_V$- and $SU(2)_A$-transformations, we have to find the respective transformation properties of the quark and anti-quark fields. From Eq. (3.142), we obtain

$$\Psi(x) \stackrel{SU(2)_V}{\longrightarrow} \Psi'(x) = \left(1 - i\alpha_{V,i} T_V^i\right) \Psi(x) \ , \tag{4.83}$$

$$\Psi(x) \stackrel{SU(2)_A}{\longrightarrow} \Psi'(x) = \left(1 - i\alpha_{A,i} \gamma_5 T_V^i\right) \Psi(x) \ , . \tag{4.84}$$

Then, the transformation behavior of the anti-quark fields follows immediately

$$\bar{\Psi}(x) \stackrel{SU(2)_V}{\longrightarrow} \bar{\Psi}'(x) = \left[\left(1 - i\alpha_{V,i} T_V^i\right) \Psi(x)\right]^\dagger \gamma_0 = \bar{\Psi}(x) \left(1 + i\alpha_{V,i} T_V^i\right) \ , \tag{4.85}$$

$$\bar{\Psi}(x) \stackrel{SU(2)_A}{\longrightarrow} \bar{\Psi}'(x) = \left[\left(1 - i\alpha_{A,i} \gamma_5 T_V^i\right) \Psi(x)\right]^\dagger \gamma_0 = \bar{\Psi}(x) \left(1 - i\alpha_{A,i} \gamma_5 T_V^i\right) \ , \tag{4.86}$$

where we used Eqs. (4.83), (4.84), as well as (6.29). Using the above relations, the $a = 0$ component of Eq.

[1] Unless indicated otherwise, all masses are obtained from Ref. [PDG].

(4.82) transforms as

$$\begin{aligned} S^0(x) \xrightarrow{SU(2)_V} S^{0\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\Big(1 + i\alpha_{V,i}T_V^i\Big)T_V^0\Big(1 - i\alpha_{V,j}T_V^j\Big)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[1 - i\alpha_{V,j}T_V^j + i\alpha_{V,i}T_V^i + \mathcal{O}(\alpha_{V,i}^2)\right]\Psi(x) \\ &= S^0(x) \ , \end{aligned} \tag{4.87}$$

and

$$\begin{aligned} S^0(x) \xrightarrow{SU(2)_A} S^{0\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\Big(1 - i\alpha_{A,i}\gamma_5T_V^i\Big)T_V^0\Big(1 - i\alpha_{A,j}\gamma_5T_V^j\Big)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)T_V^0\left[1 - i\alpha_{A,j}\gamma_5T_V^j - i\alpha_{A,i}\gamma_5T_V^i + \mathcal{O}(\alpha_{A,i}^2)\right]\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)T_V^0\Psi(x) - \alpha_{A,i}\sqrt{2}\bar{\Psi}(x)i\gamma_5T_V^i\Psi(x) \\ &= S^0(x) - \vec{\alpha}_A\cdot\vec{P}(x) \ , \end{aligned} \tag{4.88}$$

where we defined the vector $\vec{P}(x) = \sqrt{2}\bar{\Psi}(x)i\gamma_5\vec{T}_V\Psi(x)$ and neglected terms of quadratic or higher order in the group parameters. In the upcoming subsection, we will see that this vector belongs to a similar four-vector like Eq. (4.82), which can be assigned to pseudoscalar fields. From Eq. (4.87), we observe that the zeroth component of Eq. (4.82) indeed transforms as an isoscalar. On the other hand, the axial-vector transformation induce a mixing of the scalar singlet and the pseudoscalar triplet. A similar result can be found for the triplet component of Eq. (4.82)

$$\begin{aligned} S^i(x) \xrightarrow{SU(2)_V} S^{i\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\Big(1 + i\alpha_{V,j}T_V^j\Big)T_V^i\Big(1 - i\alpha_{V,k}T_V^k\Big)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[T_V^i - i\alpha_{V,k}T_V^iT_V^k + i\alpha_{V,j}T_V^jT_V^i + \mathcal{O}(\alpha_{V,i}^2)\right]\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)T_V^i\Psi(x) - \sqrt{2}\bar{\Psi}(x)i\alpha_{V,j}\left[T_V^i, T_V^j\right]_-\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)T_V^i\Psi(x) + \epsilon^{ij}{}_k\alpha_{V,j}\sqrt{2}\bar{\Psi}(x)T_V^k\Psi(x) \\ &= \Big(\vec{S}(x) + \vec{\alpha}_V\times\vec{S}(x)\Big)^i \ , \end{aligned} \tag{4.89}$$

$$\begin{aligned} S^i(x) \xrightarrow{SU(2)_A} S^{i\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\Big(1 - i\alpha_{A,j}\gamma_5T_V^j\Big)T_V^i\Big(1 - i\alpha_{A,k}\gamma_5T_V^k\Big)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[T_V^i - i\alpha_{A,j}\gamma_5T_V^iT_V^k - i\alpha_{A,j}\gamma_5T_V^jT_V^i + \mathcal{O}(\alpha_{A,i}^2)\right]\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)T_V^i\Psi(x) - \sqrt{2}\bar{\Psi}(x)i\alpha_{A,j}\gamma_5\left[T_V^i, T_V^j\right]_+\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)T_V^i\Psi(x) - \alpha_A^i\sqrt{2}\bar{\Psi}(x)i\gamma_5T_V^0\Psi(x) \\ &= \Big(\vec{S}(x) - \vec{\alpha}_AP^0(x)\Big)^i \ , \end{aligned} \tag{4.90}$$

where we used Eqs. (6.7), (6.8) in combination with Eqs. (6.13) and (6.15) and defined $P^0(x) = \sqrt{2}\bar{\Psi}(x)i\gamma_5T_V^0\Psi(x)$. This object defines the zeroth component of a four-vector describing the pseudoscalar isosinglet and isotriplet fields. From Eq. (4.89) we obtain that the transformation behavior of $\vec{S}$ with respect to $SU(2)_V$ transformations looks like the infinitesimal rotation of a vector in isospin space. This result is not surprising, since the triplet was initially defined as an isovector. Similar to the singlet component of Eq. (4.82), the axial-vector transformations mix the scalar isotriplet with the pseudoscalar isosinglet.

Using Eqs. (3.86), (3.87), (3.94), (3.95) as well as Eqs. (3.101) and (3.102), it is easy to show that Eq. (4.82) transforms as a scalar object. For proper orthochronous Lorentz transformations, we find

$$S^a(x) \xrightarrow{SO^+(1,3)} S^{a\prime}(x') = \sqrt{2}\bar{\Psi}(\Lambda^{-1}x)S^{-1}(\Lambda)T_V^aS(\Lambda)\Psi(\Lambda^{-1}x) = S^a(\Lambda^{-1}x) \ . \tag{4.91}$$

For the discrete symmetry operations, we obtain

$$S^a(x) \xrightarrow{C} S^{a\prime}(x) = \sqrt{2}\Psi^t(x)CT_V^aC\bar{\Psi}^t(x) = \sqrt{2}\left(\bar{\Psi}(x)T^a\Psi(x)\right)^t = S^a(x) \ , \tag{4.92}$$

$$S^a(t,\mathbf{r}) \xrightarrow{P} S^{a\prime}(t,\mathbf{r}') = \sqrt{2}\bar{\Psi}(t,-\mathbf{r})\gamma_0T_V^a\gamma_0\Psi(t,-\mathbf{r}) = \sqrt{2}\bar{\Psi}(t,-\mathbf{r})T_V^a\Psi(t,-\mathbf{r}) = S^a(t,-\mathbf{r}) \ , \tag{4.93}$$

where we used that the ^{t} only acts on spinor space. In addition to that, the second step of Eq. (4.92) involves two sign changes. The first change of sign originates from the square of the charge conjugation operator, $CC = -1$, while the second one arises from the interchange of the two fermionic fields. The above Eqs. (4.91)-(4.93) show that Eq. (4.82) really describes a $J^{PC} = 0^{++}$ object.

4.2.1.2 Pseudoscalar Mesons

Similar to the scalar mesons of the previous subsection, the pseudoscalar mesons also carry a total angular momentum of $J = 0$. But in contrast to the scalar ones, here this total angular momentum quantum number arises from a different coupling of the angular momentum L and spin S. To be particular, the pseudoscalar mesons are particles with spin $S = 0$ and angular momentum $L = 0$. Due to Eqs. (4.78) and (4.79), the pseudoscalar mesons are $I(0)^{-+}$ particles. In accordance with the discussion of the previous subsection, the possible isospin quantum numbers are $I = 0$ and $I = 1$. The isosinglet pseudoscalar field that enters the eLSM will be denoted as η_N, while the isotriplet will be described by $\vec{\pi} = (\pi_1, \pi_2, \pi_3)^T$. Again, we have to identify these fields with physical particles. The possible $0(0^{-+})$ states are given by the η-meson which has a mass of $m_\eta = (547.862 \pm 0.018)$ MeV; the $\eta'(958)$ with a mass of $m_{\eta'(958)} = (957.78 \pm 0.06)$ MeV; the $\eta(1295)$ with a mass of $m_{\eta(1295)} = (1294 \pm 4)$ MeV; the $\eta(1405)$ with a mass of $m_{\eta(1405)} = (1408.8 \pm 1.8)$ MeV and finally the $\eta(1475)$ with a mass $m_{\eta(1475)} = (1476 \pm 4)$ Mev. In this work, we want to identify the η_N field with the lightest η-mesons. In this context, the strange/non-strange isoscalar mixing has to be taken into account, since the mixing in the pseudoscalar sector is not negligible. This implies that η_N-field contains information about the η-meson as well as the $\eta'(958)$. For further information concerning the η-η'-mixing, see Ref. [Par2] and refs. therein.

Now, we turn to the $1(0^{-+})$ states. The isovector $\vec{\pi}$ shall be identified with the pion triplet $\{\pi^0, \pi^\pm\}$[2]. Due to isospin-breaking and electromagnetic effects, the masses of the neutral and the charged pions are slightly different. While the neutral pion has a mass of $m_{\pi^0} = (134.9766 \pm 0.0006)$ MeV, the mass of the charged pions is given by $m_{\pi^\pm} = (139.57018 \pm 0.00035)$ MeV. As already mentioned several times, the three pions are considered as the (pseudo-)Nambu-Goldstone bosons of spontaneous chiral symmetry breaking. Therefore, they will play a key role in the low-energy analysis of the eLSM in Sec. [4.3]. For the sake of completeness, we want to summarize the two remaining $1(0^{-+})$ resonances which could be described by $\vec{\pi}$: The $\pi(1300)$ with a mass of $m_{\pi(1300)} = (1300 \pm 100)$ MeV and the $\pi(1800)$ with a mass of $m_{\pi(1800)} = (1812 \pm 12)$ MeV.

As already introduced in the previous subsection, the pseudoscalar fields can be assigned to a quark/anti-quark bilinear form

$$P^a(x) = \sqrt{2}\bar{\Psi}(x) i\gamma_5 T_V^a \Psi(x) \ , \tag{4.94}$$

which describes the isosinglet for $a = 0$ and the isotriplet for $a = i = 1, 2, 3$. In analogy to the scalar case, we want to derive the transformation behavior of Eq. (4.94) under vector and axial-vector transformations. Using Eqs. (4.83)-(4.86), the singlet component transforms as

$$\begin{aligned} P^0(x) \stackrel{SU(2)_V}{\longrightarrow} P^{0\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\left(1 + i\alpha_{V,i} T_V^i\right) i\gamma_5 T_V^0 \left(1 - i\alpha_{V,j} T_V^j\right)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[i\gamma_5 T_V^0 - i\gamma_5 T_V^0 i\alpha_{V,j} T_V^j + i\alpha_{V,i} T_V^i i\gamma_5 T_V^0 + \mathcal{O}(\alpha_{V,i}^2)\right]\Psi(x) \\ &= P^0(x) \ , \end{aligned} \tag{4.95}$$

and

$$\begin{aligned} P^0(x) \stackrel{SU(2)_A}{\longrightarrow} P^{0\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\left(1 - i\alpha_{A,i}\gamma_5 T_V^i\right) i\gamma_5 T_V^0 \left(1 - i\alpha_{A,j}\gamma_5 T_V^j\right)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[i\gamma_5 T_V^0 - i\gamma_5 T_V^0 i\alpha_{A,j}\gamma_5 T_V^j - i\alpha_{A,i}\gamma_5 T_V^i i\gamma_5 T_V^0 + \mathcal{O}(\alpha_{A,i}^2)\right]\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x) i\gamma_5 T_V^0 \Psi(x) + \alpha_{A,j}\sqrt{2}\bar{\Psi}(x) T_V^j \Psi(x) \\ &= P^0(x) + \vec{\alpha}_A \cdot \vec{S}(x) \ , \end{aligned} \tag{4.96}$$

where we used $\gamma_5^2 = \mathbb{1}_{4\times 4}$ as well as $\left[T_V^0, T_V^i\right]_- = 0$, for $i = 1, 2, 3$. Again, we find that the zeroth component transforms as a singlet with respect to vector transformations. Furthermore, the axial-vector transformations

[2] This formulation is a bit inaccurate, since the isovector $\vec{\pi}$ consists of neutral pseudoscalar fields. The physical pion fields can be obtained as linear combinations of the components of the isovector $\vec{\pi}$. The same statement holds for all other charged resonances.

induce a mixing of a pseudoscalar object with a scalar one. The triplet components transform according to

$$\begin{aligned}P^i(x) \overset{SU(2)_V}{\longrightarrow} P^{i\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\left(1+i\alpha_{V,j}T_V^j\right)i\gamma_5 T_V^i\left(1-i\alpha_{V,k}T_V^k\right)\Psi(x)\\ &= \sqrt{2}\bar{\Psi}(x)\left[i\gamma_5T_V^i - i\gamma_5T_V^i i\alpha_{V,k}T_V^k + i\alpha_{V,j}T_V^j i\gamma_5 T_V^i + \mathcal{O}(\alpha_{V,i}^2)\right]\Psi(x)\\ &= \sqrt{2}\bar{\Psi}(x)i\gamma_5T_V^i\Psi(x) + \alpha_{V,j}\sqrt{2}\bar{\Psi}(x)\gamma_5\left[T_V^i, T_V^j\right]_-\Psi(x)\\ &= \sqrt{2}\bar{\Psi}(x)i\gamma_5T_V^i\Psi(x) + \epsilon^{ij}{}_k\alpha_{V,j}\sqrt{2}\bar{\Psi}(x)i\gamma_5T_V^k\Psi(x)\\ &= \left(\vec{P}(x) + \vec{\alpha}_V\times\vec{P}(x)\right)^i\;,\end{aligned} \tag{4.97}$$

and

$$\begin{aligned}P^i(x) \overset{SU(2)_A}{\longrightarrow} P^{i\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\left(1-i\alpha_{A,j}\gamma_5T_V^j\right)i\gamma_5 T_V^i\left(1-i\alpha_{A,k}\gamma_5T_V^k\right)\Psi(x)\\ &= \sqrt{2}\bar{\Psi}(x)\left[i\gamma_5T_V^i - i\gamma_5T_V^i i\alpha_{A,k}\gamma_5T_V^k - i\alpha_{A,j}\gamma_5T_V^j i\gamma_5 T_V^i + \mathcal{O}(\alpha_{A,i}^2)\right]\Psi(x)\\ &= \sqrt{2}\bar{\Psi}(x)i\gamma_5T_V^i\Psi(x) + \alpha_{A,j}\sqrt{2}\bar{\Psi}(x)\left[T_V^i, T_V^j\right]_+\Psi(x)\\ &= \sqrt{2}\bar{\Psi}(x)i\gamma_5T_V^i\Psi(x) + \alpha_A^j\sqrt{2}\bar{\Psi}(x)T_V^0\Psi(x)\\ &= \left(\vec{P}(x) + \vec{\alpha}_A S^0(x)\right)^i\;,\end{aligned} \tag{4.98}$$

where we made use of the commutation and anticommutation relations of the Pauli-matrices, Eqs. (6.7), (6.8). Finally, we want turn to the space-time transformations. Using similar manipulations as in Eqs. (4.91)-(4.93), we find

$$P^a(x) \overset{SO^+(1,3)}{\longrightarrow} P^{a\prime}(x') = \sqrt{2}\bar{\Psi}(\Lambda^{-1}x)S^{-1}(\Lambda)i\gamma_5T_V^aS(\Lambda)\Psi(\Lambda^{-1}x) = P^a(\Lambda^{-1}x)\;, \tag{4.99}$$

$$P^a(x) \overset{C}{\longrightarrow} P^{a\prime}(x) = \sqrt{2}\Psi^t(x)Ci\gamma_5T_V^aC\bar{\Psi}^t(x) = \sqrt{2}\left(\bar{\Psi}(x)i\gamma_5T_V^a\Psi(x)\right)^t = P^a(x)\;, \tag{4.100}$$

$$P^a(t,\mathbf{r}) \overset{P}{\longrightarrow} P^{a\prime}(t,\mathbf{r}') = \sqrt{2}\bar{\Psi}(t,-\mathbf{r})\gamma_0 i\gamma_5T_V^a\gamma_0\Psi(t,-\mathbf{r}) = -P^a(t,-\mathbf{r})\;. \tag{4.101}$$

4.2.1.3 Vector Mesons

As already mentioned in the introduction of this subsection, the usual linear sigma models can be extended in order to include vector and axial-vector degrees of freedom. This possibility is quite important for a meaningful description of QCD in terms of a hadronic model, since it is well known that the scalar and pseudoscalar degrees of freedom not only interact among themselves, but also with the vector and axial-vector ones. In Sec. [4.3], it will be shown that these interactions also have strong influences on the low-energy properties of the eLSM.

Before we turn to the axial-vector mesons in the upcoming Subsection, we want to consider the vector mesons. These mesons are characterized as $I(1^{--})$ particles, where the total angular momentum $J=1$ is obtained by the coupling of angular momentum $L=0$ with spin $S=1$. The negative parity and charge conjugation then follows from Eqs. (4.78) and (4.79). Similar to the scalar and pseudoscalar mesons, the eLSM includes an isosinglet vector meson, denoted as ω_N^μ, as well as an isotriplet of vector mesons, which will be denoted as $\vec{\rho}^\mu$. The latter fields can be identified with the $\rho(770)$ mesons which have a mass of $m_{\rho(770)} = (775.26\pm0.25)$ MeV. Similar to the previous subsections, it is possible to find heavier vector mesons with the same quantum numbers. Those are given by the $\rho(1450)$ with a mass of $m_{\rho(1450)} = (1465\pm25)$ MeV and the $\rho(1700)$ with a mass of $m_{\rho(1700)} = (1720\pm20)$ MeV. In the isosinglet sector, the ω_N will be identified with the $\omega(782)$ which has a mass of $m_{\omega(782)} = (782.65\pm0.12)$ MeV. Other possible isosinglet states are given by the $\phi(1020)$ with a mass of $m_{\phi(1020)} = (1019.461\pm0.019)$ MeV; the $\omega(1420)$ with a mass of $m_{\omega(1420)} = (1400-1450)$ MeV; the $\omega(1650)$ with a mass of $m_{\omega(1650)} = (1670\pm30)$ MeV and the $\phi(2170)$ with a mass of $m_{\phi(2170)} = (2175\pm15)$ MeV. At this point, it has to be taken into account that we neglect the small strange/non-strange mixing in the isosinglet sector, so that the $\omega(782)$ will be considered as a purely non-strange state.

Also the vector mesons can be assigned to a quark/anti-quark quadratic form which will be defined as

$$V^{\mu a}(x) = \sqrt{2}\bar{\Psi}(x)\gamma^\mu T_V^a\Psi(x)\;, \tag{4.102}$$

where the $a = 0$ component represents the isosinglet, while the isovector is described by the $a = i = 1, 2, 3$ components. In analogy to the previous subsections, we now want to study the transformation behavior of the singlet and triplet components of Eq. (4.102) with respect to vector and axial-vector transformations. While the singlet transforms according to

$$\begin{aligned} V^{\mu 0}(x) \overset{SU(2)_V}{\longrightarrow} V^{\mu 0\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\Big(1 + i\alpha_{V,i}T_V^i\Big)\gamma^\mu T_V^0\left(1 - i\alpha_{V,j}T_V^j\right)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[\gamma^\mu T_V^0 - \gamma^\mu T_V^0 i\alpha_{V,j}T_V^j + i\alpha_{V,i}T_V^i\gamma^\mu T_V^0 + \mathcal{O}(\alpha_{V,i}^2)\right]\Psi(x) \\ &= V^{\mu 0}(x) \ , \end{aligned} \tag{4.103}$$

and

$$\begin{aligned} V^{\mu 0}(x) \overset{SU(2)_A}{\longrightarrow} V^{\mu 0\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\Big(1 - i\alpha_{A,i}\gamma_5T_V^i\Big)\gamma^\mu T_V^0\left(1 - i\alpha_{A,j}\gamma_5T_V^j\right)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[\gamma^\mu T_V^0 - \gamma^\mu T_V^0 i\alpha_{V,j}\gamma_5T_V^j - i\alpha_{A,i}\gamma_5T_V^i\gamma^\mu T_V^0 + \mathcal{O}(\alpha_{A,i}^2)\right] \\ &= V^{\mu 0}(x) \ , \end{aligned} \tag{4.104}$$

the transformation behavior of the triplet components is given by

$$\begin{aligned} V^{\mu i}(x) \overset{SU(2)_V}{\longrightarrow} V^{\mu i\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\Big(1 + i\alpha_{V,j}T_V^j\Big)\gamma^\mu T_V^i\Big(1 - i\alpha_{V,k}T_V^k\Big)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[\gamma^\mu T_V^i - i\alpha_{V,k}\gamma^\mu T_V^iT_V^k + i\alpha_{V,j}\gamma^\mu T_V^jT_V^i + \mathcal{O}(\alpha_{V,i}^2)\right]\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\gamma^\mu T_V^i\Psi(x) - i\alpha_{V,j}\sqrt{2}\bar{\Psi}(x)\gamma^\mu\left[T_V^i, T_V^j\right]_-\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\gamma^\mu T_V^i\Psi(x) - \epsilon^{ij}{}_k\alpha_{V,j}\sqrt{2}\bar{\Psi}(x)\gamma^\mu T_V^k\Psi(x) \\ &= \left(\vec{V}^\mu(x) + \vec{\alpha}_V \times \vec{V}^\mu(x)\right)^i \ , \end{aligned} \tag{4.105}$$

$$\begin{aligned} V^{\mu i}(x) \overset{SU(2)_A}{\longrightarrow} V^{\mu i\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\Big(1 - i\alpha_{A,j}\gamma_5T_V^j\Big)\gamma^\mu T_V^i\Big(1 - i\alpha_{A,k}\gamma_5T_V^k\Big)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[\gamma^\mu T_V^i - i\alpha_{A,k}\gamma^\mu T_V^i\gamma_5T_V^k - i\alpha_{A,j}\gamma_5T_V^j\gamma^\mu T_V^i + \mathcal{O}(\alpha_{A,i}^2)\right]\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\gamma^\mu T_V^i\Psi(x) - i\alpha_{A,j}\sqrt{2}\bar{\Psi}(x)\gamma^\mu\gamma_5\left[T_V^i, T_V^j\right]_-\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\gamma^\mu T_V^i\Psi(x) + \epsilon^{ij}{}_k\alpha_{A,j}\sqrt{2}\bar{\Psi}(x)\gamma^\mu\gamma_5T_V^k\Psi(x) \\ &= \left(\vec{V}^\mu(x) + \vec{\alpha}_A \times \vec{A}^\mu\right)^i \ , \end{aligned} \tag{4.106}$$

where we defined the quark/anti-quark quadratic form $\vec{A}^\mu = \sqrt{2}\bar{\Psi}(x)\gamma^\mu\gamma_5\vec{T}_V\Psi(x)$. In the upcoming subsection we will see that this object describes the triplet components of a four-vector ($A^{\mu a}$), which can be assigned to axial-vector mesons. In contrast to Eqs. (4.88) and (4.96), the axial-vector transformations do not induce a mixing of the singlet component of Eq. (4.103) with the triplet components of another bilinear form. In the triplet sector, we obtain a similar transformation behavior as in the previous cases. Finally, we have to check the space-time transformations. Using $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$, the transformation property of the vector current (4.102) is given by

$$V^{\mu a}(x) \overset{SO^+(1,3)}{\longrightarrow} V^{\mu a\prime}(x') = \sqrt{2}\bar{\Psi}(\Lambda^{-1}x)S^{-1}(\Lambda)\gamma^\mu T_V^a S(\Lambda)\Psi(\Lambda^{-1}x) = \Lambda^\mu{}_\nu V^{\nu a}(\Lambda^{-1}x) \ , \tag{4.107}$$

which shows that the vector current indeed transforms as a Lorentz vector with respect to proper orthochronous Lorentz transformations. Furthermore, using similar manipulations as in the previous Subsections, the transformation behavior under charge-conjugation and parity transformations can be obtained as

$$\begin{aligned} V^{\mu a}(x) \overset{C}{\longrightarrow} V^{\mu a\prime}(x) &= \sqrt{2}\Psi^t(x)C\gamma^\mu T_V^a C\bar{\Psi}^t(x) = \sqrt{2}\Psi^t(x)\gamma^{\mu,t}T_V^a\bar{\Psi}^t(x) = -\sqrt{2}\left(\bar{\Psi}(x)\gamma^\mu T_V^a\Psi(x)\right)^t \\ &= -V^{\mu a}(x) \ , \end{aligned} \tag{4.108}$$

$$V^{\mu a}(t,\mathbf{r}) \overset{P}{\longrightarrow} V^{\mu a\prime}(t,\mathbf{r}') = \sqrt{2}\bar{\Psi}(t,-\mathbf{r})\gamma^0\gamma^\mu T_V^a\gamma^0\Psi(t,-\mathbf{r}\ ) = (-1)^{(\mu)}V^{\mu a}(t,-\mathbf{r}) \ , \tag{4.109}$$

where we used $C^{-1} = C^\dagger = -C$ and $C\gamma^\mu C^\dagger = -\gamma^{\mu,t}$. The factor $(-1)^{(\mu)}$ was already introduced in the discussion of Eq. (3.96) and indicates that the time-like and space-like components of Eq. (4.102) transform with a different sign with respect to spatial reflections.

4.2.1.4 Axial-Vector Mesons

The last type of mesons, which is included in the eLSM, is given by the axial-vector mesons. As already indicated at the beginning of the last subsection, the interaction of scalars and pseudoscalars with axial-vector mesons is well known, so that these degrees of freedom also have to be taken into account. In the usual convention, the axial-vector mesons are classified as $I(1^{++})$ states. In contrast to the vector mesons of the last subsection, the total angular momentum of $J = 1$ of the axial-vector mesons is obtained by the coupling of the angular momentum $L = 1$ with the spin $S = 1$. Then, these angular-momentum and spin quantum numbers immediately lead to a positive parity and charge conjugation. The isosinglet state, denoted as $f^{\mu}_{1,N}$, will be identified with the $f_1(1285)$ which has a mass of $m_{f_1(1285)} = (1281.9 \pm 0.5)$ MeV. Like in the previous subsection, we neglect the small strange/non-strange mixing of axial-vector isosinglets. Thus, $f_1(1285)$ will be regarded as a purely non-strange state. This implies that the heavier $f_1(1420)$, with a mass of $m_{f_1(1420)} = (1426.4 \pm 0.9)$ MeV, will be identified as a purely strange state. The isotriplet fields which enter the eLSM will be denoted by $\vec{a}^{\mu}_1$. These fields will be identified with the $a_1(1260)$ which has a mass of $m_{a_1(1260)} = (1230 \pm 40)$ MeV.

The triplet axial-vector current $\vec{A}^{\mu}$ which we already introduced in the previous part can also be extended to a four-vector that also involves the axial-vector isosinglet state. This four-vector is given by

$$A^{\mu a}(x) = \sqrt{2}\bar{\Psi}(x)\gamma^{\mu}\gamma_5 T^a_V \Psi(x) \ . \tag{4.110}$$

The singlet and triplet components transform in an analogous manner as those of the vector current (4.102). We obtain

$$\begin{aligned} A^{\mu 0}(x) \overset{SU(2)_V}{\longrightarrow} A^{\mu 0\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\left(1 + i\alpha_{V,i}T^i_V\right)\gamma^{\mu}\gamma_5 T^0_V \left(1 - i\alpha_{V,j}T^j_V\right)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[\gamma^{\mu}\gamma_5 T^0_V - i\alpha_{V,j}\gamma^{\mu}\gamma_5 T^0_V T^j_V + i\alpha_{V,i}T^i_V\gamma^{\mu}\gamma_5 T^0_V + \mathcal{O}(\alpha^2_{V,i})\right]\Psi(x) \\ &= A^{\mu 0}(x) \ , \end{aligned} \tag{4.111}$$

$$\begin{aligned} A^{\mu 0}(x) \overset{SU(2)_A}{\longrightarrow} A^{\mu 0\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\left(1 - i\alpha_{A,i}\gamma_5 T^i_V\right)\gamma^{\mu}\gamma_5 T^0_V \left(1 - i\alpha_{A,j}\gamma_5 T^j_V\right)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[\gamma^{\mu}\gamma_5 T^0_V - i\alpha_{A,j}\gamma^{\mu}\gamma_5 T^0_V\gamma_5 T^j - i\alpha_{A,i}\gamma_5 T^i_V\gamma^{\mu}\gamma_5 T^0_V + \mathcal{O}(\alpha^2_{A,i})\right]\Psi(x) \\ &= A^{\mu 0}(x) \ , \end{aligned} \tag{4.112}$$

and

$$\begin{aligned} A^{\mu i}(x) \overset{SU(2)_V}{\longrightarrow} A^{\mu i\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\left(1 + i\alpha_{V,j}T^j_V\right)\gamma^{\mu}\gamma_5 T^i_V\left(1 - i\alpha_{V,k}T^k_V\right)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[\gamma^{\mu}\gamma_5 T^i_V - i\alpha_{V,k}\gamma^{\mu}\gamma_5 T^i_V T^k_V + i\alpha_{V,j}T^j_V\gamma^{\mu}\gamma_5 T^i_V + \mathcal{O}(\alpha^2_{V,i})\right]\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\gamma^{\mu}\gamma_5 T^i_V\Psi(x) - i\alpha_{V,j}\sqrt{2}\bar{\Psi}(x)\gamma^{\mu}\gamma_5\left[T^i_V, T^j_V\right]_-\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\gamma^{\mu}\gamma_5 T^i_V\Psi(x) + \epsilon^{ij}{}_k\alpha_{V,j}\sqrt{2}\bar{\Psi}(x)\gamma^{\mu}\gamma_5 T^k_V\Psi(x) \\ &= \left(\vec{A}^{\mu}(x) + \vec{\alpha}_V \times \vec{A}^{\mu}(x)\right)^i \ , \end{aligned} \tag{4.113}$$

$$\begin{aligned} A^{\mu i}(x) \overset{SU(2)_A}{\longrightarrow} A^{\mu i\prime}(x) &= \sqrt{2}\bar{\Psi}(x)\left(1 - i\alpha_{A,j}\gamma_5 T^j_V\right)\gamma^{\mu}\gamma_5 T^i_V\left(1 - i\alpha_{A,k}\gamma_5 T^k_V\right)\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\left[\gamma^{\mu}\gamma_5 T^i_V - i\alpha_{A,k}\gamma^{\mu}\gamma_5 T^i_V\gamma_5 T^k_V - i\alpha_{A,j}\gamma_5 T^j_V\gamma^{\mu}\gamma_5 T^i_V + \mathcal{O}(\alpha^2_{A,i})\right]\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\gamma^{\mu}\gamma_5 T^i_V\Psi(x) - i\alpha_{A,j}\sqrt{2}\bar{\Psi}(x)\gamma^{\mu}\left[T^i_V, T^j_V\right]_-\Psi(x) \\ &= \sqrt{2}\bar{\Psi}(x)\gamma^{\mu}\gamma_5 T^i_V\Psi(x) + \epsilon^{ij}{}_k\alpha_{A,j}\sqrt{2}\bar{\Psi}(x)\gamma^{\mu}T^k_V\Psi(x) \\ &= \left(\vec{A}^{\mu}(x) + \vec{\alpha}_A \times \vec{V}^{\mu}(x)\right)^i \ . \end{aligned} \tag{4.114}$$

The transformation behavior of the axial-vector current with respect to the different space-time transformations can be obtained in a similar way as in the previous case. We find

$$A^{\mu a}(x) \overset{SO^+(1,3)}{\longrightarrow} A^{\mu a\prime}(x') = \sqrt{2}\bar{\Psi}(\Lambda^{-1}x)S^{-1}(\Lambda)\gamma^\mu\gamma_5 T_V^a S(\Lambda)\Psi(\Lambda^{-1}x) = \Lambda^\mu{}_\nu A^{\nu,a}(\Lambda^{-1}x) \ , \tag{4.115}$$

$$\begin{aligned} A^{\mu a}(x) \overset{C}{\longrightarrow} A^{\mu a\prime}(x) &= \sqrt{2}\Psi^t(x)C\gamma^\mu\gamma_5 T_V^a C\bar{\Psi}^t(x) = \sqrt{2}\Psi^t(x)\gamma^{\mu t}\gamma_5^t T_V^a \bar{\Psi}^t(x) = \sqrt{2}\left(\bar{\Psi}(x)\gamma^\mu\gamma_5 T_V^a \Psi(x)\right)^t \\ &= A^{\mu a}(x) \ , \end{aligned} \tag{4.116}$$

$$A^{\mu a}(t,\mathbf{r}) \overset{P}{\longrightarrow} A^{\mu a\prime}(t,\mathbf{r}') = \sqrt{2}\bar{\Psi}(t,-\mathbf{r})\gamma^0\gamma^\mu\gamma_5 T^a\gamma^0\Psi(t,-\mathbf{r}) = -(-1)^{(\mu)}A^{\mu,a}(t,-\mathbf{r}) \ , \tag{4.117}$$

where we again made use of the factor $(-1)^{(\mu)}$.

4.2.2 Terms of the eLSM

In the previous Section, we briefly introduced the physical content of the eLSM. In addition to that, we assigned different quark/anti-quark currents which can be used to represent the different kinds of mesons. The main focus of this section lies on the theoretical implementation of these fields into the eLSM. Therefore, we define the main building blocks of the eLSM Lagrangian and show that these objects are related to the quark/anti-quark quadratic forms of the previous section. In addition to that, we will use these objects to construct all terms of the eLSM.

4.2.2.1 The Basic Objects of the eLSM

In order to introduce the main building blocks of the eLSM, we will proceed in a similar manner as in the discussion of Sec. [3.3]. To be particular, we start with the introduction of the basic objects of the eLSM Lagrangian. Then, we will use these objects to construct the most general mesonic Lagrangian with global chiral symmetry up to order four in the fields. Since this model should describe the dynamics of strong interactions, it has to contain the basic properties of QCD. The main feature of QCD is given by its $SU(3)_C$ color gauge symmetry. This symmetry will be trivially fulfilled, since the fields that enter the Lagrangian describe mesons which are colorless objects by definition. Therefore, the fields transform as singlets under $SU(3)_C$. As already introduced in Sec. [2.3.2], the quark part of the two-flavor QCD Lagrangian possesses a global $U(2)_L \times U(2)_R$ symmetry. We already showed that this symmetry is only exact, if the quarks are considered as massless particles. In the "real" world, the different quark flavors are massive fermions, which leads to an explicit breaking of the $U(2)_L \times U(2)_R$ symmetry. Furthermore, we argued that, in addition to the explicit breaking, chiral symmetry must be spontaneously broken down to its diagonal subgroup $SU(2)_V$, compare Sec. [2.3.2]. Finally, we mentioned that the $U(1)_A$ symmetry of the classical QCD Lagrangian is not present anymore at quantum level. This so-called $U(1)_A$ anomaly arose from non-perturbative quantum effects due to instantons. In order to model the anomaly as well as the explicit symmetry breaking, we have to introduce special term structures which are able to describe these properties. The spontaneous breakdown of chiral symmetry will be obtained by a specific choice of a model parameter which realizes the potential density in its Nambu-Goldstone configuration, compare Sec. [2.2.2]. Before we discuss those term structures in detail, we have to introduce the basic building blocks of the eLSM, i.e., the mathematical objects that describe the different mesonic fields. To this end, we introduce the matrix

$$\Phi_{ij}(x) = \sqrt{2}\bar{\Psi}_{j,R}(x)\Psi_{i,L}(x) \ , \tag{4.118}$$

where the $\Psi_{L/R}(x)$ define the left- and right-handed quark and anti-quark fields. In the following, we want to show that Eq. (4.118) contains the scalar and the pseudoscalar current (4.82), (4.94) of the previous subsections. To this end, we first rewrite the above expression by using the left-handed projection operator

$$\begin{aligned} \Phi_{ij}(x) &= \sqrt{2}\bar{\Psi}_j(x)\mathcal{P}_L\mathcal{P}_L\Psi_i(x) = \sqrt{2}\bar{\Psi}_j(x)\frac{\mathbb{1}_{4\times4}-\gamma_5}{2}\Psi_i \\ &= S_{ij}(x) + iP_{ij}(x) \ , \end{aligned} \tag{4.119}$$

where we defined the scalar and pseudoscalar matrices

$$S_{ij}(x) = \frac{1}{\sqrt{2}}\bar{\Psi}_j(x)\Psi_i(x) \ , \qquad P_{ij}(x) = \frac{1}{\sqrt{2}}\bar{\Psi}_j(x)i\gamma_5\Psi_i(x) \ . \tag{4.120}$$

Since the above matrices as well as Eq. (4.118) are hermitian, we are able to expand them in the algebra of $U(2)$, so that

$$\Phi(x) = \Phi_a(x)T^a = \left[S_a(x) + iP_a(x)\right]T^a \ . \tag{4.121}$$

In order to find explicit expressions for the coefficients of the above expansion, we use Eqs. (4.119), (4.120) and multiply Eq. (4.121) from the right by T^b

$$\begin{aligned}
[S_a(x) + iP_a(x)] (T^a)_{ij} \left(T^b\right)^j{}_k &= [S_{ij}(x) + iP_{ij}(x)] \left(T^b\right)^j{}_k \\
&= \left[\frac{1}{\sqrt{2}}\bar{\Psi}_j(x)\Psi_i(x) + i\frac{1}{\sqrt{2}}\bar{\Psi}_j(x)i\gamma_5\Psi_i(x)\right] \left(T^b\right)^j{}_k \\
&= \frac{1}{\sqrt{2}}\bar{\Psi}_j(x) \left(T^b\right)^j{}_k \Psi_i(x) + i\frac{1}{\sqrt{2}}\bar{\Psi}_j(x)i\gamma_5 \left(T^b\right)^j{}_k \Psi_i(x) \; . \qquad (4.122)
\end{aligned}$$

Taking the trace of Eq. (4.122) finally yields the desired coefficients

$$S^a(x) = \sqrt{2}\bar{\Psi}(x)T^a\Psi(x) \; , \qquad P^a(x) = \sqrt{2}\bar{\Psi}(x)i\gamma_5 T^a\Psi(x) \; , \qquad (4.123)$$

which are obviously identical to Eqs. (4.82) and (4.94). The transformation behavior of Eq. (4.118) with respect to global $U(2)_L \times U(2)_R$ transformations can easily be obtained from the transformation properties of the left- and right-handed quark fields (2.149) and (2.150). We find

$$\Phi(x) \overset{U(2)_L\times U(2)_R}{\longrightarrow} \Phi'(x) = U_L\Phi(x)U_R^\dagger \; . \qquad (4.124)$$

The transformation behavior of $\Phi(x)$ with respect to $SO^+(1,3)$- as well as to CP-transformations can be obtained in the same manner as in the previous subsections. For proper orthochronous Lorentz transformations, we find

$$\begin{aligned}
\Phi_{ij}(x) \overset{SO^+(1,3)}{\longrightarrow} \Phi'_{ij}(x') &= \sqrt{2}\bar{\Psi}_j(\Lambda^{-1}x)S^{-1}(\Lambda)\mathcal{P}_L\mathcal{P}_L S(\Lambda)\Psi_i(\Lambda^{-1}x) \\
&= \Phi_{ij}(\Lambda^{-1}x) \; , \qquad (4.125)
\end{aligned}$$

where we made use of $[\gamma^\mu, \gamma_5]_+ = 0$. For the transformation properties with respect to the discrete symmetry operations, we obtain

$$\begin{aligned}
\Phi_{ij}(x) \overset{C}{\longrightarrow} \Phi'_{ij}(x) &= \sqrt{2}\Psi^t_j(x)C\mathcal{P}_L\mathcal{P}_L C\bar{\Psi}^t_i(x) \\
&= -\sqrt{2}\Psi^t_j(x)\mathcal{P}^t_L\mathcal{P}^t_L\bar{\Psi}^t_i(x) \\
&= \sqrt{2}\left[\bar{\Psi}_i(x)\mathcal{P}_L\mathcal{P}_L\Psi_j(x)\right]^t \\
&= \Phi_{ji}(x) \; , \qquad (4.126)
\end{aligned}$$

where we used $CC = -\mathbb{1}_{4\times4}$ and the fact that the interchange of two fermionic fields causes an additional change of sign. In the case of spatial reflections, we are left with

$$\begin{aligned}
\Phi_{ij}(t,\mathbf{r}) \overset{C}{\longrightarrow} \Phi'_{ij}(t,\mathbf{r}') &= \sqrt{2}\bar{\Psi}_j(t,-\mathbf{r})\gamma^0\mathcal{P}_L\mathcal{P}_L\gamma^0\Psi_i(t,-\mathbf{r}) \\
&= \sqrt{2}\bar{\Psi}_j(t,-\mathbf{r})\mathcal{P}_R\mathcal{P}_R\Psi_i(t,-\mathbf{r}) \\
&= \left(\Phi^\dagger\right)_{ij}(t,-\mathbf{r}) \; , \qquad (4.127)
\end{aligned}$$

where the last equality can easily be seen by using Eq. (4.119). As already mentioned in Secs. [4.2.1.1] and [4.2.1.2], we want to identify the quark/anti-quark currents with fields which describe physical particles. Then, using (4.121) and the discussion of the previous Subsections, Eq. (4.118) can also be written as

$$\begin{aligned}
\Phi(x) &= [S_a(x) + iP_a(x)]\, T^a \\
&= [S_0(x) + iP_0(x)]\, T^0 + [S_i(x) + iP_i(x)]\, T^i \\
&= [\sigma_N(x) + i\eta_N(x)]\, T^0 + [a_{0,i}(x) + i\pi_i(x)]\, T^i \; . \qquad (4.128)
\end{aligned}$$

Since the scalar and pseudoscalar fields are now implemented in the meson matrix $\Phi(x)$, we have to find similar objects which are able to describe the vector and axial-vector degrees of freedom. Similar to Eq. (4.118), we define the left- and the right-handed vector currents

$$L^\mu_{ij}(x) = \sqrt{2}\bar{\Psi}_{j,L}(x)\gamma^\mu\Psi_{i,L}(x) \; , \qquad (4.129)$$

$$R^\mu_{ij}(x) = \sqrt{2}\bar{\Psi}_{j,R}(x)\gamma^\mu\Psi_{i,R}(x) \; . \qquad (4.130)$$

In analogy to the discussion of $\Phi(x)$, we want to relate the above currents to the vector- and axial-vector quadratic forms of the previous Subsections, Eqs. (4.102), (4.110). To this end, we begin with Eq. (4.129) and use the left- and right-handed projection operators in order to find

$$\begin{aligned} L^\mu_{ij}(x) &= \sqrt{2}\bar{\Psi}_j(x)\mathcal{P}_R\gamma^\mu\mathcal{P}_L\Psi_i(x) = \sqrt{2}\bar{\Psi}_j(x)\gamma^\mu\frac{\mathbb{1}_{4\times4}-\gamma_5}{2}\Psi_i(x) \\ &= V^\mu_{ij}(x) - A^\mu_{ij}(x) \ , \end{aligned} \tag{4.131}$$

with

$$V^\mu_{ij}(x) = \frac{1}{\sqrt{2}}\bar{\Psi}_j(x)\gamma^\mu\Psi_i(x) \ , \qquad A^\mu_{ij}(x) = \frac{1}{\sqrt{2}}\bar{\Psi}_j(x)\gamma^\mu\gamma_5\Psi_i(x) \ . \tag{4.132}$$

Using the above definitions, the right-handed current can be written as

$$\begin{aligned} R^\mu_{ij}(x) &= \sqrt{2}\bar{\Psi}_j(x)\mathcal{P}_L\gamma^\mu\mathcal{P}_R\Psi_i(x) = \sqrt{2}\bar{\Psi}_j(x)\gamma^\mu\frac{\mathbb{1}_{4\times4}+\gamma_5}{2}\Psi_i(x) \\ &= V^\mu_{ij}(x) + A^\mu_{ij}(x) \ . \end{aligned} \tag{4.133}$$

It is quite obvious, that Eqs. (4.129) and (4.130) are also hermitian matrices. Therefore, we are able to write them in terms of $U(2)$ generators

$$L^\mu(x) = L^\mu_a(x)T^a = [V^\mu_a(x) - A^\mu_a(x)]\,T^a \ , \qquad R^\mu(x) = R^\mu_a(x)T^a = [V^\mu_a(x) + A^\mu_a(x)]\,T^a \ . \tag{4.134}$$

The coefficients of these expansions can be obtained similarly to the previous case. In order to determine the left-handed coefficient, we use Eqs. (4.134) and (4.131) and multiply the complete equation from the right by T^b

$$\begin{aligned} [V^\mu_a(x) - A^\mu_a(x)]\,(T^a)_{ij}\left(T^b\right)^j{}_k &= \left[V^\mu_{ij}(x) - A^\mu_{ij}(x)\right]\left(T^b\right)^j{}_k \\ &= \left[\frac{1}{\sqrt{2}}\bar{\Psi}_j(x)\gamma^\mu\Psi_i(x) - \frac{1}{\sqrt{2}}\bar{\Psi}_j(x)\gamma^\mu\gamma_5\Psi_i(x)\right]\left(T^b\right)^j{}_k \\ &= \frac{1}{\sqrt{2}}\bar{\Psi}_j(x)\gamma^\mu\left(T^b\right)^j{}_k\Psi_i(x) - \frac{1}{\sqrt{2}}\bar{\Psi}_j(x)\gamma^\mu\gamma_5\left(T^b\right)^j{}_k\Psi_i(x) \ . \end{aligned} \tag{4.135}$$

The trace of the above equation finally yields

$$V^{\mu a}(x) = \sqrt{2}\bar{\Psi}(x)\gamma^\mu T^a\Psi(x) \ , \ A^{\mu a}(x) = \sqrt{2}\bar{\Psi}(x)\gamma^\mu\gamma_5 T^a\Psi(x) \ . \tag{4.136}$$

But these are exactly the vector and axial-vector quadratic forms which we introduced in Secs. [4.2.1.3] and [4.2.1.4]. At this point it is clear that we would have obtained the same results, if we had used the expansion of the right-handed current instead of the left-handed one. The transformation behavior of the two currents (4.129) and (4.130) follows immediately from those of the left- and right-handed quark fields

$$L^\mu(x) \overset{U(2)_L\times U(2)_R}{\longrightarrow} L^{\mu,\prime}(x) = U_L L^\mu(x) U_L^\dagger \ , \tag{4.137}$$

$$R^\mu(x) \overset{U(2)_L\times U(2)_R}{\longrightarrow} R^{\mu,\prime}(x) = U_R R^\mu(x) U_R^\dagger \ . \tag{4.138}$$

Again, the transformation properties of these objects with respect to proper orthochronous Lorentz transformations and CP-transformations can be obtained in a similar way as in Secs. [4.2.1.3] and [4.2.1.4]. Therefore, we only present the calculation for the transformation properties of $L^\mu_{ij}(x)$ and simply quote the final results for the right-handed current $R^\mu_{ij}(x)$. In the case of $SO^+(1,3)$, we obtain

$$\begin{aligned} L^\mu_{ij}(x) \overset{SO^+(1,3)}{\longrightarrow} L^{\mu\prime}_{ij}(x') &= \sqrt{2}\bar{\Psi}_j(\Lambda^{-1}x)S^{-1}(\Lambda)\mathcal{P}_R\gamma^\mu\mathcal{P}_L S(\Lambda)\Psi_i(\Lambda^{-1}x) \\ &= \sqrt{2}\bar{\Psi}_j(\Lambda^{-1}x)\mathcal{P}_R S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\mathcal{P}_L\Psi_i(\Lambda^{-1}x) \\ &= \Lambda^\mu{}_\nu L^\nu_{ij}(\Lambda^{-1}x) \ , \end{aligned} \tag{4.139}$$

and similarly

$$R^\mu_{ij}(x) \overset{SO^+(1,3)}{\longrightarrow} R^{\mu\prime}_{ij}(x') = \Lambda^\mu{}_\nu R^\nu_{ij}(\Lambda^{-1}x) \ . \tag{4.140}$$

For charge-conjugation transformations, we find

$$\begin{aligned} L^\mu_{ij}(x) \overset{C}{\longrightarrow} L^{\mu\prime}_{ij}(x) &= \sqrt{2}\Psi^t_j(x)C\mathcal{P}_R\gamma^\mu\mathcal{P}_L C\bar{\Psi}^t_i(x) \\ &= \sqrt{2}\Psi^t_j(x)\mathcal{P}_R\gamma^{\mu,t}\mathcal{P}_L\bar{\Psi}^t_i(x) \\ &= -\sqrt{2}\left[\bar{\Psi}_i(x)\mathcal{P}_L\gamma^\mu\mathcal{P}_R\Psi_j(x)\right]^t \\ &= -R^\mu_{ji}(x) \end{aligned} \tag{4.141}$$

and similarly

$$R_{ij}^{\mu}(x) \xrightarrow{C} R_{ij}^{\mu\prime}(x) = -L_{ij}^{\mu}(x) \ . \tag{4.142}$$

And finally, under spatial reflections, the left- and right-handed currents transform as

$$\begin{aligned} L_{ij}^{\mu}(t,\mathbf{r}) \xrightarrow{P} L_{ij}^{\mu,\prime}(t,\mathbf{r}') &= \sqrt{2}\bar{\Psi}_j(t,-\mathbf{r})\gamma^0\mathcal{P}_R\gamma^\mu\mathcal{P}_L\gamma^0\Psi_i(t,-\mathbf{r}) \\ &= (-1)^{(\mu)}\sqrt{2}\bar{\Psi}_j(t,-\mathbf{r})\mathcal{P}_L\gamma^\mu\mathcal{P}_R\Psi_i(t,-\mathbf{r}) \\ &= R_{\mu,ij}(t,-\mathbf{r}) \end{aligned} \tag{4.143}$$

and

$$R_{ij}^{\mu}(t,\mathbf{r}) \xrightarrow{P} R_{ij}^{\mu,\prime}(t,\mathbf{r}') = L_{\mu,ij}(t,-\mathbf{r}) \ . \tag{4.144}$$

Similar to the scalar/pseudoscalar case, we want to relate the left- and right-handed currents with vector and axial-vector fields which will describe the physical particles. Using the discussion of Secs. [4.2.1.3] and [4.2.1.4] as well as Eqs. (4.133) and (4.134), we identify

$$L^\mu(x) = \left[\omega_N^\mu(x) + f_{1,N}^\mu(x)\right]T^0 + \left[\rho_i^\mu(x) + a_{1,i}^\mu(x)\right]T^i \ , \tag{4.145}$$

$$R^\mu(x) = \left[\omega_N^\mu(x) - f_{1,N}^\mu(x)\right]T^0 + \left[\rho_i^\mu(x) - a_{1,i}^\mu(x)\right]T^i \ . \tag{4.146}$$

Now[3], with Eqs. (4.118) and (4.129), (4.130), we have the basic building blocks for the eLSM. In the following Subsections, we will use them to construct chirally invariant terms with a maximum of four fields in each term.

4.2.2.2 Kinetic and Mass Terms

In this Subsection, we want to write down the kinetic and mass terms for all types of mesonic fields. In addition to that, we also derive the explicit expressions of these terms, since we will need them in the upcoming sections.

4.2.2.2.1 Kinetic Term of Scalar and Pseudoscalar Mesons

In order to define a kinetic part for the scalar and pseudoscalar degrees of freedom, one could use the simplest possibility and use

$$\mathrm{Tr}\left\{[\partial^\mu\Phi(x)]^\dagger\,\partial_\mu\Phi(x)\right\} \tag{4.147}$$

as kinetic term. With Eq. (4.124), it can be immediately seen that this term is globally chiral invariant. But, as already mentioned at the beginning of this section, it is well known that the scalar and pseudoscalar mesons interact with the vector and axial-vector degrees of freedom. Therefore, it seems to be quite natural to introduce a covariant derivative which directly couples the different types of mesons,

$$D_\mu\Phi(x) = \partial_\mu\Phi(x) - ig_1\left[L^\mu(x)\Phi(x) - \Phi(x)R^\mu(x)\right] \ , \tag{4.148}$$

where the coupling constant g_1 has dimension one. Another advantage of the above definition is given by the fact that one can use this covariant derivative to construct a gauged linear sigma model. For details of this approach, see Ref. [Par1]. Furthermore, it is possible to extend this covariant derivative in order to include electromagnetic interactions. For details, compare Ref. [Par2]. This covariant derivative, of course transforms in the same manner as the scalar/pseudoscalar matrix $\Phi(x)$

$$\begin{aligned} D_\mu\Phi(x) \overset{U(2)_L\times U(2)_R}{\longrightarrow} [D_\mu\Phi(x)]' &= \partial_\mu\left[U_L\Phi(x)U_R^\dagger\right] - ig_1\left[U_L L_\mu(x)U_L^\dagger U_L\Phi(x)U_R^\dagger - U_L\Phi(x)U_R^\dagger U_R\Phi(x)R_\mu(x)U_R^\dagger\right] \\ &= U_L D_\mu\Phi(x)U_R^\dagger \ . \end{aligned} \tag{4.149}$$

Now, with this covariant derivative, the simplest term is given by a straightforward extension of Ref. (4.147)

$$\mathrm{Tr}\left\{[D^\mu\Phi(x)]^\dagger\,D_\mu\Phi(x)\right\} \ . \tag{4.150}$$

From Eq. (4.148) it is clear that this expression not only involves the usual kinetic terms for the scalar and pseudoscalar fields, but also contains derivative interactions of those fields with vector and axial-vector fields. This can be shown by deriving the explicit expression of Eq. (4.150). To this end, it will be advantageous to

[3]In the above equation we used the convention of Ref. [Par2] in order to define the left- and right-handed fields.

first write down the explicit expressions of the covariant derivatives. Since these expressions become quite long, we suppress the space-time dependence of the different fields from now on. Using Eqs. (4.128), (4.145), and (4.146) we find

$$\begin{aligned}
D_\mu \Phi &= \partial_\mu (\sigma_N + i\eta_N) T^0 + \partial_\mu (a_{0,i} + i\pi_i) T^i - ig_1 \left\{ \left[(\omega_{N,\mu} + f_{1N,\mu}) T^0 + (\rho_{\mu,j} + a_{1\mu,j}) T^j \right] \right. \\
&\quad \left. \times \left[(\sigma_N + i\eta_N) T^0 + (a_{0,k} + i\pi_k) T^k \right] - \left[(\sigma_N + i\eta_N) T^0 + (a_{0,j} + i\pi_j) T^j \right] \left[(\omega_{N,\mu} - f_{1N,\mu}) T^0 + (\rho_{\mu,k} - a_{1\mu,k}) T^k \right] \right\} \\
&= (\partial_\mu \sigma_N + i\partial_\mu \eta_N) T^0 + (\partial_\mu a_{0,i} + i\partial_\mu \pi_i) T^i - ig_1 \Big\{ f_{1N,\mu} \left[(\sigma_N + i\eta_N) T^0 + (a_{0,i} + i\pi_i) T^i \right] \\
&\quad + a_{1\mu,i} \left[(\sigma_N + i\eta_N) T^i + (a_{0,j} + i\pi_j) \delta^{ij} T^0 \right] + i\rho_{\mu,i} (a_{0,j} + i\pi_j) \epsilon^{ij}{}_k T^k \Big\} \ , \qquad (4.151)
\end{aligned}$$

where we used Eqs. (6.17), (6.18) and the antisymmetry of the Levi-Civita tensor. By simply taking the hermitian conjugate of the above result, we obtain

$$\begin{aligned}
\left[D^\mu \Phi \right]^\dagger &= (\partial^\mu \sigma_N - i\partial^\mu \eta_N) T^0 + (\partial^\mu a_{0,i} - i\partial^\mu \pi_i) T^i + ig_1 \Big\{ f_{1N}^\mu \left[(\sigma_N - i\eta_N) T^0 + (a_{0,i} - i\pi_i) T^i \right] \\
&\quad + a_{1,i}^\mu \left[(\sigma_N - i\eta_N) T^i + (a_{0,j} - i\pi_j) \delta^{ij} T^0 \right] - i\rho_i^\mu (a_{0,j} - i\pi_j) \epsilon^{ij}{}_k T^k \Big\} \ . \qquad (4.152)
\end{aligned}$$

In the following, we have to calculate the product of Eq. (4.151) with (4.152) and finally take the trace of the resulting expression. Since this calculation is quite tedious and not very enlightening, we only want to illustrate the calculational steps and quote the final result. First of all, it is clear that the product of the above will result in terms that are either proportional to $T^0 T^0$, to $T^i T^j$, or to $T^0 T^i$. The latter terms will vanish by taking the trace, since the trace of a Pauli matrix is equal to zero. Using the trace relations of the $U(2)$ generators, Eqs. (6.17) - (6.20), the remaining terms can be ordered by using simple algebraic operations, so that we are finally left with

$$\begin{aligned}
&\mathrm{Tr} \left\{ \left[D^\mu \Phi(x) \right]^\dagger D_\mu \Phi(x) \right\} \\
&= \frac{1}{2} \left\{ \partial_\mu \sigma_N + g_1 \left(f_{1N,\mu} \eta_N + \vec{a}_{1\mu} \cdot \vec{\pi} \right) \right\}^2 + \frac{1}{2} \left\{ \partial_\mu \eta_N - g_1 \left(f_{1N,\mu} \sigma_N + \vec{a}_{1\mu} \cdot \vec{a}_0 \right) \right\}^2 \\
&\quad + \frac{1}{2} \left\{ \partial_\mu \vec{a}_0 + g_1 \left(f_{1N,\mu} \vec{\pi} + \vec{a}_{1\mu} \eta_N + \vec{\rho}_\mu \times \vec{a}_0 \right) \right\}^2 + \frac{1}{2} \left\{ \partial_\mu \vec{\pi} - g_1 \left(f_{1N,\mu} \vec{a}_0 + \vec{a}_{1\mu} \sigma_N + \vec{\pi} \times \vec{\rho}_\mu \right) \right\}^2 \ . \qquad (4.153)
\end{aligned}$$

4.2.2.2.2 Kinetic Term of Vector and Axial-Vector Mesons

In order to write down a kinetic term for the left- and right-handed currents, Eqs. (4.129) and (4.130), we have to define left- and right-handed field-strength tensors. In the case of global chiral symmetry, these field-strength tensors are given by

$$L_{\mu\nu}(x) = \partial_\mu L_\nu(x) - \partial_\nu L_\mu(x) \ , \qquad (4.154)$$
$$R_{\mu\nu}(x) = \partial_\mu R_\nu(x) - \partial_\nu R_\mu(x) \ . \qquad (4.155)$$

In the case of local chiral symmetry, these field-strength tensors have to be extended by a third term which is proportional to the commutator of two left- or right-handed fields, see Ref. [Par1]. On the other hand, if a global chiral symmetry is sufficient, but we want study electromagnetic interactions, the field-strength tensors have to be extended in a completely different way, compare Ref. [Par2]. The transformation behavior of the above field strength tensors with respect to global chiral rotations follows immediately from that of the left- and right-handed currents

$$L_{\mu\nu}(x) \stackrel{U(2)_L \times U(2)_R}{\longrightarrow} L'_{\mu\nu}(x) = U_L L_{\mu\nu}(x) U_L^\dagger \ , \qquad (4.156)$$
$$R_{\mu\nu}(x) \stackrel{U(2)_L \times U(2)_R}{\longrightarrow} R'_{\mu\nu}(x) = U_R R_{\mu\nu}(x) U_R^\dagger \ . \qquad (4.157)$$

The kinetic term of the vector and axial-vector fields then has the usual form, i.e.,

$$-\frac{1}{4} \mathrm{Tr} \left\{ L^{\mu\nu}(x) L_{\mu\nu}(x) + R^{\mu\nu}(x) R_{\mu\nu}(x) \right\} \ . \qquad (4.158)$$

In order to derive the explicit expression of the above kinetic term, it will be advantageous to first write down the explicit expressions for the square of the left- and right-handed field-strength tensors. Using the antisymmetry of the field-strength tensors as well as Eqs. (4.145) and (4.146), we find

$$\begin{aligned}
L^{\mu\nu} L_{\mu\nu} &= 2 \left(\partial^\mu L^\nu \right) \left(\partial_\mu L_\nu - \partial_\nu L_\mu \right) \\
&= 2 \left\{ \left(\partial^\mu \omega_N^\nu + \partial^\mu f_{1N}^\nu \right) \left(\partial_\mu \omega_{N,\nu} + \partial_\mu f_{1N,\nu} \right) T^0 T^0 - \left(\partial^\mu \omega_N^\nu + \partial^\mu f_{1N}^\nu \right) \left(\partial_\nu \omega_{N,\mu} + \partial_\nu f_{1N,\mu} \right) T^0 T^0 \right. \\
&\quad \left. + \left(\partial^\mu \rho_i^\nu + \partial^\mu a_{1,i}^\nu \right) \left(\partial_\mu \rho_{\nu,j} + \partial_\mu a_{1\nu,i} \right) T^i T^j - \left(\partial^\mu \rho_i^\nu + \partial^\mu a_{1,i}^\nu \right) \left(\partial_\nu \rho_{\mu,j} + \partial_\nu a_{1\mu,i} \right) T^i T^j + \text{terms} \propto T^0 T^i \right\} \ , \\
&\qquad (4.159)
\end{aligned}$$

where we again suppressed the space-time dependence of the fields. In addition to that, we neglected the terms which are proportional to only one Pauli matrix since these terms vanish by taking the trace. For the right-handed fields, we analogously obtain

$$\begin{aligned} R^{\mu\nu}R_{\mu\nu} &= 2\,(\partial^\mu R^\nu)\,(\partial_\mu R_\nu - \partial_\nu R_\mu) \\ &= 2\left\{(\partial^\mu \omega_N^\nu - \partial^\mu f_{1N}^\nu)\,(\partial_\mu \omega_{N,\nu} - \partial_\mu f_{1N,\nu})\,T^0T^0 - (\partial^\mu \omega_N^\nu - \partial^\mu f_{1N}^\nu)\,(\partial_\nu \omega_{N,\mu} - \partial_\nu f_{1N,\mu})\,T^0T^0 \right. \\ &\quad \left. + \left(\partial^\mu \rho_i^\nu - \partial^\mu a_{1,i}^\nu\right)(\partial_\mu \rho_{\nu,j} - \partial_\mu a_{1\nu,i})\,T^iT^j - \left(\partial^\mu \rho_i^\nu - \partial^\mu a_{1,i}^\nu\right)(\partial_\nu \rho_{\mu,j} - \partial_\nu a_{1\mu,i})\,T^iT^j + \text{terms} \propto T^0T^i\right\} \ . \end{aligned} \tag{4.160}$$

Inserting these results into Eq. (4.158) and using Eqs. (6.17), (6.18), we obtain

$$\begin{aligned} -\frac{1}{4}\mathrm{Tr}\left\{L^{\mu\nu}(x)L_{\mu\nu}(x) + R^{\mu\nu}(x)R_{\mu\nu}(x)\right\} &= -\frac{1}{4}\Big\{(\partial_\mu \omega_{N,\nu} - \partial_\nu \omega_{N,\mu})^2 + (\partial_\mu f_{1N,\nu} - \partial_\nu f_{1N,\mu})^2 + (\partial_\mu \vec{\rho}_\nu - \partial_\nu \vec{\rho}_\mu)^2 \\ &\quad + (\partial_\mu \vec{a}_{1,\nu} - \partial_\nu \vec{a}_{1,\mu})^2\Big\} \\ &= -\frac{1}{4}\left\{\omega_N^{\mu\nu}\omega_{N,\mu\nu} + f_{1N}^{\mu\nu} f_{1N,\mu\nu} + \vec{\rho}^{\,\mu\nu}\cdot\vec{\rho}_{\mu\nu} + \vec{a}_1^{\,\mu\nu}\cdot\vec{a}_{1,\mu\nu}\right\} \ , \end{aligned} \tag{4.161}$$

where we defined the field-strength tensors of the particular fields in the last line.

4.2.2.2.3 Mass Term of Scalar and Pseudoscalar Mesons

After defining kinetic terms for both types of mesons, we now have to find mass terms for the scalar/pseudoscalar and vector/axial-vector mesons. Later, it will be shown that the tree-level masses of all mesons will have further contributions which arise from the spontaneous breakdown of chiral symmetry. The simplest term which is allowed by all symmetry constraints is given by

$$-\,m_0^2\mathrm{Tr}\left\{\Phi^\dagger(x)\Phi(x)\right\} \ , \tag{4.162}$$

where the mass parameter m_0^2 is of dimension [Energy^2]. For the moment, we will not fix the sign of this parameter. In the discussion of the next Section, we will see that a particular choice of the sign can be used to model the spontaneous breakdown of global chiral symmetry. By calculating the expression (4.162) explicitly, we find the usual mass terms for neutral scalar/pseudoscalar fields, i.e.,

$$\begin{aligned} -m_0^2\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\} &= -m_0^2\mathrm{Tr}\left\{\left[(\sigma_N - i\eta_N)\,T^0 + (a_{0,i} - i\pi_i)\,T^i\right]\left[(\sigma_N + i\eta_N)\,T^0 + (a_{0,j} + i\pi_j)\,T^j\right]\right\} \\ &= -\frac{m_0^2}{2}\mathrm{Tr}\left\{\sigma_N^2 + \eta_N^2 + \vec{a}_0^{\,2} + \vec{\pi}^2\right\} \ . \end{aligned} \tag{4.163}$$

4.2.2.2.4 Mass Term of Vector and Axial-Vector Mesons

Similar to the previous paragraph, the simplest choice of a mass term for the vector/axial-vector mesons is given by a trace which contains the square of both, the left- and the right-handed, field matrices. In contrast to the scalar/pseudoscalar case, the mass term of a massive vector field needs another sign in order to lead to the correct equations of motion. Therefore, the mass term for the vector particles is given by

$$\mathrm{Tr}\left\{\frac{m_1^2}{2}\mathbb{1}_{2\times 2}\left[L^\mu(x)L_\mu(x) + R^\mu(x)R_\mu(x)\right]\right\} \ , \tag{4.164}$$

where the mass parameter m_1^2 is a positive real number of dimension [Energy^2]. Opposite to the scalar/pseudoscalar case, we have to require that this mass parameter is larger than zero, because a negative value will, in a special case, lead to imaginary masses for the ρ- and the a_1-meson. In the upcoming section we will come back to this point.

For the moment, we return to Eq. (4.164). The reason why we incorporated the mass parameter in this unusual way is that the above mass term will be extended in Sec. [4.2.2.6], when we introduce an additional object under the trace, in order to model the explicit breaking of global chiral symmetry in the vector/axial-vector sector. The explicit form of this mass term is given by

$$\begin{aligned} \mathrm{Tr}\left\{\frac{m_1^2}{2}\mathbb{1}_{2\times 2}\left[L^\mu L_\mu + R^\mu R_\mu\right]\right\} &= \frac{m_1^2}{2}\mathrm{Tr}\left\{\left[(\omega_N^\mu + f_{1N}^\mu)\,T^0 + \left(\rho_i^\mu + a_{1,i}^\mu\right)T^i\right]\left[(\omega_{N,\mu} + f_{1N,\mu})\,T^0 + (\rho_{\mu,j} + a_{1\mu,j})\,T^j\right]\right. \\ &\quad \left. + \left[(\omega_N^\mu - f_{1N}^\mu)\,T^0 + \left(\rho_i^\mu - a_{1,i}^\mu\right)T^i\right]\left[(\omega_{N,\mu} - f_{1N,\mu})\,T^0 + (\rho_{\mu,j} - a_{1\mu,j})\,T^j\right]\right\} \\ &= \frac{m_1^2}{2}\left\{(\omega_{N,\mu})^2 + (f_{1N,\mu})^2 + (\vec{\rho}_\mu)^2 + (\vec{a}_{1,\mu})^2\right\} \end{aligned} \tag{4.165}$$

4.2.2.3 Scalar and Pseudoscalar Self-Interaction Terms

After defining the basic terms to obtain a free theory for all four types of mesons of the previous Subsection, we now have to define interaction terms[4]. To this end, we start with interactions that only involve scalar and pseudoscalar degrees of freedom.

4.2.2.3.1 First Self-Interaction Term of Scalar and Pseudoscalar Mesons

The first possible interaction term which only involves scalar and pseudoscalar fields is given by a straightforward extension of the mass term (4.162)

$$-\lambda_1 \left(\mathrm{Tr}\left\{\Phi^\dagger(x)\Phi(x)\right\}\right)^2 \ , \tag{4.166}$$

where the coupling constant λ_1 is a dimensionless parameter. This coupling also has to satisfy a special constraint, in order to obtain a bounded potential. However, it will be more meaningful to fix this constraint in the next section, since there are many more possible interaction terms which contribute to the potential density of the eLSM. Using Eq. (4.163), we immediately obtain

$$-\lambda_1 \left(\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}\right)^2 = -\frac{\lambda_1}{4}\left\{\sigma_N^4 + \eta_N^4 + \left(\vec{a}_0^2\right)^2 + \left(\vec{\pi}^2\right)^2 + 2\sigma_N^2\eta_N^2 + 2\sigma_N^2\vec{a}_0^2 + 2\sigma_N^2\vec{\pi}^2 + 2\eta_N^2\vec{a}_0^2 + 2\eta_N^2\vec{\pi}^2 + 2\vec{a}_0^2\vec{\pi}^2\right\} \tag{4.167}$$

as explicit expression for Eq. (4.166).

4.2.2.3.2 Second Self-Interaction Term of Scalar and Pseudoscalar Mesons

Another possible term structure which is consistent with all required symmetries can be obtained by first taking the square of $\Phi^\dagger\Phi$ and then calculating the trace, i.e.,

$$-\lambda_2 \mathrm{Tr}\left\{\left[\Phi^\dagger(x)\Phi(x)\right]^2\right\} \ , \tag{4.168}$$

where λ_2 defines a positive coupling constant with energy dimension one. The explicit form of the above interaction term is given by

$$\begin{aligned}
-\lambda_2 \mathrm{Tr}\left\{\left[\Phi^\dagger\Phi\right]^2\right\} &= -\lambda_2 \mathrm{Tr}\left\{\left[\left(\sigma_N - i\eta_N\right)T^0 + \left(a_{0,i} - i\pi_i\right)T^i\right]\left[\left(\sigma_N + i\eta_N\right)T^0 + \left(a_{0,j} + i\pi_j\right)T^j\right]\right.\\
&\quad \left.\times \left[\left(\sigma_N - i\eta_N\right)T^0 + \left(a_{0,k} - i\pi_k\right)T^k\right]\left[\left(\sigma_N + i\eta_N\right)T^0 + \left(a_{0,l} + i\pi_l\right)T^l\right]\right\}\\
&= -\frac{\lambda_2}{8}\left\{\sigma_N^4 + \eta_N^4 + \left(\vec{a}_0^2\right)^2 + \left(\vec{\pi}^2\right)^2 + 2\sigma_N^2\eta_N^2 + 2\sigma_N^2\vec{\pi}^2 + 2\eta_N^2\vec{a}_0^2 - 4\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right.\\
&\quad \left. + 6\sigma_N^2\vec{a}_0^2 + 6\eta_N^2\vec{\pi}^2 + 6\vec{a}_0^2\vec{\pi}^2 + 8\sigma_N\eta_N\left(\vec{a}_0\cdot\vec{\pi}\right)\right\} \ ,
\end{aligned} \tag{4.169}$$

where we used Eqs. (6.17)-(6.20) and the antisymmetry of the Levi-Civita tensor.

4.2.2.4 Vector and Axial-Vector Self-Interaction Terms

After introducing the self-interaction terms for the scalar and pseudoscalar degrees of freedom, we now turn to interaction terms which only involve vector particles.

4.2.2.4.1 Derivatively Coupled Self-Interaction Term of Vector and Axial-Vector Mesons

In contrast to the scalar and pseudoscalar mesons, we now have the possibility to define interaction terms which only involve three field variables. In the previous case, such a term was not possible since chiral invariance always requires powers of $\Phi^\dagger\Phi$. In the case of the left- and right-handed fields matrices (4.145), (4.146) this situation changes, due to the definition of the left- and right-handed field-strength tensors (4.154), (4.155). This fact then leads to derivatively coupled interaction terms which can be trivially seen from the definition of the field-strength tensors. On the other hand, the emergence of derivatives in interaction terms with only three vector fields is required by Lorentz symmetry. Now, the derivatively coupled vector/axial-vector interation term is given by

$$i\frac{g_2}{2}\left[\mathrm{Tr}\left\{L^{\mu\nu}(x)\left[L_\mu(x), L_\nu(x)\right]_-\right\} + \mathrm{Tr}\left\{R^{\mu\nu}(x)\left[R_\mu(x), R_\nu(x)\right]_-\right\}\right] \ , \tag{4.170}$$

[4] This statement is actually not true, since the kinetic term (4.150) also contains interactions, due to the covariant derivative (4.148).

where g_2 is a dimensionless coupling constant. In a moment it will be shown that the i-factor will be needed to compensate an additional i-factor that arises from the trace relation (6.19). In order to calculate the expression (4.170) explicitly, it will be useful to consider both traces separately. Using the antisymmetry of the left-handed field-strength tensor we find

$$\begin{aligned}
i\frac{g_2}{2}\mathrm{Tr}\Big\{L^{\mu\nu}\,[L_\mu,L_\nu]_-\Big\} &= ig_2\mathrm{Tr}\Big\{(\partial^\mu L^\nu)\,[L_\mu,L_\nu]_-\Big\}\\
&= ig_2\mathrm{Tr}\Big\{\left[(\partial^\mu\omega_N^\nu+\partial^\mu f_{1N}^\nu)\,T^0+\left(\partial^\mu\rho_i^\nu+\partial^\mu a_{1,i}^\nu\right)T^i\right]\left\{\left[(\omega_{N,\mu}+f_{1N,\mu})\,T^0+(\rho_{\mu,j}+a_{1\mu,j})\,T^j\right]\right.\\
&\quad\times\left[(\omega_{N,\nu}+f_{1N,\nu})\,T^0+(\rho_{\nu,k}+a_{1\nu,k})\,T^j\right]-\left[(\omega_{N,\nu}+f_{1N,\nu})\,T^0+(\rho_{\nu,j}+a_{1\nu,j})\,T^j\right]\\
&\quad\left.\times\left[(\omega_{N,\mu}+f_{1N,\mu})\,T^0+(\rho_{\mu,k}+a_{1\mu,k})\,T^k\right]\right\}\Big\}\\
&= \frac{g_2}{2}\left\{(\partial^\mu\vec\rho^\nu)\cdot(\vec\rho_\nu\times\vec\rho_\mu+\vec\rho_\nu\times\vec a_{1,\mu}+\vec a_{1,\nu}\times\vec\rho_\mu+\vec a_{1,\nu}\times\vec a_{1,\mu})\right.\\
&\quad\left.+\,(\partial^\mu\vec a_1^\nu)\cdot(\vec\rho_\nu\times\vec\rho_\mu+\vec\rho_\nu\times\vec a_{1,\mu}+\vec a_{1,\nu}\times\vec\rho_\mu+\vec a_{1,\nu}\times\vec a_{1,\mu})\right\}\ ,
\end{aligned}\tag{4.171}$$

where we used Eq. (6.19). In a similar way, we find

$$\begin{aligned}
i\frac{g_2}{2}\mathrm{Tr}\Big\{R^{\mu\nu}\,[R_\mu,R_\nu]_-\Big\} &= ig_2\mathrm{Tr}\Big\{(\partial^\mu R^\nu)\,[R_\mu,R_\nu]_-\Big\}\\
&= ig_2\mathrm{Tr}\Big\{\left[(\partial^\mu\omega_N^\nu-\partial^\mu f_{1N}^\nu)\,T^0+\left(\partial^\mu\rho_i^\nu-\partial^\mu a_{1,i}^\nu\right)T^i\right]\left\{\left[(\omega_{N,\mu}-f_{1N,\mu})\,T^0+(\rho_{\mu,j}-a_{1\mu,j})\,T^j\right]\right.\\
&\quad\times\left[(\omega_{N,\nu}-f_{1N,\nu})\,T^0+(\rho_{\nu,k}-a_{1\nu,k})\,T^j\right]-\left[(\omega_{N,\nu}-f_{1N,\nu})\,T^0+(\rho_{\nu,j}-a_{1\nu,j})\,T^j\right]\\
&\quad\left.\times\left[(\omega_{N,\mu}-f_{1N,\mu})\,T^0+(\rho_{\mu,k}-a_{1\mu,k})\,T^k\right]\right\}\Big\}\\
&= \frac{g_2}{2}\left\{(\partial^\mu\vec\rho^\nu)\cdot(\vec\rho_\nu\times\vec\rho_\mu-\vec\rho_\nu\times\vec a_{1,\mu}-\vec a_{1,\nu}\times\vec\rho_\mu+\vec a_{1,\nu}\times\vec a_{1,\mu})\right.\\
&\quad\left.+\,(\partial^\mu\vec a_1^\nu)\cdot(-\vec\rho_\nu\times\vec\rho_\mu+\vec\rho_\nu\times\vec a_{1,\mu}+\vec a_{1,\nu}\times\vec\rho_\mu-\vec a_{1,\nu}\times\vec a_{1,\mu})\right\}\ .
\end{aligned}\tag{4.172}$$

Combining the above results, we finally obtain

$$\begin{aligned}
&i\frac{g_2}{2}\left[\mathrm{Tr}\Big\{L^{\mu\nu}(x)\,[L_\mu(x),L_\nu(x)]_-\Big\}+\mathrm{Tr}\Big\{R^{\mu\nu}(x)\,[R_\mu(x),R_\nu(x)]_-\Big\}\right]\\
&= g_2\left\{(\partial^\mu\vec\rho^\nu)\cdot(\vec\rho_\nu\times\vec\rho_\mu+\vec a_{1,\nu}\times\vec a_{1,\mu})+(\partial^\mu\vec a_1^\nu)\cdot(\vec\rho_\nu\times\vec a_{1,\mu}+\vec a_{1,\nu}\times\vec\rho_\mu)\right\}\ .
\end{aligned}\tag{4.173}$$

4.2.2.4.2 Second Self-Interaction Term of Vector and Axial-Vector Mesons

Another chirally invariant structure can be obtained by combining four left- or right-handed field matrices. This kind of interaction then involves four vector/axial-vector fields without derivatives. A possible realization of such a term is given by

$$g_3\left[\mathrm{Tr}\left\{L^\mu(x)L^\nu(x)L_\mu(x)L_\nu(x)\right\}+\mathrm{Tr}\left\{R^\mu(x)R^\nu(x)R_\mu(x)R_\nu(x)\right\}\right]\ ,\tag{4.174}$$

with the dimensionless coupling constant g_3. At this point it has to taken into account that the order of the Lorentz indices is important, since the commutator of two left- or right-handed field matrices is different from zero. Since the interaction terms of the following three paragraphs have similar structures like Eq. (4.174), it will be useful to analyze the explicit expressions of those interactions term by term. Then, at the end of this subsection, we collect all expressions and simplify them as much as possible. To this end, we start with the first term in Eq. (4.174)

$$\begin{aligned}
g_3\mathrm{Tr}\left\{L^\mu L^\nu L_\mu L_\nu\right\} &= g_3\mathrm{Tr}\left\{\left[(\omega_N^\mu+f_{1N}^\mu)\,T^0+\left(\rho_i^\mu+a_{1,i}^\mu\right)T^i\right]\left[(\omega_N^\nu+f_{1N}^\nu)\,T^0+\left(\rho_j^\nu+a_{1,j}^\nu\right)T^j\right]\right.\\
&\quad\left.\times\left[(\omega_{N,\mu}+f_{1N,\mu})\,T^0+(\rho_{\mu,k}+a_{1\mu,k})\,T^j\right]\left[(\omega_{N,\nu}+f_{1N,\nu})\,T^0+(\rho_{\nu,l}+a_{1\nu,l})\,T^l\right]\right\}\\
&= \frac{g_3}{8}\Big\{(\omega_{N,\mu}+f_{1N,\mu})^2\,(\omega_{N,\nu}+f_{1N,\nu})^2+4\,(\omega_N^\mu+f_{1N}^\mu)\,(\omega_N^\nu+f_{1N}^\nu)\,(\vec\rho_\mu+\vec a_{1,\mu})\cdot(\vec\rho_\nu+\vec a_{1,\nu})\\
&\quad+2\,(\omega_{N,\mu}+f_{1N,\mu})^2\,(\vec\rho_\nu+\vec a_{1,\nu})^2+2\,(\vec\rho^\mu+\vec a_1^\mu)\cdot(\vec\rho^\nu+\vec a_1^\nu)\,(\vec\rho_\mu+\vec a_{1,\mu})\cdot(\vec\rho_\nu+\vec a_{1,\nu})\\
&\quad-(\vec\rho_\mu+\vec a_{1,\mu})^2\,(\vec\rho_\nu+\vec a_{1,\nu})^2\Big\}\ ,
\end{aligned}\tag{4.175}$$

where we again made use of the antisymmetry of the Levi-Civita tensor. The explicit expression for the second part of Eq. (4.174) is obtained by using the same techniques. We find

$$
\begin{aligned}
g_3 \mathrm{Tr}\{R^\mu R^\nu R_\mu R_\nu\} &= g_3 \mathrm{Tr}\left\{\left[(\omega_N^\mu - f_{1N}^\mu)\,T^0 + \left(\rho_i^\mu - a_{1,i}^\mu\right)T^i\right]\left[(\omega_N^\nu - f_{1N}^\nu)\,T^0 + \left(\rho_j^\nu - a_{1,j}^\nu\right)T^j\right]\right. \\
&\quad \left.\times\left[(\omega_{N,\mu} - f_{1N,\mu})\,T^0 + (\rho_{\mu,k} - a_{1\mu,k})\,T^j\right]\left[(\omega_{N,\nu} - f_{1N,\nu})\,T^0 + (\rho_{\nu,l} - a_{1\nu,l})\,T^l\right]\right\} \\
&= \frac{g_3}{8}\left\{(\omega_{N,\mu} - f_{1N,\mu})^2(\omega_{N,\nu} - f_{1N,\nu})^2 + 4(\omega_N^\mu - f_{1N}^\mu)(\omega_N^\nu - f_{1N}^\nu)(\vec{\rho}_\mu - \vec{a}_{1,\mu})\cdot(\vec{\rho}_\nu - \vec{a}_{1,\nu})\right. \\
&\quad + 2(\omega_{N,\mu} - f_{1N,\mu})^2(\vec{\rho}_\nu - \vec{a}_{1,\nu})^2 + 2(\vec{\rho}^\mu - \vec{a}_1^\mu)\cdot(\vec{\rho}^\nu - \vec{a}_1^\nu)(\vec{\rho}_\mu - \vec{a}_{1,\mu})\cdot(\vec{\rho}_\nu - \vec{a}_{1,\nu}) \\
&\quad \left. - (\vec{\rho}_\mu - \vec{a}_{1,\mu})^2(\vec{\rho}_\nu - \vec{a}_{1,\nu})^2\right\} \;. \qquad (4.176)
\end{aligned}
$$

4.2.2.4.3 Third Self-Interaction Term of Vector and Axial-Vector Mesons

As already mentioned in the previous paragraph, the order of the left- and right-handed field matrices in Eq. (4.174) can be interchanged in order to obtain an independent term structure which also fulfills all symmetry constraints. Therefore, another possible four-vector/axial-vector interaction is given by

$$
g_4\left[\mathrm{Tr}\{L^\mu(x)L_\mu(x)L^\nu(x)L_\nu(x)\} + \mathrm{Tr}\{R^\mu(x)R_\mu(x)R^\nu(x)R_\nu(x)\}\right] \;. \qquad (4.177)
$$

Again, the coupling constant g_4 has energy dimension one. The explicit expression of the left-handed part can be calculated as

$$
\begin{aligned}
g_4 \mathrm{Tr}\{L^\mu L_\mu L^\nu L_\nu\} &= g_4 \mathrm{Tr}\left\{\left[(\omega_N^\mu + f_{1N}^\mu)\,T^0 + \left(\rho_i^\mu + a_{1,i}^\mu\right)T^i\right]\left[(\omega_{N,\mu} + f_{1N,\mu})\,T^0 + (\rho_{\mu,j} + a_{1\mu,j})\,T^j\right]\right. \\
&\quad \left.\times\left[(\omega_N^\nu + f_{1N}^\nu)\,T^0 + \left(\rho_k^\nu + a_{1,k}^\nu\right)T^j\right]\left[(\omega_{N,\nu} + f_{1N,\nu})\,T^0 + (\rho_{\nu,l} + a_{1\nu,l})\,T^l\right]\right\} \\
&= \frac{g_4}{8}\left\{(\omega_{N,\mu} + f_{1N,\mu})^2(\omega_{N,\nu} + f_{1N,\nu})^2 + 4(\omega_N^\mu + f_{1N}^\mu)(\omega_N^\nu + f_{1N}^\nu)(\vec{\rho}_\mu + \vec{a}_{1,\mu})\cdot(\vec{\rho}_\nu + \vec{a}_{1,\nu})\right. \\
&\quad \left. + 2(\omega_{N,\mu} + f_{1N,\mu})^2(\vec{\rho}_\nu + \vec{a}_{1,\nu})^2 + (\vec{\rho}_\mu + \vec{a}_{1,\mu})^2(\vec{\rho}_\nu + \vec{a}_{1,\nu})^2\right\} \;. \qquad (4.178)
\end{aligned}
$$

The right-handed part yields

$$
\begin{aligned}
g_4 \mathrm{Tr}\{R^\mu R_\mu R^\nu R_\nu\} &= g_4 \mathrm{Tr}\left\{\left[(\omega_N^\mu - f_{1N}^\mu)\,T^0 + \left(\rho_i^\mu - a_{1,i}^\mu\right)T^i\right]\left[(\omega_{N,\mu} - f_{1N,\mu})\,T^0 + (\rho_{\mu,j} - a_{1\mu,j})\,T^j\right]\right. \\
&\quad \left.\times\left[(\omega_N^\nu - f_{1N}^\nu)\,T^0 + \left(\rho_k^\nu - a_{1,k}^\nu\right)T^j\right]\left[(\omega_{N,\nu} - f_{1N,\nu})\,T^0 + (\rho_{\nu,l} - a_{1\nu,l})\,T^l\right]\right\} \\
&= \frac{g_4}{8}\left\{(\omega_{N,\mu} - f_{1N,\mu})^2(\omega_{N,\nu} - f_{1N,\nu})^2 + 4(\omega_N^\mu - f_{1N}^\mu)(\omega_N^\nu - f_{1N}^\nu)(\vec{\rho}_\mu - \vec{a}_{1,\mu})\cdot(\vec{\rho}_\nu - \vec{a}_{1,\nu})\right. \\
&\quad \left. + 2(\omega_{N,\mu} - f_{1N,\mu})^2(\vec{\rho}_\nu - \vec{a}_{1,\nu})^2 + (\vec{\rho}_\mu - \vec{a}_{1,\mu})^2(\vec{\rho}_\nu - \vec{a}_{1,\nu})^2\right\} \;. \qquad (4.179)
\end{aligned}
$$

4.2.2.4.4 Fourth Self-Interaction Term of Vector and Axial-Vector Mesons

Furthermore, it is possible to combine left- and right-handed field matrices in one interaction term. Since the left-handed field matrices only transform with left-handed $U(2)$ rotations, Eq. (4.137), and the right-handed fields only transform with right-handed $U(2)$ matrices, Eq. (4.138), the only way to combine them is given by

$$
g_5 \mathrm{Tr}\{L^\mu(x)L_\mu(x)\}\,\mathrm{Tr}\{R^\nu(x)R_\nu(x)\} \;, \qquad (4.180)
$$

with the dimensionless coupling constant g_5. The explicit form of the above interaction term is then given by

$$
\begin{aligned}
g_5 \mathrm{Tr}\{L^\mu L_\mu\}\,\mathrm{Tr}\{R^\nu R_\nu\} &= g_5 \mathrm{Tr}\left\{\left[(\omega_N^\mu + f_{1N}^\mu)\,T^0 + \left(\rho_i^\mu + a_{1,i}^\mu\right)T^i\right]\left[(\omega_{N,\mu} + f_{1N,\mu})\,T^0 + (\rho_{\mu,j} + a_{1\mu,j})\,T^j\right]\right\} \\
&\quad \times \mathrm{Tr}\left\{\left[(\omega_N^\nu - f_{1N}^\nu)\,T^0 + \left(\rho_i^\nu - a_{1,i}^\nu\right)T^i\right]\left[(\omega_{N,\nu} - f_{1N,\nu})\,T^0 + (\rho_{\nu,j} - a_{1\nu,j})\,T^j\right]\right\} \\
&= \frac{g_5}{4}\left\{(\omega_{N,\mu} + f_{1N,\mu})^2(\omega_{N,\nu} - f_{1N,\nu})^2 + (\vec{\rho}_\mu + \vec{a}_{1,\mu})^2(\vec{\rho}_\nu - \vec{a}_{1,\nu})^2\right. \\
&\quad \left. + (\omega_{N,\mu} + f_{1N,\mu})^2(\vec{\rho}_\nu - \vec{a}_{1,\nu})^2 + (\omega_{N,\nu} - f_{1N,\nu})^2(\vec{\rho}_\mu + \vec{a}_{1,\mu})^2\right\} \;. \qquad (4.181)
\end{aligned}
$$

4.2.2.4.5 Fifth Self-Interaction Term of Vector and Axial-Vector Mesons

The last possible interaction term which contains four vector/axial-vector fields can be contructed by combining the techniques of the last two paragraphs. To be particular, we can construct terms which contain either four left- or right-handed fields, but instead of using one trace, we use the product of two invariant traces. The resulting interaction term is then given by

$$g_6 \left[\mathrm{Tr}\left\{L^\mu(x)L_\mu(x)\right\}\mathrm{Tr}\left\{L^\nu(x)L_\nu(x)\right\}+\mathrm{Tr}\left\{R^\mu(x)R_\mu(x)\right\}\mathrm{Tr}\left\{R^\nu(x)R_\nu(x)\right\}\right] \ , \tag{4.182}$$

where we introduced the dimensionless coupling parameter g_6. The explicit form of this interaction term immediately follows from Eq. (4.181), so that

$$\begin{aligned} &g_6 \left[\mathrm{Tr}\left\{L^\mu L_\mu\right\}\mathrm{Tr}\left\{L^\nu L_\nu\right\}+\mathrm{Tr}\left\{R^\mu R_\mu\right\}\mathrm{Tr}\left\{R^\nu R_\nu\right\}\right] \\ &= \frac{g_6}{4}\Big\{\left(\omega_{N,\mu}+f_{1N,\mu}\right)^2\left(\omega_{N,\nu}+f_{1N,\nu}\right)^2+\left(\omega_{N,\mu}-f_{1N,\mu}\right)^2\left(\omega_{N,\nu}-f_{1N,\nu}\right)^2+2\left(\omega_{N,\mu}+f_{1N,\mu}\right)^2\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}\right)^2 \\ &\quad +2\left(\omega_{N,\mu}-f_{1N,\mu}\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}\right)^2+\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}\right)^2\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}\right)^2+\left(\vec{\rho}_\mu-\vec{a}_{1,\mu}\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}\right)^2\Big\} \ . \end{aligned} \tag{4.183}$$

4.2.2.4.6 Complete Interaction Part of Vector and Axial-Vector Mesons

For later purposes, we want to summarize the four vector/axial-vector interaction terms in a separate part of the Lagrangian. This also results in a simplification of these terms, since all of these interaction terms are of a similar form. Then, due to the explicit expressions (4.174), (4.177), (4.180), and (4.182), this term can be written as

$$\begin{aligned} &\mathscr{L}_{g_3,g_4,g_5,g_6} \\ &= \left(\frac{g_3}{8}+\frac{g_4}{8}+\frac{g_6}{4}\right)\Big\{\left(\omega_{N,\mu}+f_{1N,\mu}\right)^2\left(\omega_{N,\nu}+f_{1N,\nu}\right)^2+\left(\omega_{N,\mu}-f_{1N,\mu}\right)^2\left(\omega_{N,\nu}-f_{1N,\nu}\right)^2+2\left(\omega_{N,\mu}+f_{1N,\mu}\right)^2\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}\right)^2 \\ &\quad +2\left(\omega_{N,\mu}-f_{1N,\mu}\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}\right)^2\Big\} \\ &\quad +\left(\frac{g_3}{2}+\frac{g_4}{2}\right)\Big\{\left(\omega_N^\mu+f_{1N}^\mu\right)\left(\omega_N^\nu+f_{1N}^\nu\right)\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}\right)\cdot\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}\right)+\left(\omega_N^\mu-f_{1N}^\mu\right)\left(\omega_N^\nu-f_{1N}^\nu\right)\left(\vec{\rho}_\mu-\vec{a}_{1,\mu}\right)\cdot\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}\right)\Big\} \\ &\quad +\left(-\frac{g_3}{8}+\frac{g_4}{8}+\frac{g_6}{4}\right)\Big\{\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}\right)^2\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}\right)^2+\left(\vec{\rho}_\mu-\vec{a}_{1,\mu}\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}\right)^2\Big\} \\ &\quad +\frac{g_3}{4}\Big\{\left(\vec{\rho}^\mu+\vec{a}_1^\mu\right)\cdot\left(\vec{\rho}^\nu+\vec{a}_1^\nu\right)\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}\right)\cdot\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}\right)+\left(\vec{\rho}^\mu-\vec{a}_1^\mu\right)\cdot\left(\vec{\rho}^\nu-\vec{a}_1^\nu\right)\left(\vec{\rho}_\mu-\vec{a}_{1,\mu}\right)\cdot\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}\right)\Big\} \\ &\quad +\frac{g_5}{4}\Big\{\left(\omega_{N,\mu}+f_{1N,\mu}\right)^2\left(\omega_{N,\nu}-f_{1N,\nu}\right)^2+\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}\right)^2+\left(\omega_{N,\mu}+f_{1N,\mu}\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}\right)^2 \\ &\quad +\left(\omega_{N,\nu}-f_{1N,\nu}\right)^2\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}\right)^2\Big\} \ . \end{aligned} \tag{4.184}$$

4.2.2.5 Scalar/Pseudoscalar and Vector/Axial-Vector Interaction Terms

Finally, the last type of possible interactions is given by terms which contain $\Phi(x)$ and $\Phi^\dagger(x)$ as well as the left- and right-handed matrices $L^\mu(x)$, $R^\mu(x)$. These terms can again be realized as a product of two invariant traces or as a single trace. After introducing the three possible interaction terms, we want to summarize them in a separate part of the Lagrangian.

4.2.2.5.1 First Mixed-Interaction Term of Scalar, Pseudoscalar, Vector, and Axial-Vector Mesons

As already mentioned, one possibility to write down an invariant term which contains scalar and pseudoscalar as well as vector and axial-vector degrees of freedom is given by the product of two invariant traces. Since we are only interested in terms which contain four field matrices, there is only one possible term structure

$$\frac{h_1}{2}\mathrm{Tr}\left\{\Phi^\dagger(x)\Phi(x)\right\}\mathrm{Tr}\left\{L^\mu(x)L_\mu(x)+R^\mu(x)R_\mu(x)\right\} \ , \tag{4.185}$$

where we introduced the dimensionless coupling h_1. Using Eqs. (4.163) and (4.165), the explicit form of the above interaction term can be written as

$$\frac{h_1}{2}\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}\mathrm{Tr}\left\{L^\mu L_\mu+R^\mu R_\mu\right\}=\frac{h_1}{4}\left(\sigma_N^2+\eta_N^2+\vec{a}_0^2+\vec{\pi}^2\right)\left[\left(\omega_{N,\mu}\right)^2+\left(f_{1N,\mu}\right)^2+\left(\vec{\rho}_\mu\right)^2+\left(\vec{a}_{1,\mu}\right)^2\right] \ . \tag{4.186}$$

4.2.2.5.2 Second Mixed-Interaction Term of Scalar, Pseudoscalar, Vector, and Axial-Vector Mesons

Another possible structure is given by

$$h_2 \mathrm{Tr}\left\{|L^\mu(x)\Phi(x)|^2 + |\Phi(x)R^\mu(x)|^2\right\} \ , \tag{4.187}$$

where h_2 is a coupling parameter of energy dimension one. In order to find the explicit expression (4.187), we rewrite the above interaction term according to

$$\begin{aligned} h_2 \mathrm{Tr}\left\{|L^\mu(x)\Phi(x)|^2 + |\Phi(x)R^\mu(x)|^2\right\} &= h_2 \mathrm{Tr}\left\{[L^\mu(x)\Phi(x)]^\dagger L^\mu(x)\Phi(x) + [\Phi(x)R^\mu(x)]^\dagger \Phi(x)R^\mu(x)\right\} \\ &= h_2 \mathrm{Tr}\left\{\Phi(x)\Phi^\dagger(x)L^\mu(x)L_\mu(x) + \Phi^\dagger(x)\Phi(x)R^\mu(x)R_\mu(x)\right\} \ , \end{aligned} \tag{4.188}$$

where we used the cyclic property of the trace in the last line. Now, for the sake of simplicity, we consider both terms in Eq. (4.188) separately. The explicit expression of the first term is then given by

$$\begin{aligned} h_2 \mathrm{Tr}\left\{\Phi\Phi^\dagger L^\mu L_\mu\right\} &= h_2 \mathrm{Tr}\left\{\left[(\sigma_N + i\eta_N)\, T^0 + (a_{0,i} + i\pi_i)\, T^i\right]\left[(\sigma_N - i\eta_N)\, T^0 + (a_{0,j} - i\pi_j)\, T^j\right]\right. \\ &\quad \times \left.\left[(\omega_N^\mu + f_{1N}^\mu)\, T^0 + \left(\rho_k^\mu + a_{1,k}^\mu\right) T^k\right]\left[(\omega_{N,\mu} + f_{1N,\mu})\, T^0 + (\rho_{\mu,l} + a_{1\mu,l})\, T^l\right]\right\} \\ &= \frac{h_2}{8}\left\{\left[(\omega_{N,\mu} + f_{1N,\mu})^2 + (\vec{\rho}_\mu + \vec{a}_{1,\mu})^2\right]\left(\sigma_N^2 + \eta_N^2 + \vec{a}_0^2 + \vec{\pi}^2\right)\right. \\ &\quad \left. +4\,(\omega_{N,\mu} + f_{1N,\mu})\,(\vec{\rho}_\mu + \vec{a}_{1,\mu}) \cdot (\sigma_N\vec{a}_0 + \eta_N\vec{\pi} + \vec{a}_0 \times \vec{\pi})\right\} \ . \end{aligned} \tag{4.189}$$

Similarly, we find

$$\begin{aligned} h_2 \mathrm{Tr}\left\{\Phi^\dagger\Phi R^\mu R_\mu\right\} &= h_2 \mathrm{Tr}\left\{\left[(\sigma_N - i\eta_N)\, T^0 + (a_{0,i} - i\pi_i)\, T^i\right]\left[(\sigma_N + i\eta_N)\, T^0 + (a_{0,j} + i\pi_j)\, T^j\right]\right. \\ &\quad \times \left.\left[(\omega_N^\mu - f_{1N}^\mu)\, T^0 + \left(\rho_k^\mu - a_{1,k}^\mu\right) T^k\right]\left[(\omega_{N,\mu} - f_{1N,\mu})\, T^0 + (\rho_{\mu,l} - a_{1\mu,l})\, T^l\right]\right\} \\ &= \frac{h_2}{8}\left\{\left[(\omega_{N,\mu} - f_{1N,\mu})^2 + (\vec{\rho}_\mu - \vec{a}_{1,\mu})^2\right]\left(\sigma_N^2 + \eta_N^2 + \vec{a}_0^2 + \vec{\pi}^2\right)\right. \\ &\quad \left. +4\,(\omega_{N,\mu} - f_{1N,\mu})\,(\vec{\rho}_\mu - \vec{a}_{1,\mu}) \cdot (\sigma_N\vec{a}_0 + \eta_N\vec{\pi} + \vec{\pi} \times \vec{a}_0)\right\} \ . \end{aligned} \tag{4.190}$$

4.2.2.5.3 Third Mixed-Interaction Term of Scalar, Pseudoscalar, Vector, and Axial-Vector Mesons

Finally, the last invariant term structure which includes all mesonic degrees of freedom is given by

$$2h_3 \mathrm{Tr}\left\{\Phi(x)R^\mu(x)\Phi^\dagger(x)L_\mu(x)\right\} \ , \tag{4.191}$$

where h_3 defines a dimensionless coupling constant. The explicit form of the above term can be calculated as

$$\begin{aligned} 2h_3 \mathrm{Tr}\left\{\Phi R^\mu \Phi^\dagger L_\mu\right\} &= 2h_3 \mathrm{Tr}\left\{\left[(\sigma_N + i\eta_N)\, T^0 + (a_{0,i} + i\pi_i)\, T^i\right]\left[(\omega_N^\mu - f_{1N}^\mu)\, T^0 + \left(\rho_j^\mu - a_{1,j}^\mu\right) T^j\right]\right. \\ &\quad \times \left.\left[(\sigma_N - i\eta_N)\, T^0 + (a_{0,k} - i\pi_k)\, T^k\right]\left[(\omega_{N,\mu} + f_{1N,\mu})\, T^0 + (\rho_{\mu,l} + a_{1\mu,l})\, T^l\right]\right\} \\ &= \frac{h_3}{4}\left\{\left[(\omega_{N,\mu})^2 - (f_{1N,\mu})^2 + (\vec{\rho}_\mu)^2 - (\vec{a}_{1,\mu})^2\right]\left(\sigma_N^2 + \eta_N^2 + \vec{a}_0^2 + \vec{\pi}^2\right)\right. \\ &\quad + 4\,(\omega_N^\mu\vec{\rho}_\mu - f_{1N}^\mu\vec{a}_{1,\mu}) \cdot (\sigma_N\vec{a}_0 + \eta_N\vec{\pi}) + 4\,(\omega_N^\mu\vec{a}_{1,\mu} - f_{1N}^\mu\vec{\rho}_\mu) \cdot (\vec{a}_0 \times \vec{\pi}) \\ &\quad \left. + 4\,(\sigma_N\vec{\pi} - \eta_N\vec{a}_0) \cdot (\vec{a}_1^\mu \times \vec{\rho}_\mu) + 2\left[(\vec{a}_0 \times \vec{a}_{1,\mu})^2 + (\vec{\pi} \times \vec{a}_{1,\mu})^2 - (\vec{a}_0 \times \vec{\rho}_\mu)^2 - (\vec{\pi} \times \vec{\rho}_\mu)^2\right]\right\} \ , \end{aligned} \tag{4.192}$$

where we made use of the well-known Lagrange identity

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = \left(\vec{a} \cdot \vec{c}\right)\left(\vec{b} \cdot \vec{d}\right) - \left(\vec{b} \cdot \vec{c}\right)\left(\vec{a} \cdot \vec{d}\right) \ . \tag{4.193}$$

4.2.2.5.4 Complete Mixed-Interaction Part of Scalar, Pseudoscalar, Vector, and Axial-Vector Mesons

Again, for later purposes, it will be useful to summarize the three mixed-interaction terms of the previous paragraphs. Using the results (4.186), (4.189), (4.190) and (4.192), we find

$$\begin{aligned}\mathscr{L}_{h_1,h_2,h_3} = &\left(\frac{h_1}{4}+\frac{h_2}{4}\right)\left\{\left(\sigma_N^2+\eta_N^2+\vec{a}_0^2+\vec{\pi}^2\right)\left[\left(\omega_{N,\mu}\right)^2+\left(f_{1N,\mu}\right)^2+\left(\vec{\rho}_\mu\right)^2+\left(\vec{a}_{1,\mu}\right)^2\right]\right\}\\ &+\left(h_2+h_3\right)\omega_N^\mu\left[\vec{\rho}_\mu\cdot\left(\sigma_N\vec{a}_0+\eta_N\vec{\pi}\right)+\vec{a}_{1,\mu}\cdot\left(\vec{a}_0\times\vec{\pi}\right)\right]+\left(h_2-h_3\right)f_{1N}^\mu\left[\vec{a}_{1,\mu}\cdot\left(\sigma_N\vec{a}_0+\eta_N\vec{\pi}\right)+\vec{\rho}_\mu\cdot\left(\vec{a}_0\times\vec{\pi}\right)\right]\\ &+\frac{h_3}{4}\left\{\left[\left(\omega_{N,\mu}\right)^2-\left(f_{1N,\mu}\right)^2+\left(\vec{\rho}_\mu\right)^2-\left(\vec{a}_{1,\mu}\right)^2\right]\left(\sigma_N^2+\eta_N^2+\vec{a}_0^2+\vec{\pi}^2\right)\right.\\ &\left.+4\left(\sigma_N\vec{\pi}-\eta_N\vec{a}_0\right)\cdot\left(\vec{a}_1^\mu\times\vec{\rho}_\mu\right)+2\left[\left(\vec{a}_0\times\vec{a}_{1,\mu}\right)^2+\left(\vec{\pi}\times\vec{a}_{1,\mu}\right)^2-\left(\vec{a}_0\times\vec{\rho}_\mu\right)^2-\left(\vec{\pi}\times\vec{\rho}_\mu\right)^2\right]\right\}\ .\end{aligned} \tag{4.194}$$

4.2.2.6 Explicit Symmetry Breaking and Anomaly Terms

In the previous subsections, we introduced the basic terms of the eLSM. As already mentioned, these terms describe the kinetics, contributions to the tree-level masses and the different types of interactions of all four types of mesons, which are contained in the eLSM. In this section, we want to introduce another important class of terms, which is very important for a realistic description of strong hadronic processes. To be particular, the aim of this subsection is the introduction of different symmetry breaking terms. While the spontaneous breakdown of chiral symmetry can easily be modeled by a particular choice of the mass parameter m_0^2, the explicit breaking of this symmetry can only be modeled by introducing new term structures. Furthermore, we have to introduce another term in order to describe the $U(1)_A$ anomaly.

4.2.2.6.1 Explicit Symmetry Breaking in the Scalar/Pseudoscalar Sector

The explicit breaking of $SU(2)_L \times SU(2)_R$ influences the scalar/pseudoscalar as well as the vector/axial-vector sector. Therefore, we introduce an explicit symmetry breaking term for each sector. In the case of the scalar/pseudoscalar mesons, this term is of the form

$$\operatorname{Tr}\left\{H\left[\Phi^\dagger(x)+\Phi(x)\right]\right\}\ , \tag{4.195}$$

where

$$H = h_{N,0}T^0 = \operatorname{diag}\left(\frac{h_{N,0}}{2},\frac{h_{N,0}}{2}\right) \tag{4.196}$$

and $h_{N,0} \sim m_u = m_d$, so that this parameter has dimension [Energy]. By using Eq. (4.128) and its hermitian conjugate, we recognize that this term in principle tilts the potential density into the σ_N-direction

$$\operatorname{Tr}\left\{H\left[\Phi^\dagger+\Phi\right]\right\} = \operatorname{Tr}\left\{H\left[\left(\sigma_N-i\eta_N\right)T^0+\left(a_{0,i}-i\pi_i\right)T^i+\left(\sigma_N+i\eta_N\right)T^0+\left(a_{0,j}+i\pi_j\right)T^j\right]\right\} = h_{N,0}\sigma_N\ , \tag{4.197}$$

where we made use of the fact that the trace of a Pauli matrix vanishes. In general, the definition of the symmetry-breaking matrix could also be $H = h_{N,a}T^a$. But, as already indicated in Sec.[4.2.1.1], the σ_N field has the same quantum numbers as the vacuum. Therefore, the σ-meson is allowed to acquire a nonvanishing vacuum expectation value. The remaining indices, $a = i = 1, 2, 3$, refer to the $\vec{a}_0$-field and allow to tilt the potential also in the $\vec{a}_0$-direction. However, this field describes an isotriplet state which has isospin quantum number $I = 1$. Therefore, these fields are not allowed to have a non-vanishing vacuum expectation value, since the vacuum has isospin quantum number $I = 0$. These considerations suggest to set the parameters $h_{N,i}, i = 1, 2, 3$ to zero, so that we are finally left with Eq. (4.196).

4.2.2.6.2 Explicit Symmetry Breaking in the Vector/Axial-Vector Sector

In order to model the explicit breaking of chiral symmetry in the vector and axial-vector sector, we introduce the following symmetry breaking term

$$\operatorname{Tr}\left\{\Delta\left[L^\mu(x)L_\mu(x)+R^\mu(x)R_\mu(x)\right]\right\}\ , \tag{4.198}$$

where the matrix Δ is given by

$$\Delta = 2\delta_N T^0 = \operatorname{diag}\left(\delta_N,\delta_N\right)\ , \tag{4.199}$$

with $\delta_N \sim m_u^2 = m_d^2$. Thus, the physical dimension of this parameter is $\left[\text{Energy}^2\right]$. Now, the symmetry breaking term (4.198) is of a similar form as the mass term (4.164). Therefore, we are able to summarize them into only one term which incorporates a mass contribution of the vector/axial-vector mesons as well as the explicit symmetry breaking in this sector. The resulting term is given by

$$\mathrm{Tr}\left\{\left(\frac{m_1^2}{2}+\Delta\right)\left[L^\mu(x)L_\mu(x)+R^\mu(x)R_\mu(x)\right]\right\} \ . \tag{4.200}$$

With a similar calculation as in Eq. (4.165), the explicit expression of this term can be written as

$$\mathrm{Tr}\left\{\left(\frac{m_1^2}{2}+\Delta\right)\left(L^\mu L_\mu+R^\mu R_\mu\right)\right\}=\left(\frac{m_1^2}{2}+\delta_N\right)\left[\left(\omega_{N,\mu}\right)^2+\left(f_{1N,\mu}\right)^2+\left(\vec{\rho}_\mu\right)^2+\left(\vec{a}_{1,\mu}\right)^2\right] \ . \tag{4.201}$$

4.2.2.6.3 Modelling the $U(1)_A$ Anomaly

Another form of a broken symmetry is given by a so-called anomaly. This term refers to a symmetry which is present at the classical level of theory, but not at the quantum level. In the case of QCD, the classical $U(1)_A$ symmetry does not survive quantization. As already mentioned in Sec. [2.3.2], this anomaly is due to nonperturbative quantum effects which are associated with instantons. Since the eLSM has to describe the dynamics of QCD, we also have to model this type of broken symmetry. As already shown in Ref. [tHoo], the $U(1)_A$ anomaly may be described by

$$c_1\left\{\det\left[\Phi^\dagger(x)\right]+\det\left[\Phi(x)\right]\right\} \ , \tag{4.202}$$

where the parameter c_1 has dimension $\left[\text{Energy}^2\right]$. Reference [Par2] also introduces another term structure that can be used to model the $U(1)_A$ anomaly. This work also discusses the implications and consequences of both terms. In the present work, we only consider Eq. (4.202) as anomaly term. In order to derive the explicit form of the above anomaly term, we use the fact that the i-th Pauli matrix can be written as

$$\sigma^i=\begin{pmatrix}\delta^i{}_3 & \delta^i{}_1-i\delta^i{}_2\\ \delta^i{}_1+i\delta^i{}_2 & -\delta^i{}_3\end{pmatrix} \ . \tag{4.203}$$

Then, the matrix form of (4.128) and its hermitian conjugate can easily be obtained

$$\Phi(x)=\frac{1}{2}\begin{pmatrix}\sigma_N(x)+a_{0,3}(x)+i\left[\eta_N(x)+\pi_3(x)\right] & a_{0,1}(x)+\pi_2(x)-i\left[a_{0,2}(x)-\pi_1(x)\right]\\ a_{0,1}(x)-\pi_2(x)+i\left[a_{0,2}(x)+\pi_1(x)\right] & \sigma_N(x)-a_{0,3}(x)+i\left[\eta_N(x)-\pi_3(x)\right]\end{pmatrix} \ , \tag{4.204}$$

$$\Phi^\dagger(x)=\frac{1}{2}\begin{pmatrix}\sigma_N(x)+a_{0,3}(x)-i\left[\eta_N(x)+\pi_3(x)\right] & a_{0,1}(x)-\pi_2(x)-i\left[a_{0,2}(x)+\pi_1(x)\right]\\ a_{0,1}(x)+\pi_2(x)+i\left[a_{0,2}(x)-\pi_1(x)\right] & \sigma_N(x)-a_{0,3}(x)-i\left[\eta_N(x)-\pi_3(x)\right]\end{pmatrix} \ . \tag{4.205}$$

Finally, we only have to calculate the determinants of the above matrices,

$$c_1\left\{\det\left[\Phi^\dagger\right]+\det\left[\Phi\right]\right\}=\frac{c_1}{2}\left(\sigma_N^2-\eta_N^2-\vec{a}_0^{\,2}+\vec{\pi}^2\right) \ , \tag{4.206}$$

in order to obtain an explicit expression for the anomaly term. From Eq. (4.206), we recognize that this term gives another contribution to the tree-level masses of the scalar and pseudoscalar mesons. In the next section, we will see that this contribution has an important effect on the tree-level masses of the η_N and $\vec{\pi}$ fields. To be particular, the different signs in front of those fields in Eq. (4.206) will shift their masses with respect to each other. While the positive sign decreases the tree-level pion mass, the tree-level mass of the η-meson increases due to the negative sign. Finally, we want to show that Eq. (4.202) indeed breaks the $U(1)_A$ symmetry. To this end, we remember that the group parameters of the left- and right-handed $U(2)$ transformations are connected to those of the vector and axial-vector transformations

$$\alpha_{L,a}=\frac{\alpha_{V,a}-\alpha_{A,a}}{2} \ , \qquad \alpha_{R,a}=\frac{\alpha_{V,a}+\alpha_{A,a}}{2} \ . \tag{4.207}$$

Then, the left- and right-handed chiral rotations (2.148) may be rewritten as

$$U_{L/R}=\exp\left\{-i\frac{\alpha_{V,a}\mp\alpha_{A,a}}{2}T^a\right\}=\exp\left\{-i\frac{\alpha_{V,0}\mp\alpha_{A,0}}{2}T^0\right\}\exp\left\{-i\frac{\alpha_{V,i}\mp\alpha_{A,i}}{2}T^i\right\} \ , \tag{4.208}$$

where we used that $U(2) = SU(2) \times U(1)$. Using Eq. (4.124), the transformation behavior of the anomaly term with respect to $U(2)_L \times U(2)_R$ is given by

$$\begin{aligned} c_1 \left\{ \det\left[\Phi^\dagger(x)\right] + \det\left[\Phi(x)\right] \right\} &\stackrel{U(2)_L\times U(2)_R}{\longrightarrow} c_1 \left\{ \det\left[\Phi^{\dagger\prime}(x)\right] + \det\left[\Phi'(x)\right] \right\} \\ &= c_1 \left\{ \det\left[U_R \Phi^\dagger(x) U_L^\dagger\right] + \det\left[U_L \Phi(x) U_R^\dagger\right] \right\} \\ &= c_1 \left\{ \det\left[U_R\right] \det\left[\Phi^\dagger(x)\right] \det\left[U_L^\dagger\right] + \det\left[U_L\right] \det\left[\Phi(x)\right] \det\left[U_L^\dagger\right] \right\} \\ &= c_1 \left\{ \det\left[e^{-i\alpha_{A,0}T^0} \Phi^\dagger(x)\right] + \det\left[e^{i\alpha_{A,0}T^0} \Phi(x)\right] \right\} \ , \end{aligned} \tag{4.209}$$

where we used that the left- and right-handed $SU(2)$ transformations have a unit determinant. By identifying the complex phases in Eq. (4.209) as $U(1)_A$ transformations, we finally recognize that the anomaly term breaks the $U(1)_A$ invariance of the theory.

4.2.2.7 The Full eLSM Lagrangian

At the end of this Section, we want to collect all terms of the previous Subsections in order to write down the full mesonic Lagrangian of the eLSM. Omitting all space-time dependences and using the trace notation, the full Lagrangian is given by

$$\begin{aligned} \mathscr{L}_{eLSM} = &\,\mathrm{Tr}\left\{\left[D^\mu\Phi\right]^\dagger D_\mu\Phi\right\} - m_0^2 \mathrm{Tr}\left\{\Phi^\dagger\Phi\right\} - \lambda_1 \left(\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}\right)^2 - \lambda_2 \mathrm{Tr}\left\{\left[\Phi^\dagger\Phi\right]^2\right\} \\ &- \frac{1}{4}\mathrm{Tr}\left\{L^{\mu\nu}L_{\mu\nu} + R^{\mu\nu}R_{\mu\nu}\right\} + \mathrm{Tr}\left\{\left(\frac{m_1^2}{2} + \Delta\right)\left[L^\mu L_\mu + R^\mu R_\mu\right]\right\} + \mathrm{Tr}\left\{H\left[\Phi^\dagger + \Phi\right]\right\} \\ &+ c_1\left[\det\Phi + \det\Phi^\dagger\right] + i\frac{g_2}{2}\left[\mathrm{Tr}\left\{L^{\mu\nu}\left[L_\mu, L_\nu\right]_-\right\} + \mathrm{Tr}\left\{R^{\mu\nu}\left[R_\mu, R_\nu\right]_-\right\}\right] \\ &+ \frac{h_1}{2}\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}\mathrm{Tr}\left\{L^\mu L_\mu + R^\mu R_\mu\right\} + h_2 \mathrm{Tr}\left\{\left|L^\mu\Phi\right|^2 + \left|\Phi R^\mu\right|^2\right\} + 2h_3 \mathrm{Tr}\left\{\Phi R^\mu \Phi^\dagger L_\mu\right\} \\ &+ g_3\left[\mathrm{Tr}\left\{L^\mu L^\nu L_\mu L_\nu\right\} + \mathrm{Tr}\left\{R^\mu R^\nu R_\mu R_\nu\right\}\right] + g_4\left[\mathrm{Tr}\left\{L^\mu L_\mu L^\nu L_\nu\right\} + \mathrm{Tr}\left\{R^\mu R_\mu R^\nu R_\nu\right\}\right] \\ &+ g_5 \mathrm{Tr}\left\{L^\mu L_\mu\right\}\mathrm{Tr}\left\{R^\mu R_\mu\right\} + g_6\left[\mathrm{Tr}\left\{L^\mu L_\mu\right\}\mathrm{Tr}\left\{L^\nu L_\nu\right\} + \mathrm{Tr}\left\{R^\mu R_\mu\right\}\mathrm{Tr}\left\{R^\nu R_\nu\right\}\right] \ . \end{aligned} \tag{4.210}$$

In the above form, the eLSM Lagrangian contains 13 parameters, which were already introduced in previous subsections

$$g_1, m_0^2, m_1^2, \lambda_1, \lambda_2, g_2, g_3, g_4, g_5, g_6, h_1, h_2, h_3 \ . \tag{4.211}$$

In the case of three dynamical quark flavors, the number of parameters increases to 15, since the symmetry breaking matrices H and Δ then contain an additional parameter which corresponds to the strange degree of freedom. The numerical values of all parameters have been determined in Ref. [PKWGR] by considering the decay processes of the scalar/pseudoscalar and vector/axial-vector degrees of freedom. The numerical input which we will use for the calculations of upcoming Section can be found in Sec. [4.3.6]. An important property of these parameter is given by their large-N_C behavior, because this scaling behavior allows us to draw conclusions about the quark/anti-quark nature of the states which are included in the eLSM. The large-N_C scaling of the parameters (4.211) can be determined by using the fact that a vertex of n $\bar{q}q$-mesons scales as $N_C^{1-\frac{n}{2}}$, compare Refs. [tHWC]. We obtain

$$h_{N,0} \sim N_C^{\frac{1}{2}} \ , \tag{4.212}$$

$$m_0^2, m_1^2, \delta_N \sim N_C^0 \ , \tag{4.213}$$

$$g_1, g_2 \sim N_C^{-\frac{1}{2}} \ , \tag{4.214}$$

$$\lambda_2, h_2, h_3, g_3, g_4, c_1 \sim N_C^{-1} \ , \tag{4.215}$$

$$\lambda_1, h_1, g_5, g_6 \sim N_C^{-2} \ . \tag{4.216}$$

For more details of the scaling of these parameters compare Ref. [Par2]. At the end of this section, we want to quote the explicit expression of the mesonic eLSM Lagrangian (4.210). Combining all results of Secs.

[4.2.2.2]-[4.2.2.6], we find

$$
\begin{aligned}
\mathscr{L}_{eLSM} = & \frac{1}{2}\left[\partial_\mu \sigma_N + g_1\left(f_{1N,\mu}\eta_N + \vec{a}_{1\mu}\cdot\vec{\pi}\right)\right]^2 + \frac{1}{2}\left[\partial_\mu \eta_N - g_1\left(f_{1N,\mu}\sigma_N + \vec{a}_{1\mu}\cdot\vec{a}_0\right)\right]^2 \\
& + \frac{1}{2}\left[\partial_\mu \vec{a}_0 + g_1\left(f_{1N,\mu}\vec{\pi} + \vec{a}_{1\mu}\eta_N + \vec{\rho}_\mu \times \vec{a}_0\right)\right]^2 + \frac{1}{2}\left[\partial_\mu \vec{\pi} - g_1\left(f_{1N,\mu}\vec{a}_0 + \vec{a}_{1\mu}\sigma_N + \vec{\pi}\times\vec{\rho}_\mu\right)\right]^2 \\
& + \left(\frac{c_1}{2} - \frac{m_0^2}{2}\right)\left(\sigma_N^2 + \vec{\pi}^2\right) + \left(-\frac{c_1}{2} - \frac{m_0^2}{2}\right)\left(\eta_N^2 + \vec{a}_0^2\right) - \frac{1}{4}\left(\omega_N^{\mu\nu}\omega_{N,\mu\nu} + f_{1N}^{\mu\nu} f_{1N,\mu\nu} + \vec{\rho}^{\mu\nu}\cdot\vec{\rho}_{\mu\nu}\right. \\
& \left. + \vec{a}_1^{\mu\nu}\cdot\vec{a}_{1,\mu\nu}\right) + \left(\frac{m_1^2}{2} + \delta_N\right)\left[\left(\omega_{N,\mu}\right)^2 + \left(f_{1N,\mu}\right)^2 + \left(\vec{\rho}_\mu\right)^2 + \left(\vec{a}_{1,\mu}\right)^2\right] + \left(-\frac{\lambda_1}{4} - \frac{\lambda_2}{8}\right)\left[\sigma_N^4 + \eta_N^4\right. \\
& \left. + \left(\vec{a}_0^2\right)^2 + \left(\vec{\pi}^2\right)^2 + 2\sigma_N^2\eta_N^2 + 2\sigma_N^2\vec{\pi}^2 + 2\eta_N^2\vec{a}_0^2\right] + \left(-\frac{\lambda_1}{2} - \frac{3\lambda_2}{4}\right)\left(\sigma_N^2\vec{a}_0^2 + \eta_N^2\vec{\pi}^2 + \vec{a}_0^2\vec{\pi}^2\right) \\
& - \frac{\lambda_2}{8}\left[8\sigma_N\eta_N\vec{a}_0\cdot\vec{\pi} - 4\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right] + h_{N,0}\sigma_N + g_2\left[\left(\partial^\mu\vec{\rho}^\nu\right)\cdot\left(\vec{\rho}_\nu\times\vec{\rho}_\mu + \vec{a}_{1,\nu}\times\vec{a}_{1,\mu}\right)\right. \\
& \left. + \left(\partial^\mu\vec{a}_1^\nu\right)\cdot\left(\vec{\rho}_\nu\times\vec{a}_{1,\mu} + \vec{a}_{1,\nu}\times\vec{\rho}_\mu\right)\right] + \mathscr{L}_{h_1,h_2,h_3} + \mathscr{L}_{g_3,g_4,g_5,g_6}\ ,
\end{aligned}
\tag{4.217}
$$

where we used Eqs. (4.184) and (4.194).

4.3 The Low-Energy Constants of the Extended Linear Sigma Model

In the previous section, we briefly introduced the mesonic part of the eLSM Lagrangian. Now, we want to investigate the low-energy regime of the model by calculating the LECs of the eLSM. To this end, we want to introduce the basic ideas of our approach in the upcoming Sec. [4.3.1]. But before we are able to calculate the LECs of the eLSM, we have to perform different preliminary manipulations of the Lagrangian (4.210). To be particular, we first have to model the spontaneous breakdown of chiral symmetry. Then, as a consequence of this operation, we will find that the resulting Lagrangian contains non-diagonal contributions to the kinetic part of the η-meson and the pion, which can be removed by performing a suitable shift of the axial-vector fields $f_{1N,\mu}$ and $\vec{a}_{1,\mu}$. The details of this calculation will be in the focus of Sec. [4.3.2]. Then, in Secs. [4.3.3]-[4.3.5], we want to consider different versions of the eLSM and calculate the respective LECs as functions of the model parameters (4.211). Finally, in the last Sec. [4.3.6], we use the results of Secs. [4.3.3]-[4.3.5] in order to obtain numerical values for the LECs of the eLSM.

4.3.1 The Basic Idea

As already mentioned in the introduction of this section, in this Subsection we introduce the basic idea behind the approach which we use for the determination of the LECs. To this end, we have to remember the discussion of Sec. [3.3.3] where we expanded the NLO chiral Lagrangian (3.218), (3.219) up to $\mathcal{O}(\pi^6)$ in pion fields. The resulting Lagrangian (3.234) includes all possible types of interaction terms which include exactly four pions and up to four space-time derivatives. These interaction terms are coupled by five coupling constants (3.236)-(3.239). The goal of our approach is now to bring the Lagrangian of the eLSM in the same form as Eq. (3.234). From this Lagrangian, we are then able to obtain five constants $C_{i,eLSM}$, $i = 1, \ldots 5$, that are in principle functions of the model parameters (4.211). Finally, we have to compare these constants to those obtained from ChPT. This procedure will make a statement, how well - at tree-level - the low-energy regime of QCD can be described by the eLSM.

In order to bring the eLSM Lagrangian into the desired form (3.234) we have to integrate all heavy fields out of the theory. This aim can be reached by considering the transition amplitude $\langle f, \infty | f, -\infty\rangle$, where $f = \{\sigma_N, \eta_N, \vec{a}_0, \vec{\pi}, \omega_{N,\mu}, f_{1N,\mu}, \vec{\rho}_\mu, \vec{a}_{1,\mu}\}$. Using the techniques of Sec. [2.2.1], this transition amplitude can be written as a functional integral with respect to all field variables. According to the discussion of Sec. [4.1.2], we then solve the functional integrals in order to obtain a low-energy effective description of the eLSM. In contrast to the simple toy model of Sec. [4.1], the Lagrangian of the eLSM contains a much more complicated term structure, so that, in general, the various functional integrals are coupled and not in the form of a Gaussian integral. At first sight, this does not allow an analytic approach to solve the functional integrals. But the fact that we are only interested in interaction terms which contain exactly four pion fields and not more than four space-time derivatives allows us to make three assumptions, which simplify the eLSM Lagrangian remarkably. To be particular, we assume that neglecting all interaction terms that contain

(A1) less than two $\vec{\pi}$-fields,

(A2) exactly three $\vec{\pi}$-fields, and

(A3) exactly two $\vec{\pi}$-fields and two fields which correspond to different resonances

does not influence the structure of the constants $C_{i,eLSM}$ at tree-level. It is clear that these assumptions have to be supported by physical arguments.

The first assumption affects all interaction terms that either contain one or zero $\vec{\pi}$-fields. The latter terms, of course, will not affect a four-pion interaction term, since they do not contain free pion legs. Therefore, we are able to neglect them. Then, with $H_1, H_2, H_3 \in \{\sigma_N, \eta_N, \vec{a}_0, \omega_{N,\mu}, f_{1N,\mu}, \vec{\rho}_\mu, \vec{a}_{1,\mu}\}$, a term that contains exactly one pion field can, in general, be written as $H_1 H_2 H_3 \vec{\pi}$. It is clear that the "heavy" fields H_i have to be chosen in a way that all symmetries are fulfilled and all indices are contracted. Then, we can consider the case where all three heavy fields are identical. Neglecting such a term corresponds to a saddle-point expansion around the minimum of the potential. On the other hand, this type of interaction term does not contribute to four-pion interaction terms, so we are able to neglect it. Furthermore, we have the case where two heavy fields are identical, i.e., a term of the form $H_1 H_1 H_3 \vec{\pi}$. Then, depending on the masses of H_1 and H_3, we have to consider two scenarios. In the first scenario, we assume that the mass of H_1 is larger than the mass of H_3. In this case, we first have to integrate H_1 out of the Lagrangian. It is obvious that the $H_1 H_1 H_3 \vec{\pi}$ term itself cannot contribute to a four-pion interaction, so that we have to combine two of these terms by connecting the H_1 legs in order to obtain a process which is of higher order in the respective coupling constant. This term obviously corresponds to a one-loop contribution of $H_3\pi$-scattering. If we now integrate H_1 out of the theory, the two H_1 propagators shrink together, so that the resulting interaction vertex then corresponds to contact interaction term. It is clear that the contribution of such a term is strongly suppressed, since each frozen propagator is proportional to the inverse square of the heavy particle mass. Nevertheless, if we then combine two of these interaction terms, we finally obtain a one-loop contribution to four-pion interactions. This situation is also illustrated at the level of Feynman diagrams in Figs. [4.7] and [4.8].

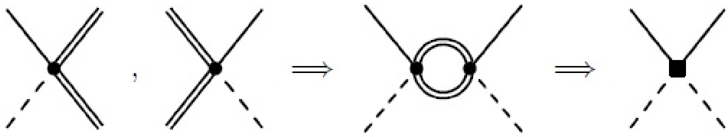


Figure 4.7: In the first step, we have two four-point interaction vertices of the type $H_1 H_1 H_3 \vec{\pi}$. Here, the double lines correspond to H_1, the ordinary line corresponds to H_3, and the dashed line describes the $\vec{\pi}$-field. In the second step, we combine both diagrams to a one-loop correction to $H_3\pi$-scattering. Finally, if we integrate out H_1, the loop freezes to a new vertex for the $H_3\pi$-scattering.

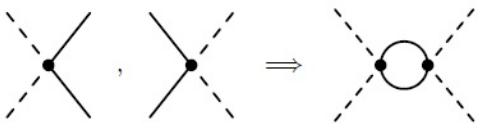


Figure 4.8: Two $H_3\pi$-scattering vertices can be combined to a one-loop contribution to $\pi\pi$-scattering.

In the second case, we assume H_3 to be the heavier resonance. When we integrate this field out of the theory, the resulting contact interaction vertex contains four H_1-fields and two $\vec{\pi}$-fields. Then, combining two of those vertices gives rise to higher-order loop contributions to four-pion interactions.

In addition to that, we have to consider the case where all heavy fields correspond to different resonances. Again, we first have to integrate the heaviest resonance out of the theory. To this end, we declare the following mass ordering $m_{H_1} > m_{H_2} > m_{H_3}$. Similar to the previous case, only one $H_1 H_2 H_3 \vec{\pi}$ term does not contribute to four-pion interactions, so that we have to consider processes of higher order in the coupling constant. If we then connect two of those vertices via the two H_1legs and integrate H_1 out of the Lagrangian, the H_1 propagator will shrink. The resulting vertex will be given by a contact interaction of two H_2, two H_3, and two π-fields. Even without integrating the remaining heavy degrees of freedom out of the theory, it is clear that this type of interaction will not have a tree-level contribution to four-pion interactions. Therefore, the first assumption seems to be verified.

Now, we consider the second assumption (A2) that effects all terms with exactly three $\vec{\pi}$-fields and an arbitrary heavy field, i.e., $H_1\vec{\pi}\vec{\pi}\vec{\pi}$ terms. Integrating the H_1-field out of the theory will result in a $6\vec{\pi}$ contact interaction term which does not contribute to tree-level four-pion interactions. This situation is depicted in Fig. [4.9].

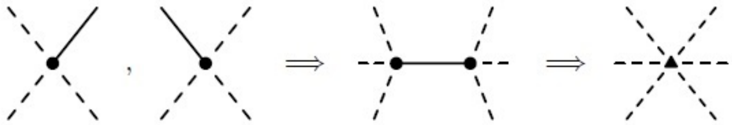


Figure 4.9: $H_1H_2H_3\vec{\pi}$-interaction terms will lead to a tree-level 6-point function.

Finally, we have to discuss the third assumption (A3). This assumption affects all terms that are of the following form $H_1H_2\vec{\pi}\vec{\pi}$, with $H_1 \neq H_2$. Again, it is clear, that these terms have to fulfill all symmetry constraints. Furthermore, we assume that H_1 corresponds to the heavier resonance, so that this field has to be integrated out of the theory first. Similar to the previous cases, we have to consider a process which is at least quadratic in the respective coupling constant. To be particular, we consider the process that can be obtained by combining two $H_1H_2\vec{\pi}\vec{\pi}$ diagrams by closing the respective H_1 lines. If we now eliminate the H_1-field from the theory, the H_1 propagator freezes, so that we obtain a contact interaction term of six fields, which also does not contribute to tree-level four-pion interactions. This situation is illustrated in Fig. [4.10].

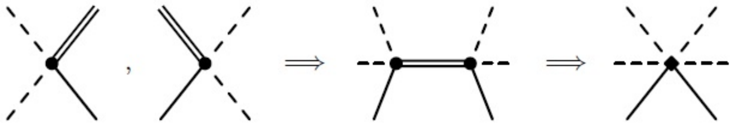


Figure 4.10: The third assumption eliminates one-loop contributions to the 8-point function of the pion.

Obviously, the three assumptions do not influence the four-pion interaction terms which we are interested in. In the upcoming sections, we will see that these assumptions simplify the Lagrangian of the eLSM in a way, so that all functional integrals of the heavy fields decouple from each other. Furthermore, all of these integrals will be of a Gaussian type, which permits an analytic solution of them. But before we are in the position to apply the simplifications (A1)-(A3), we have to perform some preliminary manipulations of the Lagrangian, which are due to the spontaneous breakdown of chiral symmetry. These manipulations will recast the Lagrangian (4.217) in a completely different form and allow new interaction terms.

4.3.2 Prerequisites

After introducing the basic ideas of our approach in the previous subsection, we now turn back to Eq. (4.217). As already mentioned in the discussion of the various terms of eLSM, we are able to model the spontaneous breakdown of chiral symmetry by choosing the sign of the mass parameter m_0^2. But this is nothing new, since in principle we did the same in the discussion of Secs. [2.2.2] and [2.2.3]. There, we choose the mass parameter m^2 in a way that the potential density of the toy models was transformed to its Nambu-Goldstone realization. This means that the potential densities are in a configuration, which allows them to have degenerate vacuum states. In the case of the eLSM, we want to proceed in the same manner. The only difference to these simple toy models arises from the much more complicated term structure of the eLSM which will lead to non-diagonal contributions to the kinetic terms of the η_N- and the $\vec{\pi}$-fields. But before we are able to see this, we have to reorder the terms of the eLSM Lagrangian. By calculating the squares of the kinetic parts of Eq. (4.217), the Lagrangian can be recast into the form

$$\mathscr{L}_{eLSM} = \mathscr{L}_{kin.} + \mathscr{L}_{deriv.} - \mathscr{V} \ . \tag{4.218}$$

In this notation, the first term only contains the canonically normalized kinetic terms of all resonances which are present in the eLSM

$$\mathscr{L}_{kin.} = \frac{1}{2}\left[(\partial_\mu\sigma_N)^2 + (\partial_\mu\eta_N)^2 + (\partial_\mu\vec{a}_0)^2 + (\partial_\mu\vec{\pi})^2\right] - \frac{1}{4}\Big(\omega_N^{\mu\nu}\omega_{N,\mu\nu} + f_{1N}^{\mu\nu}f_{1N,\mu\nu} + \vec{\rho}^{\mu\nu}\cdot\vec{\rho}_{\mu\nu} + \vec{a}_1^{\mu\nu}\cdot\vec{a}_{1,\mu\nu}\Big) \ . \tag{4.219}$$

The second and third term of Eq. (4.218) then contain the tree-level mass contributions of all fields and the different types of interactions. For later convenience, we collected all derivatively coupled interaction terms in $\mathscr{L}_{deriv.}$

$$\begin{aligned}\mathscr{L}_{deriv.} = &\ g_1\left(\partial^\mu\sigma_N\right)\left(f_{1N,\mu}\eta_N + \vec{a}_{1,\mu}\cdot\vec{\pi}\right) - g_1\left(\partial^\mu\eta_N\right)\left(f_{1N,\mu}\sigma_N + \vec{a}_{1,\mu}\cdot\vec{a}_0\right)\\ &+ g_1\left(\partial^\mu\vec{a}_0\right)\cdot\left(f_{1N,\mu}\vec{\pi} + \vec{a}_{1,\mu}\eta_N + \vec{\rho}_\mu\times\vec{a}_0\right) - g_1\left(\partial^\mu\vec{\pi}\right)\cdot\left(f_{1N,\mu}\vec{a}_0 + \vec{a}_{1,\mu}\sigma_N + \vec{\pi}\times\vec{\rho}_\mu\right)\\ &+ g_2\left[\left(\partial^\mu\vec{\rho}^\nu\right)\cdot\left(\vec{\rho}_\nu\times\vec{\rho}_\mu + \vec{a}_{1,\nu}\times\vec{a}_{1,\mu}\right) + \left(\partial^\mu\vec{a}_1^\nu\right)\cdot\left(\vec{\rho}_\nu\times\vec{a}_{1,\mu} + \vec{a}_{1,\nu}\times\vec{\rho}_\mu\right)\right]\ .\end{aligned} \tag{4.220}$$

Then, the "potential density" $\mathscr{V}$ is given by

$$\begin{aligned}\mathscr{V} = &-\left(\frac{c_1}{2} - \frac{m_0^2}{2}\right)\left(\sigma_N^2 + \vec{\pi}^2\right) - \left(-\frac{c_1}{2} - \frac{m_0^2}{2}\right)\left(\eta_N^2 + \vec{a}_0^2\right) - \left(\frac{m_1^2}{2} + \delta_N\right)\left[\left(\omega_{N,\mu}\right)^2 + \left(f_{1N,\mu}\right)^2 + \left(\vec{\rho}_\mu\right)^2 + \left(\vec{a}_{1,\mu}\right)^2\right]\\ &-\frac{g_1^2}{2}\left(f_{1N,\mu}\vec{\pi} + \vec{a}_{1,\mu}\eta_N + \vec{\rho}_\mu\times\vec{a}_0\right)^2 - \frac{g_1^2}{2}\left(f_{1N,\mu}\vec{a}_0 + \vec{a}_{1,\mu}\sigma_N + \vec{\pi}\times\vec{\rho}_\mu\right)^2 - \frac{g_1^2}{2}\left(f_{1N,\mu}\eta_N + \vec{a}_{1,\mu}\cdot\vec{\pi}\right)^2\\ &-\frac{g_1^2}{2}\left(f_{1N,\mu}\sigma_N + \vec{a}_{1,\mu}\cdot\vec{a}_0\right)^2 - \left(-\frac{\lambda_1}{4} - \frac{\lambda_2}{8}\right)\left[\sigma_N^4 + \eta_N^4 + \left(\vec{a}_0^2\right)^2 + \left(\vec{\pi}^2\right)^2 + 2\sigma_N^2\eta_N^2 + 2\sigma_N^2\vec{\pi}^2 + 2\eta_N^2\vec{a}_0^2\right]\\ &-\left(-\frac{\lambda_1}{2} - \frac{3\lambda_2}{4}\right)\left(\sigma_N^2\vec{a}_0^2 + \eta_N^2\vec{\pi}^2 + \vec{a}_0^2\vec{\pi}^2\right) + \frac{\lambda_2}{8}\left[8\sigma_N\eta_N\vec{a}_0\cdot\vec{\pi} - 4\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right] - h_{N,0}\sigma_N\\ &-\mathscr{L}_{h_1,h_2,h_3} - \mathscr{L}_{g_3,g_4,g_5,g_6}\ .\end{aligned} \tag{4.221}$$

Now, in order to realize the above potential density in its Nambu-Goldstone mode, we choose the mass parameter m_0^2 to be negative, i.e.,

$$m_0^2 \longrightarrow -m_0^2\ , \tag{4.222}$$

with $m_0^2 > 0$. With this choice, only two terms of the potential density change

$$\mathscr{V} = -\left(\frac{c_1}{2} + \frac{m_0^2}{2}\right)\left(\sigma_N^2 + \vec{\pi}^2\right) - \left(-\frac{c_1}{2} + \frac{m_0^2}{2}\right)\left(\eta_N^2 + \vec{a}_0^2\right) - \dots\ , \tag{4.223}$$

where the "..." stand for all other terms of the potential density, that remain unchanged under the replacement (4.222). At this point, it will be useful to remember the discussion of the first scalar/pseudoscalar meson self-interaction term Sec. [4.2.2.3.1]. In this paragraph, we mentioned that the coupling constant λ_1 has to fulfill a constraint in order to obtain a bounded potential density. When we now look at Eq. (4.221), this constraint can easily be read off,

$$\lambda_1 > -\frac{\lambda_2}{2}\ . \tag{4.224}$$

Without the above requirement, the last term of the third line of Eq. (4.221) would be allowed to have a positive sign, so that the quartic powers of scalar/pseudoscalar fields would lead to a potential density that is not bounded from below. With the choice (4.222), the potential density now has the usual "mexican hat"-like shape. But due to the explicit symmetry breaking term (4.197), this "mexican hat" is tilted in σ_N-direction. Therefore, we only have to consider the first derivative of $\mathscr{V}$ with respect to σ_N, in order to find the minimum of the potential. This minimum then satisfies the following condition

$$\begin{aligned}0 &\stackrel{!}{=} \left.\frac{\partial\mathscr{V}}{\partial\sigma_N}\right|_{\sigma_N=\phi_N, f=0}\\ &= -\left(m_0^2 + c_1\right)\phi_N + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^3 - h_{N,0}\ ,\end{aligned} \tag{4.225}$$

where we defined the vacuum expectation value of the σ_N-field as $\phi_N \equiv \langle\Omega|\sigma_N|\Omega\rangle$ and f represents the remaining fields. Of course, it would be possible to solve this cubic equation in order to obtain a solution for ϕ_N, but it will be shown in a moment that the above condition will be of greater importance.

The vacuum expectation value now represents the ground state of our physical system. At this point, we could start to quantize the theory around this new ground state. The first excited states would, of course, be left and the right of ϕ_N inside the "brim" of the mexican hat. But, according to the discussions of Secs. [2.2.2] and [2.2.3], the usual approach takes place at a classical level. To this end, we decompose the σ_N-field into two parts

$$\sigma_N \longrightarrow \phi_N + \sigma_N\ . \tag{4.226}$$

The first contribution is obviously given by the vacuum expectation values itself. This can be seen as the lowest order of an expansion of the initial σ_N-field around the minimum of the potential density. The second contribution is a bit confusing at first sight, since we also labeled it as σ_N. This second contribution describes a fluctuation field which parametrizes the excitations around the ground state. Therefore, this field incorporates higher orders of this expansion of the initial σ_N-field. In the following, we have to insert the decomposition (4.226) into Eq. (4.218). For the sake of clarity, we want to study in detail the changes of the Lagrangian, that are introduced by Eq. (4.226). As mentioned before, we are able to solve the above cubic equation for ϕ_N. But already from Eq. (4.225), we observe that the solution of this equation will be a function of the model parameters $m_0^2, c_1, \lambda_1, \lambda_2$, and $h_{N,0}$. It directly follows that the vacuum expectation value ϕ_N is a constant with respect to space-time, so that the kinetic term in Eq. (4.218) does not change under Eq. (4.226)

$$\begin{aligned}\mathscr{L}_{kin.} \stackrel{\sigma_N \longrightarrow \phi_N+\sigma_N}{\longrightarrow} \mathscr{L}'_{kin.} &= \frac{1}{2}\left\{\left[\partial_\mu\left(\phi_N+\sigma_N\right)\right]^2+\left(\partial_\mu \eta_N\right)^2+\left(\partial_\mu \vec{a}_0\right)^2+\left(\partial_\mu \vec{\pi}\right)^2\right\}-\frac{1}{4}\Big\{\omega_N^{\mu\nu}\omega_{N,\mu\nu}+f_{1N}^{\mu\nu}f_{1N,\mu\nu} \\ &\quad +\vec{\rho}^{\mu\nu}\cdot\vec{\rho}_{\mu\nu}+\vec{a}_1^{\mu\nu}\cdot\vec{a}_{1,\mu\nu}\Big\} \\ &= \mathscr{L}_{kin.}\ . \end{aligned} \tag{4.227}$$

An important change takes place in $\mathscr{L}_{deriv.}$, since

$$\begin{aligned}\mathscr{L}_{deriv.} \stackrel{\sigma_N \longrightarrow \phi_N+\sigma_N}{\longrightarrow} \mathscr{L}'_{deriv.} &= g_1\left[\partial^\mu\left(\phi_N+\sigma_N\right)\right]\left(f_{1N,\mu}\eta_N+\vec{a}_{1,\mu}\cdot\vec{\pi}\right)-g_1\left(\partial^\mu\eta_N\right)\left(f_{1N,\mu}\left(\phi_N+\sigma_N\right)+\vec{a}_{1,\mu}\cdot\vec{a}_0\right) \\ &\quad + g_1\left(\partial^\mu\vec{a}_0\right)\cdot\left[f_{1N,\mu}\vec{\pi}+\vec{a}_{1,\mu}\eta_N+\vec{\rho}_\mu\times\vec{a}_0\right]-g_1\left(\partial^\mu\vec{\pi}\right)\cdot\left[f_{1N,\mu}\vec{a}_0+\vec{a}_{1,\mu}\left(\phi_N+\sigma_N\right)\right. \\ &\quad \left.+\vec{\pi}\times\vec{\rho}_\mu\right]+g_2\left[\left(\partial^\mu\vec{\rho}^\nu\right)\cdot\left(\vec{\rho}_\nu\times\vec{\rho}_\mu+\vec{a}_{1,\nu}\times\vec{a}_{1,\mu}\right)+\left(\partial^\mu\vec{a}_1^\nu\right)\cdot\left(\vec{\rho}_\nu\times\vec{a}_{1,\mu}+\vec{a}_{1,\nu}\times\vec{\rho}_\mu\right)\right] \\ &= \mathscr{L}_{deriv.}-g_1\phi_N\left(\partial^\mu\eta_N\right)f_{1N,\mu}-g_1\phi_N\left(\partial^\mu\vec{\pi}\right)\cdot\vec{a}_{1,\mu}\ . \end{aligned} \tag{4.228}$$

Obviously, the two additional terms in the transformation of $\mathscr{L}_{deriv.}$ induce a mixing of the η_N- and $f_{1N,\mu}$-fields and of the $\vec{\pi}$- and $\vec{a}_{1,\mu}$-fields. This mixing introduces non-diagonal contributions to the kinetic parts of the η_N- and $\vec{\pi}$-fields, which will result in non-diagonal scattering matrix elements. In a moment, it will be shown, that we are able to eliminate these terms from the Lagrangian by introducing an appropriate redefinition of the $f_{1N,\mu}$- and $\vec{a}_{1,\mu}$-fields. But before that we have to study the impacts of Eq. (4.226) on the potential density $\mathscr{V}$. We find

$$\begin{aligned}&\mathscr{V} \stackrel{\sigma_N \longrightarrow \phi_N+\sigma_N}{\longrightarrow} \mathscr{V}' \\ &\quad = -\left(\frac{c_1}{2}-\frac{m_0^2}{2}\right)\left[\left(\phi_N+\sigma_N\right)^2+\vec{\pi}^2\right]-\left(-\frac{c_1}{2}-\frac{m_0^2}{2}\right)\left(\eta_N^2+\vec{a}_0^2\right)-\left(\frac{m_1^2}{2}+\delta_N\right)\left[\left(\omega_{N,\mu}\right)^2+\left(f_{1N,\mu}\right)^2\right. \\ &\quad \left.+\left(\vec{\rho}_\mu\right)^2+\left(\vec{a}_{1,\mu}\right)^2\right]-\frac{g_1^2}{2}\left(f_{1N,\mu}\vec{\pi}+\vec{a}_{1,\mu}\eta_N+\vec{\rho}_\mu\times\vec{a}_0\right)^2-\frac{g_1^2}{2}\left[f_{1N,\mu}\vec{a}_0+\vec{a}_{1,\mu}\left(\phi_N+\sigma_N\right)+\vec{\pi}\times\vec{\rho}_\mu\right]^2 \\ &\quad -\frac{g_1^2}{2}\left(f_{1N,\mu}\eta_N+\vec{a}_{1,\mu}\cdot\vec{\pi}\right)^2-\frac{g_1^2}{2}\left[f_{1N,\mu}\left(\phi_N+\sigma_N\right)+\vec{a}_{1,\mu}\cdot\vec{a}_0\right]^2-\left[-\frac{\lambda_1}{4}-\frac{\lambda_2}{8}\right]\left[\left(\phi_N+\sigma_N\right)^4+\eta_N^4\right. \\ &\quad \left.+\left(\vec{a}_0^2\right)^2+\left(\vec{\pi}^2\right)^2+2\left(\phi_N+\sigma_N\right)^2\eta_N^2+2\left(\phi_N+\sigma_N\right)^2\vec{\pi}^2+2\eta_N^2\vec{a}_0^2\right]-\left(-\frac{\lambda_1}{2}-\frac{3\lambda_2}{4}\right)\left[\left(\phi_N+\sigma_N\right)^2\vec{a}_0^2\right. \\ &\quad \left.+\eta_N^2\vec{\pi}^2+\vec{a}_0^2\vec{\pi}^2\right]+\frac{\lambda_2}{8}\left[8\left(\phi_N+\sigma_N\right)\eta_N\vec{a}_0\cdot\vec{\pi}-4\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right]-h_{N,0}\left(\phi_N+\sigma_N\right) \\ &\quad -\mathscr{L}'_{h_1,h_2,h_3}-\mathscr{L}'_{g_3,g_4,g_5,g_6}\ , \end{aligned} \tag{4.229}$$

where

$$\begin{aligned}\mathscr{L}'_{h_1,h_2,h_3} &= \left(\frac{h_1}{4}+\frac{h_2}{4}\right)\left\{\left[\left(\phi_N+\sigma_N\right)^2+\eta_N^2+\vec{a}_0^2+\vec{\pi}^2\right]\left[\left(\omega_{N,\mu}\right)^2+\left(f_{1N,\mu}\right)^2+\left(\vec{\rho}_\mu\right)^2+\left(\vec{a}_{1,\mu}\right)^2\right]\right\} \\ &\quad +\left(h_2+h_3\right)\omega_N^\mu\Big\{\vec{\rho}_\mu\cdot\left[\left(\phi_N+\sigma_N\right)\vec{a}_0+\eta_N\vec{\pi}\right]+\vec{a}_{1,\mu}\cdot\left(\vec{a}_0\times\vec{\pi}\right)\Big\}+\left(h_2-h_3\right)f_{1N}^\mu\Big\{\vec{a}_{1,\mu}\cdot\left[\left(\phi_N+\sigma_N\right)\vec{a}_0+\eta_N\vec{\pi}\right] \\ &\quad +\vec{\rho}_\mu\cdot\left(\vec{a}_0\times\vec{\pi}\right)\Big\}+\frac{h_3}{4}\left\{\left[\left(\omega_{N,\mu}\right)^2-\left(f_{1N,\mu}\right)^2+\left(\vec{\rho}_\mu\right)^2-\left(\vec{a}_{1,\mu}\right)^2\right]\left[\left(\phi_N+\sigma_N\right)^2+\eta_N^2+\vec{a}_0^2+\vec{\pi}^2\right]\right. \\ &\quad \left.+4\left[\left(\phi_N+\sigma_N\right)\vec{\pi}-\eta_N\vec{a}_0\right]\cdot\left(\vec{a}_1^\mu\times\vec{\rho}_\mu\right)+2\left[\left(\vec{a}_0\times\vec{a}_{1,\mu}\right)^2+\left(\vec{\pi}\times\vec{a}_{1,\mu}\right)^2-\left(\vec{a}_0\times\vec{\rho}_\mu\right)^2-\left(\vec{\pi}\times\vec{\rho}_\mu\right)^2\right]\right\} \end{aligned} \tag{4.230}$$

and

$$\mathscr{L}'_{g_3,g_4,g_5,g_6} = \mathscr{L}_{g_3,g_4,g_5,g_6}\ , \tag{4.231}$$

which directly follows from the fact that Eq. (4.184) only contains interaction terms that involve four vector/axial-vector mesons. Obviously, the shift (4.226) introduces new three-point interactions as well as new contributions to the tree-level masses of the scalar/pseudoscalar and vector/axial-vector mesons. If we simplify Eqs. (4.229) and (4.230), we obtain

$$\begin{aligned}
\mathscr{V}' = &\left[-\frac{c_1}{2} - \frac{m_0^2}{2} + \frac{3}{2}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2\right]\sigma_N^2 + \left[\frac{c_1}{2} - \frac{m_0^2}{2} + \frac{1}{2}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2\right]\eta_N^2 + \left[\frac{c_1}{2} - \frac{m_0^2}{2} + \frac{1}{2}\left(\lambda_1 + \frac{3\lambda_2}{2}\right)\phi_N^2\right]\vec{a}_0^2 \\
&+ \left[-\frac{c_1}{2} - \frac{m_0^2}{2} + \frac{1}{2}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2\right]\vec{\pi}^2 - \left[\frac{m_1^2}{2} + \delta_N + \frac{1}{4}(h_1 + h_2 + h_3)\phi_N^2\right](\omega_{N,\mu})^2 \\
&- \left[\frac{m_1^2}{2} + \delta_N + \frac{1}{4}(h_1 + h_2 - h_3)\phi_N^2 + \frac{g_1^2}{2}\phi_N^2\right](f_{1N,\mu})^2 - \left[\frac{m_1^2}{2} + \delta_N + \frac{1}{4}(h_1 + h_2 + h_3)\phi_N^2\right](\vec{\rho}_\mu)^2 \\
&- \left[\frac{m_1^2}{2} + \delta_N + \frac{1}{4}(h_1 + h_2 - h_3)\phi_N^2 + \frac{g_1^2}{2}\phi_N^2\right](\vec{a}_{1,\mu})^2 - \frac{g_1^2}{2}\left(f_{1N,\mu}\eta_N + \vec{a}_{1,\mu}\cdot\vec{\pi}\right)^2 - \frac{g_1^2}{2}\left(f_{1N,\mu}\vec{\pi} + \vec{a}_{1,\mu}\eta_N\right. \\
&\left.+ \vec{\rho}_\mu \times \vec{a}_0\right)^2 - \frac{g_1^2}{2}\left(f_{1N,\mu}\sigma_N + \vec{a}_{1,\mu}\cdot\vec{a}_0\right)^2 - \frac{g_1^2}{2}\left(f_{1N,\mu}\vec{a}_0 + \sigma_N\vec{a}_{1,\mu} + \vec{\pi}\times\vec{\rho}_\mu\right)^2 - g_1^2\phi_N f_{1N}^\mu\left(f_{1N,\mu}\sigma_N + \vec{a}_{1,\mu}\cdot\vec{a}_0\right) \\
&- g_1^2\phi_N\vec{a}_{1,\mu}\left(f_{1N,\mu}\vec{a}_0 + \sigma_N\vec{a}_{1,\mu} + \vec{\pi}\times\vec{\rho}_\mu\right) + \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left[\sigma_N^4 + 4\phi_N\sigma_N^3 + \eta_N^4 + \left(\vec{a}_0^2\right)^2 + \left(\vec{\pi}^2\right)^2 + 2\sigma_N^2\eta_N^2\right. \\
&\left.+ 4\phi_N\sigma_N\eta_N^2 + 2\sigma_N^2\vec{\pi}^2 + 4\phi_N\sigma_N\vec{\pi}^2 + 2\eta_N^2\vec{a}_0^2\right] + \frac{1}{2}\left(\lambda_1 + \frac{3\lambda_2}{2}\right)\left[\sigma_N^2\vec{a}_0^2 + 2\phi_N\sigma_N\vec{a}_0^2 + \eta_N^2\vec{\pi}^2 + \vec{a}_0^2\vec{\pi}^2\right] \\
&+ \frac{\lambda_2}{8}\left[8\sigma_N\eta_N\vec{a}_0\cdot\vec{\pi} + 8\phi_N\eta_N\vec{a}_0\cdot\vec{\pi} - 4\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right] - \frac{1}{4}(h_1 + h_2)\left\{\left(\sigma_N^2 + \eta_N^2 + \vec{a}_0^2 + \vec{\pi}^2\right)\left[(\omega_{N,\mu})^2 + (f_{1N,\mu})^2\right.\right. \\
&\left.\left.+ (\vec{\rho}_\mu)^2 + (\vec{a}_{1,\mu})^2\right]\right\} - \frac{1}{2}(h_1 + h_2)\phi_N\sigma_N\left[(\omega_{N,\mu})^2 + (f_{1N,\mu})^2 + (\vec{\rho}_\mu)^2 + (\vec{a}_{1,\mu})^2\right] - (h_2 + h_3)\,\omega_N^\mu\left[\vec{\rho}_\mu\cdot\left(\sigma_N\vec{a}_0\right.\right. \\
&\left.\left.+ \eta_N\vec{\pi} + \phi_N\vec{a}_0\right) + \vec{a}_{1,\mu}\cdot\left(\vec{a}_0\times\vec{\pi}\right)\right] - (h_2 - h_3)\,f_{1N}^\mu\left[\vec{a}_{1,\mu}\cdot\left(\sigma_N\vec{a}_0 + \eta_N\vec{\pi} + \phi_N\vec{a}_0\right) + \vec{\rho}_\mu\cdot\left(\vec{a}_0\times\vec{\pi}\right)\right] - \frac{h_3}{4}\left[(\omega_{N,\mu})^2\right. \\
&\left.- (f_{1N,\mu})^2 + (\vec{\rho}_\mu)^2 - (\vec{a}_{1,\mu})^2\right]\left(\sigma_N^2 + \eta_N^2 + \vec{a}_0^2 + \vec{\pi}^2\right) - \frac{h_3}{2}\phi_N\sigma_N\left[(\omega_{N,\mu})^2 - (f_{1N,\mu})^2 + (\vec{\rho}_\mu)^2 - (\vec{a}_{1,\mu})^2\right] \\
&- h_3\left(\sigma_N\vec{\pi} - \eta_N\vec{a}_0 + \phi_N\vec{\pi}\right)\cdot\left(\vec{a}_1^\mu\times\vec{\rho}_\mu\right) - \frac{h_3}{2}\left[\left(\vec{a}_0\times\vec{a}_{1,\mu}\right)^2 + \left(\vec{\pi}\times\vec{a}_{1,\mu}\right)^2 - \left(\vec{a}_0\times\vec{\rho}_\mu\right)^2 - \left(\vec{\pi}\times\vec{\rho}_\mu\right)^2\right] - \mathscr{L}_{g_3,g_4,g_5,g_6}\;.
\end{aligned} \tag{4.232}$$

At this point, one might ask two questions: (i) What happened to the terms which are linear in σ_N and (ii) what happened to the constant contributions which appear due to the shift (4.226) ? The first question can easily be answered, by considering the explicit form of these terms

$$\left\{-\left(m_0^2 + c_1\right)\phi_N + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^3 - h_{N,0}\right\}\sigma_N\;. \tag{4.233}$$

If we now compare the coefficient of the σ_N-field in Eq. (4.233) with the condition that has to be fulfilled by the vacuum expectation value ϕ_N, Eq. (4.225), we observe that both expressions are equivalent. Therefore, the linear contributions of the σ_N-field vanish. Also the second question has an easy answer. The constant terms of the "shifted" potential density are given by

$$\mathscr{V}(\phi_N) = -\left(\frac{c_1}{2} + \frac{m_0^2}{2}\right)\phi_N^2 + \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^4 - h_{N,0}\phi_N\;. \tag{4.234}$$

The term "$\mathscr{V}(\phi_N)$" makes sense, since the constant terms exactly correspond to the case where we evaluate the potential density at $\sigma_N = \phi_N$ with all other fields set to zero, i.e., this contribution is equal to the potential, evaluated at its minimum. But, as already mentioned, these contributions are constant and therefore do not affect the equations of motion. The only consequence of this constant term would be a shift of the zero of the energy scale, so that we are able to neglect the contributions (4.234).

Another important aspect of Eq. (4.232) concerns the tree-level masses of the mesons. At this point, one might think that the squares of the tree-level masses are, up to a constant factor, given by the square brackets in the first four lines of Eq. (4.232). But, as we will see in a moment, this is not the case, since the tree-level masses of two fields will pick up another constant factor. This factor will arise from the manipulations that are needed to remove the mixing terms (4.228) from the Lagrangian. Using Eqs. (4.227), (4.228), and (4.232), the complete Lagrangian of the eLSM is given by

$$\begin{aligned}
\mathscr{L}_{eLSM} &= \mathscr{L}_{kin.} + \mathscr{L}_{deriv.} - g_1\phi_N\left(\partial^\mu\eta_N\right)f_{1N,\mu} - g_1\phi_N\left(\partial^\mu\vec{\pi}\right)\cdot\vec{a}_{1,\mu} - \mathscr{V}' \\
&= \mathscr{L}_{kin.} + \mathscr{L}_{deriv.} - g_1\phi_N\left(\partial^\mu\eta_N\right)f_{1N,\mu} - g_1\phi_N\left(\partial^\mu\vec{\pi}\right)\cdot\vec{a}_{1,\mu} - \mathscr{V}\;,
\end{aligned} \tag{4.235}$$

where we renamed $\mathscr{V}' \to \mathscr{V}$, since Eq. (4.232) describes the new potential density of the eLSM, that is obtained by considering the fluctuations around the minimum of the initial potential density (4.221).

Before we proceed, it will be useful to summarize briefly, what we have done so far: In Eq. (4.218), we started with the explicit form of the eLSM Lagrangian, which we derived in the previous section. Due to the explicit symmetry breaking term (4.195), the potential density $\mathscr{V}$ corresponds to a "mexican hat"-like potential that is tilted in the direction of σ_N in field space. Since the σ_N-field has the same quantum numbers as the vacuum, it is allowed to acquire a non-vanishing vacuum expectation value. This vacuum expectation value is clearly given by the minimum of the potential and was determined in Eq. (4.225). Then, we studied the fluctuations around this minimum by expanding the initial σ_N-field around the vacuum expectation value, compare Eq. (4.226). At this point, it is important to understand that the spontaneous breakdown of chiral symmetry only occurs, since the vacuum expectation value ϕ_N is different from zero. Otherwise, the fluctuation field σ_N in Eq. (4.226) would exactly correspond to the initial σ_N-field. In this case, the Lagrangian would stay chirally symmetric. As already mentioned in the discussion of Eq. (2.88), the initial symmetry is not really broken, but hidden. In fact, the eLSM is still symmetric with respect to chiral transformations, but, in general, in a nonlinear realization of the symmetry, which arises from fluctuation field σ_N. The occurrence of this field then introduced new interaction vertices which are proportional to ϕ_N. In addition to these interaction vertices, the decomposition (4.226) also introduced two mixing terms, see Eq. (4.235). In the following, we want to eliminate these mixing terms. To this end, it will be useful to reorder the terms of the Lagrangian (4.235) in a different way. If we arrange all terms with respect to their coupling constants, the Lagrangian (4.235) can be written as

$$\mathscr{L}_{eLSM} = \mathscr{L}_{kin.} + \mathscr{L}_{mass,g_1,g_2} + \mathscr{L}_{\lambda_1,\lambda_2} + \mathscr{L}_{h_1,h_2,h_3} + \mathscr{L}_{g_3,g_4,g_5,g_6} \ , \tag{4.236}$$

where the second part $\mathscr{L}_{mass,g_1,g_2}$ also contains the tree-level mass contributions of Eq. (4.232). For the sake of clarity, we want to quote the explicit expressions of these terms. Since the kinetic part $\mathscr{L}_{kin.}$ and the four vector/axial-vector meson interaction part $\mathscr{L}_{g_3,g_4,g_5,g_6}$ are given by Eqs. (4.219) and (4.184), the other three terms have to be defined again. The second part $\mathscr{L}_{mass,g_1,g_2}$ is given by

$$\mathscr{L}_{mass,g_1,g_2} = \mathscr{L}_{mass} + \mathscr{L}_{g_1,g_2} \ , \tag{4.237}$$

with

$$\begin{aligned}
\mathscr{L}_{mass} = &-\frac{1}{2}\left[-c_1 - m_0^2 + 3\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2\right]\sigma_N^2 - \frac{1}{2}\left[c_1 - m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2\right]\eta_N^2 \\
&- \frac{1}{2}\left[c_1 - m_0^2 + \left(\lambda_1 + \frac{3\lambda_2}{2}\right)\phi_N^2\right]\vec{a}_0^2 - \frac{1}{2}\left[-c_1 - m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2\right]\vec{\pi}^2 \\
&+ \frac{1}{2}\left[m_1^2 + 2\delta_N + \frac{1}{2}(h_1+h_2+h_3)\phi_N^2\right](\omega_{N,\mu})^2 + \frac{1}{2}\left[m_1^2 + 2\delta_N + \frac{1}{2}(h_1+h_2-h_3)\phi_N^2 + g_1^2\phi_N^2\right](f_{1N,\mu})^2 \\
&+ \frac{1}{2}\left[m_1^2 + 2\delta_N + \frac{1}{2}(h_1+h_2+h_3)\phi_N^2\right](\vec{\rho}_\mu)^2 + \frac{1}{2}\left[m_1^2 + 2\delta_N + \frac{1}{2}(h_1+h_2-h_3)\phi_N^2 + g_1^2\phi_N^2\right](\vec{a}_{1,\mu})^2
\end{aligned} \tag{4.238}$$

and

$$\begin{aligned}
\mathscr{L}_{g_1,g_2} = &-g_1\phi_N\left[(\partial^\mu\eta_N) f_{1N,\mu} + (\partial^\mu\vec{\pi})\cdot\vec{a}_{1,\mu}\right] + g_1(\partial^\mu\sigma_N)(f_{1N,\mu}\eta_N + \vec{a}_{1,\mu}\cdot\vec{\pi}) - g_1(\partial^\mu\eta_N)(f_{1N,\mu}\sigma_N + \vec{a}_{1,\mu}\cdot\vec{a}_0) \\
&+ g_1(\partial^\mu\vec{a}_0)\cdot(f_{1N,\mu}\vec{\pi} + \vec{a}_{1,\mu}\eta_N + \vec{\rho}_\mu\times\vec{a}_0) - g_1(\partial^\mu\vec{\pi})\cdot(f_{1N,\mu}\vec{a}_0 + \vec{a}_{1,\mu}\sigma_N + \vec{\pi}\times\vec{\rho}_\mu) + \frac{g_1^2}{2}(f_{1N,\mu}\eta_N + \vec{a}_{1,\mu}\cdot\vec{\pi})^2 \\
&+ \frac{g_1^2}{2}(f_{1N,\mu}\vec{\pi} + \vec{a}_{1,\mu}\eta_N + \vec{\rho}_\mu\times\vec{a}_0)^2 + \frac{g_1^2}{2}(f_{1N,\mu}\sigma_N + \vec{a}_{1,\mu}\cdot\vec{a}_0)^2 + \frac{g_1^2}{2}(f_{1N,\mu}\vec{a}_0 + \sigma_N\vec{a}_{1,\mu} + \vec{\pi}\times\vec{\rho}_\mu)^2 \\
&+ g_1^2\phi_N f_{1N}^\mu(f_{1N,\mu}\sigma_N + \vec{a}_{1,\mu}\cdot\vec{a}_0) + g_1^2\phi_N\vec{a}_{1,\mu}\cdot(f_{1N,\mu}\vec{a}_0 + \sigma_N\vec{a}_{1,\mu} + \vec{\pi}\times\vec{\rho}_\mu) \\
&+ g_2\left[(\partial^\mu\vec{\rho}^\nu)\cdot(\vec{\rho}_\nu\times\vec{\rho}_\mu + \vec{a}_{1,\nu}\times\vec{a}_{1,\mu}) + (\partial^\mu\vec{a}_1^\nu)\cdot(\vec{\rho}_\nu\times\vec{a}_{1,\mu} + \vec{a}_{1,\nu}\times\vec{\rho}_\mu)\right] \ .
\end{aligned} \tag{4.239}$$

The third term in Eq. (4.236) only contains the interactions of the scalar and pseudoscalar mesons among themselves. This term is given by

$$\begin{aligned}
\mathscr{L}_{\lambda_1,\lambda_2} = &-\frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left[\sigma_N^4 + 4\phi_N\sigma_N^3 + \eta_N^4 + \left(\vec{a}_0^2\right)^2 + \left(\vec{\pi}^2\right)^2 + 2\sigma_N^2\eta_N^2 + 4\phi_N\sigma_N\eta_N^2 + 2\sigma_N^2\vec{\pi}^2 + 4\phi_N\sigma_N\vec{\pi}^2 + 2\eta_N^2\vec{a}_0^2\right] \\
&- \frac{1}{2}\left(\lambda_1 + \frac{3\lambda_2}{2}\right)\left[\sigma_N^2\vec{a}_0^2 + 2\phi_N\sigma_N\vec{a}_0^2 + \eta_N^2\vec{\pi}^2 + \vec{a}_0^2\vec{\pi}^2\right] - \frac{\lambda_2}{8}\left[8\sigma_N\eta_N\vec{a}_0\cdot\vec{\pi} + 8\phi_N\eta_N\vec{a}_0\cdot\vec{\pi} - 4\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right] \ .
\end{aligned} \tag{4.240}$$

Finally, the fourth term is defined as

$$\begin{aligned}
\mathscr{L}_{h_1,h_2,h_3} &= \frac{1}{4}(h_1+h_2)\left\{\left(\sigma_N^2+\eta_N^2+\vec{a}_0^2+\vec{\pi}^2\right)\left[(\omega_{N,\mu})^2+(f_{1N,\mu})^2+(\vec{\rho}_\mu)^2+(\vec{a}_{1,\mu})^2\right]\right\} \\
&+ \frac{1}{2}(h_1+h_2)\,\phi_N\sigma_N\left[(\omega_{N,\mu})^2+(f_{1N,\mu})^2+(\vec{\rho}_\mu)^2+(\vec{a}_{1,\mu})^2\right]+(h_2+h_3)\,\omega_N^\mu\Big[\vec{\rho}_\mu\cdot(\sigma_N\vec{a}_0+\eta_N\vec{\pi}+\phi_N\vec{a}_0) \\
&+ \vec{a}_{1,\mu}\cdot(\vec{a}_0\times\vec{\pi})\Big]+(h_2-h_3)\,f_{1N}^\mu\Big[\vec{a}_{1,\mu}\cdot(\sigma_N\vec{a}_0+\eta_N\vec{\pi}+\phi_N\vec{a}_0)+\vec{\rho}_\mu\cdot(\vec{a}_0\times\vec{\pi})\Big] \\
&+ \frac{h_3}{4}\left[(\omega_{N,\mu})^2-(f_{1N,\mu})^2+(\vec{\rho}_\mu)^2-(\vec{a}_{1,\mu})^2\right]\left(\sigma_N^2+\eta_N^2+\vec{a}_0^2+\vec{\pi}^2\right)+\frac{h_3}{2}\phi_N\sigma_N\Big[(\omega_{N,\mu})^2-(f_{1N,\mu})^2 \\
&+ (\vec{\rho}_\mu)^2-(\vec{a}_{1,\mu})^2\Big]+h_3\,(\sigma_N\vec{\pi}-\eta_N\vec{a}_0+\phi_N\vec{\pi})\cdot(\vec{a}_1^\mu\times\vec{\rho}_\mu)+\frac{h_3}{2}\Big[(\vec{a}_0\times\vec{a}_{1,\mu})^2+(\vec{\pi}\times\vec{a}_{1,\mu})^2-(\vec{a}_0\times\vec{\rho}_\mu)^2 \\
&- (\vec{\pi}\times\vec{\rho}_\mu)^2\Big]\ . \qquad (4.241)
\end{aligned}$$

Now, in order to eliminate the mixing terms in Eq. (4.239), it is clear that we only have to consider the mass terms (4.238) in Eq. (4.237), since these terms are the only ones that also contain only two fields. It is therefore quite intuitive to introduce the following shifts of the $f_{1N,\mu}$- and $\vec{a}_{1,\mu}$-fields

$$f_{1N,\mu}\longrightarrow f'_{1N,\mu}=f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\ , \qquad (4.242)$$

$$\vec{a}_{1,\mu}\longrightarrow \vec{a}'_{1,\mu}=\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\ , \qquad (4.243)$$

where the parameters $w_{f_{1N}}$ and $w_{\vec{a}_1}$ will be determined in a moment. As a consequence of the above shifts, the initial mass terms of the $f_{1N,\mu}$- and $\vec{a}_{1,\mu}$-fields will generate similar mixing terms as in Eq. (4.239). The basic idea is now to adjust the new parameters w_i, $i=f_{1N},\vec{a}_1$, in a way that all mixing terms cancel out of the Lagrangian. In order to see this, we have to insert the shifts (4.242) and (4.243) into all parts of the Lagrangian (4.236) and study their consequences. Using the Schwarz theorem and the antisymmetry of the field-strength tensors, it is easy to see that the kinetic part (4.219) remains invariant with respect to Eqs. (4.242), (4.243)

$$\begin{aligned}
\mathscr{L}_{kin.} &\xrightarrow[\vec{a}_{1,\mu}\longrightarrow\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}]{f_{1N,\mu}\longrightarrow f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N,} \mathscr{L}'_{kin.} \\
&= \frac{1}{2}\left\{(\partial_\mu\sigma_N)^2+(\partial_\mu\eta_N)^2+(\partial_\mu\vec{a}_0)^2+(\partial_\mu\vec{\pi})^2\right\}-\frac{1}{4}\Big\{\omega_N^{\mu\nu}\omega_{N,\mu\nu}+\vec{\rho}^{\mu\nu}\cdot\vec{\rho}_{\mu\nu} \\
&+ \left[\partial^\mu\left(f_{1N}^\nu+w_{f_{1N}}\partial^\nu\eta_N\right)-\partial^\nu\left(f_{1N}^\mu+w_{f_{1N}}\partial^\mu\eta_N\right)\right]\Big[\partial_\mu\left(f_{1N,\nu}+w_{f_{1N}}\partial_\nu\eta_N\right) \\
&- \partial_\nu\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\Big]+\left[\partial^\mu\left(\vec{a}_1^\nu+w_{\vec{a}_1}\partial^\nu\vec{\pi}\right)-\partial^\nu\left(\vec{a}_1^\mu+w_{\vec{a}_1}\partial^\mu\vec{\pi}\right)\right] \\
&\cdot \left[\partial_\mu\left(\vec{a}_{1,\nu}+w_{\vec{a}_1}\partial_\nu\vec{\pi}\right)-\partial_\nu\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\right]\Big\} \\
&= \mathscr{L}_{kin.}\ . \qquad (4.244)
\end{aligned}$$

Then, the second contribution to Eq. (4.236) can be written as

$$\begin{aligned}
\mathscr{L}_{mass,g_1,g_2} &\xrightarrow[\vec{a}_{1,\mu}\longrightarrow\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}]{f_{1N,\mu}\longrightarrow f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N,} \mathscr{L}'_{mass,g_1,g_2} \\
&= \mathscr{L}'_{mass}+\mathscr{L}'_{g_1,g_2} \\
&= \mathscr{L}_{mass}+\mathscr{L}''_{g_1,g_2}+\left\{\left[m_1^2+2\delta_N+\frac{1}{2}(h_1+h_2-h_3)\,\phi_N^2+g_1^2\phi_N^2\right]w_{f_{1N}}-g_1\phi_N\right\} \\
&\times(\partial^\mu\eta_N)\,f_{1N,\mu}+\left\{\left[m_1^2+2\delta_N+\frac{1}{2}(h_1+h_2-h_3)\,\phi_N^2+g_1^2\phi_N^2\right]w_{\vec{a}_1}-g_1\phi_N\right\} \\
&\times(\partial^\mu\vec{\pi})\cdot\vec{a}_{1,\mu}+\frac{1}{2}\left\{\left[m_1^2+2\delta_N+\frac{1}{2}(h_1+h_2-h_3)\,\phi_N^2+g_1^2\phi_N^2\right]w_{f_{1N}}^2-2g_1\phi_N w_{f_{1N}}\right\} \\
&\times(\partial_\mu\eta_N)^2+\frac{1}{2}\left\{\left[m_1^2+2\delta_N+\frac{1}{2}(h_1+h_2-h_3)\,\phi_N^2+g_1^2\phi_N^2\right]w_{\vec{a}_1}^2-2g_1\phi_N w_{\vec{a}_1}\right\} \\
&\times(\partial_\mu\vec{\pi})^2\ , \qquad (4.245)
\end{aligned}$$

where $\mathscr{L}_{mass}$ is given by Eq. (4.238). This mass term is, of course, not invariant with respect to the field redefinitions (4.242) and (4.243). However, it is possible, to collect the additional contributions to the initial mass term in a different way. To be particular, the four additional terms are contained in the last four terms of Eq. (4.245). In the case of $\mathscr{L}_{g_1,g_2}$ such a procedure is not possible, since the shifts of the axial-vector fields affect all terms of this part of the Lagrangian. In addition to that, we had to pull both contributions of the shifted mixing terms out of $\mathscr{L}_{g_1,g_2}$ and introduce $\mathscr{L}''_{g_1,g_2}$, which is given by

$$\begin{aligned}\mathscr{L}''_{g_1,g_2} &= g_1\left(\partial^\mu\sigma_N\right)\left[\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\eta_N+\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\cdot\vec{\pi}\right]-g_1\left(\partial^\mu\eta_N\right)\left[\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\sigma_N\right.\\&\quad\left.+\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\cdot\vec{a}_0\right]+g_1\left(\partial^\mu\vec{a}_0\right)\cdot\left[\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\vec{\pi}+\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\eta_N+\vec{\rho}_\mu\times\vec{a}_0\right]\\&\quad-g_1\left(\partial^\mu\vec{\pi}\right)\cdot\left[\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\vec{a}_0+\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\sigma_N+\vec{\pi}\times\vec{\rho}_\mu\right]+\frac{g_1^2}{2}\left\{\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\eta_N\right.\\&\quad\left.+\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\cdot\vec{\pi}\right\}^2+\frac{g_1^2}{2}\left\{\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\vec{\pi}+\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\eta_N+\vec{\rho}_\mu\times\vec{a}_0\right\}^2\\&\quad+\frac{g_1^2}{2}\left\{\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\sigma_N+\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\cdot\vec{a}_0\right\}^2+\frac{g_1^2}{2}\left\{\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\vec{a}_0\right.\\&\quad\left.+\sigma_N\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)+\vec{\pi}\times\vec{\rho}_\mu\right\}^2+g_1^2\phi_N\left(f^\mu_{1N}+w_{f_{1N}}\partial^\mu\eta_N\right)\left\{\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\sigma_N\right.\\&\quad\left.+\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\cdot\vec{a}_0\right\}+g_1^2\phi_N\left(\vec{a}^\mu_1+w_{\vec{a}_1}\partial^\mu\vec{\pi}\right)\cdot\left\{\left(f_{1N,\mu}+w_{f_{1N}}\partial_\mu\eta_N\right)\vec{a}_0+\sigma_N\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\right.\\&\quad\left.+\vec{\pi}\times\vec{\rho}_\mu\right\}+g_2\left\{\left(\partial^\mu\vec{\rho}^\nu\right)\cdot\left[\vec{\rho}_\nu\times\vec{\rho}_\mu+\left(\vec{a}_{1,\nu}+w_{\vec{a}_1}\partial_\nu\vec{\pi}\right)\times\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)\right]\right.\\&\quad\left.+\left[\partial^\mu\left(\vec{a}^\nu_1+w_{\vec{a}_1}\partial^\nu\vec{\pi}\right)\right]\cdot\left[\vec{\rho}_\nu\times\left(\vec{a}_{1,\mu}+w_{\vec{a}_1}\partial_\mu\vec{\pi}\right)+\left(\vec{a}_{1,\nu}+w_{\vec{a}_1}\partial_\nu\vec{\pi}\right)\times\vec{\rho}_\mu\right]\right\}\;, \qquad (4.246)\end{aligned}$$

in order to rewrite $\mathscr{L}_{mass,g_1,g_2}$ according to Eq. (4.245). The importance of the form of Eq. (4.245) now derives from the fact, that the third and fourth term yield conditions for the shape of the parameters $w_{f_{1N}}$ and $w_{\vec{a}_1}$, which help us to eliminate the mixing terms

$$w_{f_{1N}}=w_{\vec{a}_1}=\frac{g_1\phi_N}{m_1^2+2\delta_N+\frac{1}{2}\left(h_1+h_2-h_3\right)\phi_N^2+g_1^2\phi_N^2}\equiv w\;. \qquad (4.247)$$

Using this definition, we are able to simplify the coefficients of the fifth and sixth term in Eq. (4.245). We find

$$\frac{1}{2}\left\{\left[m_1^2+2\delta_N+\frac{1}{2}\left(h_1+h_2-h_3\right)\phi_N^2+g_1^2\phi_N^2\right]w_{f_{1N}}^2-2g_1\phi_Nw_{f_{1N}}\right\}=\frac{1}{2}\frac{g_1^2\phi_N^2}{m_1^2+2\delta_N+\frac{1}{2}\left(h_1+h_2-h_3\right)\phi_N^2+g_1^2\phi_N^2}\;, \qquad (4.248)$$

$$\frac{1}{2}\left\{\left[m_1^2+2\delta_N+\frac{1}{2}\left(h_1+h_2-h_3\right)\phi_N^2+g_1^2\phi_N^2\right]w_{\vec{a}_1}^2-2g_1\phi_Nw_{\vec{a}_1}\right\}=\frac{1}{2}\frac{g_1^2\phi_N^2}{m_1^2+2\delta_N+\frac{1}{2}\left(h_1+h_2-h_3\right)\phi_N^2+g_1^2\phi_N^2}\;. \qquad (4.249)$$

It is quite obvious that these terms describe additional contributions to the kinetic parts of the η_N- and the $\vec{\pi}$-fields. The problem with this additional contribution is the interpretation of the creation and annihilation operators which emerge in the Fourier decompositions of these fields. To be particular, these Fourier decompositions are obtained as general solutions of the free Klein-Gordon equations of these fields. If we now have an additional contribution to the usual factor of $1/2$ in the kinetic part of the free Klein-Gordon Lagrangian, the Fourier components cannot be interpreted as creation and annihilation operators of normalized one meson states. In order to solve this problem, we summarize Eqs. (4.244) and (4.245) into only one term

$$\mathscr{L}_{kin.,mass,g_1,g_2}=\mathscr{L}_{kin.,mass}+\mathscr{L}''_{g_1,g_2}\;, \qquad (4.250)$$

where

$$\begin{aligned}\mathscr{L}_{kin.,mass}&=\frac{1}{2}\left\{\left(\partial_\mu\sigma_N\right)^2+\left(\partial_\mu\vec{a}_0\right)^2\right\}+\frac{1}{2}\left\{1-\frac{g_1^2\phi_N^2}{m_1^2+2\delta_N+\frac{1}{2}\left(h_1+h_2-h_3\right)\phi_N^2+g_1^2\phi_N^2}\right\}\left(\partial_\mu\eta_N\right)^2\\&\quad+\frac{1}{2}\left\{1-\frac{g_1^2\phi_N^2}{m_1^2+2\delta_N+\frac{1}{2}\left(h_1+h_2-h_3\right)\phi_N^2+g_1^2\phi_N^2}\right\}\left(\partial_\mu\vec{\pi}\right)^2-\frac{1}{4}\left\{\omega_N^{\mu\nu}\omega_{N,\mu\nu}+f_{1N}^{\mu\nu}f_{1N,\mu\nu}\right.\\&\quad\left.+\vec{\rho}^{\mu\nu}\cdot\vec{\rho}_{\mu\nu}+\vec{a}_1^{\mu\nu}\cdot\vec{a}_{1,\mu\nu}\right\}+\mathscr{L}_{mass}\;. \qquad (4.251)\end{aligned}$$

Now, in order to obtain a canonical normalization of the kinetic terms, we have to introduce a field renormalization of the η_N- and the $\vec{\pi}$-fields

$$\eta_N\longrightarrow Z_{\eta_N}\eta_N\;, \qquad (4.252)$$
$$\vec{\pi}\longrightarrow Z_{\vec{\pi}}\vec{\pi}\;, \qquad (4.253)$$

with

$$Z_{\eta_N} = Z_{\vec{\pi}} = \left[1 - \frac{g_1^2 \phi_N^2}{m_1^2 + 2\delta_N + \frac{1}{2}(h_1 + h_2 - h_3)\phi_N^2 + g_1^2 \phi_N^2}\right]^{-\frac{1}{2}} \equiv Z \ . \tag{4.254}$$

It is quite obvious that the kinetic terms of the renormalized η_N- and $\vec{\pi}$-fields now have the right prefactor. In the following, we have to introduce the renormalized fields in all terms of the eLSM Lagrangian. Starting with Eq. (4.250), we find

$$\begin{aligned}
\mathscr{L}_{kin.,mass,g_1,g_2} \quad \overset{\substack{\eta_N \longrightarrow Z_{\eta_N}\eta_N, \\ \vec{\pi} \longrightarrow Z_{\vec{\pi}}\vec{\pi}}}{\longrightarrow} \quad & \mathscr{L}'_{kin.,mass,g_1,g_2} \\
&= \mathscr{L}'_{kin.,mass} + \mathscr{L}'''_{g_1,g_2} \\
&\equiv \mathscr{L}_{kin.,mass} + \mathscr{L}_{g_1,g_2} \ ,
\end{aligned} \tag{4.255}$$

where we renamed $\mathscr{L}'''_{g_1,g_2} \to \mathscr{L}_{g_1,g_2}$ and with

$$\begin{aligned}
\mathscr{L}_{kin.,mass} \equiv\ & \frac{1}{2}(\partial_\mu \sigma_N)^2 - \frac{1}{2} m_{\sigma_N}^2 \sigma_N^2 + \frac{1}{2}(\partial_\mu \eta_N)^2 - \frac{1}{2} m_{\eta_N}^2 \eta_N^2 + \frac{1}{2}(\partial_\mu \vec{a}_0)^2 - \frac{1}{2} m_{\vec{a}_0}^2 \vec{a}_0^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 - \frac{1}{2} m_{\vec{\pi}}^2 \vec{\pi}^2 \\
& - \frac{1}{4}\omega_N^{\mu\nu}\omega_{N,\mu\nu} + \frac{1}{2} m_{\omega_N}^2 (\omega_{N,\mu})^2 - \frac{1}{4} f_{1N}^{\mu\nu} f_{1N,\mu\nu} + \frac{1}{2} m_{f_{1N}}^2 (f_{1N,\mu})^2 - \frac{1}{4}\vec{\rho}^{\mu\nu}\cdot\vec{\rho}_{\mu\nu} + \frac{1}{2} m_{\vec{\rho}}^2 (\vec{\rho}_\mu)^2 \\
& - \frac{1}{4}\vec{a}_1^{\mu\nu}\cdot\vec{a}_{1,\mu\nu} + \frac{1}{2} m_{\vec{a}_1}^2 (\vec{a}_{1,\mu})^2 \ ,
\end{aligned} \tag{4.256}$$

where we defined the tree-level masses of the mesons as

$$m_{\sigma_N}^2 = -c_1 - m_0^2 + 3\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 \ , \tag{4.257}$$

$$m_{\eta_N}^2 = \left[c_1 - m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2\right] Z^2 \ , \tag{4.258}$$

$$m_{\vec{a}_0}^2 = c_1 - m_0^2 + \left(\lambda_1 + \frac{3\lambda_2}{2}\right)\phi_N^2 \ , \tag{4.259}$$

$$m_{\vec{\pi}}^2 = \left[-c_1 - m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2\right] Z^2 \ , \tag{4.260}$$

$$m_{\omega_N}^2 = m_{\vec{\rho}}^2 = m_1^2 + 2\delta_N + \frac{1}{2}(h_1 + h_2 + h_3)\phi_N^2 \ , \tag{4.261}$$

$$m_{f_{1N}}^2 = m_{\vec{a}_1}^2 = m_1^2 + 2\delta_N + \frac{1}{2}(h_1 + h_2 - h_3)\phi_N^2 + g_1^2\phi_N^2 \ , \tag{4.262}$$

and

$$\begin{aligned}
\mathscr{L}_{g_1,g_2} \equiv\ & g_1(\partial^\mu \sigma_N)\Big[\Big(f_{1N,\mu} + Zw\partial_\mu \eta_N\Big)Z\eta_N + \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)\cdot Z\vec{\pi}\Big] - g_1 Z(\partial^\mu \eta_N)\Big[\Big(f_{1N,\mu} \\
& + Zw\partial_\mu \eta_N\Big)\sigma_N + \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)\cdot \vec{a}_0\Big] + g_1(\partial^\mu \vec{a}_0)\cdot\Big[\Big(f_{1N,\mu} + Zw\partial_\mu \eta_N\Big)Z\vec{\pi} \\
& + \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)Z\eta_N + \vec{\rho}_\mu \times \vec{a}_0\Big] - g_1 Z(\partial^\mu \vec{\pi})\cdot\Big[\Big(f_{1N,\mu} + Zw\partial_\mu \eta_N\Big)\vec{a}_0 + \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)\sigma_N \\
& + Z\vec{\pi}\times\vec{\rho}_\mu\Big] + \frac{g_1^2}{2}\Big\{\Big(f_{1N,\mu} + Zw\partial_\mu \eta_N\Big)Z\eta_N + \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)\cdot Z\vec{\pi}\Big\}^2 + \frac{g_1^2}{2}\Big\{\Big(f_{1N,\mu} \\
& + Z\partial_\mu \eta_N\Big)Z\vec{\pi} + \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)Z\eta_N + \vec{\rho}_\mu \times \vec{a}_0\Big\}^2 + \frac{g_1^2}{2}\Big\{\Big(f_{1N,\mu} + Zw\partial_\mu \eta_N\Big)\sigma_N \\
& + \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)\cdot\vec{a}_0\Big\}^2 + \frac{g_1^2}{2}\Big\{\Big(f_{1N,\mu} + Zw\partial_\mu \eta_N\Big)\vec{a}_0 + \sigma_N\Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big) + Z\vec{\pi}\times\vec{\rho}_\mu\Big\}^2 \\
& + g_1^2\phi_N\Big(f_{1N}^\mu + Zw\partial^\mu \eta_N\Big)\Big\{\Big(f_{1N,\mu} + Zw\partial_\mu \eta_N\Big)\sigma_N + \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)\cdot\vec{a}_0\Big\} + g_1^2\phi_N \\
& \times\Big(\vec{a}_1^\mu + Zw\partial^\mu \vec{\pi}\Big)\cdot\Big\{\Big(f_{1N,\mu} + Zw\partial_\mu \eta_N\Big)\vec{a}_0 + \sigma_N\Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big) + Z\vec{\pi}\times\vec{\rho}_\mu\Big\} \\
& + g_2\Big\{(\partial^\mu \vec{\rho}^\nu)\cdot\Big[\vec{\rho}_\nu \times \vec{\rho}_\mu + \Big(\vec{a}_{1,\nu} + Zw\partial_\nu \vec{\pi}\Big)\times\Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big)\Big] + \Big[\partial^\mu\Big(\vec{a}_1^\nu + Zw\partial^\nu \vec{\pi}\Big)\Big] \\
& \cdot\Big[\vec{\rho}_\nu \times \Big(\vec{a}_{1,\mu} + Zw\partial_\mu \vec{\pi}\Big) + \Big(\vec{a}_{1,\nu} + Zw\partial_\nu \vec{\pi}\Big)\times\vec{\rho}_\mu\Big]\Big\} \ .
\end{aligned} \tag{4.263}$$

At this point, it should be mentioned that the definition of the tree-level masses enables us to simplify the expressions for Eqs. (4.247) and (4.254),

$$w = \frac{g_1\phi_N}{m_{f_{1N}}^2} = \frac{g_1\phi_N}{m_{\vec{a}_1}^2} \ , \qquad Z = (1 - g_1\phi_N w)^{-\frac{1}{2}} \ . \tag{4.264}$$

Now, since the $\eta_N/f_{1N,\mu}$- and $\vec{\pi}/\vec{a}_{1,\mu}$-mixing terms are eliminated from the eLSM Lagrangian and all kinetic terms are normalized correctly, we can study the influences of the shifts (4.242), (4.243) and the field renormalizations (4.252), (4.253) on the remaining interaction terms of the eLSM Lagrangian. Since Eq. (4.240) describes the self-interactions of scalar/pseudoscalar mesons, it is not affected by the shifts in the axial-vector sector, so that

$$\begin{aligned}
\mathscr{L}_{\lambda_1,\lambda_2} &\overset{(4.242),(4.243),}{\underset{(4.252),(4.253)}{\longrightarrow}} \mathscr{L}'_{\lambda_1,\lambda_2} \\
&= -\frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right)\left[\sigma_N^4+4\phi_N\sigma_N^3+Z^4\eta_N^4+\left(\vec{a}_0^2\right)^2+Z^4\left(\vec{\pi}^2\right)^2+2Z^2\sigma_N^2\eta_N^2+4\phi_N Z^2\sigma_N\eta_N^2\right. \\
&\quad \left.+2Z^2\sigma_N^2\vec{\pi}^2+4\phi_N Z^2\sigma_N\vec{\pi}^2+2Z^2\eta_N^2\vec{a}_0^2\right]-\frac{1}{2}\left(\lambda_1+\frac{3\lambda_2}{2}\right)\left[\sigma_N^2\vec{a}_0^2+2\phi_N\sigma_N\vec{a}_0^2+Z^4\eta_N^2\vec{\pi}^2+Z^2\vec{a}_0^2\vec{\pi}^2\right] \\
&\quad -\frac{\lambda_2}{8}\left[8Z^2\sigma_N\eta_N\vec{a}_0\cdot\vec{\pi}+8\phi_N Z^2\eta_N\vec{a}_0\cdot\vec{\pi}-4Z^2\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right] \\
&\equiv \mathscr{L}_{\lambda_1,\lambda_2}\ ,
\end{aligned} \tag{4.265}$$

where we renamed $\mathscr{L}'_{\lambda_1,\lambda_2}\to\mathscr{L}_{\lambda_1,\lambda_2}$ in the last line. Then, there is the mixed interaction term that contains the interactions of the scalar/pseudoscalar mesons with the vector/axial-vector mesons. This term is, of course, affected by the shifts as well as the field renormalizations. We find

$$\begin{aligned}
\mathscr{L}_{h_1,h_2,h_3} &\overset{(4.242),(4.243),}{\underset{(4.252),(4.253)}{\longrightarrow}} \mathscr{L}'_{h_1,h_2,h_3} \\
&= \frac{1}{4}(h_1+h_2)\left\{\left(\sigma_N^2+Z^2\eta_N^2+\vec{a}_0^2+Z^2\vec{\pi}^2\right)\left[(\omega_{N,\mu})^2+\left(f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2+(\vec{\rho}_\mu)^2\right.\right. \\
&\quad \left.\left.+\left(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)^2\right]\right\}+\frac{1}{2}(h_1+h_2)\,\phi_N\sigma_N\left\{(\omega_{N,\mu})^2+\left(f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2\right. \\
&\quad \left.+(\vec{\rho}_\mu)^2+\left(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)^2\right\}+(h_2+h_3)\,\omega_N^\mu\left[\vec{\rho}_\mu\cdot\left(\sigma_N\vec{a}_0+Z^2\eta_N\vec{\pi}+\phi_N\vec{a}_0\right)\right. \\
&\quad \left.+\left(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)\cdot\left(\vec{a}_0\times Z\vec{\pi}\right)\right]+(h_2-h_3)\left(f_{1N}^\mu+Zw\partial^\mu\eta_N\right)\left\{\left(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)\cdot\left[\sigma_N\vec{a}_0\right.\right. \\
&\quad \left.\left.+Z^2\eta_N\vec{\pi}+\phi_N\vec{a}_0\right]+\vec{\rho}_\mu\cdot\left(\vec{a}_0\times Z\vec{\pi}\right)\right\}+\frac{h_3}{4}\left[(\omega_{N,\mu})^2-\left(f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2+(\vec{\rho}_\mu)^2-\left(\vec{a}_{1,\mu}\right.\right. \\
&\quad \left.\left.+Zw\partial_\mu\vec{\pi}\right)^2\right]\left(\sigma_N^2+Z^2\eta_N^2+\vec{a}_0^2+Z^2\vec{\pi}^2\right)+\frac{h_3}{2}\phi_N\sigma_N\left[(\omega_{N,\mu})^2-\left(f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2+(\vec{\rho}_\mu)^2\right. \\
&\quad \left.-\left(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)^2\right]+h_3\left(Z\sigma_N\vec{\pi}-Z\eta_N\vec{a}_0+\phi_N Z\vec{\pi}\right)\cdot\left[\left(\vec{a}_1^\mu+Zw\partial^\mu\vec{\pi}\right)\times\vec{\rho}_\mu\right] \\
&\quad +\frac{h_3}{2}\left\{\left[\vec{a}_0\times\left(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)\right]^2+\left[Z\vec{\pi}\times\left(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)\right]^2-\left(\vec{a}_0\times\vec{\rho}_\mu\right)^2-\left(Z\vec{\pi}\times\vec{\rho}_\mu\right)^2\right\} \\
&\equiv \mathscr{L}_{h_1,h_2,h_3}\ ,
\end{aligned} \tag{4.266}$$

where we renamed $\mathscr{L}'_{h_1,h_2,h_3}\to\mathscr{L}_{h_1,h_2,h_3}$ in the last line. At this point, it should be mentioned that the expression "mixed interaction term" is not very meaningful anymore in order to describe Eq. (4.266), since the redefinitions (4.242) and (4.243) will also introduce additional interaction vertices in $\mathscr{L}_{g_3,g_4,g_5,g_6}$. To be particular, performing all shifts and field renormalizations, (4.184) becomes

$$\begin{aligned}
\mathscr{L}_{g_3,g_4,g_5,g_6} &\overset{(4.242),(4.243),}{\underset{(4.252),(4.253)}{\longrightarrow}} \mathscr{L}'_{g_3,g_4,g_5,g_6} \\
&= \left(\frac{g_3}{8}+\frac{g_4}{8}+\frac{g_6}{4}\right)\left[\left(\omega_{N,\mu}+f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2\left(\omega_{N,\nu}+f_{1N,\nu}+Zw\partial_\nu\eta_N\right)^2\right. \\
&\quad +\left(\omega_{N,\mu}-f_{1N,\mu}-Zw\partial_\mu\eta_N\right)^2\left(\omega_{N,\nu}-f_{1N,\nu}-Zw\partial_\nu\eta_N\right)^2 \\
&\quad +2\left(\omega_{N,\mu}+f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi}\right)^2 \\
&\quad \left.+2\left(\omega_{N,\mu}-f_{1N,\mu}-Zw\partial_\mu\eta_N\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi}\right)^2\right] \\
&\quad +\left(\frac{g_3}{2}+\frac{g_4}{2}\right)\left[\left(\omega_N^\mu+f_{1N}^\mu+Zw\partial^\mu\eta_N\right)\left(\omega_N^\nu+f_{1N}^\nu+Zw\partial^\nu\eta_N\right)\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)\right. \\
&\quad \cdot\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi}\right)+\left(\omega_N^\mu-f_{1N}^\mu-Zw\partial^\mu\eta_N\right)\left(\omega_N^\nu-f_{1N}^\nu-Zw\partial^\nu\eta_N\right) \\
&\quad \left.\times\left(\vec{\rho}_\mu-\vec{a}_{1,\mu}-Zw\partial_\mu\vec{\pi}\right)\cdot\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi}\right)\right]+\left(-\frac{g_3}{8}+\frac{g_4}{8}+\frac{g_6}{4}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times\Big[(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi})^2(\vec{\rho}_\nu+\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi})^2+(\vec{\rho}_\mu-\vec{a}_{1,\mu}-Zw\partial_\mu\vec{\pi})^2\\
&\times(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi})^2\Big]+\frac{g_3}{4}\Big[(\vec{\rho}^\mu+\vec{a}_1^\mu+Zw\partial^\mu\vec{\pi})\cdot(\vec{\rho}^\nu+\vec{a}_1^\nu+Zw\partial^\nu\vec{\pi})\\
&\times(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi})\cdot(\vec{\rho}_\nu+\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi})+(\vec{\rho}^\mu-\vec{a}_1^\mu-Zw\partial^\mu\vec{\pi})\\
&\cdot(\vec{\rho}^\nu-\vec{a}_1^\nu-Zw\partial^\nu\vec{\pi})(\vec{\rho}_\mu-\vec{a}_{1,\mu}-Zw\partial_\mu\vec{\pi})\cdot(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi})\Big]\\
&+\frac{g_5}{4}\Big[(\omega_{N,\mu}+f_{1N,\mu}+Zw\partial_\mu\eta_N)^2(\omega_{N,\nu}-f_{1N,\nu}-Zw\partial_\nu\eta_N)^2\\
&+(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi})^2(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi})^2+(\omega_{N,\mu}+f_{1N,\mu}+Zw\partial_\mu\eta_N)^2\\
&\times(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi})^2+(\omega_{N,\nu}-f_{1N,\nu}-Zw\partial_\nu\eta_N)^2(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi})^2\Big]\\
\equiv\;&\mathscr{L}_{g_3,g_4,g_5,g_6}\;,
\end{aligned}
\tag{4.267}
$$

where we renamed $\mathscr{L}'_{g_3,g_4,g_5,g_6}\to\mathscr{L}_{g_3,g_4,g_5,g_6}$ in the last line. Finally, combining the results (4.256), (4.263), and (4.265)-(4.267), the full eLSM Lagrangian is given by

$$
\begin{aligned}
&\mathscr{L}_{eLSM}\\
&\equiv\mathscr{L}_{kin.,mass}+\mathscr{L}_{g_1,g_2}+\mathscr{L}_{\lambda_1,\lambda_2}+\mathscr{L}_{h_1,h_2,h_3}+\mathscr{L}_{g_3,g_4,g_5,g_6}\\
&=\frac{1}{2}(\partial_\mu\sigma_N)^2-\frac{1}{2}m_{\sigma_N}^2\sigma_N^2+\frac{1}{2}(\partial_\mu\eta_N)^2-\frac{1}{2}m_{\eta_N}^2\eta_N^2+\frac{1}{2}(\partial_\mu\vec{a}_0)^2-\frac{1}{2}m_{\vec{a}_0}^2\vec{a}_0^2+\frac{1}{2}(\partial_\mu\vec{\pi})^2-\frac{1}{2}m_{\vec{\pi}}^2\vec{\pi}^2-\frac{1}{4}\omega_N^{\mu\nu}\omega_{N,\mu\nu}\\
&\quad+\frac{1}{2}m_{\omega_N}^2(\omega_{N,\mu})^2-\frac{1}{4}f_{1N}^{\mu\nu}f_{1N,\mu\nu}+\frac{1}{2}m_{f_{1N}}^2(f_{1N,\mu})^2-\frac{1}{4}\vec{\rho}^{\mu\nu}\cdot\vec{\rho}_{\mu\nu}+\frac{1}{2}m_{\vec{\rho}}^2(\vec{\rho}_\mu)^2-\frac{1}{4}\vec{a}_1^{\mu\nu}\cdot\vec{a}_{1,\mu\nu}+\frac{1}{2}m_{\vec{a}_1}^2(\vec{a}_{1,\mu})^2\\
&\quad+g_1(\partial^\mu\sigma_N)\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)Z\eta_N+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\cdot Z\vec{\pi}\Big]-g_1Z(\partial^\mu\eta_N)\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)\\
&\quad\times\sigma_N+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\cdot\vec{a}_0\Big]+g_1(\partial^\mu\vec{a}_0)\cdot\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)Z\vec{\pi}+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)Z\eta_N+\vec{\rho}_\mu\times\vec{a}_0\Big]\\
&\quad-g_1Z(\partial^\mu\vec{\pi})\cdot\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)\vec{a}_0+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\sigma_N+Z\vec{\pi}\times\vec{\rho}_\mu\Big]+\frac{g_1^2}{2}\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)\\
&\quad\times Z\eta_N+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\cdot Z\vec{\pi}\Big]^2+\frac{g_1^2}{2}\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)Z\vec{\pi}+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)Z\eta_N+\vec{\rho}_\mu\times\vec{a}_0\Big]^2\\
&\quad+\frac{g_1^2}{2}\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)\sigma_N+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\cdot\vec{a}_0\Big]^2+\frac{g_1^2}{2}\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)\vec{a}_0+\sigma_N\\
&\quad\times\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)+Z\vec{\pi}\times\vec{\rho}_\mu\Big]^2+g_1^2\phi_N\Big(f_{1N}^\mu+Zw\partial^\mu\eta_N\Big)\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)\sigma_N+\Big(\vec{a}_{1,\mu}\\
&\quad+Zw\partial_\mu\vec{\pi}\Big)\cdot\vec{a}_0\Big]+g_1^2\phi_N\Big(\vec{a}_1^\mu+Zw\partial^\mu\vec{\pi}\Big)\cdot\Big[\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)\vec{a}_0+\sigma_N\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)+Z\vec{\pi}\times\vec{\rho}_\mu\Big]\\
&\quad+g_2\Big\{(\partial^\mu\vec{\rho}^\nu)\cdot\Big[\vec{\rho}_\nu\times\vec{\rho}_\mu+\Big(\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi}\Big)\times\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\Big]+\Big[\partial^\mu\Big(\vec{a}_1^\nu+Zw\partial^\nu\vec{\pi}\Big)\Big]\\
&\quad\cdot\Big[\vec{\rho}_\nu\times\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)+\Big(\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi}\Big)\times\vec{\rho}_\mu\Big]\Big\}-\frac{1}{4}\Big(\lambda_1+\frac{\lambda_2}{2}\Big)\Big[\sigma_N^4+4\phi_N\sigma_N^3+Z^4\eta_N^4+(\vec{a}_0^2)^2\\
&\quad+Z^4(\vec{\pi}^2)^2+2Z^2\sigma_N^2\eta_N^2+4\phi_NZ^2\sigma_N\eta_N^2+2Z^2\sigma_N^2\vec{\pi}^2+4\phi_NZ^2\sigma_N\vec{\pi}^2+2Z^2\eta_N^2\vec{a}_0^2\Big]-\frac{1}{2}\Big(\lambda_1+\frac{3\lambda_2}{2}\Big)\Big[\sigma_N^2\vec{a}_0^2\\
&\quad+2\phi_N\sigma_N\vec{a}_0^2+Z^4\eta_N^2\vec{\pi}^2+Z^2\vec{a}_0^2\vec{\pi}^2\Big]-\frac{\lambda_2}{8}\Big[8Z^2\sigma_N\eta_N\vec{a}_0\cdot\vec{\pi}+8\phi_NZ^2\eta_N\vec{a}_0\cdot\vec{\pi}-4Z^2(\vec{a}_0\cdot\vec{\pi})^2\Big]\\
&\quad+\frac{1}{4}(h_1+h_2)\Big\{(\sigma_N^2+Z^2\eta_N^2+\vec{a}_0^2+Z^2\vec{\pi}^2)\Big[(\omega_{N,\mu})^2+\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)^2+(\vec{\rho}_\mu)^2+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)^2\Big]\Big\}\\
&\quad+\frac{1}{2}(h_1+h_2)\phi_N\sigma_N\Big[(\omega_{N,\mu})^2+\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)^2+(\vec{\rho}_\mu)^2+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)^2\Big]+(h_2+h_3)\omega_N^\mu\\
&\quad\times\Big[\vec{\rho}_\mu\cdot(\sigma_N\vec{a}_0+Z^2\eta_N\vec{\pi}+\phi_N\vec{a}_0)+\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\cdot(\vec{a}_0\times Z\vec{\pi})\Big]+(h_2-h_3)\Big(f_{1N}^\mu+Zw\partial^\mu\eta_N\Big)\Big[\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\\
&\quad\cdot(\sigma_N\vec{a}_0+Z^2\eta_N\vec{\pi}+\phi_N\vec{a}_0)+\vec{\rho}_\mu\cdot(\vec{a}_0\times Z\vec{\pi})\Big]+\frac{h_3}{4}\Big[(\omega_{N,\mu})^2-\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)^2+(\vec{\rho}_\mu)^2-\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)^2\Big]\\
&\quad\times(\sigma_N^2+Z^2\eta_N^2+\vec{a}_0^2+Z^2\vec{\pi}^2)+\frac{h_3}{2}\phi_N\sigma_N\Big[(\omega_{N,\mu})^2-\Big(f_{1N,\mu}+Zw\partial_\mu\eta_N\Big)^2+(\vec{\rho}_\mu)^2-\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)^2\Big]\\
&\quad+h_3\left[Z\sigma_N\vec{\pi}-Z\eta_N\vec{a}_0+\phi_NZ\vec{\pi}\right]\cdot\Big[\Big(\vec{a}_1^\mu+Zw\partial^\mu\vec{\pi}\Big)\times\vec{\rho}_\mu\Big]+\frac{h_3}{2}\Big\{\Big[\vec{a}_0\times\Big(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\Big)\Big]^2
\end{aligned}
$$

$$+\left[Z\vec{\pi}\times\left(\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)\right]^2-\left(\vec{a}_0\times\vec{\rho}_\mu\right)^2-\left(Z\vec{\pi}\times\vec{\rho}_\mu\right)^2\Big\}+\left(\frac{g_3}{8}+\frac{g_4}{8}+\frac{g_6}{4}\right)\Big[\left(\omega_{N,\mu}+f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2$$
$$\times\left(\omega_{N,\nu}+f_{1N,\nu}+Zw\partial_\nu\eta_N\right)^2+\left(\omega_{N,\mu}-f_{1N,\mu}-Zw\partial_\mu\eta_N\right)^2\left(\omega_{N,\nu}-f_{1N,\nu}-Zw\partial_\nu\eta_N\right)^2$$
$$+2\left(\omega_{N,\mu}+f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi}\right)^2+2\left(\omega_{N,\mu}-f_{1N,\mu}-Zw\partial_\mu\eta_N\right)^2$$
$$\times\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi}\right)^2\Big]+\left(\frac{g_3}{2}+\frac{g_4}{2}\right)\Big[\left(\omega_N^\mu+f_{1N}^\mu+Zw\partial^\mu\eta_N\right)\left(\omega_N^\nu+f_{1N}^\nu+Zw\partial^\nu\eta_N\right)$$
$$\times\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)\cdot\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi}\right)+\left(\omega_N^\mu-f_{1N}^\mu-Zw\partial^\mu\eta_N\right)\left(\omega_N^\nu-f_{1N}^\nu-Zw\partial^\nu\eta_N\right)$$
$$\times\left(\vec{\rho}_\mu-\vec{a}_{1,\mu}-Zw\partial_\mu\vec{\pi}\right)\cdot\left(\vec{\rho}_\nu-\left(\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi}\right)\right]+\left(-\frac{g_3}{8}+\frac{g_4}{8}+\frac{g_6}{4}\right)\Big[\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)^2$$
$$\times\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi}\right)^2+\left(\vec{\rho}_\mu-\vec{a}_{1,\mu}-Zw\partial_\mu\vec{\pi}\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi}\right)^2\Big]+\frac{g_3}{4}\Big[\left(\vec{\rho}^\mu+\vec{a}_1^\mu+Zw\partial^\mu\vec{\pi}\right)$$
$$\cdot\left(\vec{\rho}^\nu+\vec{a}_1^\nu+Zw\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)\cdot\left(\vec{\rho}_\nu+\vec{a}_{1,\nu}+Zw\partial_\nu\vec{\pi}\right)+\left(\vec{\rho}^\mu-\vec{a}_1^\mu-Zw\partial^\mu\vec{\pi}\right)$$
$$\cdot\left(\vec{\rho}^\nu-\vec{a}_1^\nu-Zw\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\mu-\vec{a}_{1,\mu}-Zw\partial_\mu\vec{\pi}\right)\cdot\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi}\right)\Big]$$
$$+\frac{g_5}{4}\Big[\left(\omega_{N,\mu}+f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2\left(\omega_{N,\nu}-f_{1N,\nu}-Zw\partial_\nu\eta_N\right)^2+\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)^2$$
$$\times\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi}\right)^2+\left(\omega_{N,\mu}+f_{1N,\mu}+Zw\partial_\mu\eta_N\right)^2\left(\vec{\rho}_\nu-\vec{a}_{1,\nu}-Zw\partial_\nu\vec{\pi}\right)^2$$
$$+\left(\omega_{N,\nu}-f_{1N,\nu}-Zw\partial_\nu\eta_N\right)^2\left(\vec{\rho}_\mu+\vec{a}_{1,\mu}+Zw\partial_\mu\vec{\pi}\right)^2\Big]\ . \tag{4.268}$$

At this point, we have made all preparations in order to calculate the tree-level LECs of the eLSM. In the following three subsections, we will therefore consider three different versions of the eLSM and calculate the LECs for each of them. In the first version of the eLSM, we set all fields except the σ_N- and the $\vec{\pi}$-fields to zero. Then, apart from the anomaly and the explicit symmetry breaking term, the eLSM will reduce to the usual $O(4)$-model. In the second version, we keep all scalar/pseudoscalar fields and only set the vector and axial-vector degrees of freedom to zero. In this case, the eLSM model already has its $U(2)_L\times U(2)_R$ symmetry. And finally in the last subsection, we consider the full eLSM Lagrangian (4.268). This procedure will show us some kind of "evolution" for the LECs, depending on the level of "completeness" of the eLSM.

4.3.3 The $O(4)$ Linear Sigma Model

As mentioned before, we want to use this subsection to calculate the LECs of a reduced version of the eLSM. To this end, we consider the scalar/pseudoscalar matrix $\Phi(x)$ with the scalar isotriplet and the pseudoscalar isosinglet set to zero, i.e.,

$$\Phi(x)=\sigma_N(x)T^0+i\pi_i(x)T^i\ . \tag{4.269}$$

In addition to that, we set the left- and right-handed vector/axial-vector matrices to zero, so that

$$L^\mu(x)=R^\mu(x)=0\qquad\Longrightarrow\qquad D_\mu\Phi(x)\longrightarrow\partial_\mu\Phi(x)\ . \tag{4.270}$$

In this case, the eLSM Lagrangian reduces to

$$\begin{aligned}\mathscr{L}_{eLSM}&=\mathrm{Tr}\left\{\left(\partial^\mu\Phi\right)^\dagger\partial_\mu\Phi\right\}-m_0^2\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}-\lambda_1\left(\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}\right)^2-\lambda_2\mathrm{Tr}\left\{\left(\Phi^\dagger\Phi\right)^2\right\}\\&\quad+\mathrm{Tr}\left\{H\left(\Phi^\dagger+\Phi\right)\right\}+c_1\left(\det\Phi+\det\Phi^\dagger\right)\\&\equiv\mathscr{L}_{O(4)}\ .\end{aligned} \tag{4.271}$$

At this point, we can use the discussion of the previous subsection. Normally, we would now derive the explicit form of the above Lagrangian and then spontaneously break its $O(4)$ symmetry by setting $m_0^2\to-m_0^2$, with $m_0^2>0$. Then, we would have to find the minimum of the potential, which is, due to the explicit symmetry breaking term (4.195), of course, again in σ_N-direction. It is easy to see that the vacuum expectation value ϕ_N of the σ_N-field fulfills the same cubic equation as in Eq. (4.225). Finally, we would have to expand the initial σ_N-field around its vacuum expectation value ϕ_N and insert this expansion into the Lagrangian. Now, since the covariant derivative reduces to the usual 4-gradient, the mixing terms (4.235) do not enter the Lagrangian. Therefore, the explicit form of the Lagrangian can be obtained from Eq. (4.268) by setting all vector/axial-vector fields as well as $\vec{a}_0$ and η_N to zero

$$\begin{aligned}\mathscr{L}_{O(4)}&=\frac{1}{2}\left(\partial_\mu\sigma_N\right)^2-\frac{1}{2}m_{\sigma_N}^2\sigma_N^2+\frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2-\frac{1}{2}m_{\vec{\pi}}^2\vec{\pi}^2-\frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right)\Big[\sigma_N^4+4\phi_N\sigma_N^3+\left(\vec{\pi}^2\right)^2+2\sigma_N^2\vec{\pi}^2\\&\qquad+4\phi_N\sigma_N\vec{\pi}^2\Big]\ ,\end{aligned} \tag{4.272}$$

where we used that $g_1 = w = 0$ and therefore $Z = 1$. Then, the tree-level masses of the σ_N- and the $\vec{\pi}$-fields are given by

$$m_{\sigma_N}^2 = -c_1 - m_0^2 + 3\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 \; , \tag{4.273}$$

$$m_{\vec{\pi}}^2 = -c_1 - m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 \; . \tag{4.274}$$

At this point, we are able to calculate the LECs of Eq. (4.272). Following the discussion of Sec. [4.3.1], we use assumption (A1) and neglect the cubic and quartic powers of σ_N. The functional integral of this theory is then given by

$$\begin{aligned}\langle \sigma_N, \vec{\pi}, \infty | \sigma_N, \vec{\pi}, -\infty\rangle &= \mathcal{N} \int \mathscr{D}\sigma_N(x)\mathscr{D}\vec{\pi}(x)\; \exp\left\{ i \int \mathrm{d}^4x \; \mathscr{L}_{O(4)} \right\} \\ &= \mathcal{N}\int \mathscr{D}\vec{\pi}(x)\; \exp\left\{ i\int \mathrm{d}^4x\; \left[\frac{1}{2}\left(\partial_\mu \vec{\pi}\right)^2 - \frac{1}{2}m_{\vec{\pi}}^2\vec{\pi}^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left(\vec{\pi}^2\right)^2\right]\right\} I_{\sigma_N}[\vec{\pi}] \; ,\end{aligned} \tag{4.275}$$

where we defined

$$I_{\sigma_N}[\vec{\pi}] = \int \mathscr{D}\sigma_N(x)\; \exp\left\{ i\int \mathrm{d}^4x\; \left[\frac{1}{2}\left(\partial_\mu\sigma_N\right)^2 - \frac{1}{2}m_{\sigma_N}^2\sigma_N^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left(2\sigma_N^2\vec{\pi}^2 + 4\phi_N\sigma_N\vec{\pi}^2\right)\right]\right\} \; . \tag{4.276}$$

The above functional integral is of a Gaussian type and can be solved analytically. To this end, we have to rewrite the exponential by using an integration by parts

$$\begin{aligned} I_{\sigma_N}[\vec{\pi}] &= \int \mathscr{D}\sigma_N(x)\; \exp\left\{ -\frac{i}{2}\int \mathrm{d}^4x\sigma_N\left[\Box + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2\right]\sigma_N + 2\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N\vec{\pi}^2\sigma_N\right\} \\ &= \int \mathscr{D}\sigma_N(x)\; \exp\left\{ -\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\; \sigma_N(x)\mathscr{O}_{\vec{\pi}}(x,y)\sigma_N(y) + i\int \mathrm{d}^4x\; J_{\vec{\pi}}(x)\sigma_N(x)\right\} \; ,\end{aligned} \tag{4.277}$$

where we neglected the surface term and defined the operator

$$\mathscr{O}_{\vec{\pi}}(x,y) = \left[\Box_x + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2(x)\right]\delta^{(4)}(x-y) \tag{4.278}$$

and the "source"

$$J_{\vec{\pi}}(x) = -\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N\vec{\pi}^2 \; . \tag{4.279}$$

Now, in order to solve Eq. (4.277) by using Eq. (6.39), we have to perform an analytic continuation of the integral. Using Eqs. (4.18), (4.20), and (4.21), the Euclidean functional integral is given by

$$I_{\sigma_N}[\vec{\pi}] = \int \mathscr{D}\sigma_N(x_E)\; \exp\left\{ -\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\; \sigma_N(x_E)\mathscr{O}_{\vec{\pi},E}(x_E,y_E)\sigma_N(y_E) + \int \mathrm{d}^4x_E\; J_{\vec{\pi}}(x_E)\sigma_N(x_E)\right\} \tag{4.280}$$

with the Euclidean operator

$$\mathscr{O}_{\vec{\pi},E}(x_E,y_E) = \left[-\Box_{x,E} + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2(x_E)\right]\delta^{(4)}(x_E - y_E) \; . \tag{4.281}$$

Using Eq. (6.39), the solution of Eq. (4.280) is given by

$$\begin{aligned} I_{\sigma_N}[\vec{\pi}] &= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\vec{\pi},E}(x_E,y_E)\right]^{-1/2}\exp\left\{\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\; J_{\vec{\pi}}(x_E)\mathscr{O}^{-1}_{\vec{\pi},E}(x_E,y_E)J_{\vec{\pi}}(y_E)\right\} \\ &= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\vec{\pi}}(x,y)\right]^{-1/2}\exp\left\{\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\; J_{\vec{\pi}}(x)\mathscr{O}^{-1}_{\vec{\pi}}(x,y)J_{\vec{\pi}}(y)\right\} \\ &= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\vec{\pi}}(x,y)\right]^{-1/2}\exp\left\{ i\int \mathrm{d}^4x\mathrm{d}^4y\; \frac{\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2}{2}\phi_N^2\vec{\pi}^2(x)\left[\Box_x + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2(x)\right]^{-1} \right. \\ &\quad \left. \times\, \delta^{(4)}(x-y)\vec{\pi}^2(y)\right\} \\ &= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\vec{\pi}}(x,y)\right]^{-1/2}\exp\left\{ i\int \mathrm{d}^4x\; \frac{\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2}{2}\phi_N^2\vec{\pi}^2\left[\Box + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2\right]^{-1}\vec{\pi}^2\right\} \; ,\end{aligned} \tag{4.282}$$

where we performed the analytic continuation back to Minkowski space-time in the second line and introduced the inverse operator

$$\mathscr{O}_{\vec{\pi}}^{-1}(x,y) = \left[\Box + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2\right]^{-1} \delta^{(4)}(x-y) \ . \tag{4.283}$$

Inserting the solution (4.282) back into the transition amplitude (4.275), we find

$$\begin{aligned}
\langle \sigma_N, \vec{\pi}, \infty | \sigma_N, \vec{\pi}, -\infty \rangle &= \mathcal{N} \int \mathscr{D}\vec{\pi}(x) \ \exp\left\{ i \int \mathrm{d}^4x \ \left[\frac{1}{2}\left(\partial_\mu \vec{\pi}\right)^2 - \frac{1}{2} m_{\vec{\pi}}^2 \vec{\pi}^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left(\vec{\pi}^2\right)^2\right]\right\} \\
&\quad \times \mathcal{N}_{\sigma_N} \left[\det \mathscr{O}_{\vec{\pi}}(x,y)\right]^{-1/2} \exp\left\{ i \int \mathrm{d}^4x \ \frac{\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2}{2} \phi_N^2 \vec{\pi}^2 \left[\Box + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2\right]^{-1} \vec{\pi}^2 \right\} \\
&= \mathcal{N}_{eff} \int \mathscr{D}\vec{\pi}(x) \ \left[\det \mathscr{O}_{\vec{\pi}}(x,y)\right]^{-1/2} \exp\left\{ i S_{O(4),eff}[\vec{\pi}]\right\} \ ,
\end{aligned} \tag{4.284}$$

with $\mathcal{N}_{eff} \equiv \mathcal{N}\mathcal{N}_{\sigma_N}$ and where

$$S_{O(4),eff}[\vec{\pi}] = \int \mathrm{d}^4x \ \left\{ \frac{1}{2}\left(\partial_\mu \vec{\pi}\right)^2 - \frac{1}{2} m_{\vec{\pi}}^2 \vec{\pi}^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left(\vec{\pi}^2\right)^2 + \frac{\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2}{2} \phi_N^2 \vec{\pi}^2 \left[\Box + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2\right]^{-1} \vec{\pi}^2 \right\} \tag{4.285}$$

defines the tree-level effective action of Eq. (4.272). Now, in order to obtain a local effective action, we have to expand the inverse operator. According to the discussion of Sec. [4.3.1], we are only interested in four-pion interaction terms with a maximum number of four space-time derivatives. Keeping this constraint in mind, the inverse operator is given by

$$\begin{aligned}
\left[\Box + m_{\sigma_N}^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2\right]^{-1} &= \left\{ \left(\Box + m_{\sigma_N}^2\right) \left[1 + \left(\Box + m_{\sigma_N}^2\right)^{-1} \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2\right]\right\}^{-1} \\
&= \sum_{m=0}^{\infty} (-1)^m \left[\left(\Box + m_{\sigma_N}^2\right)^{-1} \left(\lambda_1 + \frac{\lambda_2}{2}\right)\vec{\pi}^2\right]^m \frac{1}{m_{\sigma_N}^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\Box}{m_{\sigma_N}^2}\right)^n \\
&= \frac{1}{m_{\sigma_N}^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\Box}{m_{\sigma_N}^2}\right)^n + \text{terms with two or more } \vec{\pi}\text{-fields} \ .
\end{aligned} \tag{4.286}$$

Now, we have to consider all terms of the first sum up to $n = 2$ in order to obtain terms with four derivatives, so that the local effective action is given by

$$\begin{aligned}
S_{O(4),eff}^{n=2,m=0}[\vec{\pi}] &= \int \mathrm{d}^4x \ \left\{ \frac{1}{2}\left(\partial_\mu \vec{\pi}\right)^2 - \frac{1}{2} m_{\vec{\pi}}^2 \vec{\pi}^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left(\vec{\pi}^2\right)^2 + \frac{\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2}{2} \phi_N^2 \vec{\pi}^2 \frac{1}{m_{\sigma_N}^2} \left(1 - \frac{\Box}{m_{\sigma_N}^2} + \frac{\Box^2}{m_{\sigma_N}^4}\right)\vec{\pi}^2 \right\} \\
&= \int \mathrm{d}^4x \ \left\{ \frac{1}{2}\left(\partial_\mu \vec{\pi}\right)^2 - \frac{1}{2} m_{\vec{\pi}}^2 \vec{\pi}^2 + \left[\frac{\phi_N^2}{2 m_{\sigma_N}^2}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right) - \frac{2\phi_N^2 m_{\vec{\pi}}^4}{m_{\sigma_N}^6}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2\right]\left(\vec{\pi}^2\right)^2 \right. \\
&\quad \left. + \left[\frac{2\phi_N^2}{m_{\sigma_N}^4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 + \frac{8\phi_N^2 m_{\vec{\pi}}^2}{m_{\sigma_N}^6}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2\right]\left(\vec{\pi} \cdot \partial_\mu \vec{\pi}\right)^2 + \frac{2\phi_N^2}{m_{\sigma_N}^6}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 \left(\partial_\mu \vec{\pi}\right)^2 \left(\partial_\nu \vec{\pi}\right)^2 \right\} \\
&= \int \mathrm{d}^4x \ \left[\frac{1}{2}\left(\partial_\mu \vec{\pi}\right)^2 - \frac{1}{2} m_{\vec{\pi}}^2 \vec{\pi}^2 + C_{1,O(4)}\left(\vec{\pi}^2\right)^2 + C_{2,O(4)}\left(\vec{\pi} \cdot \partial_\mu \vec{\pi}\right)^2 + C_{3,O(4)}\left(\partial_\mu \vec{\pi}\right)^2 \left(\partial_\nu \vec{\pi}\right)^2 \right. \\
&\quad \left. + C_{4,O(4)}\left(\partial_\mu \vec{\pi} \cdot \partial_\nu \vec{\pi}\right)^2\right] \ ,
\end{aligned} \tag{4.287}$$

where we used

$$\int \mathrm{d}^4x \ \vec{\pi}^2 \Box \vec{\pi}^2 = -\int \mathrm{d}^4x \ \left(\partial_\mu \vec{\pi}^2\right) \partial^\mu \vec{\pi}^2 = -4 \int \mathrm{d}^4x \ \left(\vec{\pi} \cdot \partial_\mu \vec{\pi}\right)^2 \tag{4.288}$$

as well as

$$\begin{aligned}
\int \mathrm{d}^4x \ \vec{\pi}^2 \Box^2 \vec{\pi}^2 &= \int \mathrm{d}^4x \ \left(\Box \vec{\pi}^2\right) \Box \vec{\pi}^2 \\
&= \int \mathrm{d}^4x \ \left[\left(\partial_\mu \vec{\pi}\right)^2 \left(\partial_\nu \vec{\pi}\right)^2 - 2 m_{\vec{\pi}}^2 \vec{\pi}^2 \left(\partial_\mu \vec{\pi}\right)^2 + m_{\vec{\pi}}^4 \left(\vec{\pi}^2\right)^2\right] \\
&= \int \mathrm{d}^4x \ \left[\left(\partial_\mu \vec{\pi}\right)^2 \left(\partial_\nu \vec{\pi}\right)^2 + 4 m_{\vec{\pi}}^2 \left(\vec{\pi} \cdot \partial_\mu \vec{\pi}\right)^2 - m_{\vec{\pi}}^4 \left(\vec{\pi}^2\right)^2\right] \ ,
\end{aligned} \tag{4.289}$$

the free Klein-Gordon equation for the $\vec{\pi}$-fields, and identified the LECs of the $O(4)$ version of the eLSM as

$$C_{1,O(4)} = \frac{\phi_N^2}{2m_{\sigma_N}^2}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right) - \frac{2\phi_N^2 m_{\vec{\pi}}^4}{m_{\sigma_N}^6}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 , \tag{4.290}$$

$$C_{2,O(4)} = \frac{2\phi_N^2}{m_{\sigma_N}^4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 + \frac{8\phi_N^2 m_{\vec{\pi}}^2}{m_{\sigma_N}^6}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 , \tag{4.291}$$

$$C_{3,O(4)} = \frac{2\phi_N^2}{m_{\sigma_N}^6}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 , \tag{4.292}$$

$$C_{4,O(4)} = 0 . \tag{4.293}$$

At this point, we want to discuss the above results at a qualitative level, since the numerical evaluation of Eqs. (4.290)-(4.293) is part of the discussion in Sec. [4.3.6]. First of all, we observe that one of the above constants vanishes. Due to Eq. (4.293), we at least qualitatively conclude that the $O(4)$ version of the eLSM is not sufficient in order to describe the low-energy regime of QCD in the correct way, because $C_{4,\chi PT}$ is different from zero. It should be taken into account that this statement is only verified at tree level. A more detailed discussion of these results can be found in Sec. [4.3.6]. Finally, before we turn to a more complete version of the eLSM, we consider the $(\vec{\pi}^2)^2$ term: In the discussion of Sec. [3.2], we concluded that, in the case of an exact symmetry, Nambu-Goldstone bosons may only interact among themselves through derivatively coupled vertices. Therefore, we want to study the chiral limit of Eq. (4.287) in order to show that $C_{1,O(4)}$ indeed vanishes in the case of massless pions. To this end, we observe that the $\vec{\pi}$ and the σ_N masses can be written as

$$m_{\sigma_N}^2 = m_{\vec{\pi}}^2 + 2\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 , \tag{4.294}$$

$$m_{\vec{\pi}}^2 = \frac{h_{N,0}}{\phi_N} , \tag{4.295}$$

where we used Eq. (4.225) in the second line and Eq. (4.274) in order to rewrite the σ_N mass. Now it is obvious that in the case of an exact $O(4)$ symmetry ($h_{N,0} = 0$), the pion becomes massless. In this limit, the σ_N mass becomes

$$m_{\sigma_N, h_{N,0}=0}^2 = 2\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 . \tag{4.296}$$

Now, we insert the above σ_N mass and $m_{\vec{\pi}} = 0$ into Eq. (4.290) and find

$$C_{1,O(4)} = \frac{\phi_N^2}{2m_{\sigma_N, h_{N,0}=0}^2}\left(\lambda_1 + \frac{\lambda_2}{2}\right)^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right) = 0 , \tag{4.297}$$

i.e., the $(\vec{\pi}^2)^2$ interaction term indeed vanishes. This kind of cross-check can be used to show, that our approach and the assumptions (A1)-(A3) do not destroy the symmetry properties of the effective Lagrangian.

4.3.4 The $U(2)_L \times U(2)_R$ Linear Sigma Model without Vector/Axial-Vector Mesons

In this Subsection, we want to consider a more complete version of the eLSM. To this end, we involve all scalar and pseudoscalar degrees of freedom in our considerations, i.e., the scalar/pseudoscalar matrix $\Phi(x)$ is then given by

$$\Phi(x) = (\sigma_N + i\eta_N)\, T^0 + (a_{0,i} + i\pi_i)\, T^i . \tag{4.298}$$

Similar to the previous subsection, we set all vector and axial-vector mesons to zero

$$L^\mu(x) = R^\mu(x) = 0 , \tag{4.299}$$

so that the covariant derivative (4.148) again reduces to a usual 4-gradient

$$D_\mu \Phi(x) \longrightarrow \partial_\mu \Phi(x) . \tag{4.300}$$

Due to Eqs. (4.299) and (4.300), the eLSM Lagrangian takes the same form as in the considerations of the previous Subsection, i.e.,

$$\begin{aligned}\mathscr{L}_{eLSM} &= \mathrm{Tr}\left\{(\partial^\mu\Phi)^\dagger\,\partial_\mu\Phi\right\} - m_0^2\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\} - \lambda_1\left(\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}\right)^2 - \lambda_2\mathrm{Tr}\left\{\left(\Phi^\dagger\Phi\right)^2\right\}\\ &\quad + \mathrm{Tr}\left\{H\left(\Phi^\dagger + \Phi\right)\right\} + c_1\left(\det\Phi + \det\Phi^\dagger\right)\\ &\equiv \mathscr{L}_{U(2)_L\times U(2)_R} .\end{aligned} \tag{4.301}$$

The important difference to the previous case is now given by the form of the scalar/pseudoscalar matrix $\Phi(x)$. While Eq. (4.298) is an object that has a well-defined transformation behavior with respect to $U(2)_L \times U(2)_R$, the symmetry of the previous version of the eLSM is smaller. This can be seen by the explicit expression (4.271). There, it is possible to sort the scalar isosinglet σ_N and the pseudoscalar isotriplet $\vec{\pi}$ into a four-dimensional vector in field space. The resulting Lagrangian is then symmetric with respect to global $O(4)$ rotations. Now it is well known that $O(4)$ is at least locally isomorphic to $SU(2) \times SU(2)$, so that we have a smaller symmetry as in Eq. (4.301).

Similar to the previous consideration, the absence of the vector/axial-vecotr sector and therefore of the covariant derivative does not require any shifts of field variables in order to diagonalize the Lagrangian. Therefore, the explicit form of Eq. (4.301) can be obtained from Eq. (4.268) by neglecting the vector/axial-vector degrees of freedom as well as their couplings and setting $g_1 = w = 0$, $Z = 1$

$$\begin{aligned}\mathscr{L}_{U(2)_L\times U(2)_R} =& \frac{1}{2}\left(\partial_\mu\sigma_N\right)^2 - \frac{1}{2}m^2_{\sigma_N}\sigma_N^2 + \frac{1}{2}\left(\partial_\mu\eta_N\right)^2 - \frac{1}{2}m^2_{\eta_N}\eta_N^2 + \frac{1}{2}\left(\partial_\mu\vec{a}_0\right)^2 - \frac{1}{2}m^2_{\vec{a}_0}\vec{a}_0^2 + \frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2 - \frac{1}{2}m^2_{\vec{\pi}}\vec{\pi}^2 \\ &- \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left[\sigma_N^4 + 4\phi_N\sigma_N^3 + \eta_N^4 + \left(\vec{a}_0^2\right)^2 + \left(\vec{\pi}^2\right)^2 + 2\sigma_N^2\eta_N^2 + 4\phi_N\sigma_N\eta_N^2 + 2\sigma_N^2\vec{\pi}^2\right. \\ &\left. + 4\phi_N\sigma_N\vec{\pi}^2 + 2\eta_N^2\vec{a}_0^2\right] - \frac{1}{2}\left(\lambda_1 + \frac{3\lambda_2}{2}\right)\left[\sigma_N^2\vec{a}_0^2 + 2\phi_N\sigma_N\vec{a}_0^2 + \eta_N^2\vec{\pi}^2 + \vec{a}_0^2\vec{\pi}^2\right] \\ &- \frac{\lambda_2}{8}\left[8\sigma_N\eta_N\vec{a}_0\cdot\vec{\pi} + 8\phi_N\eta_N\vec{a}_0\cdot\vec{\pi} - 4\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right] \;, \end{aligned} \tag{4.302}$$

where the vacuum expectation value of the σ_N-field again fulfills Eq. (4.225). The tree-level masses of the scalar/pseudoscalar mesons are now given by

$$m^2_{\sigma_N} = -c_1 - m_0^2 + 3\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 \;, \tag{4.303}$$

$$m^2_{\eta_N} = c_1 - m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 \;, \tag{4.304}$$

$$m^2_{\vec{a}_0} = c_1 - m_0^2 + \left(\lambda_1 + \frac{3\lambda_2}{2}\right)\phi_N^2 \;, \tag{4.305}$$

$$m^2_{\vec{\pi}} = -c_1 - m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 \;. \tag{4.306}$$

Again, we can use assumption (A1) in order to eliminate redundant terms from the Lagrangian (4.302). With $f = \{\sigma_N, \eta_N, \vec{a}_0, \vec{\pi}\}$, the transition amplitude can be written as

$$\begin{aligned}\langle f,\infty|f,-\infty\rangle =& \mathcal{N}\int \mathscr{D}\sigma_N(x)\mathscr{D}\eta_N(x)\mathscr{D}\vec{a}_0(x)\mathscr{D}\vec{\pi}(x)\;\exp\left\{i\int \mathrm{d}^4x\;\mathscr{L}_{U(2)_L\times U(2)_R}\right\} \\ =& \mathcal{N}\int \mathscr{D}\vec{\pi}(x)\exp\left\{i\int \mathrm{d}^4x\;\left[\frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2 - \frac{1}{2}m^2_{\vec{\pi}}\vec{\pi}^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left(\vec{\pi}^2\right)^2\right]\right\} I_{\sigma_N}[\vec{\pi}]I_{\eta_N}[\vec{\pi}]I_{\vec{a}_0}[\vec{\pi}] \;, \end{aligned} \tag{4.307}$$

where

$$I_{\sigma_N}[\vec{\pi}] = \int \mathscr{D}\sigma_N(x)\;\exp\left\{i\int \mathrm{d}^4x\;\left[\frac{1}{2}\left(\partial_\mu\sigma_N\right)^2 - \frac{1}{2}m^2_{\sigma_N}\sigma_N^2 - \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right)\left(2\sigma_N^2\vec{\pi}^2 + 4\phi_N\sigma_N\vec{\pi}^2\right)\right]\right\} \;, \tag{4.308}$$

$$I_{\eta_N}[\vec{\pi}] = \int \mathscr{D}\eta_N(x)\;\exp\left\{i\int \mathrm{d}^4x\;\left[\frac{1}{2}\left(\partial_\mu\eta_N\right)^2 - \frac{1}{2}m^2_{\eta_N}\eta_N^2 - \frac{1}{2}\left(\lambda_1 + \frac{3\lambda_2}{2}\right)\eta_N^2\vec{\pi}^2\right]\right\} \;, \tag{4.309}$$

$$I_{\vec{a}_0}[\vec{\pi}] = \int \mathscr{D}\vec{a}_0(x)\;\exp\left\{i\int \mathrm{d}^4x\;\left[\frac{1}{2}\left(\partial_\mu\vec{a}_0\right)^2 - \frac{1}{2}m^2_{\vec{a}_0}\vec{a}_0^2 - \frac{1}{2}\left(\lambda_1 + \frac{3\lambda_2}{2}\right)\vec{a}_0^2\vec{\pi}^2 + \frac{\lambda_2}{2}\left(\vec{a}_0\cdot\vec{\pi}\right)^2\right]\right\} \;. \tag{4.310}$$

Our first observation is that the use of assumption (A1) decouples all Gaussian functional integrals that correspond to the "heavy fields", which of course simplifies the following calculation. Furthermore, due to Eqs. (4.309) and (4.310), the only functional integral that, at tree-level, will introduce four-pion interaction terms is given by Eq. (4.308). This directly follows from the fact, that the functional integrals with respect to η_N and $\vec{a}_0$ are not shifted, so that their solution only corresponds to a functional determinant. We also observe that Eq. (4.308) is exactly the same Gaussian functional integral that we already solved in the previous Subsection. For the sake of completeness, we quote the final result of this integral

$$I_{\sigma_N}[\vec{\pi}] = \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\vec{\pi}}(x,y)\right]^{-1/2}\exp\left\{i\int \mathrm{d}^4x\,\frac{\left(\lambda_1+\frac{\lambda_2}{2}\right)^2}{2}\phi_N^2\vec{\pi}^2\left[\Box+m_{\sigma_N}^2+\left(\lambda_1+\frac{\lambda_2}{2}\right)\vec{\pi}^2\right]^{-1}\vec{\pi}^2\right\}\,. \tag{4.311}$$

Now, we focus on the two remaining functional integrals. The η_N integral can be written as

$$\begin{aligned}I_{\eta_N}[\vec{\pi}] &= \int \mathscr{D}\eta_N(x)\;\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\;\eta_N\left[\Box+m_{\eta_N}^2+\left(\lambda_1+\frac{3\lambda_2}{2}\right)\vec{\pi}^2\right]\eta_N\right\}\\ &= \int \mathscr{D}\eta_N(x)\;\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\;\eta_N(x)\mathscr{O}_{\vec{\pi},\eta_N}(x,y)\eta_N(y)\right\}\,,\end{aligned} \tag{4.312}$$

where we integrated the kinetic term by parts and introduced the operator

$$\mathscr{O}_{\vec{\pi},\eta_N}(x,y) = \left[\Box_x+m_{\eta_N}^2+\left(\lambda_1+\frac{3\lambda_2}{2}\right)\vec{\pi}^2(x)\right]\delta^{(4)}(x-y)\,. \tag{4.313}$$

In order to solve Eq. (4.312), we have to Wick-rotate the functional integral by using Eqs. (4.18), (4.20), and (4.21). The Euclidean functional integral is then given by

$$\begin{aligned}I_{\eta_N}[\vec{\pi}] &= \int \mathscr{D}\eta_N(x_E)\;\exp\left\{-\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\;\eta_N(x_E)\mathscr{O}_{\vec{\pi},\eta_N,E}(x_E,y_E)\eta_N(y_E)\right\}\\ &= \mathcal{N}_{\eta_N}\left[\det\mathscr{O}_{\vec{\pi},\eta_N,E}(x_E,y_E)\right]^{-1/2}\\ &= \mathcal{N}_{\eta_N}\left[\det\mathscr{O}_{\vec{\pi},\eta_N}(x,y)\right]^{-1/2}\,,\end{aligned} \tag{4.314}$$

where we used Eq. (6.39), introduced the Euclidean operator

$$\mathscr{O}_{\vec{\pi},\eta_N,E}(x_E,y_E) = \left[-\Box_{x,E}+m_{\eta_N}^2+\left(\lambda_1+\frac{3\lambda_2}{2}\right)\vec{\pi}^2(x_E)\right]\delta^{(4)}(x_E-y_E)\,, \tag{4.315}$$

and performed the analytic continuation back to Minkowski space-time in the last line of Eq. (4.314). Finally, we are left with Eq. (4.310). In order to rewrite this functional integral, we have to introduce isospin indices, since the resulting operator is not diagonal in isospin space. We find

$$\begin{aligned}I_{\vec{a}_0}[\vec{\pi}] &= \int \mathscr{D}\vec{a}_0(x)\;\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\;a_{0,i}\left\{\left[\Box+m_{\vec{a}_0}^2+\frac{1}{2}\left(\lambda_1+\frac{3\lambda_2}{2}\right)\pi_k\pi^k\right]g^{ij}-\lambda_2\pi^i\pi^j\right\}a_{0,j}\right\}\\ &= \int \mathscr{D}\vec{a}_0(x)\;\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\;a_{0,i}(x)\mathscr{O}^{ij}_{\vec{\pi},\vec{a}_0}(x,y)a_{0,j}(y)\right\}\,,\end{aligned} \tag{4.316}$$

where

$$\mathscr{O}^{ij}_{\vec{\pi},\vec{a}_0}(x,y) = \left\{\left[\Box_x+m_{\vec{a}_0}^2+\frac{1}{2}\left(\lambda_1+\frac{3\lambda_2}{2}\right)\pi_k(x)\pi^k(x)\right]g^{ij}-\lambda_2\pi^i(x)\pi^j(x)\right\}\delta^{(4)}(x-y)\,. \tag{4.317}$$

In the upcoming Subsection, we will encounter this type of non-diagonal operator again, when we have to solve the functional integral with respect to the $\vec{\rho}$-fields. For the moment, we return to Eq. (4.316) and transform the functional integral to Euclidean space-time

$$\begin{aligned}I_{\vec{a}_0}[\vec{\pi}] &= \int \mathscr{D}\vec{a}_0(x_E)\;\exp\left\{-\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\;a_{0,i}(x_E)\mathscr{O}^{ij}_{\vec{\pi},\vec{a}_0,E}(x_E,y_E)a_{0,j}(y_E)\right\}\\ &= \mathcal{N}_{\vec{a}_0}\left[\det\mathscr{O}_{\vec{\pi},\vec{a}_0,E}(x_E,y_E)\right]^{-1/2}\\ &= \mathcal{N}_{\vec{a}_0}\left[\det\mathscr{O}_{\vec{\pi},\vec{a}_0}(x,y)\right]^{-1/2}\,,\end{aligned} \tag{4.318}$$

where

$$\mathscr{O}_{\vec{\pi},\vec{a}_0,E}(x_E,y_E) = \left\{\left[\Box_{x,E}+m_{\vec{a}_0}^2+\frac{1}{2}\left(\lambda_1+\frac{3\lambda_2}{2}\right)\pi_k(x_E)\pi^k(x_E)\right]g^{ij}-\lambda_2\pi^i(x_E)\pi^j(x_E)\right\}\delta^{(4)}(x_E-y_E) \tag{4.319}$$

defines the Euclidean version of the operator (4.317). Finally, combining Eqs. (4.311), (4.314), and (4.318), the transition amplitude (4.307) becomes

$$
\begin{aligned}
\langle f,\infty|f,-\infty\rangle &= \mathcal{N}\int \mathscr{D}\vec{\pi}(x)\exp\left\{i\int \mathrm{d}^4x\,\left[\frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2-\frac{1}{2}m_{\vec{\pi}}^2\vec{\pi}^2-\frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right)\left(\vec{\pi}^2\right)^2\right]\right\}\\
&\quad\times\mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\vec{\pi}}(x,y)\right]^{-1/2}\exp\left\{i\int \mathrm{d}^4x\,\frac{\left(\lambda_1+\frac{\lambda_2}{2}\right)^2}{2}\phi_N^2\vec{\pi}^2\left[\Box+m_{\sigma_N}^2+\left(\lambda_1+\frac{\lambda_2}{2}\right)\vec{\pi}^2\right]^{-1}\vec{\pi}^2\right\}\\
&\quad\times\mathcal{N}_{\eta_N}\left[\det\mathscr{O}_{\vec{\pi},\eta_N}(x,y)\right]^{-1/2}\mathcal{N}_{\vec{a}_0}\left[\det\mathscr{O}_{\vec{\pi},\vec{a}_0}(x,y)\right]^{-1/2}\\
&=\mathcal{N}_{eff}\int\mathscr{D}\vec{\pi}(x)\,\left[\det\mathscr{O}_{\vec{\pi}}(x,y)\right]^{-1/2}\left[\det\mathscr{O}_{\vec{\pi},\eta_N}(x,y)\right]^{-1/2}\left[\det\mathscr{O}_{\vec{\pi},\vec{a}_0}(x,y)\right]^{-1/2}\exp\left\{iS_{U(2)_L\times U(2)_R,eff}[\vec{\pi}]\right\}\ ,
\end{aligned}
\tag{4.320}
$$

where the tree-level effective action $S_{U(2)_L\times U(2)_R,eff}[\vec{\pi}]$ is given by Eq. (4.285), since only the σ_N has a tree-level contribution to the four-pion interaction terms. Performing the same expansion as in Eq. (4.286) and using an integration by parts, one finally obtains the same result as in Eq. (4.287)

$$
\begin{aligned}
&S^{n=2,m=0}_{U(2)_L\times U(2)_R,eff}[\vec{\pi}]\\
&=\int \mathrm{d}^4x\,\left\{\frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2-\frac{1}{2}m_{\vec{\pi}}^2\vec{\pi}^2-\frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right)\left(\vec{\pi}^2\right)^2+\frac{\left(\lambda_1+\frac{\lambda_2}{2}\right)^2}{2}\phi_N^2\vec{\pi}^2\frac{1}{m_{\sigma_N}^2}\left(1-\frac{\Box}{m_{\sigma_N}^2}+\frac{\Box^2}{m_{\sigma_N}^4}\right)\vec{\pi}^2\right\}\\
&=\int \mathrm{d}^4x\,\left\{\frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2-\frac{1}{2}m_{\vec{\pi}}^2\vec{\pi}^2+\left[\frac{\phi_N^2}{2m_{\sigma_N}^2}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2-\frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right)-\frac{2\phi_N^2m_{\vec{\pi}}^4}{m_{\sigma_N}^6}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2\right]\left(\vec{\pi}^2\right)^2\right.\\
&\quad\left.+\left[\frac{2\phi_N^2}{m_{\sigma_N}^4}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2+\frac{8\phi_N^2m_{\vec{\pi}}^2}{m_{\sigma_N}^6}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2\right]\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2+\frac{2\phi_N^2}{m_{\sigma_N}^6}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2\right\}\\
&=\int \mathrm{d}^4x\,\left[\frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2-\frac{1}{2}m_{\vec{\pi}}^2\vec{\pi}^2+C_{1,U(2)_L\times U(2)_R}\left(\vec{\pi}^2\right)^2+C_{2,U(2)_L\times U(2)_R}\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2+C_{3,U(2)_L\times U(2)_R}\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2\right.\\
&\quad\left.+C_{4,U(2)_L\times U(2)_R}\left(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi}\right)^2\right]\ ,
\end{aligned}
\tag{4.321}
$$

which yields the same tree-level LECs as in the previous discussion

$$
C_{1,SU(2)_L\times SU(2)_R}=\frac{\phi_N^2}{2m_{\sigma_N}^2}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2-\frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right)-\frac{2\phi_N^2m_{\vec{\pi}}^4}{m_{\sigma_N}^6}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2\ , \tag{4.322}
$$

$$
C_{2,SU(2)_L\times SU(2)_R}=\frac{2\phi_N^2}{m_{\sigma_N}^4}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2+\frac{8\phi_N^2m_{\vec{\pi}}^2}{m_{\sigma_N}^6}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2\ , \tag{4.323}
$$

$$
C_{3,SU(2)_L\times SU(2)_R}=\frac{2\phi_N^2}{m_{\sigma_N}^6}\left(\lambda_1+\frac{\lambda_2}{2}\right)^2\ , \tag{4.324}
$$

$$
C_{4,SU(2)_L\times SU(2)_R}=0\ . \tag{4.325}
$$

It is therefore clear that the final discussion of Sec. [4.3.3] concerning the $(\vec{\pi}^2)^2$ interaction terms also applies to this case. Finally, we have to finish this calculation with a brief qualitative discussion of the results. It is obvious that the extension of the previous $O(4)$ symmetry to $U(2)_L\times U(2)_R$ is not sufficient in order to describe the low-energy dynamics of QCD at tree-level in an appropriate way, if we neglect the vector/axial-vector degrees of freedom. But, in contrast to the previous case, at one-loop order, we obtain more contributions which arise from the inclusion of the η_N- and the $\vec{a}_0$-fields.

4.3.5 The $U(2)_L\times U(2)_R$ Extended Linear Sigma Model

In the last two subsections, we considered slimmed versions of the eLSM and found that, at tree-level, the absence of vector/axial-vector degrees of freedom leads to a wrong description of the low-energy regime of QCD. In the following, we therefore include all vector/axial-vector mesons and calculate the LECs for the complete eLSM. This means that the scalar/pseudoscalar matrix $\Phi(x)$ and the left- and right-handed matrices $L^\mu(x)$ and $R^\mu(x)$ are given by

$$\Phi(x) = [\sigma_N(x) + i\eta_N(x)]\,T^0 + [a_{0,i}(x) + i\pi_i(x)]\,T^i \ , \tag{4.326}$$
$$L^\mu(x) = \left[\omega_N^\mu(x) + f_{1,N}^\mu(x)\right] T^0 + \left[\rho_i^\mu(x) + a_{1,i}^\mu(x)\right] T^i \ , \tag{4.327}$$
$$R^\mu(x) = \left[\omega_N^\mu(x) - f_{1,N}^\mu(x)\right] T^0 + \left[\rho_i^\mu(x) - a_{1,i}^\mu(x)\right] T^i \ . \tag{4.328}$$

Therefore, the covariant derivative, defined in Sec. [4.2.2.2.1], takes its usual form, i.e.,

$$D_\mu\Phi(x) = \partial_\mu\Phi(x) - ig_1\left[L^\mu(x)\Phi(x) - \Phi(x)R^\mu(x)\right] \ . \tag{4.329}$$

As already introduced in detail in Sec. [4.2.2], the full eLSM Lagrangian is given by

$$\begin{aligned}
\mathscr{L}_{eLSM} =\ & \mathrm{Tr}\left\{[D^\mu\Phi]^\dagger\,[D_\mu\Phi]\right\} - m_0^2\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\} - \lambda_1\left(\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}\right)^2 - \lambda_2\mathrm{Tr}\left\{\left[\Phi^\dagger\Phi\right]^2\right\} \\
& - \frac{1}{4}\mathrm{Tr}\left\{L^{\mu\nu}L_{\mu\nu} + R^{\mu\nu}R_{\mu\nu}\right\} + \mathrm{Tr}\left\{\left(\frac{m_1^2}{2} + \Delta\right)[L^\mu L_\mu + R^\mu R_\mu]\right\} + \mathrm{Tr}\left\{H\left[\Phi^\dagger + \Phi\right]\right\} \\
& + c_1\left[\det\Phi + \det\Phi^\dagger\right] + i\frac{g_2}{2}\left[\mathrm{Tr}\left\{L^{\mu\nu}\left[L_\mu, L_\nu\right]_-\right\} + \mathrm{Tr}\left\{R^{\mu\nu}\left[R_\mu, R_\nu\right]_-\right\}\right] \\
& + \frac{h_1}{2}\mathrm{Tr}\left\{\Phi^\dagger\Phi\right\}\mathrm{Tr}\left\{L^\mu L_\mu + R^\mu R_\mu\right\} + h_2\mathrm{Tr}\left\{\left|L^\mu\Phi\right|^2 + \left|\Phi R^\mu\right|^2\right\} + 2h_3\mathrm{Tr}\left\{\Phi R^\mu\Phi^\dagger L_\mu\right\} \\
& + g_3\left[\mathrm{Tr}\left\{L^\mu L^\nu L_\mu L_\nu\right\} + \mathrm{Tr}\left\{R^\mu R^\nu R_\mu R_\nu\right\}\right] + g_4\left[\mathrm{Tr}\left\{L^\mu L_\mu L^\nu L_\nu\right\} + \mathrm{Tr}\left\{R^\mu R_\mu R^\nu R_\nu\right\}\right] \\
& + g_5\mathrm{Tr}\left\{L^\mu L_\mu\right\}\mathrm{Tr}\left\{R^\mu R_\mu\right\} + g_6\left[\mathrm{Tr}\left\{L^\mu L_\mu\right\}\mathrm{Tr}\left\{L^\nu L_\nu\right\} + \mathrm{Tr}\left\{R^\mu R_\mu\right\}\mathrm{Tr}\left\{R^\nu R_\nu\right\}\right] \ .
\end{aligned} \tag{4.330}$$

In Sec. [4.3.2], we saw that the modeling of spontaneous chiral symmetry breaking leads to non-diagonal terms in the eLSM Lagrangian, which had to be eliminated by shifting the axial-vector field variables according to Eqs. (4.242) and (4.243). These field redefinitions gave rise to a wrong normalization of the kinetic terms of the η_N- and the $\vec{\pi}$-fields. To this end, we had to renormalize the fields by introducing an appropriate scaling factor (4.264). After performing these manipulations the explicit form of the eLSM Lagrangian is given by Eq. (4.268). This Lagrangian now yields the basis for the following discussion.

Similar to the discussions of the previous subsections, we now have to apply the assumptions (A1)-(A3) in order to get rid of all interaction terms, that are redundant for a tree-level analysis of the LECs. In principle, we now have to write out all terms of Eq. (4.268) explicitly and then apply (A1)-(A3). Since this procedure is quite tedious and not very meaningful, we only quote the final result. It can be shown that the eLSM Lagrangian (4.268) can be cast into the form

$$\mathscr{L}_{eLSM} = \mathscr{L}_{kin.,mass} + \mathscr{L}_{R\pi\pi} + \mathscr{L}_{RR\pi\pi} + \mathscr{L}_{4\pi} \ , \tag{4.331}$$

where $\mathscr{L}_{kin.,mass}$ is given by Eq. (4.256) and the tree-level masses are given by Eqs. (4.257)-(4.262). The second and third term in Eq. (4.331) contain interaction terms that include one or two resonances and two pion fields. The explicit form of these terms is given by

$$\begin{aligned}
\mathscr{L}_{R\pi\pi} =\ & -\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N Z^2\sigma_N\vec{\pi}^2 + g_1 w Z^2\left(\partial^\mu\sigma_N\right)\left(\partial_\mu\vec{\pi}\right)\cdot\vec{\pi} + \left\{\left[g_1^2\phi_N + (h_1 + h_2 - h_3)\,\frac{\phi_N}{2}\right] w^2 Z^2 - g_1 w Z^2\right\} \\
& \times\sigma_N\left(\partial_\mu\vec{\pi}\right)^2 + g_2 w^2 Z^2\left(\partial^\mu\vec{\rho}^\nu\right)\cdot\left(\partial_\nu\vec{\pi}\times\partial_\mu\vec{\pi}\right) + \left[\left(g_1^2\phi_N - h_3\phi_N\right) w Z^2 - g_1 Z^2\right]\vec{\rho}^\mu\cdot\left(\partial_\mu\vec{\pi}\times\vec{\pi}\right)
\end{aligned} \tag{4.332}$$

and

$$\begin{aligned}
\mathscr{L}_{RR\pi\pi} =\ & \left[\frac{g_1^2}{2} + \frac{1}{4}(h_1 + h_2 - h_3)\right] w^2 Z^2\sigma_N^2\left(\partial_\mu\vec{\pi}\right)^2 - \frac{1}{2}\left(\lambda_1 + \frac{\lambda_2}{2}\right) Z^2\sigma_N^2\vec{\pi}^2 + \frac{1}{2}\left(g_1^2 - h_3\right) Z^2\left(\vec{\pi}\times\vec{\rho}_\mu\right)^2 \\
& + \frac{1}{4}(h_1 + h_2 + h_3)\, Z^2\vec{\rho}_\mu^{\,2}\vec{\pi}^2 + \left(-\frac{g_3}{2} + \frac{g_4}{2} + \frac{g_5}{2} + g_6\right) w^2 Z^2\vec{\rho}_\mu^{\,2}\left(\partial_\nu\vec{\pi}\right)^2 + \left(-g_3 + g_4 - g_5 + 2g_6\right) w^2 Z^2 \\
& \times\left(\vec{\rho}^\mu\cdot\partial_\mu\vec{\pi}\right)\left(\vec{\rho}^\nu\cdot\partial_\nu\vec{\pi}\right) + g_3 w^2 Z^2\left[\left(\vec{\rho}^\mu\cdot\vec{\rho}^\nu\right)\left(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi}\right) + \left(\vec{\rho}^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\mu\cdot\partial_\nu\vec{\pi}\right) + \left(\vec{\rho}^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\nu\cdot\partial_\mu\vec{\pi}\right)\right] \\
& + \left[\frac{g_1^2}{2} + \frac{1}{4}(h_1 + h_2 - h_3)\right] w^2 Z^4\left(\partial_\mu\eta_N\right)^2\vec{\pi}^2 + \left[\frac{g_1^2}{2} + \frac{1}{4}(h_1 + h_2 - h_3)\right] w^2 Z^4\eta_N^2\left(\partial_\mu\vec{\pi}\right)^2 \\
& - \frac{1}{2}\left(\lambda_1 + \frac{3\lambda_2}{2}\right) Z^4\eta_N^2\vec{\pi}^2 + \left(2g_1^2 + h_2 - h_3\right) w^2 Z^4\eta_N\left(\partial^\mu\eta_N\right)\vec{\pi}\cdot\left(\partial_\mu\vec{\pi}\right) + \left(g_3 + g_4\right) w^4 Z^4
\end{aligned}$$

$$\times \left(\partial^{\mu}\eta_N\right)\left(\partial^{\nu}\eta_N\right)\left(\partial_{\mu}\vec{\pi}\right)\cdot\left(\partial_{\nu}\vec{\pi}\right) + \left(\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right) w^4 Z^4 \left(\partial_{\mu}\eta_N\right)^2\left(\partial_{\nu}\vec{\pi}\right)^2 + \frac{\lambda_2}{2} Z^2 \left(\vec{a}_0\cdot\vec{\pi}\right)^2$$
$$-\frac{1}{2}\left(\lambda_1+\frac{3\lambda_2}{2}\right) Z^2 \vec{a}_0^2\vec{\pi}^2 + \frac{1}{4}\left(h_1+h_2-h_3\right) w^2 Z^2 \vec{a}_0^2\left(\partial_{\mu}\vec{\pi}\right)^2 + \frac{g_1^2}{2} w^2 Z^2\left(\vec{a}_0\cdot\partial_{\mu}\vec{\pi}\right)^2 + \frac{h_3}{2} w^2 Z^2\left(\vec{a}_0\times\partial_{\mu}\vec{\pi}\right)^2$$
$$+\frac{1}{4}\left(h_1+h_2+h_3\right) Z^2 \omega_{N,\mu}^2\vec{\pi}^2 + \left(\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right) w^2 Z^2 \omega_{N,\mu}^2\left(\partial_{\nu}\vec{\pi}\right)^2 + \left(g_3+g_4\right) w^2 Z^2$$
$$\times\, \omega_N^{\mu}\omega_N^{\nu}\left(\partial_{\mu}\vec{\pi}\cdot\partial_{\nu}\vec{\pi}\right) + \left[\frac{g_1^2}{2}+\frac{1}{4}\left(h_1+h_2-h_3\right)\right] Z^2 f_{1N,\mu}^2\vec{\pi}^2 + \left(\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right) w^2 Z^2$$
$$\times\, f_{1N,\mu}^2\left(\partial_{\nu}\vec{\pi}\right)^2 + \left(g_3+g_4\right) w^2 Z^2 f_{1N}^{\mu} f_{1N}^{\nu}\left(\partial_{\mu}\vec{\pi}\right)\cdot\left(\partial_{\nu}\vec{\pi}\right) + \frac{g_1^2}{2} Z^2\left(\vec{a}_{1,\mu}\cdot\vec{\pi}\right)^2 + \frac{1}{4}\left(h_1+h_2-h_3\right) Z^2\vec{a}_{1,\mu}^2\vec{\pi}^2$$
$$+\left(-\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right) w^2 Z^2 \vec{a}_{1,\mu}^2\left(\partial_{\nu}\vec{\pi}\right)^2 + \frac{h_3}{2} Z^2\left(\vec{a}_{1,\mu}\times\vec{\pi}\right)^2 + \left(-g_3+g_4+g_5+2g_6\right) w^2 Z^2$$
$$\times\left(\vec{a}_1^{\mu}\cdot\partial_{\mu}\vec{\pi}\right)\left(\vec{a}_1^{\nu}\cdot\partial_{\nu}\vec{\pi}\right) + g_3 w^2 Z^2\left[\left(\vec{a}_1^{\mu}\cdot\partial^{\nu}\vec{\pi}\right)\left(\vec{a}_{1,\mu}\cdot\partial_{\nu}\vec{\pi}\right) + \left(\vec{a}_1^{\mu}\cdot\partial^{\nu}\vec{\pi}\right)\left(\vec{a}_{1,\nu}\cdot\partial_{\mu}\vec{\pi}\right) + \left(\vec{a}_1^{\mu}\cdot\vec{a}_1^{\nu}\right)\left(\partial_{\mu}\vec{\pi}\right)\cdot\left(\partial_{\nu}\vec{\pi}\right)\right] . \tag{4.333}$$

Finally, the last term of Eq. (4.331) contains all terms, which contain four pion fields with up to four space-time derivatives. The explicit form of this term is given by

$$\mathscr{L}_{4\pi} = \frac{g_1^2}{2} w^2 Z^4\left(\vec{\pi}\cdot\partial_{\mu}\vec{\pi}\right)^2 - \frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right) Z^4\left(\vec{\pi}^2\right)^2 + \frac{1}{4}\left(h_1+h_2-h_3\right) w^2 Z^4\vec{\pi}^2\left(\partial_{\mu}\vec{\pi}\right)^2 + \frac{h_3}{2} w^2 Z^4\left(\vec{\pi}\times\partial_{\mu}\vec{\pi}\right)^2$$
$$+\left(-\frac{g_3}{4}+\frac{g_4}{4}+\frac{g_5}{4}+\frac{g_6}{2}\right) w^4 Z^4\left(\partial_{\mu}\vec{\pi}\right)^2\left(\partial_{\nu}\vec{\pi}\right)^2 + \frac{g_3}{2} w^4 Z^4\left(\partial^{\mu}\vec{\pi}\cdot\partial^{\nu}\vec{\pi}\right)\left(\partial_{\mu}\vec{\pi}\cdot\partial_{\nu}\vec{\pi}\right) . \tag{4.334}$$

From Eq. (4.332) we can make an important observation: In contrast to the previous cases, we now have another resonance which couples in the form $R\pi\pi$ to two pion fields. This, of course, will result in new four-pion interaction terms. In addition to that, also the σ_N-field obtains new interaction vertices with two pion fields. All other resonances only couple in the form $RR\pi\pi$ to the pion field. Therefore, all other mesons only contribute at one-loop order to the LECs of the eLSM. Another important observation is, that the assumptions (A1)-(A3) again decouple all heavy-field interactions. Therefore, the functional integrals of the heavy fields also decouple, so that we are able to solve them independently of each other.

With $f = \{\sigma_N, \eta_N, \vec{a}_0, \vec{\pi}, \omega_{N,\mu}, f_{1N,\mu}, \vec{\rho}_{\mu}, \vec{a}_{1,\mu}\}$, the transition amplitude can be written as

$$\langle f,\infty|f,-\infty\rangle = \mathcal{N}\int \mathscr{D}\sigma_N(x)\mathscr{D}\eta_N(x)\mathscr{D}\vec{a}_0(x)\mathscr{D}\vec{\pi}(x)\mathscr{D}\omega_{N,\mu}(x)\mathscr{D}f_{1N,\mu}(x)\mathscr{D}\vec{\rho}_{\mu}(x)\mathscr{D}\vec{a}_{1,\mu}(x)\ \exp\Big\{iS_{eLSM}$$
$$+ i\int \mathrm{d}^4x\ \mathscr{L}_{GF}\Big\} , \tag{4.335}$$

where

$$S_{eLSM} = \int \mathrm{d}^4x\ \mathscr{L}_{eLSM}$$
$$= \int \mathrm{d}^4x\ \left(\mathscr{L}_{kin.,mass} + \mathscr{L}_{R\pi\pi} + \mathscr{L}_{RR\pi\pi} + \mathscr{L}_{4\pi}\right) \tag{4.336}$$

and

$$\mathscr{L}_{GF} = -\frac{\xi_{\omega_N}}{2}\left(\partial_{\mu}\omega_N^{\mu}\right)^2 - \frac{\xi_{f_{1N}}}{2}\left(\partial_{\mu}f_{1N}^{\mu}\right)^2 - \frac{\xi_{\vec{\rho}}}{2}\left(\partial_{\mu}\vec{\rho}^{\mu}\right)^2 - \frac{\xi_{\vec{a}_1}}{2}\left(\partial_{\mu}\vec{a}_1^{\mu}\right)^2 \tag{4.337}$$

defines a gauge-fixing term. In principle, this term is not necessary, since the Lorentz gauge-like condition $(\partial_{\mu}A^{\mu})$, with $A^{\mu}\in\{\omega_N^{\mu}, f_{1N}^{\mu}, \vec{\rho}^{\mu}, \vec{a}_1^{\mu}\}$, which can be obtained from the equation of motion of a Proca field, is sufficient in order to eliminate the redundant degree of freedom. But it will be shown that the introduction of this so-called Stueckelberg term will be quite useful, when we consider the differential operator corresponding to the $\vec{\rho}^{\mu}$-fields. Before we continue with the functional integral (4.335), it will be useful to reorganize the terms in S_{eLSM}. To this end, we use the fact that all heavy fields decouple, so that the eLSM Lagrangian can be written as

$$\mathscr{L}_{eLSM} = \mathscr{L}_{\vec{\pi}} + \mathscr{L}_{\sigma_N\vec{\pi}} + \mathscr{L}_{\eta_N\vec{\pi}} + \mathscr{L}_{\vec{a}_0\vec{\pi}} + \mathscr{L}_{\omega_N\vec{\pi}} + \mathscr{L}_{f_{1N}\vec{\pi}} + \mathscr{L}_{\vec{\rho}\vec{\pi}} + \mathscr{L}_{\vec{a}_1\vec{\pi}} , \tag{4.338}$$

with

$$\mathscr{L}_{\vec{\pi}} = \frac{1}{2}\left(\partial_{\mu}\vec{\pi}\right)^2 - \frac{1}{2}m_{\vec{\pi}}^2\vec{\pi}^2 + \frac{g_1^2}{2} w^2 Z^4\left(\vec{\pi}\cdot\partial_{\mu}\vec{\pi}\right)^2 - \frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right) Z^4\left(\vec{\pi}^2\right)^2 + \frac{1}{4}\left(h_1+h_2-h_3\right) w^2 Z^4\vec{\pi}^2\left(\partial_{\mu}\vec{\pi}\right)^2$$
$$+\frac{h_3}{2} w^2 Z^4\left(\vec{\pi}\times\partial_{\mu}\vec{\pi}\right)^2 + \left(-\frac{g_3}{4}+\frac{g_4}{4}+\frac{g_5}{4}+\frac{g_6}{2}\right) w^4 Z^4\left(\partial_{\mu}\vec{\pi}\right)^2\left(\partial_{\nu}\vec{\pi}\right)^2 + \frac{g_3}{2} w^4 Z^4\left(\partial^{\mu}\vec{\pi}\right)\cdot\left(\partial^{\nu}\vec{\pi}\right)\left(\partial_{\mu}\vec{\pi}\right)\cdot\left(\partial_{\nu}\vec{\pi}\right) , \tag{4.339}$$

$$\mathscr{L}_{\sigma_N\vec{\pi}} = \frac{1}{2}\left(\partial_\mu\sigma_N\right)^2 - \frac{1}{2}m^2_{\sigma_N}\sigma_N^2 - \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N Z^2\sigma_N\vec{\pi}^2 + g_1 w Z^2\left(\partial^\mu\sigma_N\right)\left(\partial_\mu\vec{\pi}\right)\cdot\vec{\pi} + \left\{\left[g_1^2\phi_N + \left(h_1+h_2-h_3\right)\frac{\phi_N}{2}\right]\right.$$
$$\left.\times\, w^2 Z^2 - g_1 w Z^2\right\}\sigma_N\left(\partial_\mu\vec{\pi}\right)^2 + \left[\frac{g_1^2}{2} + \frac{1}{4}\left(h_1+h_2-h_3\right)\right]w^2 Z^2\sigma_N^2\left(\partial_\mu\vec{\pi}\right)^2 - \frac{1}{2}\left(\lambda_1+\frac{\lambda_2}{2}\right)Z^2\sigma_N^2\vec{\pi}^2\ , \tag{4.340}$$

$$\mathscr{L}_{\eta_N\vec{\pi}} = \frac{1}{2}\left(\partial_\mu\eta_N\right)^2 - \frac{1}{2}m^2_{\eta_N}\eta_N^2 + \left[\frac{g_1^2}{2} + \frac{1}{4}\left(h_1+h_2-h_3\right)\right]w^2 Z^4\left(\partial_\mu\eta_N\right)^2\vec{\pi}^2 + \left[\frac{g_1^2}{2} + \frac{1}{4}\left(h_1+h_2-h_3\right)\right]w^2 Z^4$$
$$\times\,\eta_N^2\left(\partial_\mu\vec{\pi}\right)^2 - \frac{1}{2}\left(\lambda_1 + \frac{3\lambda_2}{2}\right)Z^4\eta_N^2\vec{\pi}^2 + \left(2g_1^2 + h_2 - h_3\right)w^2 Z^4\eta_N\left(\partial^\mu\eta_N\right)\vec{\pi}\cdot\left(\partial_\mu\vec{\pi}\right) + \left(g_3+g_4\right)w^4 Z^4$$
$$\times\left(\partial^\mu\eta_N\right)\left(\partial^\nu\eta_N\right)\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial_\nu\vec{\pi}\right) + \left(\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right)w^4 Z^4\left(\partial_\mu\eta_N\right)^2\left(\partial_\nu\vec{\pi}\right)^2\ , \tag{4.341}$$

$$\mathscr{L}_{\vec{a}_0\vec{\pi}} = \frac{1}{2}\left(\partial_\mu\vec{a}_0\right)^2 - \frac{1}{2}m^2_{\vec{a}_0}\vec{a}_0^2 + \frac{\lambda_2}{2}Z^2\left(\vec{a}_0\cdot\vec{\pi}\right)^2 - \frac{1}{2}\left(\lambda_1+\frac{3\lambda_2}{2}\right)Z^2\vec{a}_0^2\vec{\pi}^2 + \frac{1}{4}\left(h_1+h_2-h_3\right)w^2 Z^2\vec{a}_0^2\left(\partial_\mu\vec{\pi}\right)^2$$
$$+\,\frac{g_1^2}{2}w^2 Z^2\left(\vec{a}_0\cdot\partial_\mu\vec{\pi}\right)^2 + \frac{h_3}{2}w^2 Z^2\left(\vec{a}_0\times\partial_\mu\vec{\pi}\right)^2\ , \tag{4.342}$$

$$\mathscr{L}_{\omega_N\vec{\pi}} = -\frac{1}{4}\omega_N^{\mu\nu}\omega_{N,\mu\nu} + \frac{1}{2}m^2_{\omega_N}\omega^2_{N,\mu} - \frac{\xi_{\omega_N}}{2}\left(\partial_\mu\omega_N^\mu\right)^2 + \frac{1}{4}\left(h_1+h_2+h_3\right)Z^2\omega^2_{N,\mu}\vec{\pi}^2 + \left(\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right)w^2 Z^2$$
$$\times\,\omega^2_{N,\mu}\left(\partial_\nu\vec{\pi}\right)^2 + \left(g_3+g_4\right)w^2 Z^2\omega_N^\mu\omega_N^\nu\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial_\nu\vec{\pi}\right)\ , \tag{4.343}$$

$$\mathscr{L}_{f_{1N}\vec{\pi}} = -\frac{1}{4}f_{1N}^{\mu\nu}f_{1N,\mu\nu} + \frac{1}{2}m^2_{f_{1N}}f^2_{1N,\mu} - \frac{\xi_{f_{1N}}}{2}\left(\partial_\mu f_{1N}^\mu\right)^2 + \left[\frac{g_1^2}{2}+\frac{1}{4}\left(h_1+h_2-h_3\right)\right]Z^2 f^2_{1N,\mu}\vec{\pi}^2$$
$$+\left(\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right)w^2 Z^2 f^2_{1N,\mu}\left(\partial_\nu\vec{\pi}\right)^2 + \left(g_3+g_4\right)w^2 Z^2 f_{1N}^\mu f_{1N}^\nu\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial_\nu\vec{\pi}\right)\ , \tag{4.344}$$

$$\mathscr{L}_{\vec{\rho}\vec{\pi}} = -\frac{1}{4}\vec{\rho}^{\mu\nu}\cdot\vec{\rho}_{\mu\nu} + \frac{1}{2}m^2_{\vec{\rho}}\vec{\rho}^2_\mu - \frac{\xi_{\vec{\rho}}}{2}\left(\partial_\mu\vec{\rho}^\mu\right)^2 + g_2 w^2 Z^2\left(\partial^\mu\vec{\rho}^\nu\right)\cdot\left(\partial_\nu\vec{\pi}\times\partial_\mu\vec{\pi}\right) + \left[\left(g_1^2\phi_N - h_3\phi_N\right)wZ^2 - g_1 Z^2\right]$$
$$\times\,\vec{\rho}^\mu\cdot\left(\partial_\mu\vec{\pi}\times\vec{\pi}\right) + \frac{1}{2}\left(g_1^2 - h_3\right)Z^2\left(\vec{\pi}\times\vec{\rho}_\mu\right)^2 + \frac{1}{4}\left(h_1+h_2+h_3\right)Z^2\vec{\rho}^2_\mu\vec{\pi}^2 + \left(-\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right)w^2 Z^2$$
$$\times\,\vec{\rho}^2_\mu\left(\partial_\nu\vec{\pi}\right)^2 + \left(-g_3+g_4-g_5+2g_6\right)w^2 Z^2\left(\vec{\rho}^\mu\cdot\partial_\mu\vec{\pi}\right)\left(\vec{\rho}^\nu\cdot\partial_\nu\vec{\pi}\right) + g_3 w^2 Z^2\left[\left(\vec{\rho}^\mu\cdot\vec{\rho}^\nu\right)\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial_\nu\vec{\pi}\right)\right.$$
$$\left.+\left(\vec{\rho}^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\mu\cdot\partial_\nu\vec{\pi}\right) + \left(\vec{\rho}^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\nu\cdot\partial_\mu\vec{\pi}\right)\right]\ , \tag{4.345}$$

$$\mathscr{L}_{\vec{a}_1\vec{\pi}} = -\frac{1}{4}\vec{a}_1^{\mu\nu}\cdot\vec{a}_{1,\mu\nu} + \frac{1}{2}m^2_{\vec{a}_1}\vec{a}^2_{1,\mu} - \frac{\xi_{\vec{a}_1}}{2}\left(\partial_\mu\vec{a}_1^\mu\right)^2 + \frac{g_1^2}{2}Z^2\left(\vec{a}_{1,\mu}\cdot\vec{\pi}\right)^2 + \frac{1}{4}\left(h_1+h_2-h_3\right)Z^2\vec{a}^2_{1,\mu}\vec{\pi}^2$$
$$+\left(-\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right)w^2 Z^2\vec{a}^2_{1,\mu}\left(\partial_\nu\vec{\pi}\right)^2 + \frac{h_3}{2}Z^2\left(\vec{a}_{1,\mu}\times\vec{\pi}\right)^2 + \left(-g_3+g_4+g_5+2g_6\right)w^2 Z^2$$
$$\times\left(\vec{a}_1^\mu\cdot\partial_\mu\vec{\pi}\right)\left(\vec{a}_1^\nu\cdot\partial_\nu\vec{\pi}\right) + g_3 w^2 Z^2\left[\left(\vec{a}_1^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{a}_{1,\mu}\cdot\partial_\nu\vec{\pi}\right) + \left(\vec{a}_1^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{a}_{1,\nu}\cdot\partial_\mu\vec{\pi}\right) + \left(\vec{a}_1^\mu\cdot\vec{a}_1^\nu\right)\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial_\nu\vec{\pi}\right)\right]\ . \tag{4.346}$$

Then, using (4.338)-(4.346), the functional integral (4.335) can be written as

$$\begin{aligned}\langle f,\infty|f,-\infty\rangle &= \mathcal{N}\int\mathscr{D}\sigma_N(x)\mathscr{D}\eta_N(x)\mathscr{D}\vec{a}_0(x)\mathscr{D}\vec{\pi}(x)\mathscr{D}\omega_{N,\mu}(x)\mathscr{D}f_{1N,\mu}(x)\mathscr{D}\vec{\rho}_\mu(x)\mathscr{D}\vec{a}_{1,\mu}(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left[\mathscr{L}_{kin.,mass}\right.\right.\\ &\quad\left.\left. + \mathscr{L}_{R\pi\pi} + \mathscr{L}_{RR\pi\pi} + \mathscr{L}_{4\pi} + \mathscr{L}_{GF}\right]\right\}\\ &= \mathcal{N}\int\mathscr{D}\vec{\pi}(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{\vec{\pi}}\right\}\int\mathscr{D}\sigma_N(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{\sigma_N\vec{\pi}}\right\}\int\mathscr{D}\vec{\rho}_\mu(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{\vec{\rho}\vec{\pi}}\right\}\\ &\quad\times\int\mathscr{D}\eta_N(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{\eta_N\vec{\pi}}\right\}\int\mathscr{D}\vec{a}_0(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{\vec{a}_0\vec{\pi}}\right\}\int\mathscr{D}\omega_{N,\mu}(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{\omega_N\vec{\pi}}\right\}\\ &\quad\times\int\mathscr{D}f_{1N,\mu}(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{f_{1N}\vec{\pi}}\right\}\int\mathscr{D}\vec{a}_{1,\mu}(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{\vec{a}_1\vec{\pi}}\right\}\\ &= \mathcal{N}\int\mathscr{D}\vec{\pi}(x)\ \exp\left\{i\int\mathrm{d}^4x\ \mathscr{L}_{\vec{\pi}}\right\}I_{\sigma_N}[\vec{\pi}]I_{\vec{\rho}}[\vec{\pi}]I_{\eta_N}[\vec{\pi}]I_{\vec{a}_0}[\vec{\pi}]I_{\omega_N}[\vec{\pi}]I_{f_{1N}}[\vec{\pi}]I_{\vec{a}_1}[\vec{\pi}]\ ,\end{aligned} \tag{4.347}$$

where we defined the different functional integrals according to

$$I_{f'}[\vec{\pi}] \equiv \int \mathscr{D} f'(x) \, \exp\left\{ i \int \mathrm{d}^4x \, \mathscr{L}_{f'\vec{\pi}} \right\} , \tag{4.348}$$

for $f' = \sigma_N, \vec{\rho}_\mu, \eta_N, \vec{a}_0, \omega_{N,\mu}, f_{1N,\mu}, \vec{a}_{1,\mu}$. At this point, we are able to calculate the Gaussian functional integrals (4.347). As already mentioned, the only interesting integrals for our tree-level calculation are those with respect to σ_N and $\vec{\rho}_\mu$. All other integrals only contribute at one-loop order. The solutions of these integrals can be obtained in a similar way as those in Eqs. (4.311)-(4.319). Without specification of the exact form of various differential operators, the solutions of the these integrals are given by

$$I_{\eta_N}[\vec{\pi}] = \mathcal{N}_{\eta_N} \left[\det \mathscr{O}_{\eta_N,\vec{\pi}}(x,y) \right]^{-1/2} , \tag{4.349}$$

$$I_{\vec{a}_0}[\vec{\pi}] = \mathcal{N}_{\vec{a}_0} \left[\det \mathscr{O}^{ij}_{\vec{a}_0,\vec{\pi}}(x,y) \right]^{-1/2} , \tag{4.350}$$

$$I_{\omega_N}[\vec{\pi}] = \mathcal{N}_{\omega_N} \left[\det \mathscr{O}^{\mu\nu}_{\omega_N,\vec{\pi}}(x,y) \right]^{-1/2} , \tag{4.351}$$

$$I_{f_{1N}}[\vec{\pi}] = \mathcal{N}_{f_{1N}} \left[\det \mathscr{O}^{\mu\nu}_{f_{1N},\vec{\pi}}(x,y) \right]^{-1/2} , \tag{4.352}$$

$$I_{\vec{a}_1}[\vec{\pi}] = \mathcal{N}_{\vec{a}_1} \left[\det \left(\mathscr{O}^{\mu\nu}_{\vec{a}_1,\vec{\pi}} \right)^{ij} (x,y) \right]^{-1/2} . \tag{4.353}$$

In the following, we now have to solve the remaining two functional integrals. To this end, we start with the σ_N integral which is given by

$$\begin{aligned}
I_{\sigma_N}[\vec{\pi}] &= \int \mathscr{D}\sigma_N(x) \, \exp\left\{ i \int \mathrm{d}^4x \, \mathscr{L}_{\sigma_N\vec{\pi}} \right\} \\
&= \int \mathscr{D}\sigma_N \, \exp\left\{ i \int \mathrm{d}^4x \, \left\{ \frac{1}{2} (\partial_\mu \sigma_N)^2 - \frac{1}{2} m^2_{\sigma_N} \sigma_N^2 - \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N Z^2 \sigma_N \vec{\pi}^2 + g_1 w Z^2 (\partial^\mu \sigma_N)(\partial_\mu \vec{\pi}) \cdot \vec{\pi} \right.\right. \\
&\quad + \left\{ \left[g_1^2 \phi_N + (h_1 + h_2 - h_3) \frac{\phi_N}{2} \right] w^2 Z^2 - g_1 w Z^2 \right\} \sigma_N (\partial_\mu \vec{\pi})^2 + \left[\frac{g_1^2}{2} + \frac{1}{4}(h_1 + h_2 - h_3) \right] w^2 Z^2 \sigma_N^2 (\partial_\mu \vec{\pi})^2 \\
&\quad \left.\left. - \frac{1}{2} \left(\lambda_1 + \frac{\lambda_2}{2} \right) Z^2 \sigma_N^2 \vec{\pi}^2 \right\} \right\} .
\end{aligned} \tag{4.354}$$

In order to rewrite the above functional integral into the standard form of a shifted Gaussian integral, we have to integrate the first and the fourth term of the exponential by parts, so that

$$\begin{aligned}
I_{\sigma_N}[\vec{\pi}] &= \int \mathscr{D}\sigma_N(x) \, \exp\left\{ -\frac{i}{2} \int \mathrm{d}^4x \, \sigma_N \left\{ \Box + m^2_{\sigma_N} + \left(\lambda_1 + \frac{\lambda_2}{2} \right) Z^2 \vec{\pi}^2 + \left[g_1^2 + \frac{1}{2}(h_1 + h_2 - h_3) \right] w^2 Z^2 (\partial_\mu \vec{\pi})^2 \right\} \sigma_N \right. \\
&\quad + i \int \mathrm{d}^4x \, \left\{ - \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N Z^2 \vec{\pi}^2 \sigma_N + \left\{ \left[g_1^2 \phi_N + (h_1 + h_2 - h_3) \frac{\phi_N}{2} \right] w^2 Z^2 - g_1 w Z^2 \right\} (\partial_\mu \vec{\pi})^2 \sigma_N \right. \\
&\quad \left.\left. - g_1 w Z^2 \left[\partial^\mu (\partial_\mu \vec{\pi} \cdot \vec{\pi}) \right] \sigma_N \right\} \right\} \\
&= \int \mathscr{D}\sigma_N(x) \, \exp\left\{ -\frac{i}{2} \int \mathrm{d}^4x \mathrm{d}^4y \, \sigma_N(x) \mathscr{O}_{\sigma_N\vec{\pi}}(x,y) \sigma_N(y) + i \int \mathrm{d}^4x J_{\sigma_N,\vec{\pi}}(x) \sigma_N(x) \right\} ,
\end{aligned} \tag{4.355}$$

where

$$\mathscr{O}_{\sigma_N\vec{\pi}}(x,y) = \left\{ \Box_x + m^2_{\sigma_N} + \left(\lambda_1 + \frac{\lambda_2}{2} \right) Z^2 \vec{\pi}^2(x) + \left[g_1^2 + \frac{1}{2}(h_1 + h_2 - h_3) \right] w^2 Z^2 \left[\partial_\mu \vec{\pi}(x) \right]^2 \right\} \delta^{(4)}(x-y) \tag{4.356}$$

and

$$\begin{aligned}
J_{\sigma_N,\vec{\pi}}(x) &= \left[g_1 w Z^2 m^2_{\vec{\pi}} - \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N Z^2 \right] \vec{\pi}^2 + \left\{ \left[g_1^2 \phi_N + (h_1 + h_2 - h_3) \frac{\phi_N}{2} \right] w^2 Z^2 - 2 g_1 w Z^2 \right\} (\partial_\mu \vec{\pi})^2 \\
&\equiv c_{1,\sigma_N} \vec{\pi}^2 + c_{2,\sigma_N} (\partial_\mu \vec{\pi})^2 .
\end{aligned} \tag{4.357}$$

The above "source" was obtained from the first line of Eq. (4.355) by using the product rule

$$\partial^\mu (\partial_\mu \vec{\pi} \cdot \vec{\pi}) = (\Box \vec{\pi}) \cdot \vec{\pi} + (\partial_\mu \vec{\pi})^2 \tag{4.358}$$

and by using the free Klein-Gordon equation for the $\vec{\pi}$-fields

$$\Box\vec{\pi}(x) = -m_{\vec{\pi}}^2\vec{\pi} \ . \tag{4.359}$$

Furthermore, we defined the coefficients

$$c_{1,\sigma_N} = g_1 w Z^2 m_{\vec{\pi}}^2 - \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N Z^2 \ , \tag{4.360}$$

$$c_{2,\sigma_N} = \left[g_1^2\phi_N + (h_1+h_2-h_3)\frac{\phi_N}{2}\right] w^2 Z^2 - 2g_1 w Z^2 \ , \tag{4.361}$$

which have, to this order, dimension [Energy] and [Energy^{-1}], respectively. As a cross-check, we can now use these coefficients and the operator (4.356) and set $w = g_1 = g_2$, $Z = 1$. In these limits, we exactly obtain the Gaussian functional integral for the σ_N-field, which we had to solve in the two previous Subsections, Eq. (4.277). Similar to that discussion, we now have to perform the analytic continuation from Minkowski space-time to Euclidean space-time. With Eqs. (4.18), (4.20), (4.21), and the Euclidean operator

$$\mathscr{O}_{\sigma_N,\vec{\pi},E}(x_E,y_E) = \left\{-\Box_{x,E} + m_{\sigma_N}^2 + \left(\lambda_1+\frac{\lambda_2}{2}\right)Z^2\vec{\pi}^2(x_E) - \left[g_1^2 + \frac{1}{2}(h_1+h_2-h_3)\right]w^2Z^2\left[\partial_\mu^E\vec{\pi}(x_E)\right]^2\right\}\delta^{(4)}(x_E-y_E) \tag{4.362}$$

the Euclidean functional integral is given by

$$\begin{aligned}
I_{\sigma_N}[\vec{\pi}] &= \int\mathscr{D}\sigma_N(x_E)\ \exp\left\{-\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\ \sigma_N(x_E)\mathscr{O}_{\sigma_N,\vec{\pi},E}(x_E,y_E)\sigma_N(y_E) + \int\mathrm{d}^4x_E J_{\sigma_N,\vec{\pi},E}(x_E)\sigma_N(x_E)\right\}\\
&= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\sigma_N,\vec{\pi},E}(x_E,y_E)\right]^{-1/2}\ \exp\left\{\frac{1}{2}\int\mathrm{d}^4x_E\mathrm{d}^4y_E\ J_{\sigma_N,\vec{\pi},E}(x_E)\mathscr{O}^{-1}_{\sigma_N,\vec{\pi},E}(x_E,y_E)J_{\sigma_N,\vec{\pi},E}(y_E)\right\}\\
&= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\sigma_N,\vec{\pi}}(x,y)\right]^{-1/2}\ \exp\left\{\frac{i}{2}\int\mathrm{d}^4x\mathrm{d}^4y\ J_{\sigma_N,\vec{\pi}}(x)\mathscr{O}^{-1}_{\sigma_N,\vec{\pi}}(x,y)J_{\sigma_N,\vec{\pi}}(y)\right\}\\
&= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\sigma_N,\vec{\pi}}(x,y)\right]^{-1/2}\ \exp\left\{\frac{i}{2}\int\mathrm{d}^4x\ J_{\sigma_N,\vec{\pi}}\left\{\Box + m_{\sigma_N}^2 + \left(\lambda_1+\frac{\lambda_2}{2}\right)Z^2\vec{\pi}^2 + \left[g_1^2+\frac{1}{2}(h_1+h_2-h_3)\right]\right.\right.\\
&\quad\left.\left.\times\ w^2Z^2\left(\partial_\mu\vec{\pi}\right)^2\right\}^{-1} J_{\sigma_N,\vec{\pi}}\right\} \ ,
\end{aligned} \tag{4.363}$$

where we transformed the integral back to Minkowski space-time and introduced the inverse of Eq. (4.356),

$$\mathscr{O}^{-1}_{\sigma_N\vec{\pi}}(x,y) = \left\{\Box_x + m_{\sigma_N}^2 + \left(\lambda_1+\frac{\lambda_2}{2}\right)Z^2\vec{\pi}^2(x) + \left[g_1^2+\frac{1}{2}(h_1+h_2-h_3)\right]w^2Z^2\left[\partial_\mu\vec{\pi}(x)\right]^2\right\}^{-1}\delta^{(4)}(x-y) \ . \tag{4.364}$$

Now, in order to obtain a local result for Eq. (4.363), we have to expand the square bracket in powers of inverse σ_N masses and $\vec{\pi}$-fields. We find

$$\begin{aligned}
&\left\{\Box + m_{\sigma_N}^2 + \left(\lambda_1+\frac{\lambda_2}{2}\right)Z^2\vec{\pi}^2 + \left[g_1^2+\frac{1}{2}(h_1+h_2-h_3)\right]w^2Z^2\left(\partial_\mu\vec{\pi}\right)^2\right\}^{-1}\\
&= \left\{1 + \left(\Box+m_{\sigma_N}^2\right)^{-1}\left[\left(\lambda_1+\frac{\lambda_2}{2}\right)Z^2\vec{\pi}^2 + \left[g_1^2+\frac{1}{2}(h_1+h_2-h_3)\right]w^2Z^2\left(\partial_\mu\vec{\pi}\right)^2\right]\right\}^{-1}\left(\Box+m_{\sigma_N}^2\right)^{-1}\\
&= \sum_{m=0}^{\infty}(-1)^m\left\{\left(\Box+m_{\sigma_N}^2\right)^{-1}\left[\left(\lambda_1+\frac{\lambda_2}{2}\right)Z^2\vec{\pi}^2 + \left[g_1^2+\frac{1}{2}(h_1+h_2-h_3)\right]w^2Z^2\left(\partial_\mu\vec{\pi}\right)^2\right]\right\}^m\frac{1}{m_{\sigma_N}^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box}{m_{\sigma_N}^2}\right)^n\\
&= \frac{1}{m_{\sigma_N}^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box}{m_{\sigma_N}^2}\right)^n + \text{terms involving two or more } \vec{\pi}\text{-fields} \ ,
\end{aligned} \tag{4.365}$$

where we suppressed all terms with $m > 0$ in the first sum. Similar to the previous Sections, we are interested in four-pion interaction terms that contain up to four space-time derivatives. Therefore, we expand the first sum in Eq. (4.365) only up to $n = 2$ and neglect terms of $\mathcal{O}(\partial^6)$, so that the functional integral is given by

$$
\begin{aligned}
I_{\sigma_N}[\vec{\pi}] &= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\sigma_N,\vec{\pi}}(x,y)\right]^{-1/2}\exp\left\{\frac{i}{2}\int \mathrm{d}^4x\,\left[c_{1,\sigma_N}\vec{\pi}^2 + c_{2,\sigma_N}\left(\partial_\mu\vec{\pi}\right)^2\right]\left(\frac{1}{m^2_{\sigma_N}} - \frac{\Box}{m^4_{\sigma_N}} + \frac{\Box^2}{m^6_{\sigma_N}}\right)\left[c_{1,\sigma_N}\vec{\pi}^2\right.\right. \\
&\quad \left.\left. + c_{2,\sigma_N}\left(\partial_\nu\vec{\pi}\right)^2\right]\right\} \\
&= \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\sigma_N,\vec{\pi}}(x,y)\right]^{-1/2}\exp\left\{\frac{i}{2}\int \mathrm{d}^4x\,\left[\underbrace{\frac{c^2_{1,\sigma_N}}{m^2_{\sigma_N}}\vec{\pi}^2\left(1 - \frac{\Box}{m^2_{\sigma_N}} + \frac{\Box^2}{m^4_{\sigma_N}}\right)\vec{\pi}^2}_{(a)} + \underbrace{\frac{c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\left(\partial_\mu\vec{\pi}\right)^2\left(1 - \frac{\Box}{m^2_{\sigma_N}}\right)\vec{\pi}^2}_{(b)}\right.\right. \\
&\quad \left.\left. + \underbrace{\frac{c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\vec{\pi}^2\left(1 - \frac{\Box}{m^2_{\sigma_N}}\right)\left(\partial_\nu\vec{\pi}\right)^2}_{(c)} + \underbrace{\frac{c^2_{2,\sigma_N}}{m^2_{\sigma_N}}\left(\partial_\mu\vec{\pi}\right)^2\left(1 - \frac{\Box}{m^2_{\sigma_N}}\right)\left(\partial_\nu\vec{\pi}\right)^2}_{(d)}\right]\right\}\,. \qquad (4.366)
\end{aligned}
$$

For the sake of clarity and in order to identify the relevant terms, we consider the four terms (a)-(d) separately. Integrating the second term in (a) by parts, we find

$$
\begin{aligned}
\frac{c^2_{1,\sigma_N}}{m^2_{\sigma_N}}\int \mathrm{d}^4x\,\vec{\pi}^2\left(1 - \frac{\Box}{m^2_{\sigma_N}} + \frac{\Box^2}{m^4_{\sigma_N}}\right)\vec{\pi}^2 &= \frac{c^2_{1,\sigma_N}}{m^2_{\sigma_N}}\int \mathrm{d}^4x\,\left[\left(\vec{\pi}^2\right)^2 + \frac{1}{m^2_{\sigma_N}}\left(\partial^\mu\vec{\pi}^2\right)\left(\partial_\mu\vec{\pi}^2\right) + \frac{1}{m^4_{\sigma_N}}\left(\Box\vec{\pi}^2\right)\Box\vec{\pi}^2\right] \\
&= \frac{c^2_{1,\sigma_N}}{m^2_{\sigma_N}}\int \mathrm{d}^4x\,\left\{\left(\vec{\pi}^2\right)^2 + \frac{4}{m^2_{\sigma_N}}\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2 + \frac{4}{m^2_{\sigma_N}}\left[\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2\right.\right. \\
&\quad \left.\left. + 4m^2_{\vec{\pi}}\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2 - m^4_{\vec{\pi}}\left(\vec{\pi}^2\right)^2\right]\right\} \\
&= \int \mathrm{d}^4x\,\left[\left(\frac{c^2_{1,\sigma_N}}{m^2_{\sigma_N}} - \frac{4c^2_{1,\sigma_N}m^4_{\vec{\pi}}}{m^6_{\sigma_N}}\right)\left(\vec{\pi}^2\right)^2 + \left(\frac{4c^2_{1,\sigma_N}}{m^4_{\sigma_N}} + \frac{16c^2_{1,\sigma_N}m^2_{\vec{\pi}}}{m^6_{\sigma_N}}\right)\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2\right. \\
&\quad \left. + \frac{4c^2_{1,\sigma_N}}{m^6_{\sigma_N}}\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2\right]\,.\,, \qquad (4.367)
\end{aligned}
$$

Similarly, term (b) can be rewritten as

$$
\begin{aligned}
\frac{c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\int \mathrm{d}^4x\,\left(\partial_\mu\vec{\pi}\right)^2\left(1 - \frac{\Box}{m^2_{\sigma_N}}\right)\vec{\pi}^2 &= \frac{c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\int \mathrm{d}^4x\,\left[\left(\partial_\mu\vec{\pi}\right)^2\vec{\pi}^2 - \left(\partial_\mu\vec{\pi}\right)^2\frac{\Box}{m^2_{\sigma_N}}\vec{\pi}^2\right] \\
&= \frac{c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\int \mathrm{d}^4x\,\left\{\left(\partial_\mu\vec{\pi}\right)^2\vec{\pi}^2 - \frac{2}{m^2_{\sigma_N}}\left(\partial_\mu\vec{\pi}\right)^2\left[\left(\partial_\nu\vec{\pi}\right)^2 + \vec{\pi}\Box\vec{\pi}\right]\right\} \\
&= \int \mathrm{d}^4x\,\left[\frac{c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\left(1 + \frac{2m^2_{\vec{\pi}}}{m^2_{\sigma_N}}\right)\left(\partial_\mu\vec{\pi}\right)^2\vec{\pi}^2 - \frac{2c_{1,\sigma_N}c_{2,\sigma_N}}{m^4_{\sigma_N}}\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2\right]\,, \qquad (4.368)
\end{aligned}
$$

where we used the chain rule, the product rule and the free Klein-Gordon equation for the $\vec{\pi}$-fields, Eq. (4.359).

Performing two integrations by parts in the second term of (c) and using exactly the same manipulations as in Eq. (4.368), we are able to show that the third term yields the same result as (b),

$$
\frac{c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\int \mathrm{d}^4x\,\vec{\pi}^2\left(1 - \frac{\Box}{m^2_{\sigma_N}}\right)\left(\partial_\nu\vec{\pi}\right)^2 = \int \mathrm{d}^4x\,\left[\frac{c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\left(1 + \frac{2m^2_{\vec{\pi}}}{m^2_{\sigma_N}}\right)\left(\partial_\mu\vec{\pi}\right)^2\vec{\pi}^2 - \frac{2c_{1,\sigma_N}c_{2,\sigma_N}}{m^4_{\sigma_N}}\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2\right]\,. \qquad (4.369)
$$

Finally, in the case of (d), we are able to neglect the contribution of the second term, since it contains six space-time derivatives. We find

$$
\frac{c^2_{2,\sigma_N}}{m^2_{\sigma_N}}\int \mathrm{d}^4x\,\left(\partial_\mu\vec{\pi}\right)^2\left(1 - \frac{\Box}{m^2_{\sigma_N}}\right)\left(\partial_\nu\vec{\pi}\right)^2 = \int \mathrm{d}^4x\,\frac{c^2_{2,\sigma_N}}{m^2_{\sigma_N}}\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2 + \mathcal{O}(\partial^6)\,. \qquad (4.370)
$$

Combining the results (4.366)-(4.370) and summarizing identical terms, the functional integral $I_{\sigma_N}[\vec{\pi}]$ can be written as

$$I_{\sigma_N}[\vec{\pi}] = \mathcal{N}_{\sigma_N}\left[\det\mathscr{O}_{\sigma_N,\vec{\pi}}(x,y)\right]^{-1/2} \exp\left\{i\int \mathrm{d}^4x \,\left\{\left[\frac{c_{1,\sigma_N}^2}{2m_{\sigma_N}^2}\left(1-\frac{4m_{\vec{\pi}}^4}{m_{\sigma_N}^4}\right)+\frac{c_{1,\sigma_N}c_{2,\sigma_N}m_{\vec{\pi}}^2}{m_{\sigma_N}^2}\left(1+\frac{2m_{\vec{\pi}}^2}{m_{\sigma_N}^2}\right)\right]\left(\vec{\pi}^2\right)^2\right.\right.$$
$$+\left[\frac{2c_{1,\sigma_N}^2}{m_{\sigma_N}^4}\left(1+\frac{4m_{\vec{\pi}}^2}{m_{\sigma_N}^2}\right)-\frac{2c_{1,\sigma_N}c_{2,\sigma_N}}{m_{\sigma_N}^2}\left(1+\frac{2m_{\vec{\pi}}^2}{m_{\sigma_N}^2}\right)\right]\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2+\left[\frac{c_{2,\sigma_N}^2}{2m_{\sigma_N}^2}-\frac{2c_{1,\sigma_N}c_{2,\sigma_N}}{m_{\sigma_N}^4}+\frac{2c_{1,\sigma_N}^2}{m_{\sigma_N}^6}\right]$$
$$\left.\left.\times\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2\right\}\right\} . \tag{4.371}$$

Finally, we have to consider the $\vec{\rho}$ integral. In this case, the situation becomes more involved, since the $\vec{\rho}_\mu$- and the $\vec{\pi}$-fields are both vectorial structures in isospin space. To this end, we write the functional integral as

$$I_{\vec{\rho}}[\vec{\pi}] = \int \mathscr{D}\vec{\rho}_\mu(x)\ \exp\left\{i\int \mathrm{d}^4x\ \mathscr{L}_{\vec{\rho}\vec{\pi}}\right\}$$
$$= \int \mathscr{D}\vec{\rho}_\mu(x)\ \exp\left\{i\int \mathrm{d}^4x\ \left\{\underbrace{-\frac{1}{4}\left(\partial^\mu\vec{\rho}^\nu-\partial^\nu\vec{\rho}^\mu\right)\cdot\left(\partial_\mu\vec{\rho}_\nu-\partial_\nu\vec{\rho}_\mu\right)+\frac{1}{2}m_{\vec{\rho}}^2\vec{\rho}_\mu^2-\frac{\xi_{\vec{\rho}}}{2}\left(\partial_\mu\vec{\rho}^\mu\right)\cdot\left(\partial_\nu\vec{\rho}^\nu\right)}_{(a)}\right.\right.$$
$$+\underbrace{\left[\frac{g_1^2}{2}+\frac{1}{4}\left(h_1+h_2-h_3\right)\right]Z^2\vec{\rho}_\mu^2\vec{\pi}^2}_{(b)}-\underbrace{\frac{1}{2}\left(g_1^2-h_3\right)Z^2\left(\vec{\rho}_\mu\cdot\vec{\pi}\right)^2}_{(c)}+\underbrace{\left(-\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right)w^2Z^2\vec{\rho}_\mu^2\left(\partial_\nu\vec{\pi}\right)^2}_{(d)}$$
$$+\underbrace{g_3w^2Z^2\left[\left(\vec{\rho}^\mu\cdot\vec{\rho}^\nu\right)\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial_\nu\vec{\pi}\right)+\left(\vec{\rho}^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\mu\cdot\partial_\nu\vec{\pi}\right)+\left(\vec{\rho}^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\nu\cdot\partial_\mu\vec{\pi}\right)\right]}_{(e)}$$
$$+\underbrace{\left(-g_3+g_4-g_5+2g_6\right)w^2Z^2\left(\vec{\rho}^\mu\cdot\partial_\mu\vec{\pi}\right)\left(\vec{\rho}^\nu\cdot\partial_\nu\vec{\pi}\right)}_{(f)}+\underbrace{\left[\left(g_1^2-h_3\right)\phi_N wZ^2-g_1Z^2\right]\vec{\rho}^\mu\cdot\left(\partial_\mu\vec{\pi}\times\vec{\pi}\right)}_{(g)}$$
$$\left.\left.+\underbrace{g_2w^2Z^2\left(\partial^\mu\vec{\rho}^\nu\right)\cdot\left(\partial_\nu\vec{\pi}\times\partial_\mu\vec{\pi}\right)}_{(h)}\right\}\right\} . \tag{4.372}$$

In order to rewrite the above integral in the usual form of a shifted Gaussian integral, we have to introduce isospin indices for those terms that are quadratic in the $\vec{\rho}_\mu$-fields, i.e., (a)-(f). The remaining two terms, (g)-(h), correspond to the "source" term, which is linearly coupled to the $\vec{\rho}_\mu$-fields. But before we consider these terms, we focus on the first six terms and rewrite them in the desired way. Using the antisymmetry of the field-strength tensor and integrating the first and the third term of (a) by parts, we find

$$\int \mathrm{d}^4x\ \left[-\frac{1}{4}\left(\partial^\mu\vec{\rho}^\nu-\partial^\nu\vec{\rho}^\mu\right)\cdot\left(\partial_\mu\vec{\rho}_\nu-\partial_\nu\vec{\rho}_\mu\right)+\frac{1}{2}m_{\vec{\rho}}^2\vec{\rho}_\mu^2-\frac{\xi_{\vec{\rho}}}{2}\left(\partial_\mu\vec{\rho}^\mu\right)\cdot\left(\partial_\nu\vec{\rho}^\nu\right)\right]$$
$$= \int \mathrm{d}^4x\ \left[-\frac{1}{2}\left(\partial^\mu\vec{\rho}^\nu-\partial^\nu\vec{\rho}^\mu\right)\cdot\left(\partial_\mu\vec{\rho}_\nu\right)+\frac{1}{2}m_{\vec{\rho}}^2\vec{\rho}_\mu^2-\frac{\xi_{\vec{\rho}}}{2}\left(\partial_\mu\vec{\rho}^\mu\right)\cdot\left(\partial_\nu\vec{\rho}^\nu\right)\right]$$
$$= \int \mathrm{d}^4x\ \left[\frac{1}{2}\vec{\rho}_\mu\cdot\left(\Box g^{\mu\nu}-\partial^\mu\partial^\nu\right)\vec{\rho}_\nu+\frac{1}{2}m_{\vec{\rho}}^2\vec{\rho}_\mu^2+\frac{\xi_{\vec{\rho}}}{2}\vec{\rho}_\mu\partial^\mu\partial^\nu\cdot\vec{\rho}_\nu\right]$$
$$= \int \mathrm{d}^4x\ \frac{1}{2}\vec{\rho}_\mu\cdot\left[\left(\Box+m_{\vec{\rho}}^2\right)g^{\mu\nu}-\left(1-\xi_{\vec{\rho}}\right)\partial^\mu\partial^\nu\right]\vec{\rho}_\nu$$
$$= \int \mathrm{d}^4x\ \frac{1}{2}\rho_{\mu,i}\cdot\left[\left(\Box+m_{\vec{\rho}}^2\right)g^{\mu\nu}-\left(1-\xi_{\vec{\rho}}\right)\partial^\mu\partial^\nu\right]\delta^{ij}\rho_{\nu,j}\ , \tag{4.373}$$

where we introduced isospin-space indices in the last line. Obviously, this term corresponds to a structure which is diagonal in isospin space. This kind of structure is also present in (b) and (d), so that

$$\int \mathrm{d}^4x\ \left[\frac{g_1^2}{2}+\frac{1}{4}\left(h_1+h_2-h_3\right)\right]Z^2\vec{\rho}_\mu^2\vec{\pi}^2 = \int \mathrm{d}^4x\ \left[\frac{g_1^2}{2}+\frac{1}{4}\left(h_1+h_2-h_3\right)\right]Z^2\rho_{\mu,i}g^{\mu\nu}\delta^{ij}\vec{\pi}^2\rho_{\nu,j}\ , \tag{4.374}$$
$$\int \mathrm{d}^4x\ \left(-\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right)w^2Z^2\vec{\rho}_\mu^2\left(\partial_\nu\vec{\pi}\right)^2 = \int \mathrm{d}^4x\ \left(-\frac{g_3}{2}+\frac{g_4}{2}+\frac{g_5}{2}+g_6\right)w^2Z^2\rho_{\mu,i}g^{\mu\nu}\delta^{ij}\left(\partial_\alpha\vec{\pi}\right)^2\rho_{\nu,j}\ . \tag{4.375}$$

On the other hand, the terms (c), (e), and (f) are not diagonal in isospin space, since

$$\int \mathrm{d}^4x\,\frac{1}{2}\left(g_1^2-h_3\right)Z^2\left(\vec{\rho}_\mu\cdot\vec{\pi}\right)^2=\int \mathrm{d}^4x\,\frac{1}{2}\left(g_1^2-h_3\right)Z^2\rho_{\mu,i}g^{\mu\nu}\pi^i\pi^j\rho_{\nu,j}\;, \tag{4.376}$$

$$\begin{aligned}&\int \mathrm{d}^4x\; g_3 w^2 Z^2\left[\left(\vec{\rho}^\mu\cdot\vec{\rho}^\nu\right)\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial_\nu\vec{\pi}\right)+\left(\vec{\rho}^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\mu\cdot\partial_\nu\vec{\pi}\right)+\left(\vec{\rho}^\mu\cdot\partial^\nu\vec{\pi}\right)\left(\vec{\rho}_\nu\cdot\partial_\mu\vec{\pi}\right)\right]\\&=\int \mathrm{d}^4x\; g_3 w^2 Z^2\left[\rho_{\mu,i}g^{ij}\left(\partial^\mu\pi^k\right)\left(\partial^\nu\pi_k\right)\rho_{\nu,j}+\rho_{\mu,i}g^{\mu\nu}\left(\partial^\alpha\pi^i\right)\left(\partial_\alpha\pi^j\right)\rho_{\nu,j}+\rho_{\mu,i}\left(\partial^\nu\pi^i\right)\left(\partial^\mu\pi^j\right)\rho_{\nu,j}\right]\;,\end{aligned} \tag{4.377}$$

$$\int \mathrm{d}^4x\;\left(-g_3+g_4-g_5+2g_6\right)w^2Z^2\left(\vec{\rho}^\mu\cdot\partial_\mu\vec{\pi}\right)\left(\vec{\rho}^\nu\cdot\partial_\nu\vec{\pi}\right)=\int \mathrm{d}^4x\;\left(-g_3+g_4-g_5+2g_6\right)w^2Z^2\rho_{\mu,i}\left(\partial^\mu\pi^i\right)\left(\partial^\nu\pi^j\right)\rho_{\nu,j}\;. \tag{4.378}$$

And exactly the non-diagonal contributions of (e) and (f) will cause problems, when we have to invert the corresponding operator. This can be understood as follows: In the previous calculations all operators of the heavy fields, which we had to invert, were diagonal in isospin space as well as in space-time, compare Eqs. (4.278) and (4.356). In the case of the $\vec{\rho}_\mu$-fields this situation changes, since the terms (a)-(f) contain contributions which are non-diagonal either in isospin space, or in space-time, or in both spaces. Therefore, the process of inverting the corresponding operator becomes more involved. In our case, the outermost mathematical structure is that of isospin space, so that the inversion process would be as follows: First of all, we would introduce two projection operators

$$\mathcal{P}_\parallel^{ij}=\frac{\pi^i\pi^j}{\vec{\pi}^2}\;, \tag{4.379}$$

$$\mathcal{P}_\perp^{ij}=\delta^{ij}-\mathcal{P}_\parallel^{ij}\;, \tag{4.380}$$

which project onto the one-dimensional subspace parallel to $\vec{\pi}$ and onto the two-dimensional subspace perpendicular to $\vec{\pi}$, which can be seen as follows

$$\mathcal{P}_\parallel^{ij}\pi_j=\frac{\pi^i\pi^j}{\vec{\pi}^2}\pi_j=\pi^i\;, \tag{4.381}$$

$$\mathcal{P}_\perp^{ij}\pi_j=\left(\delta^{ij}-\mathcal{P}_\parallel^{ij}\right)\pi_j=\pi^i-\pi^i=0\;. \tag{4.382}$$

Furthermore, both operators are idempotent, orthogonal, and complete

$$\mathcal{P}^i_{\;k,\parallel}\mathcal{P}_\parallel^{kj}=\frac{\pi^i\pi_k}{\vec{\pi}^2}\frac{\pi^k\pi^j}{\vec{\pi}^2}=\frac{\pi^i\pi^j}{\vec{\pi}^2}=\mathcal{P}_\parallel^{ij}\;,\qquad \mathcal{P}^i_{\;k,\perp}\mathcal{P}_\perp^{kj}=\left(\delta^i_{\;k}-\mathcal{P}^i_{\;k,\parallel}\right)\left(\delta^{kj}-\mathcal{P}_\parallel^{kj}\right)=\delta^{ij}-\mathcal{P}_\parallel^{ij}=\mathcal{P}_\perp^{ij}\;, \tag{4.383}$$

$$\mathcal{P}^i_{\;k,\parallel}\mathcal{P}_\perp^{kj}=\mathcal{P}^i_{\;k,\parallel}\left(\delta^{kj}-\mathcal{P}_\parallel^{kj}\right)=\mathcal{P}_\parallel^{ij}-\mathcal{P}_\parallel^{ij}=0\;, \tag{4.384}$$

$$\mathcal{P}_\parallel^{ij}+\mathcal{P}_\perp^{ij}=\delta^{ij}\;. \tag{4.385}$$

With these properties, it is, in principle, possible to rewrite the $\vec{\rho}_\mu$ operator $\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi}}\right)^{ij}(x,y)$ in the following form

$$\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi}}\right)^{ij}(x,y)=\mathcal{D}^{\mu\nu}_{\vec{\pi},\parallel}(x,y)\mathcal{P}_\parallel^{ij}+\mathcal{D}^{\mu\nu}_{\vec{\pi},\perp}(x,y)\mathcal{P}_\perp^{ij}\;, \tag{4.386}$$

where the coefficients $\mathcal{D}^{\mu\nu}_{\vec{\pi},\parallel}(x,y)$, $\mathcal{D}^{\mu\nu}_{\vec{\pi},\perp}(x,y)$ define differential operators that also contain $\vec{\pi}$-fields. These differential operators are, of course, scalar objects in isospin space and tensorial objects in space-time. Using Eq. (4.386) and the various properties of the projection operators (4.379) and (4.380), the inverse operator is given by

$$\left(\mathscr{O}^{\mu\nu-1}_{\vec{\rho},\vec{\pi}}\right)^{ij}(x,y)=\mathcal{D}^{\mu\nu-1}_{\vec{\pi},\parallel}(x,y)\mathcal{P}_\parallel^{ij}+\mathcal{D}^{\mu\nu-1}_{\vec{\pi},\perp}(x,y)\mathcal{P}_\perp^{ij}\;. \tag{4.387}$$

Now, since the differential operators $\mathcal{D}^{\mu\nu}_{\vec{\pi},\parallel}(x,y)$ and $\mathcal{D}^{\mu\nu}_{\vec{\pi},\perp}(x,y)$ are, in general, not diagonal objects in space-time, we have to repeat this inversion procedure in space-time. This means that we have to transform the differential operators to momentum space and introduce two projection operators which project onto the one-dimensional subspace parallel to k^μ and onto the three-dimensional subspace perpendicular to the four-momentum k^μ. These operators have a similar structure as those that we introduced in isospin space

$$\mathcal{P}_\parallel^{\mu\nu} = \frac{k^\mu k^\nu}{k^2} \;, \tag{4.388}$$
$$\mathcal{P}_\perp^{\mu\nu} = g^{\mu\nu} - \mathcal{P}_\parallel^{\mu\nu} \;. \tag{4.389}$$

It is easy to see that these operators are also idempotent, orthogonal and complete

$$\mathcal{P}^\mu_{\lambda,\parallel}\mathcal{P}_\parallel^{\lambda\nu} = \frac{k^\mu k_\lambda}{k^2}\frac{k^\lambda k^\nu}{k^2} = \frac{k^\mu k^\nu}{k^2} = \mathcal{P}_\parallel^{\mu\nu} \;, \qquad \mathcal{P}^\mu_{\lambda\perp}\mathcal{P}_\perp^{\lambda\nu} = \left(g^\mu_\lambda - \mathcal{P}^\mu_{\lambda,\parallel}\right)\left(g^{\lambda\nu} - \mathcal{P}_\parallel^{\lambda\nu}\right) = g^{\mu\nu} - \mathcal{P}_\parallel^{\mu\nu} = \mathcal{P}_\perp^{\mu\nu} \;, \tag{4.390}$$

$$\mathcal{P}^\mu_{\lambda,\parallel}\mathcal{P}_\perp^{\lambda\nu} = \mathcal{P}^\mu_{\lambda,\parallel}\left(g^{\mu\nu} - \mathcal{P}_\parallel^{\mu\nu}\right) = \mathcal{P}_\parallel^{\mu\nu} - \mathcal{P}_\parallel^{\mu\nu} = 0 \;, \tag{4.391}$$
$$\mathcal{P}_\parallel^{\mu\nu} + \mathcal{P}_\perp^{\mu\nu} = g^{\mu\nu} \;. \tag{4.392}$$

Similar to Eq. (4.386), it is then possible to decompose $\mathcal{D}^{\mu\nu}_{\vec{\pi},\parallel}(x,y)$ and $\mathcal{D}^{\mu\nu}_{\vec{\pi},\perp}(x,y)$ according to

$$\tilde{\mathcal{D}}^{\mu\nu}_{\vec{\pi},\parallel/\perp}(k) = \tilde{\mathcal{D}}_{\vec{\pi},\parallel}(k)\mathcal{P}_\parallel^{\mu\nu} + \tilde{\mathcal{D}}_{\vec{\pi},\perp}(k)\mathcal{P}_\perp^{\mu\nu} \;, \tag{4.393}$$

where $\tilde{\mathcal{D}}_{\vec{\pi},\parallel}(k)$ and $\tilde{\mathcal{D}}_{\vec{\pi},\perp}(k)$ define Lorentz-scalar coefficients, that depend on the four-momentum k^μ. So, in principle, the procedure in order to invert the $\vec{\rho}_\mu$ operator is quite clear. The problems now derive from the terms (e) and (f), since the space-time derivatives that act on the $\vec{\pi}$-fields prevent us from rewriting the operator into the form (4.386). To be particular, by using integrations by parts, it was, up to now, not possible to isolate a structure that is proportional $\pi^i\pi^j$, so that it is possible to introduce the projection operators. Therefore, the upcoming calculation will be based on the assumption that the terms (b)-(f) will not have a tree-level contribution to four-pion interaction terms, so that the operator which corresponds to the $\vec{\rho}_\mu$-fields will be made of terms from (a). This assumption seems to be quite radical, but in the previous calculations and also in the case of the σ_N-field, it was shown that the additional terms which are proportional to $\vec{\pi}$-fields inside the operator do not correspond to tree-level four-pion interactions, since we terminated the expansion after the $m = 0$ term, compare Eqs. (4.286) and (4.365). With these assumptions the functional integral (4.372) simplifies significantly

$$\begin{aligned}
I_{\vec{\rho}}[\vec{\pi}] &= \int \mathscr{D}\vec{\rho}_\mu(x)\ \exp\Biggl\{i\int \mathrm{d}^4x\ \Biggl\{-\frac{1}{4}\left(\partial^\mu\vec{\rho}^\nu - \partial^\nu\vec{\rho}^\mu\right)\cdot\left(\partial_\mu\vec{\rho}_\nu - \partial_\nu\vec{\rho}_\mu\right) + \frac{1}{2}m_{\vec{\rho}}^2\vec{\rho}_\mu^2 - \frac{\xi_{\vec{\rho}}}{2}\left(\partial_\mu\vec{\rho}^\mu\right)\cdot\left(\partial_\nu\vec{\rho}^\nu\right) \\
&\quad + \left[\left(g_1^2 - h_3\right)\phi_N w Z^2 - g_1 Z^2\right]\vec{\rho}^\mu\cdot\left(\partial_\mu\vec{\pi}\times\vec{\pi}\right) + g_2 w^2 Z^2\left(\partial^\mu\vec{\rho}^\nu\right)\cdot\left[\left(\partial_\nu\vec{\pi}\right)\times\left(\partial_\mu\vec{\pi}\right)\right]\Biggr\}\Biggr\} \\
&= \int \mathscr{D}\vec{\rho}_\mu(x)\ \exp\Biggl\{i\int \mathrm{d}^4x\ \Biggl\{\frac{1}{2}\rho_{\mu,i}\cdot\left[\left(\Box + m_{\vec{\rho}}^2\right)g^{\mu\nu} - \left(1 - \xi_{\vec{\rho}}\partial^\mu\partial^\nu\right)\right]\delta^{ij}\rho_{\nu,j} \\
&\quad + \left[\left(g_1^2 - h_3\right)\phi_N w Z^2 - g_1 Z^2\right]\rho_i^\mu\left(\partial_\mu\vec{\pi}\times\vec{\pi}\right)^i - g_2 w^2 Z^2\rho_i^\nu\partial^\mu\left[\left(\partial_\nu\vec{\pi}\right)\times\left(\partial_\mu\vec{\pi}\right)\right]^i\Biggr\}\Biggr\} \\
&= \int \mathscr{D}\vec{\rho}_\mu(x)\ \exp\Biggl\{i\int \mathrm{d}^4x\ \Biggl\{\frac{1}{2}\rho_{\mu,i}\cdot\left[\left(\Box + m_{\vec{\rho}}^2\right)g^{\mu\nu} - \left(1 - \xi_{\vec{\rho}}\partial^\mu\partial^\nu\right)\right]g^{ij}\rho_{\nu,j} \\
&\quad + \left[\left(g_1^2 - h_3\right)\phi_N w Z^2 - g_1 Z^2 + g_2 w^2 Z^2 m_{\vec{\pi}}^2\right]\rho_{\mu,i}\left(\partial^\mu\vec{\pi}\times\vec{\pi}\right)^i - g_2 w^2 Z^2\rho_{\mu,i}\left[\left(\partial^\mu\partial^\nu\vec{\pi}\right)\times\left(\partial_\nu\vec{\pi}\right)\right]^i\Biggr\}\Biggr\} \\
&= \int \mathscr{D}\vec{\rho}_\mu(x)\ \exp\Biggl\{\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ \vec{\rho}_\mu(x)\cdot\mathscr{O}^{\mu\nu}(x,y)\vec{\rho}_\nu(y) + i\int \mathrm{d}^4x\ \vec{J}^\mu_{\vec{\rho},\vec{\pi}}(x)\cdot\vec{\rho}_\mu(x)\Biggr\} \;,
\end{aligned} \tag{4.394}$$

where we integrated the third term in the second line by parts and made use of Eq. (4.373). In the third line, we applied the product rule and used the free Klein-Gordon equation (4.359) of the $\vec{\pi}$-fields. Finally, in the last line, we introduced the operator

$$\mathscr{O}^{\mu\nu}(x,y) = i\left[\left(\Box_x + m_{\vec{\rho}}^2\right)g^{\mu\nu} - (1 - \xi_{\vec{\rho}})\partial_x^\mu\partial_x^\nu\right]\delta^{(4)}(x-y) \tag{4.395}$$

and the "source"

$$\begin{aligned}
\vec{J}^\mu_{\vec{\rho},\vec{\pi}}(x) &= \left[\left(g_1^2 - h_3\right)\phi_N w Z^2 - g_1 Z^2 + g_2 w^2 Z^2 m_{\vec{\pi}}^2\right]\left[\left(\partial^\mu\vec{\pi}(x)\right)\times\vec{\pi}(x)\right] - g_2 w^2 Z^2\left[\left(\partial^\mu\partial^\nu\vec{\pi}(x)\right)\times\partial_\nu\vec{\pi}(x)\right] \\
&\equiv c_{1,\vec{\rho}}\left[\left(\partial^\mu\vec{\pi}(x)\right)\times\vec{\pi}(x)\right] - c_{2,\vec{\rho}}\left[\left(\partial^\mu\partial^\nu\vec{\pi}(x)\right)\times\partial_\nu\vec{\pi}(x)\right] \;,
\end{aligned} \tag{4.396}$$

where we defined the coefficients

$$c_{1,\vec{\rho}} = \left(g_1^2 - h_3\right)\phi_N w Z^2 - g_1 Z^2 + g_2 w^2 Z^2 m_{\vec{\pi}}^2 \ , \tag{4.397}$$

$$c_{1,\vec{\rho}} = g_2 w^2 Z^2 \ , \tag{4.398}$$

which have dimension [1] and [Energy^{-2}]. From Eq. (4.373) it was already clear that the operator (4.395) has to be diagonal in space-time. This fact will simplify the following calculation. But before we need to invert the operator (4.395), we have to evaluate the functional integral (4.394). To this end, we have to perform the analytic continuation to Euclidean space-time by using Eqs. (4.18), (4.20), and (4.21). In contrast to the previous calculations, we now have to deal with vectorial and tensorial structures that have to be transformed to Euclidean space-time. We start with the scalar product of the source (4.396) and the $\vec{\rho}_\mu$ vector

$$\begin{aligned}\vec{J}^\mu_{\vec{\rho},\vec{\pi}}(x)\cdot\vec{\rho}_\mu(x) &= \vec{J}^0_{\vec{\rho},\vec{\pi}}(x)\cdot\vec{\rho}_0(x) - \vec{J}^i_{\vec{\rho},\vec{\pi}}(x)\cdot\vec{\rho}_i(x)\\ &= i\vec{J}^0_{\vec{\rho},\vec{\pi},E}(x_E)\cdot i\vec{\rho}_{0,E}(x_E) - \vec{J}^i_{\vec{\rho},\vec{\pi},E}(x_E)\cdot\vec{\rho}_{i,E}(x_E)\\ &= -\vec{J}^\mu_{\vec{\rho},\vec{\pi},E}(x_E)\cdot\vec{\rho}_{\mu,E}(x_E)\ ,\end{aligned} \tag{4.399}$$

where we defined the Euclidean four-vectors

$$\left(\vec{J}^\mu_{\vec{\rho},\vec{\pi},E}\right)(x_E) = \left(\vec{J}^0_{\vec{\rho},\vec{\pi},E}(x_E), \vec{J}^i_{\vec{\rho},\vec{\pi},E}(x_E)\right)^T \ , \qquad \left(\vec{\rho}^\mu_E\right)(x_E) = \left(\vec{\rho}^0_E(x_E), \vec{\rho}^i_E(x_E)\right)^T \ , \tag{4.400}$$

with $\vec{J}^i_{\vec{\rho},\vec{\pi},E} \equiv \vec{J}^i_{\vec{\rho},\vec{\pi}}$, $\vec{\rho}^i_E \equiv \vec{\rho}^i$. It should be noted that the i-indices in Eqs. (4.399) and (4.400) denote spatial directions in $\mathbb{R}^3$ and not the components in isospin space as in Eq. (4.394). Now, we also have to transform the bilinear form in Eq. (4.394)

$$\begin{aligned}&\vec{\rho}_\mu(x)\cdot\mathscr{O}^{\mu\nu}(x,y)\vec{\rho}_\nu(y)\\ &= \vec{\rho}_\mu(x)\cdot i\left[\left(\Box_x + m^2_{\vec{\rho}}\right)g^{\mu\nu} - (1-\xi_{\vec{\rho}})\partial^\mu_x\partial^\nu_x\right]\delta^{(4)}(x-y)\vec{\rho}_\nu(y)\\ &= \vec{\rho}_0(x)\cdot i\left[\left(\Box_x + m^2_{\vec{\rho}}\right)g^{00} - (1-\xi_{\vec{\rho}})\partial^0_x\partial^0_x\right]\delta^{(4)}(x-y)\vec{\rho}_0(y) - \vec{\rho}_0(x)\cdot i\left[-(1-\xi_{\vec{\rho}})\partial^0_x\partial^i_x\right]\delta^{(4)}(x-y)\vec{\rho}_i(y)\\ &\quad - \vec{\rho}_i(x)\cdot i\left[-(1-\xi_{\vec{\rho}})\partial^i_x\partial^0_x\right]\delta^{(4)}(x-y)\vec{\rho}_0(y) + \vec{\rho}_i(x)\cdot i\left[\left(\Box_x + m^2_{\vec{\rho}}\right)g^{ij} - (1-\xi_{\vec{\rho}})\partial^i_x\partial^j_x\right]\delta^{(4)}(x-y)\vec{\rho}_j(y)\\ &= i\vec{\rho}_{0,E}(x_E)\cdot i\left[\left(-\Box_{x,E} + m^2_{\vec{\rho}}\right)g^{00}_E - (1-\xi_{\vec{\rho}})i\partial^0_{x,E}i\partial^0_{x,E}\right]i\delta^{(4)}(x_E-y_E)i\vec{\rho}_0(y_E)\\ &\quad - i\vec{\rho}_{0,E}(x_E)\cdot i\left[-(1-\xi_{\vec{\rho}})i\partial^0_{x,E}\partial^i_{x,E}\right]i\delta^{(4)}(x_E-y_E)\vec{\rho}_{i,E}(y_E) - \vec{\rho}_{i,E}(x_E)\cdot i\left[-(1-\xi_{\vec{\rho}})\partial^i_{x,E}i\partial^0_{x,E}\right]i\delta^{(4)}(x_E-y_E)i\vec{\rho}_{0,E}(y)\\ &\quad + \vec{\rho}_{i,E}(x_E)\cdot i\left[\left(-\Box_{x,E} + m^2_{\vec{\rho}}\right)(-1)g^{ij}_E - (1-\xi_{\vec{\rho}})\partial^i_{x,E}\partial^j_{x,E}\right]i\delta^{(4)}(x_E-y_E)\vec{\rho}_{j,E}(y_E)\\ &= \vec{\rho}_{\mu,E}(x_E)\cdot\left[\left(-\Box_{x,E} + m^2_{\vec{\rho}}\right)g^{\mu\nu}_E + (1-\xi_{\vec{\rho}})\partial^\mu_{x,E}\partial^\nu_{x,E}\right]\delta^{(4)}(x_E-y_E)\vec{\rho}_{\nu,E}(x_E)\\ &\equiv \vec{\rho}_{\mu,E}(x_E)\cdot\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}(x_E,y_E)\vec{\rho}_{\nu,E}(x_E)\ ,\end{aligned} \tag{4.401}$$

where we made use of $g^{i0} = g^{0i} = 0$, the Euclidean 4-gradient $\partial^\mu_E = (\partial_\tau, \boldsymbol{\nabla})^T$, and introduced the Euclidean metric

$$g^{\mu\nu}_E = \mathrm{diag}\,(1,1,1,1)\ . \tag{4.402}$$

Inserting the above results into the functional integral (4.394), we find

$$\begin{aligned}I_{\vec{\rho}}[\vec{\pi}] &= \int \mathscr{D}\vec{\rho}_{\mu,E}(x_E)\ \exp\left\{-\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\ \vec{\rho}_{\mu,E}(x_E)\left[\left(-\Box_{x,E} + m^2_{\vec{\rho}}\right)g^{\mu\nu}_E + (1-\xi_{\vec{\rho}})\partial^\mu_{x,E}\partial^\nu_{x,E}\right]\delta^{(4)}(x_E-y_E)\vec{\rho}_{\nu,E}(x_E)\right.\\ &\qquad\left. + \int \mathrm{d}^4x_E\ \left[-\vec{J}^\mu_{\vec{\rho},\vec{\pi},E}(x_E)\right]\cdot\vec{\rho}_{\mu,E}(x_E)\right\}\\ &= \int \mathscr{D}\vec{\rho}_{\mu,E}(x_E)\ \exp\left\{-\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\ \vec{\rho}_{\mu,E}(x_E)\cdot\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}(x_E,y_E)\vec{\rho}_{\nu,E}(x_E) + \int \mathrm{d}^4x_E\ \left[-\vec{J}^\mu_{\vec{\rho},\vec{\pi},E}(x_E)\right]\cdot\vec{\rho}_{\mu,E}(x_E)\right\}\\ &= \mathcal{N}_{\vec{\rho}}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x_E,y_E)\right]^{-1/2}\exp\left\{\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\ \vec{J}_{\vec{\rho},\vec{\pi},\mu,E}(x_E)\cdot\mathscr{O}^{\mu\nu-1}_{\vec{\rho},\vec{\pi},E}(x_E,y_E)\vec{J}_{\vec{\rho},\vec{\pi},\mu,E}(y_E)\right\}\end{aligned} \tag{4.403}$$

with the Euclidean operator defined in Eq. (4.401). Before we perform the analytic continuation back to Minkowski space-time, we want to derive the explicit expression of the inverse operator $\mathscr{O}^{\mu\nu,-1}_{\vec{\rho},\vec{\pi},E}(x_E,y_E)$. To this end, we begin with the determination of the Fourier transform of the operator in Eq. (4.401), i.e.,

$$\begin{aligned}
&\left[\left(-\Box_{x,E}+m_{\vec\rho}^2\right)g_E^{\mu\nu}+(1-\xi_{\vec\rho})\partial_{x,E}^{\mu}\partial_{x,E}^{\nu}\right]\delta^{(4)}(x_E-y_E)\\
&=\left[\left(-\Box_{x,E}+m_{\vec\rho}^2\right)g_E^{\mu\nu}+(1-\xi_{\vec\rho})\partial_{x,E}^{\mu}\partial_{x,E}^{\nu}\right]\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\,e^{-ik_E(x_E-y_E)}\\
&=\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\,\tilde{\mathscr{O}}^{\mu\nu}_{\vec\rho,\vec\pi,E}(k_E)e^{-ik_E(x_E-y_E)}\ ,
\end{aligned}\tag{4.404}$$

where

$$\tilde{\mathscr{O}}^{\mu\nu}_{\vec\rho,\vec\pi,E}(k_E)=\left(k_E^2+m_{\vec\rho}^2\right)g_E^{\mu\nu}-(1-\xi_{\vec\rho})\,k_E^{\mu}k_E^{\nu}\ .\tag{4.405}$$

Now, in order to invert this momentum-space operator, we introduce the Euclidean versions of the projection operators (4.388) and (4.389)

$$\mathcal{P}^{\mu\nu}_{E,\parallel}=\frac{k_E^{\mu}k_E^{\nu}}{k_E^2}\ ,\tag{4.406}$$

$$\mathcal{P}^{\mu\nu}_{E,\perp}=g^{E,\mu\nu}-\mathcal{P}^{\mu\nu}_{E,\parallel}\ .\tag{4.407}$$

and rewrite the operator (4.405) as follows

$$\begin{aligned}
\tilde{\mathscr{O}}^{\mu\nu}_{\vec\rho,\vec\pi,E}(k_E)&=\left(k_E^2+m_{\vec\rho}^2\right)g_E^{\mu\nu}-k_E^2\left(1-\xi_{\vec\rho}\right)\frac{k_E^{\mu}k_E^{\nu}}{k_E^2}\\
&=\left(k_E^2+m_{\vec\rho}^2\right)\mathcal{P}^{\mu\nu}_{E,\perp}+\left(k_E^2\xi_{\vec\rho}+m_{\vec\rho}^2\right)\mathcal{P}^{\mu\nu}_{E,\parallel}\\
&=\tilde{\mathcal{D}}_{E,\parallel}(k_E)\mathcal{P}^{\mu\nu}_{\parallel}+\tilde{\mathcal{D}}_{E,\perp}(k_E)\mathcal{P}^{\mu\nu}_{\perp}\ ,
\end{aligned}\tag{4.408}$$

where we defined the scalar coefficients $\tilde{\mathcal{D}}_{E,\parallel}(k_E)$ and $\tilde{\mathcal{D}}_{E,\perp}(k_E)$. At this point we are also able to check an important property of the operator (4.405), which has to be fulfilled in order to guarantee that the integral (4.403) really exists. To be particular, the operator has to be non-singular. This property is easily verified for the previous calculations, since the eigenvalues of the Klein-Gordon operators are always different from zero, so that the determinant is different from zero. In order to check this property for Eq. (4.408), we rewrite the determinant of the momentum-space operator as follows

$$\begin{aligned}
\det\tilde{\mathscr{O}}^{\mu\nu}_{\vec\rho,\vec\pi,E}(k_E)&=\exp\left\{\ln\left[\det\tilde{\mathscr{O}}^{\mu\nu}_{\vec\rho,\vec\pi,E}(k_E)\right]\right\}\\
&=\exp\left\{\mathrm{Tr}\ln\left[\tilde{\mathcal{D}}_{E,\parallel}(k_E)\mathcal{P}^{\mu\nu}_{\parallel}+\tilde{\mathcal{D}}_{E,\perp}(k_E)\mathcal{P}^{\mu\nu}_{\perp}\right]\right\}\\
&=\exp\left\{\mathrm{Tr}\left[\ln\left(\tilde{\mathcal{D}}_{E,\perp}(k_E)g^{\mu}_{\lambda,E}\right)+\ln\left[g_E^{\lambda\nu}-\left(1-\frac{\tilde{\mathcal{D}}_{E,\parallel}(k_E)}{\tilde{\mathcal{D}}_{E,\perp}(k_E)}\right)\mathcal{P}^{\lambda\nu}_{E,\parallel}\right]\right]\right\}\\
&=\exp\left\{-\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\tilde{\mathcal{D}}_{E,\perp}(k_E)-1\right)^j\mathrm{Tr}g^{\mu}_{\lambda,E}-\sum_{j=1}^{\infty}\frac{1}{j}\left(1-\frac{\tilde{\mathcal{D}}_{E,\parallel}(k_E)}{\tilde{\mathcal{D}}_{E,\perp}(k_E)}\right)^j\mathrm{Tr}\left(\mathcal{P}^{\lambda\nu}_{E,\parallel}\right)^j\right\}\\
&=\exp\left\{\ln\left(\tilde{\mathcal{D}}^4_{E,\perp}(k_E)\right)+\ln\left(\frac{\tilde{\mathcal{D}}_{E,\parallel}(k_E)}{\tilde{\mathcal{D}}_{E,\perp}(k_E)}\right)\right\}\\
&=\tilde{\mathcal{D}}^3_{E,\perp}(k_E)\tilde{\mathcal{D}}_{E,\parallel}(k_E)\ ,
\end{aligned}\tag{4.409}$$

which is obviously unequal to zero in the case of the usual gauges (Feynman gauge: $\xi_{\vec\rho}=1$, unitary gauge: $\xi_{\vec\rho}=0$). But before we discuss the influences of the choice of the gauge parameter, we have to comment on the above calculation. In the third line of Eq. (4.409), we factorized the operator and used that the coefficient $\tilde{\mathcal{D}}_{E,\perp}(k_E)$ is different from zero. Then, we used the series expansion of the matrix logarithm

$$\ln\left(x\mathbb{1}_{n\times n}\right)=-\sum_{j=1}^{\infty}\frac{(-1)^j}{j}(x-1)^j\mathbb{1}_{n\times n}\ ,\qquad\ln\left(\mathbb{1}_{n\times n}-xM\right)=-\sum_{j=1}^{\infty}\frac{x^j}{j}M^j\tag{4.410}$$

and pulled the trace inside the series. Due to the idempotence of $\mathcal{P}^{\mu\nu}_{E,\parallel}$, the second trace in the last line is given by

$$\mathrm{Tr}\left(\mathcal{P}^{\lambda\nu}_{E,\parallel}\right)^j=\mathrm{Tr}\mathcal{P}^{\lambda\nu}_{E,\parallel}=\mathcal{P}^{\lambda}_{\lambda,E,\parallel}=\frac{k_E^{\lambda}k_{\lambda,E}}{k_E^2}=1\ .\tag{4.411}$$

Now, the inverse operator is clearly given by

$$\begin{aligned}\tilde{\mathscr{O}}^{\mu\nu,-1}_{\vec{\rho},\vec{\pi},E}(k_E) &= \tilde{\mathcal{D}}^{-1}_{E,\|}(k_E)\mathcal{P}^{\mu\nu}_{\|} + \tilde{\mathcal{D}}^{-1}_{E,\perp}(k_E)\mathcal{P}^{\mu\nu}_{\perp} \\ &= \frac{1}{k_E^2+m_{\vec{\rho}}^2}\left(g_E^{\mu\nu} - \frac{k_E^\mu k_E^\nu}{k_E^2}\right) + \frac{1}{k_E^2\xi_{\vec{\rho}}+m_{\vec{\rho}}^2}\frac{k_E^\mu k_E^\nu}{k_E^2} \\ &= \frac{g_E^{\mu\nu}}{k_E^2+m_{\vec{\rho}}^2} + \frac{1-\xi_{\vec{\rho}}}{\left(k_E^2\xi_{\vec{\rho}}+m_{\vec{\rho}}^2\right)\left(k_E^2+m_{\vec{\rho}}^2\right)}k_E^\mu k_E^\nu \ .\end{aligned} \tag{4.412}$$

The position-space equivalent of the above inverse operator is then given by

$$\mathscr{O}^{\mu\nu,-1}_{\vec{\rho},\vec{\pi},E}(x_E,y_E) = \left[\frac{g_E^{\mu\nu}}{-\Box_{x,E}+m_{\vec{\rho}}^2} - \frac{1-\xi_{\vec{\rho}}}{\left(-\Box_{x,E}\xi_{\vec{\rho}}+m_{\vec{\rho}}^2\right)\left(-\Box_{x,E}+m_{\vec{\rho}}^2\right)}\partial^\mu_{x,E}\partial^\nu_{x,E}\right]\delta^{(4)}(x_E-y_E)\ , \tag{4.413}$$

which is of course a non-local result. For the following discussion, we will choose Feynman gauge $\xi_{\vec{\rho}} = 1$, since this choice will simplify the above operator significantly. But, in the end of this subsection, we will show that it is not relevant for our purposes, if we choose Feynman or unitary gauge. In Feynman gauge, the Euclidean functional integral (4.403) is given by

$$I_{\vec{\rho}}[\vec{\pi}] = \mathcal{N}_{\vec{\rho}}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x_E,y_E)\right]^{-1/2}\exp\left\{\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\ \vec{J}_{\vec{\rho},\vec{\pi},\mu,E}(x_E)\cdot\frac{g_E^{\mu\nu}}{-\Box_{x,E}+m_{\vec{\rho}}^2}\delta^{(4)}(x_E-y_E)\vec{J}_{\vec{\rho},\vec{\pi},\mu,E}(y_E)\right\}\ . \tag{4.414}$$

In order to transform this functional integral back to Minkowski space-time, we have to perform the analytic continuation of the bilinear form in the exponential. With similar calculation steps as in Eq. (4.401), we obtain

$$\begin{aligned}&\vec{J}_{\vec{\rho},\vec{\pi},\mu,E}(x_E)\cdot\frac{g_E^{\mu\nu}}{-\Box_{x,E}+m_{\vec{\rho}}^2}\delta^{(4)}(x_E-y_E)\vec{J}_{\vec{\rho},\vec{\pi},\mu,E}(y_E) \\ &= \vec{J}_{\vec{\rho},\vec{\pi},0,E}(x_E)\cdot\frac{g_E^{00}}{-\Box_{x,E}+m_{\vec{\rho}}^2}\delta^{(4)}(x_E-y_E)\vec{J}_{\vec{\rho},\vec{\pi},0,E}(y_E) + \vec{J}_{\vec{\rho},\vec{\pi},i,E}(x_E)\cdot\frac{g_E^{i0}}{-\Box_{x,E}+m_{\vec{\rho}}^2}\delta^{(4)}(x_E-y_E)\vec{J}_{\vec{\rho},\vec{\pi},0,E}(y_E) \\ &\quad + \vec{J}_{\vec{\rho},\vec{\pi},0,E}(x_E)\cdot\frac{g_E^{0j}}{-\Box_{x,E}+m_{\vec{\rho}}^2}\delta^{(4)}(x_E-y_E)\vec{J}_{\vec{\rho},\vec{\pi},j,E}(y_E) + \vec{J}_{\vec{\rho},\vec{\pi},i,E}(x_E)\cdot\frac{g_E^{ij}}{-\Box_{x,E}+m_{\vec{\rho}}^2}\delta^{(4)}(x_E-y_E)\vec{J}_{\vec{\rho},\vec{\pi},j,E}(y_E) \\ &= (-i)\vec{J}_{\vec{\rho},\vec{\pi},0}(x)\cdot\frac{g^{00}}{\Box_x+m_{\vec{\rho}}^2}(-i)\delta^{(4)}(x-y)(-i)\vec{J}_{\vec{\rho},\vec{\pi},0}(y) + \vec{J}_{\vec{\rho},\vec{\pi},i}(x)\cdot\frac{-g^{ij}}{\Box_x+m_{\vec{\rho}}^2}(-i)\delta^{(4)}(x-y)\vec{J}_{\vec{\rho},\vec{\pi},j}(y) \\ &= \vec{J}_{\vec{\rho},\vec{\pi},\mu}(x)\cdot i\frac{g^{\mu\nu}}{\Box_x+m_{\vec{\rho}}^2}\delta^{(4)}(x-y)\vec{J}_{\vec{\rho},\vec{\pi},\nu}(y)\ .\end{aligned} \tag{4.415}$$

Expanding the inverse operator

$$\begin{aligned}g^{\mu\nu}\left(\Box_x+m_{\vec{\rho}}^2\right)^{-1} &= \frac{g^{\mu\nu}}{m_{\vec{\rho}}^2}\left(1+\frac{\Box_x}{m_{\vec{\rho}}^2}\right)^{-1} \\ &= \frac{g^{\mu\nu}}{m_{\vec{\rho}}^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box_x}{m_{\vec{\rho}}^2}\right)^n\ ,\end{aligned} \tag{4.416}$$

the Minkowski version of the functional integral (4.414) can be written as

$$\begin{aligned}I_{\vec{\rho}}[\vec{\pi}] &= \mathcal{N}_{\vec{\rho}}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\right]^{-1/2}\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ \vec{J}_{\vec{\rho},\vec{\pi},\mu}(x)\cdot\frac{g^{\mu\nu}}{m_{\vec{\rho}}^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box_x}{m_{\vec{\rho}}^2}\right)^n\delta^{(4)}(x-y)\vec{J}_{\vec{\rho},\vec{\pi},\nu}(y)\right\} \\ &= \mathcal{N}_{\vec{\rho}}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\right]^{-1/2}\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\ \left[c_{1,\vec{\rho}}\left(\partial_\mu\right)\vec{\pi}\times\vec{\pi} - c_{2,\vec{\rho}}\left(\partial_\mu\partial^\alpha\vec{\pi}\right)\times\partial_\alpha\vec{\pi}\right]\right. \\ &\quad\left.\cdot\frac{g^{\mu\nu}}{m_{\vec{\rho}}^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box_x}{m_{\vec{\rho}}^2}\right)^n\left[c_{1,\vec{\rho}}\left(\partial_\nu\vec{\pi}\right)\times\vec{\pi} - c_{2,\vec{\rho}}\left(\partial_\nu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi}\right]\right\}\end{aligned}$$

$$\overset{n=1}{\equiv} \mathcal{N}_{\vec{\rho}}\Big[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\Big]^{-1/2} \exp\Bigg\{-i\int \mathrm{d}^4x\ \frac{1}{2m^2_{\vec{\rho}}}\Bigg\{\underbrace{c^2_{1,\vec{\rho}}\left[(\partial_\mu\vec{\pi})\times\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)(\partial^\mu\vec{\pi})\times\vec{\pi}}_{(a)}$$

$$\underbrace{-\,c_{1,\vec{\rho}}c_{2,\vec{\rho}}\left[(\partial_\mu\vec{\pi})\times\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)\left(\partial^\mu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi}}_{(b)}\underbrace{-\,c_{1,\vec{\rho}}c_{2,\vec{\rho}}\left[(\partial_\mu\partial^\alpha\vec{\pi})\times\partial_\alpha\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)(\partial^\mu\vec{\pi})\times\vec{\pi}}_{(c)}$$

$$\underbrace{+\,c^2_{2,\vec{\rho}}\left[(\partial_\mu\partial^\alpha\vec{\pi})\times\partial_\alpha\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)\left(\partial^\mu\partial^\beta\vec{\pi}\right)\times\partial_\beta}_{(d)}\Bigg\}\Bigg\}\ , \tag{4.417}$$

where we terminated the expansion after the $n = 1$ term, in order to obtain four pion-interaction terms with a maximum number of four space-time derivatives. In the following, we have to rewrite the terms (a)-(d). The first term becomes

$$\begin{aligned}
c^2_{1,\vec{\rho}}\int \mathrm{d}^4x\ (\partial_\mu\vec{\pi})\times\vec{\pi}\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)(\partial^\mu\vec{\pi})\times\vec{\pi} &= c^2_{1,\vec{\rho}}\int \mathrm{d}^4x\ \left\{\left[(\partial_\mu\vec{\pi})\times\vec{\pi}\right]^2-\frac{1}{m^2_{\vec{\rho}}}\left[(\partial_\mu\vec{\pi})\times\vec{\pi}\right]\cdot\Box\,(\partial^\mu\vec{\pi})\times\vec{\pi}\right\}\\
&= c^2_{1,\vec{\rho}}\int \mathrm{d}^4x\ \left\{(\partial_\mu\vec{\pi})^2\,\vec{\pi}^2-\left[(\partial_\mu\vec{\pi})\cdot\vec{\pi}\right]^2+\frac{1}{m^2_{\vec{\rho}}}\partial_\nu\left[(\partial_\mu\vec{\pi})\times\vec{\pi}\right]\partial^\nu\left[(\partial^\mu\vec{\pi})\times\vec{\pi}\right]\right\}\\
&= c^2_{1,\vec{\rho}}\int \mathrm{d}^4x\ \Bigg\{(\partial_\mu\vec{\pi})^2\,\vec{\pi}^2-\left[(\partial_\mu\vec{\pi})\cdot\vec{\pi}\right]^2+\frac{1}{m^2_{\vec{\rho}}}\Big[(\partial_\mu\partial_\nu\vec{\pi})^2\,\vec{\pi}^2-\left[(\partial_\mu\partial_\nu\vec{\pi})\cdot\vec{\pi}\right]^2\\
&\quad+(\partial_\mu\vec{\pi})^2\,(\partial_\nu\vec{\pi})^2-\left[(\partial_\mu\vec{\pi})\cdot\partial_\nu\vec{\pi}\right]^2\Big]\Bigg\}\ ,
\end{aligned}\tag{4.418}$$

where we integrated the second term by parts and used the product rule. Furthermore, several times we made use of Eq. (4.193). The same vector identity can be used in order to rewrite (b) and (c). We find

$$\begin{aligned}
c_{1,\vec{\rho}}c_{2,\vec{\rho}}\int \mathrm{d}^4x\ \left[(\partial_\mu\vec{\pi})\times\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)\left(\partial^\mu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi} &= c_{1,\vec{\rho}}c_{2,\vec{\rho}}\int \mathrm{d}^4x\ \left[(\partial_\mu\vec{\pi})\times\vec{\pi}\right]\cdot\left[\left(\partial^\mu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi}\right]+\mathcal{O}(\partial^6)\\
&= c_{1,\vec{\rho}}c_{2,\vec{\rho}}\int \mathrm{d}^4x\ \left\{(\partial_\mu\vec{\pi})\cdot\left(\partial^\mu\partial^\beta\vec{\pi}\right)\vec{\pi}\cdot\partial_\beta\vec{\pi}\right.\\
&\quad\left.-\left[\left(\partial^\mu\partial^\beta\vec{\pi}\right)\cdot\vec{\pi}\right](\partial_\mu\vec{\pi})\cdot\partial_\beta\vec{\pi}\right\}
\end{aligned}\tag{4.419}$$

and

$$\begin{aligned}
c_{1,\vec{\rho}}c_{2,\vec{\rho}}\int \mathrm{d}^4x\ \left[(\partial_\mu\partial^\alpha\vec{\pi})\times\partial_\alpha\vec{\pi}\right)\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)(\partial^\mu\vec{\pi})\times\vec{\pi} &= c_{1,\vec{\rho}}c_{2,\vec{\rho}}\int \mathrm{d}^4x\ \left[(\partial^\mu\partial^\alpha\vec{\pi})\times\partial_\alpha\vec{\pi}\right]\cdot\left[(\partial_\mu\vec{\pi})\times\vec{\pi}\right]+\mathcal{O}(\partial^6)\\
&= c_{1,\vec{\rho}}c_{2,\vec{\rho}}\int \mathrm{d}^4x\ \left\{(\partial_\mu\partial^\alpha\vec{\pi})\cdot(\partial^\mu\vec{\pi})\,(\partial_\alpha\vec{\pi})\cdot\vec{\pi}\right.\\
&\quad\left.-(\partial_\alpha\vec{\pi})\cdot(\partial^\mu\vec{\pi})\,(\partial_\mu\partial^\alpha\vec{\pi})\cdot(\vec{\pi})\right\}\ ,
\end{aligned}\tag{4.420}$$

where we neglected terms that contain six or more space-time derivatives. Finally, for the same reason, we also neglect all terms of (d). Then, the functional integral is given by

$$\begin{aligned}
I_{\vec{\rho}}[\vec{\pi}] &= \mathcal{N}_{\vec{\rho}}\Big[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\Big]^{-1/2}\exp\Bigg\{-i\int \mathrm{d}^4x\ \Bigg\{\frac{c^2_{1,\vec{\rho}}}{2m^2_{\vec{\rho}}}\left[(\partial_\mu\vec{\pi})^2\,\vec{\pi}^2-(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2\right]+\frac{c^2_{1,\vec{\rho}}}{2m^4_{\vec{\rho}}}\Big[(\partial_\mu\partial_\nu\vec{\pi})^2\,\vec{\pi}^2-(\vec{\pi}\cdot\partial_\mu\partial_\nu\vec{\pi})^2\\
&\quad+(\partial_\mu\vec{\pi})^2\,(\partial_\nu\vec{\pi})^2-\left[(\partial_\mu\vec{\pi})\cdot\partial_\nu\vec{\pi}\right]^2\Big]-\frac{c_{1,\vec{\rho}}c_{2,\vec{\rho}}}{m^2_{\vec{\rho}}}\left[(\partial_\mu\vec{\pi})\cdot(\partial^\mu\partial^\nu\vec{\pi})\,\vec{\pi}\cdot\partial_\nu\vec{\pi}-(\partial^\mu\partial^\nu\vec{\pi})\cdot\vec{\pi}\,(\partial_\mu\vec{\pi})\cdot\partial_\nu\vec{\pi}\right]\Bigg\}\\
&= \mathcal{N}_{\vec{\rho}}\Big[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\Big]^{-1/2}\exp\Bigg\{-i\int \mathrm{d}^4x\ \Bigg\{\left(\frac{c^2_{1,\vec{\rho}}m^2_{\vec{\pi}}}{2m^2_{\vec{\rho}}}-\frac{c_{1,\vec{\rho}}c_{2,\vec{\rho}}m^4_{\vec{\pi}}}{m^2_{\vec{\rho}}}\right)\left(\vec{\pi}^2\right)^2-\left(\frac{3c^2_{1,\vec{\rho}}}{2m^2_{\vec{\rho}}}-\frac{3c_{1,\vec{\rho}}c_{2,\vec{\rho}}m^2_{\vec{\pi}}}{m^2_{\vec{\rho}}}\right)(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2\\
&\quad+\left(\frac{c^2_{1,\vec{\rho}}}{m^4_{\vec{\rho}}}+\frac{c_{1,\vec{\rho}}c_{2,\vec{\rho}}}{m^2_{\vec{\rho}}}\right)(\partial_\mu\vec{\pi})^2\,(\partial_\nu\vec{\pi})^2-\left(\frac{c^2_{1,\vec{\rho}}}{m^4_{\vec{\rho}}}+\frac{c_{1,\vec{\rho}}c_{2,\vec{\rho}}}{m^2_{\vec{\rho}}}\right)\left[(\partial_\mu\vec{\pi})\cdot\partial_\nu\vec{\pi}\right]\Bigg\}\Bigg\}\ ,
\end{aligned}\tag{4.421}$$

where we integrated the first line by parts for several times. Before we combine this result with Eq. (4.371) and Eqs. (4.349)-(4.353) in order to find the tree-level effective action of the eLSM, we want to show that we would obtain the same solution (4.421), if we had used unitary gauge $\xi_{\vec{\rho}} = 0$. To this end, we go back to Eq. (4.417). In unitary gauge and after performing the analytic continuation to Minkowski space, the functional integral would be given by

$$
\begin{aligned}
I_{\vec{\rho}}[\vec{\pi}] &= \mathcal{N}_{\vec{\rho}}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\right]^{-1/2}\exp\Bigg\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, \vec{J}_{\vec{\rho},\vec{\pi},\mu}(x)\cdot\left[\left(\Box_x+m^2_{\vec{\rho}}\right)^{-1}g^{\mu\nu}+\frac{1}{m^2_{\vec{\rho}}}\left(\Box_x+m^2_{\vec{\rho}}\right)^{-1}\partial^\mu_x\partial^\nu_x\right] \\
&\quad\times\delta^{(4)}(x-y)\vec{J}_{\vec{\rho},\vec{\pi},\nu}(y)\Bigg\} \\
&= \mathcal{N}_{\vec{\rho}}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\right]^{-1/2}\exp\Bigg\{-\frac{i}{2}\int \mathrm{d}^4x\,\left[c_{1,\vec{\rho}}\left(\partial_\mu\vec{\pi}\right)\times\vec{\pi}-c_{2,\vec{\rho}}\left(\partial_\mu\partial^\alpha\vec{\pi}\right)\times\partial_\alpha\vec{\pi}\right] \\
&\quad\cdot\left[\frac{g^{\mu\nu}}{m^2_{\vec{\rho}}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box_x}{m^2_{\vec{\rho}}}\right)^n+\frac{1}{m^4_{\vec{\rho}}}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\Box_x}{m^2_{\vec{\rho}}}\right)^m\partial^\mu_x\partial^\nu_x\right]\left[c_{1,\vec{\rho}}\left(\partial_\nu\vec{\pi}\right)\times\vec{\pi}-c_{2,\vec{\rho}}\left(\partial_\nu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi}\right]\Bigg\} \\
&\overset{n=1,\, m=0}{=} \mathcal{N}_{\vec{\rho}}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\right]^{-1/2}\exp\Bigg\{-i\int \mathrm{d}^4x\,\frac{1}{2m^2_{\vec{\rho}}}\Bigg\{\underbrace{c^2_{1,\vec{\rho}}\left[\left(\partial_\mu\vec{\pi}\right)\times\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)\left(\partial^\mu\vec{\pi}\right)\times\vec{\pi}}_{(a)} \\
&\quad\underbrace{-\,c_{1,\vec{\rho}}c_{2,\vec{\rho}}\left[\left(\partial_\mu\vec{\pi}\right)\times\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)\left(\partial^\mu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi}}_{(b)}\underbrace{-\,c_{1,\vec{\rho}}c_{2,\vec{\rho}}\left[\left(\partial_\mu\partial^\alpha\vec{\pi}\right)\times\partial_\alpha\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)\left(\partial^\mu\vec{\pi}\right)\times\vec{\pi}}_{(c)} \\
&\quad\underbrace{+\,c^2_{2,\vec{\rho}}\left[\left(\partial_\mu\partial^\alpha\vec{\pi}\right)\times\partial_\alpha\vec{\pi}\right]\cdot\left(1-\frac{\Box_x}{m^2_{\vec{\rho}}}\right)\left(\partial^\mu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi}}_{(d)}\Bigg\}+\frac{1}{2m^4_{\vec{\rho}}}\Bigg\{\underbrace{c^2_{1,\vec{\rho}}\left[\left(\partial_\mu\vec{\pi}\right)\times\vec{\pi}\right]\cdot\partial^\mu\partial^\mu\left(\partial_\nu\vec{\pi}\right)\times\vec{\pi}}_{(e)} \\
&\quad\underbrace{-\,c_{1,\vec{\rho}}c_{2,\vec{\rho}}\left[\left(\partial_\mu\vec{\pi}\right)\times\vec{\pi}\right]\cdot\partial^\mu\partial^\nu\left[\left(\partial_\nu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi}\right]}_{(f)}\underbrace{-\,c_{1,\vec{\rho}}c_{2,\vec{\rho}}\left[\left(\partial_\mu\partial^\alpha\vec{\pi}\right)\times\partial_\alpha\vec{\pi}\right]\cdot\partial^\mu\partial^\nu\left[\left(\partial_\nu\vec{\pi}\right)\times\vec{\pi}\right]}_{(g)} \\
&\quad\underbrace{+\,c^2_{2,\vec{\rho}}\left[\left(\partial_\mu\partial^\alpha\vec{\pi}\right)\times\partial_\alpha\vec{\pi}\right]\cdot\partial^\mu\partial^\nu\left[\left(\partial_\nu\partial^\beta\vec{\pi}\right)\times\partial_\beta\vec{\pi}\right]}_{(h)}\Bigg\}\Bigg\}\,,
\end{aligned}
\tag{4.422}
$$

where we introduced the inverse operator

$$
\begin{aligned}
\left(\Box_x+m^2_{\vec{\rho}}\right)^{-1}g^{\mu\nu}+\frac{1}{m^2_{\vec{\rho}}}\left(\Box_x+m^2_{\vec{\rho}}\right)^{-1}\partial^\mu_x\partial^\nu_x &= \frac{1}{m^2_{\vec{\rho}}}\left(1+\frac{\Box_x}{m^2_{\vec{\rho}}}\right)^{-1}+\frac{1}{m^4_{\vec{\rho}}}\left(1+\frac{\Box_x}{m^2_{\vec{\rho}}}\right)^{-1}\partial^\mu_x\partial^\nu_x \\
&= \frac{g^{\mu\nu}}{m^2_{\vec{\rho}}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\Box_x}{m^2_{\vec{\rho}}}\right)^n+\frac{1}{m^4_{\vec{\rho}}}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\Box}{m^2_{\vec{\rho}}}\right)^m\partial^\mu_x\partial^\nu_x
\end{aligned}
\tag{4.423}
$$

in the second line of Eq. (4.422) and expanded this operator only up to $n=1$, $m=0$. Obviously, the non-diagonal part of the operator gives four new contributions to the four-pion interaction terms. But since the terms (f)-(h) contain six space-time derivatives, we are able to neglect them directly. Finally, we are only left with (e) which can be shown to vanish identically

$$
\begin{aligned}
c^2_{1,\vec{\rho}}\int \mathrm{d}^4x\,\left[\left(\partial_\mu\vec{\pi}\right)\times\vec{\pi}\right]\cdot\partial^\mu\partial^\nu\left[\left(\partial_\nu\vec{\pi}\right)\times\vec{\pi}\right] &= c^2_{1,\vec{\rho}}\int \mathrm{d}^4x\,\left[\left(\partial_\mu\vec{\pi}\right)\times\vec{\pi}\right]\cdot\partial^\mu\left[\left(\Box\vec{\pi}\right)\times\vec{\pi}+\left(\partial_\nu\vec{\pi}\right)\times\partial^\nu\vec{\pi}\right] \\
&= c^2_{1,\vec{\rho}}\int \mathrm{d}^4x\,\left[\left(\partial_\mu\vec{\pi}\right)\times\vec{\pi}\right]\cdot\partial^\mu\left[-m^2_{\vec{\pi}}\left(\vec{\pi}\times\vec{\pi}\right)+\left(\partial_\nu\vec{\pi}\right)\times\left(\partial^\nu\vec{\pi}\right)\right] \\
&= 0\,,
\end{aligned}
\tag{4.424}
$$

where we used (4.359) and the antisymmetry of the vector product. Now, since all four contributions (e)-(h) are ether negligible or equal to zero, we are left with the same final result (4.421) as in the case of Feynman gauge. Finally, we have to combine all results of this subsection in order to obtain a solution for the transition amplitude (4.347). We find

$$
\begin{aligned}
\langle f,\infty|f,-\infty\rangle &= \mathcal{N}\int \mathscr{D}\vec{\pi}(x)\ \exp\left\{i\int \mathrm{d}^4x\ \mathscr{L}_{\vec{\pi}}\right\} I_{\sigma_N}[\vec{\pi}]I_{\vec{\rho}}[\vec{\pi}]I_{\eta_N}[\vec{\pi}]I_{\vec{a}_0}[\vec{\pi}]I_{\omega_N}[\vec{\pi}]I_{f_{1N}}[\vec{\pi}]I_{\vec{a}_1}[\vec{\pi}] \\
&= \mathcal{N}_{eff}\int \mathscr{D}\vec{\pi}(x)\ \left[\det\mathscr{O}_{\eta_N,\vec{\pi}}(x,y)\right]^{-1/2}\left[\det\mathscr{O}^{ij}_{\vec{a}_0,\vec{\pi}}(x,y)\right]^{-1/2}\left[\det\mathscr{O}^{\mu\nu}_{\omega_N,\vec{\pi}}(x,y)\right]^{-1/2} \\
&\quad\times\left[\det\mathscr{O}^{\mu\nu}_{f_{1N},\vec{\pi}}(x,y)\right]^{-1/2}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{a}_1,\vec{\pi}}\right)^{ij}(x,y)\right]^{-1/2}\left[\det\mathscr{O}_{\sigma_N,\vec{\pi}}(x,y)\right]^{-1/2}\left[\det\left(\mathscr{O}^{\mu\nu}_{\vec{\rho},\vec{\pi},E}\right)^{ij}(x,y)\right]^{-1/2} \\
&\quad\times\exp\left\{iS^{(n=1,m=0)}_{eLSM,eff}[\vec{\pi}]\right\}\ ,
\end{aligned}
\tag{4.425}
$$

where the tree-level effective action of the eLSM is given by

$$
\begin{aligned}
S^{(n=1,m=0)}_{eLSM,eff}[\vec{\pi}] = \int \mathrm{d}^4x\ \Big\{&\frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2-\frac{1}{2}m^2_{\vec{\pi}}\vec{\pi}^2+C_{1,eLSM}\left(\vec{\pi}^2\right)^2+C_{2,eLSM}\left(\partial_\mu\vec{\pi}\cdot\vec{\pi}\right)^2+C_{3,eLSM}\left(\partial_\mu\vec{\pi}\right)^2\left(\partial_\nu\vec{\pi}\right)^2 \\
&+C_{4,eLSM}\left[\left(\partial_\mu\vec{\pi}\right)\cdot\partial_\nu\vec{\pi}\right]^2\Big\}\ ,
\end{aligned}
\tag{4.426}
$$

where the LECs of the eLSM are given by

$$
\begin{aligned}
C_{1,eLSM} &= \frac{Z^4}{4}\left[\left(h_1+h_2+h_3\right)w^2m^2_{\vec{\pi}}-\left(\lambda_1+\frac{\lambda_2}{2}\right)\right]+\frac{c^2_{1,\sigma_N}}{2m^2_{\sigma_N}}\left(1-\frac{4m^4_{\vec{\pi}}}{m^4_{\sigma_N}}\right)+\frac{c_{1,\sigma_N}c_{2,\sigma_N}m^2_{\vec{\pi}}}{m^2_{\sigma_N}}\left(1+\frac{2m^2_{\vec{\pi}}}{m^2_{\sigma_N}}\right) \\
&\quad-\frac{c^2_{1,\vec{\rho}}m^2_{\vec{\pi}}}{2m^2_{\vec{\rho}}}+\frac{c_{1,\vec{\rho}}c_{2,\vec{\rho}}m^4_{\vec{\pi}}}{m^2_{\vec{\rho}}}\ ,
\end{aligned}
\tag{4.427}
$$

$$
C_{2,eLSM} = \frac{1}{2}\left(g_1^2-h_1-h_2-2h_3\right)w^2Z^4+\frac{2c^2_{1,\sigma_N}}{m^4_{\sigma_N}}\left(1+\frac{4m^2_{\vec{\pi}}}{m^2_{\sigma_N}}\right)-\frac{2c_{1,\sigma_N}c_{2,\sigma_N}}{m^2_{\sigma_N}}\left(1+\frac{2m^2_{\vec{\pi}}}{m^2_{\sigma_N}}\right)+\frac{3c_{1,\vec{\rho}}}{2m^2_{\vec{\rho}}}-\frac{3c_{1,\vec{\rho}}c_{2,\vec{\rho}}m^2_{\vec{\pi}}}{m^2_{\vec{\rho}}}\ ,
\tag{4.428}
$$

$$
C_{3,eLSM} = \left(-\frac{g_3}{4}+\frac{g_4}{4}+\frac{g_5}{4}+\frac{g_6}{2}\right)w^4Z^4+\frac{c^2_{2,\sigma_N}}{2m^2_{\sigma_N}}-\frac{2c_{1,\sigma_N}c_{2,\sigma_N}}{m^4_{\sigma_N}}+\frac{2c^2_{1,\sigma_N}}{m^6_{\sigma_N}}-\frac{c^2_{1,\vec{\rho}}}{m^4_{\vec{\rho}}}-\frac{c_{1,\vec{\rho}}c_{2,\vec{\rho}}}{m^2_{\vec{\rho}}}\ ,
\tag{4.429}
$$

$$
C_{4,eLSM} = \frac{g_3}{2}w^4Z^4+\frac{c^2_{1,\vec{\rho}}}{m^4_{\vec{\rho}}}+\frac{c_{1,\vec{\rho}}c_{2,\vec{\rho}}}{m^2_{\vec{\rho}}}\ .
\tag{4.430}
$$

Before we determine the numerical values for the above LECs in the upcoming subsection, we recognize that the presence of vector- and axial-vector mesons in the eLSM Lagrangian influences the structure of the LECs in a positive way. On the one hand, the contribution of the σ_N-field is modified by new terms which arise from the covariant derivative. On the other hand, the presence of the $\vec{\rho}_\mu$-field gives rise to new interaction vertices which were not present in the previous calculations. These new vertices result in a nonzero value of $C_{4,eLSM}$, so that we now have nonzero values for all LECs. Finally, we want to check, if the $(\vec{\pi}^2)^2$ interaction term vanishes in the chiral limit as it should be. To this end, we consider the limit $h_{N,0}\to 0$, so that the tree-level pion mass (4.260) vanishes, since

$$
m^2_{\vec{\pi}} = \left[-c_1-m_0^2+\left(\lambda_1+\frac{\lambda_2}{2}\right)\phi_N^2\right]Z^2 = \frac{h_{N,0}Z^2}{\phi_N}\ .
\tag{4.431}
$$

In this limit, the tree-level mass of the σ_N-field is given by Eq. (4.296). Then, using the definition (4.360) and taking the chiral limit, we find

$$
\begin{aligned}
C_{1,eLSM} &= \frac{c^2_{1,\sigma_N}}{2m^2_{\sigma_N}}-\frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right)Z^4 \\
&= \frac{\left(\lambda_1+\frac{\lambda_2}{2}\right)^2\phi_N^2Z^4}{4\left(\lambda_1+\frac{\lambda_2}{2}\right)\phi_N^2}-\frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{2}\right)Z^4 \\
&= 0\ ,
\end{aligned}
\tag{4.432}
$$

which beautifully demonstrates that all non-derivatively coupled interaction vertices vanish in the case of an exact chiral symmetric Lagrangian.

4.3.6 Numerical Input and Results

Now we want to determine the numerical values for the coupling constants of ChPT, (3.236)-(3.239), as well as those of the different versions of the eLSM, Eqs. (4.290)-(4.293), (4.322)-(4.325), and (4.427)-(4.430). To this end, we want to start with those of Chiral Perturbation Theory. Obviously, the first coupling constant (3.236) depends on the mass parameter defined by Eq. (3.225). At tree-level and NLO in the chiral expansion, the physical pion mass and this mass parameter are connected through Eq. (3.235). This relation yields a quartic equation that we can use to determine the mass parameter (3.225). To this end, we first of all need a numerical value for the pion mass. Since we worked in the isospin symmetric limit and neglected electromagnetic interactions, we take an isospin-averaged value for the pion mass

$$M_\pi = (138 \pm 6.9) \text{ MeV} . \tag{4.433}$$

This numerical values is taken from Ref. [PKWGR]. Furthermore, we need the numerical values of the first three LECs of two-flavor Chiral Perturbation Theory. These parameters can be obtained from Ref. [BiEc]. The first two LECs ℓ_1 and ℓ_2 are obtained from $\pi\pi$ scattering lengths

$$\ell_1 = (-4.051 \pm 0.642) \cdot 10^{-3} , \qquad \ell_2 = (1.819 \pm 0.299) \cdot 10^{-3} . \tag{4.434}$$

The third LEC is obtained from an analysis of the dependence of the pion mass on the mass parameters of ChPT

$$\ell_3 = (0.852 \pm 3.803) \cdot 10^{-3} . \tag{4.435}$$

Note that the LECs (4.434) and (4.435) have to be determined from the subtraction scale idependent ones, cf. Ref. [BiEc]. It should be mentioned that the LECs from Ref. [BiEc] are not tree-level values. These values are obtained through a two-loop analysis of ChPT, see Ref. [BiEc] and refs. therein for more details. Now, using (4.433) and (4.435), the quartic equation (4.225) can be solved, in order to obtain

$$M = (137.738 \pm 6.958) \text{ MeV} . \tag{4.436}$$

The error has been calculated by using the usual Gaussian error law. Now, the LECs (3.236)-(3.239) are also functions of the pion decay constant f_π. The numerical value of this constant is also taken from Ref. [PKWGR]

$$f_\pi = (92.2 \pm 4.6) \text{ MeV} . \tag{4.437}$$

Finally, using (4.434)-(4.437), the coupling constants of the four pion interaction terms can be calculated as

$$C_{1,\chi PT} = -0.279 \pm 1.941 , \tag{4.438}$$
$$C_{2,\chi PT} = (5.882 \pm 0.587) 10^{-5} \text{ MeV}^{-2} , \tag{4.439}$$
$$C_{3,\chi PT} = (-5.606 \pm 1.429) \cdot 10^{-11} \text{ MeV}^{-4} , \tag{4.440}$$
$$C_{4,\chi PT} = (2.517 \pm 0.651) 10^{-11} \text{ MeV}^{-4} . \tag{4.441}$$

Now we also have to calculate the LECs of the different versions of the eLSM. In the Secs. [4.3.3] and [4.3.4] it was shown, that both versions of the model lead to the same low-energy couplings. The reason for this to happen was simply given by the fact that the relevant σ_N interactions with the $\vec{\pi}$-fields remain the same in both cases. It was shown that there are only three non-vanishing LECs which are given by Eqs. (4.290), (4.291), and (4.292). These constants, of course, depend on the parameters of the eLSM. The numerical values which have been used in order to calculate the LECs of the eLSM are taken from the three-flavor fit of the model, which was presented in Ref. [PKWGR]. These values have been determined by studying the decay widths of the different mesons included in the eLSM. For more details on this calculation and on the fit procedure see Ref. [PKWGR]. For the sake of clarity, we want to quote all numerical values which have been used to calculate the LECs. First of all, the LECs (4.290)-(4.293) and (4.427)-(4.430) have a dependence on four masses, the pion mass, the mass of the σ_N-field as well as the masses of the $\vec{a}_{1,\mu}$- and the $\vec{\rho}_\mu$-fields. While we use the isospin-averaged values (4.433) for the pion mass, the mass of the σ_N has been determined in a large-N_C limit fit in Ref. [PKWGR]

$$m_{\sigma_N} = 1362.7 \text{ MeV} . \tag{4.442}$$

Also the masses of the vector and the axial-vector isotriplets have been taken from the fit results of Ref. [PKWGR]

$$m_{\vec{a}_1} = (1186 \pm 6) \text{ MeV} , \qquad m_{\vec{\rho}} = (783.1 \pm 7) \text{ MeV} . \tag{4.443}$$

Furthermore, the different LECs depend on all coupling parameters of the eLSM, which have been introduced in Sec. [4.2.2]. In the following, we also consider the large-N_C scenario, so that

$$\lambda_1 = h_1 = g_5 = g_6 = 0 \; . \tag{4.444}$$

In addition to that, we also set

$$g_3 = g_4 = 0 \; , \tag{4.445}$$

since these values could not be determined by the analysis of the decay widths in Ref. [PKWGR]. The remaining parameters of the model are

$$\phi_N, g_1, g_2, h_2, h_3 \text{ and } \lambda_2 \; . \tag{4.446}$$

From the three-flavor fit procedure one obtains

$$g_1 = 5.843 \pm 0.018 \; , \tag{4.447}$$
$$g_2 = 3.025 \pm 0.233 \; , \tag{4.448}$$
$$h_2 = 9.88 \pm 0.663 \; , \tag{4.449}$$
$$h_3 = 4.867 \pm 0.086 \; , \tag{4.450}$$
$$\lambda_2 = 68.297 \pm 0.044 \tag{4.451}$$
$$\tag{4.452}$$

for the coupling parameters. The vacuum expectation values of the σ_N-field is given by

$$\phi_N = (164.6 \pm 0.1) \text{ MeV} \; . \tag{4.453}$$

At this point, it should be taken into account that this vacuum expectation value and the pion decay constant are connected via

$$\phi_N = Z f_\pi \; . \tag{4.454}$$

The relation can obtained from the axial current, compare Ref. [Par2]. Now, since we have all numerical values which are needed in order to determine the LECs of the different versions of the eLSM, we start with those of the $O(4)$- and the $U(2)_L \times U(2)_R$-case without vector/axial-vector degrees of freedom. In this case we have

$$w = 0 \; , \qquad Z = 1 \qquad \Longrightarrow \phi_N \equiv f_\pi \; . \tag{4.455}$$

Then, the LECs (4.290)-(4.293) or (4.322)-(4.325) are given by

$$C_{1,O(4)} = C_{1,SU(2)_L\times SU(2)_R} = -5.869 \pm 0.226 \; , \tag{4.456}$$
$$C_{2,O(4)} = C_{2,SU(2)_L\times SU(2)_R} = (5.985 \pm 27.533) \cdot 10^{-6} \text{ MeV}^{-2} \; , \tag{4.457}$$
$$C_{3,O(4)} = C_{3,SU(2)_L\times SU(2)_R} = (3.096 \pm 0.309) 10^{-12} \text{ MeV}^{-4} \; , \tag{4.458}$$
$$C_{4,O(4)} = C_{4,SU(2)_L\times SU(2)_R} = 0 \; . \tag{4.459}$$

Now we come to the results of the full eLSM. In this case, the parameters w and Z are given by

$$w = (683.784 \pm 7.231) 10^{-6} \text{ MeV}^{-1} \; , \qquad Z = 1.709 \pm 0.181 \; . \tag{4.460}$$

Then, during the calculation of Sec. [4.3.5], we introduced four constants which are defined by Eqs. (4.360), (4.361), (4.397), and (4.398). These coefficients are, of course, also functions of the model parameters. With the above numerical input, these constants can be determined as

$$c_{1,\sigma_N} = (-16187.2 \pm 3425.5) \text{ MeV} \; , \tag{4.461}$$
$$c_{2,\sigma_N} = (-0.015 \pm 0.003) \text{ MeV}^{-1} \; , \tag{4.462}$$
$$c_{1,\vec{\rho}} = -7.361 \pm 1.561 \; , \tag{4.463}$$
$$c_{2,\vec{\rho}} = (41.316 \pm 9.342) \cdot 10^{-7} \text{ MeV}^{-2} \; . \tag{4.464}$$

The reason why the first constant (4.461) has such a large absolute value is simply given by the fact that this constant does not involve powers of w, which are present in all other constants. Due to the very small value of this shift parameter, all other constants (4.462)-(4.464) are strongly suppressed in comparison to Eq. (4.461). Now, with these values, the LECs of the eLSM are given by

$$C_{1,eLSM} = -0.345 \pm 69.093 \; , \tag{4.465}$$

$$C_{2,eLSM} = (5.385 \pm 8.20) \cdot 10^{-5} \text{ MeV}^{-2} \; , \tag{4.466}$$

$$C_{3,eLSM} = (-9.303 \pm 5.114) \cdot 10^{-11} \text{ MeV}^{-4} \; , \tag{4.467}$$

$$C_{4,eLSM} = (9.449 \pm 5.078) \cdot 10^{-11} \text{ MeV}^{-4} \; . \tag{4.468}$$

A discussion of this work as well as the numerical results can be found in the last Chapter [5].

Chapter 5

Discussion

The aim of this work was the determination of the low-energy constants of the eLSM at tree-level. To this end, we started in Chapter [2] with a brief introduction of the mathematical and physical basics that are essential for the understanding of this work. An important step towards the understanding of the low-energy regime of strong interactions was the development of Chiral Perturbation Theory. This framework is based on a systematic analysis of the hadronic n-point functions. Using the methods of the Effective Field Theory, it is then possible to construct the most general chiral Lagrangian which describes the interaction of the pseudo-Nambu-Goldstone bosons of spontaneous chiral symmetry breaking. The importance of Chiral Perturbation Theory for our approach then derives from the fact that the low-energy landscape of strong interactions is dominated by the interactions of these pseudo-Nambu-Goldstone bosons among themselves. Therefore, this approach yields a good possibility to compare our values of the low-energy constants to the physical ones. For more details on Chiral Perturbation Theory see Chapter [3].

Then, in Chapter [4], we introduced the mesonic part of the eLSM in detail and considered three different versions of the model. In the first two versions of the model, we only considered the interaction of scalar and pseudoscalar mesons and set all vector and axial-vector degrees of freedom to zero. In our calculation, it was shown that both versions of the model lead to the same low-energy couplings. Table [5.1] summarizes the numerical values of these LECs in comparison to the results of two-flavor ChPT. It is obvious that it

LEC	$(N_f = 2)$-χPT	$O(4)$-model/$U(2)_L \times U(2)_R$-model
$C_{1,i}$	-0.279 ± 1.941	-5.869 ± 0.266
$C_{2,i}$ [MeV^{-2}]	$(5.882 \pm 0.587) \cdot 10^{-5}$	$(5.985 \pm 27.533) \cdot 10^{-6}$
$C_{3,i}$ [MeV^{-4}]	$(-5.606 \pm 1.429) \cdot 10^{-11}$	$(3.096 \pm 0.309) \cdot 10^{-12}$
$C_{4,i}$ [MeV^{-4}]	$(2.517 \pm 0.651) \cdot 10^{-11}$	0

Table 5.1: Comparison of the low-energy couplings of two-flavor ChPT and the $O(4)$- and $U(2)_L \times U(2)_R$-version of the eLSM.

is not sufficient, if we only incorporate scalar and pseudoscalar mesons in the eLSM in order to describe the low-energy regime of strong interactions. We observe that in these versions of the model, three possible interaction structures of the $\vec{\pi}$-fields are not present at tree-level. This derives from the fact that the relevant interaction terms that contribute to four-pion interactions are only given by $\sigma_N \vec{\pi}\vec{\pi}$ couplings. Obviously, these structures are not sufficient to generate all possible chirally symmetric four-pion interactions. In the $O(4)$ case of the eLSM, these LECs may only be improved at one-loop order by contributions of the σ_N-field. In the second case where we included all scalar and pseudoscalar degrees of freedom, also the η_N- and the $\vec{a}_0$-fields will contribute at one-loop order, so that this version of the model seems to be a bit more promising than the $O(4)$ model. But, due to Tab. [5.1], it is obvious that, at tree-level, both approaches are not able to describe the low-energy end of the QCD spectrum.

In the third calculation, we considered the full mesonic part of the eLSM. Table [5.2] illustrates a comparison of the tree-level LECs of the full mesonic eLSM and two-flavor Chiral Perturbation Theory. When we consider the numerical values of the low-energy couplings of the eLSM, we observe that now all possible chirally symmetric interaction structures are present in the low-energy effective action (4.426). It is quite obvious that, in comparison to the previous cases the additional term structures have their origin in the interactions between scalar/pseudoscalar mesons and vector/axial-vector mesons. To be particular, it turns out that the presence of the vector/axial-vector mesons improves the structures of the low-energy effective

LEC	$(N_f = 2)$-χPT	eLSM
$C_{1,i}$	-0.279 ± 1.941	-0.345 ± 69.093
$C_{2,i}$ [MeV^{-2}]	$(5.882 \pm 0.587) \cdot 10^{-5}$	$(5.385 \pm 8.20) \cdot 10^{-5}$
$C_{3,i}$ [MeV^{-4}]	$(-5.606 \pm 1.429) \cdot 10^{-11}$	$(-9.303 \pm 5.114) \cdot 10^{-11}$
$C_{4,i}$ [MeV^{-4}]	$(2.517 \pm 0.651) \cdot 10^{-11}$	$(9.449 \pm 5.078) \cdot 10^{-11}$

Table 5.2: Comparison of the low-energy couplings of two-flavor ChPT and the eLSM.

action of the eLSM. This follows from the fact that, on the one hand, the covariant derivative introduces new couplings of the σ_N-field to the pions. On the other hand, the presence of the $\vec{\rho}_\mu$-fields which also couple to the pion introduces new tree-level contributions to four-pion interactions. Furthermore, the necessary shifts of the axial-vector fields give rise to derivatively coupled four-pion interactions that were not present in the previous cases of the eLSM.

When we consider the values of the eLSM, we observe that the LECs fit very well to those of two-flavor Chiral Perturbation Theory. Within the error tolerances the first three constants fit very well to those of ChPT. The last low-energy coupling differs by a factor of two. The small deviations may have different reasons. First of all, the interaction terms of two-flavor Chiral Perturbation Theory are not reparametrization-invariant with respect to the parametrization of the coset space. This means that a different coset representative leads to different interaction structures and therefore to different $C_{i,\chi PT}$. But, in principle, it is not expected that another parametrization of the coset space will influence the values of the low-energy couplings in a strong way, since the inverse powers of the pion decay constant should dominate the respective expressions. Another and more important reason is that we only consider the LECs at tree-level. In the case of the eLSM five mesons do not contribute at tree-level, but only at one-loop order. These contributions will, of course, influence the numerical values of the LECs and may improve them. Furthermore, the input data of the eLSM parameters may be improved, since the values used are obtained from a fit of the three-flavor version of the eLSM. In addition to that, nonzero values of the coupling constants g_3 and g_4 may also improve the values of the LECs. Therefore, we finally conclude that the eLSM is able to reproduce the relevant interaction structures of the pion. This tree-level analysis illustrates that the eLSM describes the low-energy regime of QCD very well, but, in order to improve the values of the respective low-energy couplings, our analysis has to be extended to one-loop order.

Chapter 6

Appendix

6.1 Pauli Matrices, Dirac Matrices, and Useful Relations

The first section of this appendix is dedicated to the so-called Pauli and Dirac matrices. To this end, we summarize the most important identities and relations of these matrices, which are used throughout this work. In addition to that, we will shortly describe the connection of the Pauli matrices to unitary and orthogonal groups.

6.1.1 The Pauli Matrices

The three hermitian and unitary matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.1}$$

define the set of the so-called Pauli matrices. Together with

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_{2\times 2} \ , \tag{6.2}$$

the set $\{\tau_a |\ a = 0, \ldots, 3\}$ spans the full vector space of hermitian (2×2)-matrices, i.e., any hermitian (2×2)-matrix M can be written as a linear combination of the following form

$$M = c_a \tau^a \ , \tag{6.3}$$

with $c_a \in \mathbb{R}$. It is quite obvious that the Pauli matrices are traceless

$$\mathrm{Tr}\,(\tau_i) = 0 \tag{6.4}$$

matrices with a determinant of

$$\det(\tau_i) = -1 \ , \qquad i = 1, 2, 3 \ . \tag{6.5}$$

Therefore, it follows that the eigenvalues of the Pauli matrices are ± 1. Furthermore, each Pauli matrix τ_i represents its own inverse, i.e., the Pauli matrices are involutory matrices. This property can be written as

$$\tau_i^2 = \mathbb{1}_{2\times 2} \ . \tag{6.6}$$

The Pauli matrices also obey the following commutation and anticommutation relations

$$[\tau_i, \tau_j]_- = 2i\epsilon_{ijk}\tau^k \ , \tag{6.7}$$

$$[\tau_i, \tau_j]_+ = 2\delta_{ij}\mathbb{1}_{2\times 2} \ . \tag{6.8}$$

Another useful relation, involving the product of two Pauli matrices, can be derived by summing the above commutation and anticommutation relations. We find

$$\tau_i\tau_j = i\epsilon_{ijk}\tau^k + \delta_{ij}\mathbb{1}_{2\times 2} \ . \tag{6.9}$$

Using the above relation (6.9) in combination with Eq. (6.4), we are able to evaluate the traces of products of Pauli matrices

$$
\begin{aligned}
&\mathrm{Tr}\left(\tau_i \tau_j\right) = \mathrm{Tr}\left(i\epsilon_{ijk}\tau^k + \delta_{ij}\mathbb{1}_{2\times 2}\right) = 2\delta_{ij} \ , && (6.10)\\
&\mathrm{Tr}\left(\tau_i \tau_j \tau_k\right) = \mathrm{Tr}\left\{\left[i\epsilon_{ijl}\tau^l + \delta_{ij}\mathbb{1}_{2\times 2}\right]\tau_k\right\} = i\epsilon_{ijl}\mathrm{Tr}\left(\tau^l\tau_k\right) = i\epsilon_{ijk} \ , && (6.11)\\
&\mathrm{Tr}\left(\tau_i \tau_j \tau_k \tau_l\right) = \mathrm{Tr}\left\{\left[i\epsilon_{ijm}\tau^m + \delta_{ij}\mathbb{1}_{2\times 2}\right]\left[i\epsilon_{kln}\tau^n + \delta_{kl}\mathbb{1}_{2\times 2}\right]\right\} = -\epsilon_{ijm}\epsilon_{kln}\mathrm{Tr}\left(\tau^m\tau^n\right) + \delta_{ij}\delta_{kl}\mathrm{Tr}\left(\mathbb{1}_{2\times 2}\right)\\
&\qquad\qquad = 2\left\{\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right\} \ , && (6.12)
\end{aligned}
$$

where we used the contraction identity $\epsilon_{ijk}\epsilon^k_{\ lm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ of two Levi-Civita tensors in the last step of Eq. (6.12).

6.1.2 The Pauli Matrices and their Connection to Unitary and Orthogonal Groups

The Pauli matrices are also related to unitary groups. More precisely, the set $\{\tau_i |\ i = 1, 2, 3\}$ spans the Lie algebra $\mathfrak{su}(2)$ of the Lie group of unitary (2×2)-matrices with unit determinant $SU(2)$. Therefore, using the conventional normalization, the three generators of $SU(2)$ are given by

$$T_i = \frac{\tau_i}{2} \ , \quad i = 1, 2, 3 \ , \tag{6.13}$$

so that each element $U \in SU(2)$ can be written as

$$U = e^{-i\alpha_i T^i} \ , \tag{6.14}$$

where the $\alpha_i \in \mathbb{R}$ are the so-called group parameters of $SU(2)$. If we also include Eq. (6.2) with the same normalization as above,

$$T_0 = \frac{\tau_0}{2} \ , \tag{6.15}$$

the set $\{T_a |\ a = 0, \dots, 3\}$ defines the generators of the Lie group of unitary $(2 \times 2)-$matrices $U(2)$. Then, each element $U \in U(2)$ can be written as

$$U = e^{-i\alpha_a T^a} \ . \tag{6.16}$$

The complete set of $U(2)$-generators satisfies the following trace identities

$$
\begin{aligned}
\mathrm{Tr}\left\{T^0T^0T^0T^0\right\} &= \frac{1}{8} \ , && (6.17)\\
\mathrm{Tr}\left\{T^0T^0T^iT^j\right\} &= \frac{1}{8}\delta^{ij} \ , && (6.18)\\
\mathrm{Tr}\left\{T^0T^iT^jT^k\right\} &= \frac{i}{8}\epsilon^{ijk} \ , && (6.19)\\
\mathrm{Tr}\left\{T^iT^jT^kT^l\right\} &= \frac{1}{8}\left[\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right] \ , && (6.20)
\end{aligned}
$$

where we used Eqs. (6.10)-(6.12). Finally, it should also be noted that the Pauli matrices are connected to the group of orthogonal $(3 \times 3)-$matrices with unit determinant $SO(3)$. This relation arises from the fact that the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$. But it can be shown that the groups itself are not isomorphic, since the kernel

$$K_\varphi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{6.21}$$

of the group homomorphism $\varphi : SU(2) \to SO(3)$ is non-trivial, since two elements of $SU(2)$ map onto the identity of $SO(3)$. This non-trivialty of K_φ has its origin in the different periodicities of the two Lie groups. The exact relation of $SU(2)$ and $SO(3)$ can be obtained from the fact that the kernel K_φ forms a Z_2 normal subgroup of $SU(2)$, so that

$$SO(3) \cong SU(2)\backslash Z_2 \ . \tag{6.22}$$

6.1.3 The Dirac Matrices

In the Dirac basis, the so-called γ- or Dirac-matrices are given by

$$\begin{aligned}\gamma_0 &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \mathbb{1}_{2\times 2} = \begin{pmatrix}\mathbb{1}_{2\times 2} & 0\\ 0 & -\mathbb{1}_{2\times 2}\end{pmatrix} \ ,\\ \gamma_i &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \otimes \tau_i = \begin{pmatrix}0 & \tau_i\\ -\tau_i & 0\end{pmatrix} \ , \quad i = 1,2,3 \ ,\end{aligned} \tag{6.23}$$

where the τ_i are the Pauli matrices (6.1) and $\otimes$ denotes the so-called Kronecker product. These matrices can be assembled into a 4-vector which is given by

$$\gamma^\mu = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)^T \ . \tag{6.24}$$

A fifth Dirac matrix, denoted as γ_5, can be defined as the product of the other four Dirac matrices. In the Dirac basis, one obtains

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}0 & \mathbb{1}_{2\times 2}\\ \mathbb{1}_{2\times 2} & 0\end{pmatrix} \ . \tag{6.25}$$

From the definitions (6.19) and (6.20) it is obvious that the zeroth and the fifth Dirac matrices are hermitian, while the remaining three matrices are anti-hermitian, i.e.,

$$\gamma^{\mu,\dagger} = \gamma_0\gamma^\mu\gamma_0 \ , \tag{6.26}$$

$$\gamma_5^\dagger = \gamma_5 \ . \tag{6.27}$$

In addition to that, the Dirac matrices fulfill the following anticommutation relations

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}\mathbb{1}_{4\times 4} \ , \tag{6.28}$$

$$[\gamma^\mu, \gamma_5]_+ = 0 \ . \tag{6.29}$$

Furthermore, it is possible to construct a second-rank Lorentz tensor with the six antisymmetric combinations of the Dirac matrices (6.23)

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]_- \ . \tag{6.30}$$

Then, from the sixteen objects $\mathbb{1}_{4\times 4}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5$, and $\sigma^{\mu\nu}$, which also have a distinct transformation behaviour under Lorentz transformations, it is possible to construct any $(4\times 4)-$matrix. Finally, it should be noted that these objects fulfill various contraction and trace identities which will be omitted here, since they do not play any role in this work.

6.2 Multidimensional Gaussian Integrals

In Chapter [4], we frequently had to evaluate Gaussian functional integrals in order to determine the LECs of various physical and unphysical models. The functional integral identities used are infinite-dimensional generalizations of ordinary shifted Gaussian integrals. The aim of this section will be the derivation of these integral identities. To this end, we start with the ordinary one-dimensional Gaussian integral and extend this case up to an n-dimensional generalization of the Gaussian integral. The approach will be first to state the final result and then to prove it by a straightforward calculation.

The one-dimensional Gaussian integral is given by

$$I = \int\limits_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}} \ , \tag{6.31}$$

with $a \in \mathbb{R} > 0$. The usual proof of this identity relies on the trick not to consider the integral I itself, but the square I^2. This ansatz allows the transformation to polar coordinates, so that

$$\begin{aligned} I^2 &= \int\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\, e^{-\frac{a}{2}(x^2+y^2)} = \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^{\infty} \mathrm{d}r\, re^{-\frac{ar^2}{2}} \\ &= \frac{2\pi}{a}\int\limits_0^{2\pi} \mathrm{d}\xi\, e^{-\xi} = \frac{2\pi}{a} \ , \end{aligned} \tag{6.32}$$

which proves Eq. (6.31). Now, it is possible to modify the exponential by a term which is linear in the integration variable. The value of this shifted Gaussian integral is given by

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{ax^2}{2}+Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \;, \tag{6.33}$$

where $a, J \in \mathbb{R}$. We immediately observe that the result (6.33) reduces to Eq. (6.31), if we set the "source" J to zero. In order to prove this result, we simply complete the square in the exponential and rewrite Eq. (6.33) in a way where we are able to use Eq. (6.31). We find

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{ax^2}{2}+Jx} = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{ax^2}{2}-\frac{J^2}{2a}+Jx+\frac{J^2}{2a}} \\ &= e^{\frac{J^2}{2a}} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{a}{2}\left(x-\frac{J}{a}\right)^2} = e^{\frac{J^2}{2a}} \int_{-\infty}^{\infty} \mathrm{d}\xi \, e^{-\frac{a\xi^2}{2}} \\ &= \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \;. \end{aligned} \tag{6.34}$$

Now, we extend our considerations to n dimensions. The n-dimensional generalization of Eq. (6.31) is given by

$$I = \int_{-\infty}^{\infty} \mathrm{d}^n \mathbf{x} \, e^{-\frac{1}{2}\mathbf{x}^T A \mathbf{x}} = \sqrt{\frac{(2\pi)^n}{\det(A)}} \;, \tag{6.35}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and $\mathbf{x}^T A \mathbf{x} = a_{ij} x^i x^j$. It is obvious that this identity cannot be fulfilled for all matrices $A \in \mathbb{R}^{n\times n}$. Therefore, we require A to be a non-singular, positive definite, and symmetric matrix with eigenvalues $\lambda_j, \; j = 1, \ldots, n$. These properties are closely connected and need some explanation: It is obvious, that Eq. (6.35) is only well defined, if the determinant of A is different from zero. On the one hand, this requirement is met by the fact, that A is non-singular, i.e. that its inverse $A^{-1} \in \mathbb{R}^{n\times n}$ exists. The existence of an inverse will also be needed in the next case, where we consider the n-dimensional generalization of Eq. (6.33). On the other hand, the positive definiteness of a symmetric matrix is equivalent to the statement that all eigenvalues of the matrix are positive and therefore different from zero, such that the invertibility of A is also satisfied. In order to prove Eq. (6.35), we exploit the fact that A is symmetric. This property ensures the existence of an orthogonal matrix $O \in \mathbb{R}^{n\times n}$, so that $D \equiv OAO^T = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. We find

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{d}^n \mathbf{x} \, e^{-\frac{1}{2}\mathbf{x}^T A \mathbf{x}} = \int_{-\infty}^{\infty} \mathrm{d}^n \mathbf{x} \, e^{-\frac{1}{2}\mathbf{x}^T O^T O A O^T O \mathbf{x}} \\ &= \int_{-\infty}^{\infty} \mathrm{d}^n \boldsymbol{\xi} \, e^{-\frac{1}{2}\boldsymbol{\xi}^T D \boldsymbol{\xi}} = \prod_{j=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}\xi_j \, e^{-\frac{\lambda_j}{2}\xi_j^2} \\ &= \prod_{j=1}^{n} \sqrt{\frac{2\pi}{\lambda_j}} = \sqrt{\frac{(2\pi)^n}{\det(A)}} \;, \end{aligned} \tag{6.36}$$

where we introduced new variables $\xi_i = O_{ij} x^j$ and used that the Jacobian $|\det(O)|$ of an orthogonal transformation is one. Finally, it should be mentioned that the application of Eq. (6.31) in the last line is only valid, since the eigenvalues of the matrix A are positive. As mentioned earlier, we also have to consider the n-dimensional generalization of the shifted Gaussian integral (6.33). The desired integral identity is given by

$$I = \int_{-\infty}^{\infty} \mathrm{d}^n \mathbf{x} \, e^{-\frac{1}{2}\mathbf{x}^T A \mathbf{x} + \mathbf{J}^T \mathbf{x}} = \sqrt{\frac{(2\pi)^n}{\det(A)}} e^{\frac{1}{2}\mathbf{J}^T A^{-1} \mathbf{J}} \;, \tag{6.37}$$

where $\mathbf{J} = (J_1, \ldots, J_n) \in \mathbb{R}^n$ and $\mathbf{J}^T \mathbf{x} = J_i x^i$. In analogy to the previous case (6.35), we require the matrix A to be non-singular, positive definite, and symmetric. Again, the eigenvalues of A are given by $\lambda_j, \; j = 1, \ldots, n$. The idea of the proof of Eq. (6.37) is the same as in the one-dimensional case: We complete the square in the exponential and rewrite the integral in a way that we are able to use Eq. (6.35). We obtain

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \mathrm{d}^n\mathbf{x}\; e^{-\frac{1}{2}\mathbf{x}^T A\mathbf{x}+\mathbf{J}^T\mathbf{x}} = \int_{-\infty}^{\infty} \mathrm{d}^n\mathbf{x}\; e^{-\frac{1}{2}\left(\mathbf{x}^T A\mathbf{x}-2\mathbf{J}^T\mathbf{x}+\mathbf{J}^T A^{-1}\mathbf{J}\right)+\frac{1}{2}\mathbf{J}^T A^{-1}\mathbf{J}} \\
&= e^{\frac{1}{2}\mathbf{J}^T A^{-1}\mathbf{J}} \int_{-\infty}^{\infty} \mathrm{d}^n\mathbf{x}\; e^{-\frac{1}{2}\left(\mathbf{x}-A^{-1}\mathbf{J}\right)^T A\left(\mathbf{x}-A^{-1}\mathbf{J}\right)} = e^{\frac{1}{2}\mathbf{J}^T A^{-1}\mathbf{J}} \int_{-\infty}^{\infty} \mathrm{d}^n\boldsymbol{\xi}\; e^{-\frac{1}{2}\boldsymbol{\xi}^T A\boldsymbol{\xi}} \\
&= \sqrt{\frac{(2\pi)^n}{\det(A)}}\, e^{\frac{1}{2}\mathbf{J}^T A^{-1}\mathbf{J}} \;, \qquad (6.38)
\end{aligned}
$$

where we used $\left(A^{-1}\right)^T = A^{-1}$, $\mathbf{x}^T\mathbf{J} = \mathbf{J}^T\mathbf{x}$, and defined $\boldsymbol{\xi} = \mathbf{x} - A^{-1}\mathbf{J}$. The Jacobian of this coordinate transformation is clearly the identity, since we simply shifted the initial integration variable by a constant. Finally, we applied Eq. (6.31) in the last line.

In order to use Eq. (6.37) in the functional integral formulation of quantum field theory for spin-0 bosons, we have to make further considerations. An important problem of the path integral approach to quantum field theory concerns the convergence of the integrals. In particular, the strongly oscillating exponential in the functional integral threatens the convergence of the integral. In order to resolve this problem, the usual approach consists of a $\pi/2$-rotation of the time-like component of the 4-vectors into the complex plane. This so-called Wick rotation transforms the usual 4-vectors of Minkowski space-time into Euclidean 4-vectors which will be indicated by the subscript $_E$. Replacing the n-dimensional "spatial" vector $\mathbf{x}$ by the scalar field $\phi(x_E)$ as well as $\mathbf{x}^T A\mathbf{x}$ and $\mathbf{J}^T\mathbf{x}$ by their continuous generalizations, the Gaussian functional integral identity is given by

$$
\begin{aligned}
&\int \mathscr{D}\phi(x_E)\; \exp\left\{-\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\; \phi(x_E)A(x_E,y_E)\phi(y_E) + \int \mathrm{d}^4x_E\; J(x_E)\phi(x_E)\right\} \\
&= \mathcal{N}\left[\det A(x_E,y_E)\right]^{-1/2} \exp\left\{\frac{1}{2}\int \mathrm{d}^4x_E\mathrm{d}^4y_E\; J(x_E)A^{-1}(x_E,y_E)J(y_E)\right\} \;. \qquad (6.39)
\end{aligned}
$$

Since Eq. (6.39) should pass into Eq. (6.37) if we consider a discretized space-time, the operator $A(x_E, y_E)$ should fulfill the same properties as the matrix A of Eq. (6.37). Therefore, we require the operator $A(x_E, y_E)$ to be nonsingular, positive definite, and symmetric. Finally, it should be mentioned that the functional determinant $\det(A)$ in Eq. (6.39) may have a physical meaning in some cases. It is also possible to generalize the presented formalism to complex-valued as well as Grassmann-valued Gaussian integrals. The latter type of Gaussian integrals plays an important role in the functional integral description of fermionic systems, since the Grassmann-valued fields realize the Fermi-Dirac statistics of fermions.

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