Johann Wolfgang Goethe-Universität
Frankfurt am Main

Master Thesis

Department of Physics - Institute for Theoretical Physics

Phenomenology of Baryons in the Extended Linear Sigma Model

Lisa Olbrich

Frankfurt am Main, September 2014

Supervisor

Priv. Doz. Dr. Francesco Giacosa

Institute of Theoretical Physics
University Frankfurt am Main

Second Assessor

Prof. Dr. Dirk H. Rischke

Institute of Theoretical Physics and
Frankfurt Institute for Advanced Studies
University Frankfurt am Main

Contents

1. Introduction

"If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis (or the atomic fact, or whatever you wish to call it) that all things are made of atoms — little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another. In that one sentence, you will see, there is an enormous amount of information about the world, if just a little imagination and thinking are applied."

Richard Feynman, "The Feynman Lectures on Physics" (1964)

Nowadays, thousands of years after Democritus and Leucippus came up with the idea of "atoms", millenniums of using imagination and thinking (and experimenting), physicists have found that atoms are made of smaller ingredients: quarks and leptons. Using the so-called Standard Model of Elementary Particle Physics with six quarks (called up, down, strange, charm, bottom, top), six leptons (called electron, electron neutrino, muon, muon neutrino, tau, tau neutrino) plus their corresponding twelve antiparticles and some symmetries (first and foremost the gauge symmetries to generate the three[1] fundamental interactions: $U(1)$ (electromagnetism), $SU(3)$ (quantum chromodynamics (QCD)), $SU(2)$ (weak interaction)), we are theoretically able to describe all matter around us - more we do not need. But, unfortunately, we are not able to analytically calculate every process. For that reason, we use approximations, such as perturbation theory or effective theories and models. For example in low-energy QCD an effective model has no longer the same degrees of freedom as QCD itself, but it is based on the same symmetries and breakings of (some of) them.

For the work presented in this thesis we use an effective model which is called extended linear sigma model (eLSM, e.g. refs. [1, 2, 3, 4]). Our aim is to enlarge this model by including baryons and their chiral partners. For two flavors this has already been done in ref. [3, 4] where the chiral partner was incorporated in the so-called mirror assignment (refs. [3, 4, 6, 7, 8]). Here we want to analyse the case where three flavors are present. To this end we study chiral transformations of these baryon fields and construct a chirally invariant Lagrangian. In a second step we reduce this Lagrangian to two flavors and fix the parameters by using experimental data for the masses of the nucleons $N(939)$, $N(1440)$, the chiral partners $N(1535)$, $N(1650)$, and five available decay widths.

In this introductory chapter we will present a short review of the basic features of the underlying theory of quarks and gluons, QCD.

[1]We ignore gravity here since it is negligibly small in the realm of microscopic physics.

1.1. Quantum Chromodynamics: Lagrangian, Symmetries, and their Breaking

Quantum chromodynamics (QCD) is a non-abelian gauge field theory with a local $SU(3)$ color symmetry. It describes the interaction between quarks (fermions) and gluons (gauge bosons). Its Lagrangian has the following form [9, 10]:

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{q}_f \left(i\gamma^\mu D_\mu - m_f\right) q_f - \frac{1}{2}\,\text{Tr}\left[\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}\right], \qquad f = u, d, s, c, t, b, \tag{1.1}$$

with

$$\begin{aligned} D_\mu &= \partial_\mu - ig\mathcal{A}_\mu, \\ \mathcal{G}_{\mu\nu} &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig\left[\mathcal{A}_\mu, \mathcal{A}_\nu\right], \\ \mathcal{A}_\mu &= A^a_\mu T_a\ , \qquad a = 1, ..., N_c^2 - 1 = 8, \end{aligned} \tag{1.2}$$

and T^a being the generators of $SU(3)$. The first terms describe the bare masses of the quarks q_f (with flavor f) and their interaction with the gluon field A^μ. The last term contains the field-strength tensor $G_{\mu\nu}$, which provides the kinematics and self interactions of the gluon field.

The Lagrangian (1.1) follows from the Lagrangian of a free Dirac field,

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi, \tag{1.3}$$

by the claim that it should be invariant under local $SU(3)$ color symmetry. The general solution of the related Dirac equation is given by a 4-spinor in Fourier space:

$$\Psi(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left(a^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-ipx} + b^{s\star}_{\boldsymbol{p}} v^s(\boldsymbol{p}) e^{ipx}\right), \tag{1.4}$$

where s labels the spin states. The field is complex-valued. Implying that the Fourier amplitudes $a^s_{\boldsymbol{p}}$ and $b^{s\star}_{\boldsymbol{p}}$ are independent. Furthermore, we distinguish between Dirac spinors for positive and negative energy. The former ones are named u^s and the latter v^s. They read in Dirac representation

$$u^s(\boldsymbol{p}) = \sqrt{E_{\boldsymbol{p}} + m}\begin{pmatrix} \mathbb{1}_{2\times2}\chi^s \\ \frac{\boldsymbol{p}\cdot\bar{\sigma}}{E_{\boldsymbol{p}}+m}\chi^s \end{pmatrix} \qquad \text{and} \qquad v^s(\boldsymbol{p}) = \sqrt{E_{\boldsymbol{p}} + m}\begin{pmatrix} \frac{\boldsymbol{p}\cdot\bar{\sigma}}{E_{\boldsymbol{p}}+m}\eta^{-s} \\ \mathbb{1}_{2\times2}\eta^{-s} \end{pmatrix}, \tag{1.5}$$

where $\chi^{+1/2} = (1,0)^T$, $\chi^{-1/2} = (0,1)^T$, $\eta^{-1/2} = (0,1)^T$, and $\eta^{+1/2} = (-1,0)^T$. For later calculations we also give them in Weyl representation:

$$u^s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{\boldsymbol{p}\cdot\sigma}\xi^s \\ \sqrt{\boldsymbol{p}\cdot\bar{\sigma}}\xi^s \end{pmatrix} \qquad \text{and} \qquad v^s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{\boldsymbol{p}\cdot\sigma}\xi^s \\ -\sqrt{\boldsymbol{p}\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \tag{1.6}$$

where the 2-spinors ξ^s are usually chosen to be $\xi^{+1/2} = (1,0)^T$ and $\xi^{-1/2} = (0,1)^T$. The 4-"vectors" σ^μ and $\bar{\sigma}^\mu$ are defined by $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ with $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$. So they contain the three Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.7}$$

In **quantum** field theory the fields Ψ in eq. (1.4) has to be quantized. So $a^s_{\boldsymbol{p}}$ and $b^{s\star}_{\boldsymbol{p}}$ become ladder operators $\hat{a}^s_{\boldsymbol{p}}$ and $\hat{b}^{s\dagger}_{\boldsymbol{p}}$ which create or annihilate a particle ($\hat{a}^{s\dagger}_{\boldsymbol{p}} |...\rangle$) or an antiparticle ($\hat{b}^{s\dagger}_{\boldsymbol{p}} |...\rangle$) with spin s and momentum $\boldsymbol{p}$. They obey certain anticommutation relations, so that they describe fermions. The non-trivial ones are

$$\left\{\hat{a}^s_{\boldsymbol{p}}, \hat{a}^{s'\dagger}_{\boldsymbol{p'}}\right\} = \left\{\hat{b}^s_{\boldsymbol{p}}, \hat{b}^{s'\dagger}_{\boldsymbol{p'}}\right\} = 2E_{\boldsymbol{p}}\delta_{s,s'}(2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}'), \tag{1.8}$$

(for more details see e.g. [9, 10]). In the following we will omit the hats over operators, since we work solely in quantised theories and there is no danger of confusion.
The QCD Lagrangian has many symmetries. In addition to continuous Lorentz symmetry and the discrete charge conjugation, parity, and time reversal (briefly CPT symmetries), it has a local (gauge) $SU(3)$ color symmetry (by construction), in the chiral limit ($m_f = 0$) of massless quarks an exact global $U(N_f)_R \times U(N_f)_L$ chiral symmetry (apart from the chiral anomaly), and also the classical dilatation (or scale) symmetry.

1.1.1. Lorentz and Poincaré Symmetry

A Lagrangian describing elementary particles has to be invariant under Lorentz transformations. Mathematically, such a transformation of a Lorentz vector is given by

$$x^\mu \quad \longrightarrow \quad \Lambda^\mu_\nu x^\nu, \tag{1.9}$$

where

$$g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma = g_{\rho\sigma}. \tag{1.10}$$

We consider only proper (orthochrone and orientation-true) transformations with $\det(\Lambda) = +1$ and $\Lambda^0_0 > 0$. These are combinations of rotations and boosts. If we perform a rotation or a boost of the reference frame, then the transformed fields in the new reference frame satisfy the same equations. We speak of a Poincaré transformation if we additionally have translations. In general, all inertial systems which are connected by proper Lorentz transformations (with $\det(\Lambda) = +1$ and $\Lambda^0_0 > 0$) and/or translations are equivalent for what concerns the physics of the standard model. Examining the invariance under continuous Lorentz transformations of a field theory is quite simple when working in the Lagrangian formulation. As an immediate consequence of the principle of least action, we only have to check that the Lagrangian is a Lorentz scalar. By determining the transformation behaviour of Ψ and A^μ it can be shown that the QCD Lagrangian (1.1) is a Lorentz scalar and the theory exhibits proper Lorentz symmetry. Since these specific transformations are not very relevant for this thesis, we refer to ref. [9, 10] for more details.

1.1.2. Discrete Symmetries (CPT)

Besides the continuous transformations there are three discrete symmetries. Two of them are spacetime operations (which are improper Lorentz transformations): parity, denoted by P, reverses the space: $(t, \boldsymbol{x}) \to (t, -\boldsymbol{x})$; and time reversal, denoted by T, inverts the time: $(t, \boldsymbol{x}) \to (-t, \boldsymbol{x})$. The third (non-spacetime) operation is charge conjugation, denoted by C. These symmetries are particularly important for this work, thus we present them in detail in the following:

- **Parity:**
  A parity operation P transforms $(t, \boldsymbol{x}) \to (t, -\boldsymbol{x})$ and thus must also reverse the momentum of a particle. Mathematically, we want P to generate the transitions, which are implemented by a unitary operator U_P as follows

$$a^s_{\boldsymbol{p}} \quad \xrightarrow{\quad P \quad} \quad U_P a^s_{\boldsymbol{p}} U_P^\dagger = \eta_a a^s_{-\boldsymbol{p}} \qquad \text{and} \qquad b^s_{\boldsymbol{p}} \quad \xrightarrow{\quad P \quad} \quad U_P b^s_{\boldsymbol{p}} U_P^\dagger = \eta_b b^s_{-\boldsymbol{p}} \tag{1.11}$$

where η_a and η_b are phases. Namely, since a second application of the parity operator should yield the original state, the square of the phase factors should be equal to plus or minus one, $|\eta_{a/b}|^2 = \pm 1$. To find the transformation of a (quark) spinor Ψ and to fix the phase factors we consider the transformed expression pf eq. (1.4). Using the transformation of the ladder operators (1.11) we have

$$\begin{aligned}\Psi(x) \quad \xrightarrow{\quad P \quad} \quad & U_P \Psi(x) U_P^\dagger = \Psi^P(x) = \\ & = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left(\eta_a a^s_{-\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-ipx} + \eta_b^\star b^{s\dagger}_{-\boldsymbol{p}} v^s(\boldsymbol{p}) e^{ipx} \right),\end{aligned} \tag{1.12}$$

where the unitary operators U_P act only on the creation and annihilation operators.
For simplicity the following calculation is performed in the **Weyl representation**, but can also be done in any other representation. Substituting the variable p to $\tilde{p} = (p^0, -\boldsymbol{p})$ we can rearrange the two 4-spinors (1.6) to

$$\begin{aligned} u^s(\boldsymbol{p}) &= \begin{pmatrix} \sqrt{\boldsymbol{p}\cdot\sigma}\,\xi^s \\ \sqrt{\boldsymbol{p}\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{\boldsymbol{p}}\cdot\bar{\sigma}}\,\xi^s \\ \sqrt{\tilde{\boldsymbol{p}}\cdot\sigma}\,\xi^s \end{pmatrix} = \gamma^0 u^s(\tilde{\boldsymbol{p}}) \\ v^s(\boldsymbol{p}) &= \begin{pmatrix} \sqrt{\boldsymbol{p}\cdot\sigma}\,\xi^s \\ -\sqrt{\boldsymbol{p}\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{\boldsymbol{p}}\cdot\bar{\sigma}}\,\xi^s \\ -\sqrt{\tilde{\boldsymbol{p}}\cdot\sigma}\,\xi^s \end{pmatrix} = -\gamma^0 v^s(\tilde{\boldsymbol{p}}). \end{aligned} \tag{1.13}$$

Therewith eq. (1.12) becomes

$$\Psi^P(x) = \int \frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{\boldsymbol{p}}}}} \sum_{s=1}^{2} \left(\eta_a a^s_{\tilde{\boldsymbol{p}}} \gamma^0 u^s(\tilde{\boldsymbol{p}}) e^{-i\tilde{p}(t,-\boldsymbol{x})} - \eta_b^\star b^{s\dagger}_{\tilde{\boldsymbol{p}}} \gamma^0 v^s(\tilde{\boldsymbol{p}}) e^{i\tilde{p}(t,-\boldsymbol{x})} \right). \tag{1.14}$$

Since the parity transformation reverses only the space, $\Psi^P(\boldsymbol{x})$ should be proportional to $\Psi(t, -\boldsymbol{x})$. This is possible if $\eta_b^\star = -\eta_a$ which implies $\eta_a \eta_b = -\eta_a \eta_a^\star = -1$ and therefore $|\eta_a|^2 = 1$. Consequently the final form of the parity transformation of the spinor is

$$\Psi \quad \xrightarrow{P} \quad \Psi^P(x) = \eta_a \gamma^0 \Psi(t, -\boldsymbol{x}). \tag{1.15}$$

This result is valid in general (i.e. for all representations).
Note, one can show that the transformations of fermion bilinears are independent of the phase η_a and therefore there is no loss of generality in setting $\eta_a = -\eta_b = 1$ from the beginning (for more details see e.g. [10]). Since we will compute parity transformations in the framework of effective models in detail later on in the text, we now only remark that the QCD Lagrangian is invariant under parity transformations.

- **Charge Conjugation:**
Another discrete symmetry of the QCD Langrangian is the particle-antiparticle symmetry implemented by the charge conjugation which transforms a fermion with given momentum and spin into an antifermion with the same momentum and spin. Thus, a Lagrangian which is invariant under charge conjugation should have an analogous form of the equation of motion for an antiparticle as for the particle. As an example, the Dirac equation of an electron Ψ (with charge $-e$) in an electromagnetic field $\mathcal{A}_\mu$ is

$$\left[i\gamma^\mu \left(\partial_\mu + ie\mathcal{A}_\mu\right) - m\right] \Psi = 0. \tag{1.16}$$

Then the Dirac equation of the positron Ψ^C (with charge $+e$) should have the same form:

$$\left[i\gamma^\mu \left(\partial_\mu - ie\mathcal{A}_\mu\right) - m\right] \Psi^C = 0. \tag{1.17}$$

In order to find a relation between Ψ and the charge-conjugate Ψ^C we take the Dirac adjoint of the Dirac equation for the electron (1.16),

$$-i\partial_\mu \bar{\Psi}\gamma^\mu - e\bar{\Psi}\mathcal{A}_\mu\gamma^\mu - m\bar{\Psi} = 0, \tag{1.18}$$

where we have used that $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$ and that the gauge field $\mathcal{A}_\mu$ is hermitian[2]. Transposing it yields

$$-i\left(\gamma^\mu\right)^T \partial_\mu \bar{\Psi}^T - e\left(\gamma^\mu\right)^T \mathcal{A}_\mu \bar{\Psi}^T - m\bar{\Psi}^T = 0. \tag{1.19}$$

If we introduce a matrix C which fulfils the relations

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \tag{1.20}$$

and

[2]This is obvious for the case of the electromagnetic field $\mathcal{A}_\mu$, which is a real quantity. In general the gauge field $\mathcal{A}_\mu$ is a combination of the generators T^a of the unitary group $SU(N)$ and various real coefficients, $\mathcal{A}_\mu = A^a_\mu T^a$. For the generators T^a holds that $T^{a\dagger} = T^a$, because the elements of the unitary group has to fulfil $U^\dagger U \simeq 1 + i\epsilon T - i\epsilon T \equiv 1$. Hence $\mathcal{A}_\mu$ is hermitian.

$$C^{-1} = C^{\dagger} = C^{T} = -C, \tag{1.21}$$

we find that

$$\left[i\gamma^{\mu}\left(\partial_{\mu} - ie\mathcal{A}_{\mu}\right) - m\right] C\bar{\Psi}^{T} = 0. \tag{1.22}$$

When we compare this with the Dirac equation for the positron (1.17), we can read off the final expression of the charge-conjugate spinor:

$$\Psi \quad \xrightarrow{C} \quad \Psi^{C} = C\bar{\Psi}^{T}. \tag{1.23}$$

In the Dirac representation the charge conjugation matrix is given by

$$C = i\gamma^{2}\gamma^{0}. \tag{1.24}$$

As can be shown, the QCD Lagrangian is invariant under charge conjugation.

- **Time Reversal and $\boldsymbol{CPT}$ theorem:**
  If a Lagrangian has to be invariant under C, P, and T separately it suffices to check only two of these three transformations, because it is generally true that one cannot build a Lorentz-invariant quantum field theory with a hermitian Hamiltonian that violates the combined symmetry CPT. This fact is called CPT theorem (see e.g. ref. [15]).

1.1.3. Chiral Symmetry

In the limit of N_f massless quarks the QCD Lagrangian is symmetric under global $U(N_f)_L \times U(N_f)_R$ transformations. This is also a particularly important symmetry for this work. In order to see how a quark spinor q_i transforms under this group, we decompose it into a left-handed component $P_L q_i = q_{i,L}$ and right-handed component $P_R q_i = q_{i,R}$ by making use of the two chiral projectors,

$$P_L = \frac{1-\gamma^5}{2} \qquad \text{and} \qquad P_R = \frac{1+\gamma^5}{2}, \tag{1.25}$$

such that $q_i = q_{i,L} + q_{i,R}$. Under chiral transformations the components behave as

$$q_{i,L} \longrightarrow q'_{i,L} = \exp\left\{-i \sum_{a=0}^{N_f^2-1} \theta_L^a T_{ij}^a\right\} q_{j,L} = U_{L,ij} q_{j,L} \qquad \text{with } U_L \in U(N_f)_L,$$
$$q_{i,R} \longrightarrow q'_{i,R} = \exp\left\{-i \sum_{a=0}^{N_f^2-1} \theta_R^a T_{ij}^a\right\} q_{j,R} = U_{R,ij} q_{j,R} \qquad \text{with } U_R \in U(N_f)_R, \tag{1.26}$$

where T^a (with $a = 1, ..., N_f^2 - 1$) are the generators of $SU(N_f)$, $T^0 = \mathbb{1}_{N_f}/\sqrt{2N_f}$ and $\theta_{R/L}^a$ are the parameters of the transformation. An invariance under these transformations is referred to as chiral symmetry. The QCD Lagrangian is invariant under such transformations only in the chiral limit, $m_f \to 0$. in order to show this, we use the chiral projection operators (1.25) and their properties

$$P_{R/L}^2 = P_{R/L} \qquad \text{and} \qquad P_R P_L = P_L P_R = 0, \tag{1.27}$$

to rearrange the QCD Lagrangian (1.1) (omitting the gluon self-interaction term) into:

$$\mathcal{L}_{qA} = \bar{q}_f \left(i\gamma^\mu D_\mu - m_f\right) q_f = \bar{q}_{f,R} i\gamma^\mu D_\mu q_{f,R} + \bar{q}_{f,L} i\gamma^\mu D_\mu q_{f,L} - \bar{q}_{f,R} m_f q_{f,L} - \bar{q}_{f,L} m_f q_{f,R}, \tag{1.28}$$

which is obviously symmetric under $U(N_f)_R \times U(N_f)_L$ only for vanishing quark masses since the mass terms mix left- and right-handed quark components.
One commonly works with the currents A^μ and V^μ, because they have definitive parity instead of L^μ and R^μ. They are related by

$$V^\mu = \frac{R^\mu + L^\mu}{2} \qquad \text{and} \qquad A = \frac{R^\mu - L^\mu}{2}. \tag{1.29}$$

The chiral group $U(N_f)_R \times U(N_f)_L$ is isomorphic to $U(N_f)_V \times U(N_f)_A$. Hence with the properties of a unitary group the relation

$$U(N_f)_L \times U(N_f)_R \equiv U(1)_V \times SU(N_f)_V \times U(1)_A \times SU(N_f)_A \tag{1.30}$$

holds. It should be noted that $SU(N_f)_A$ is a set of transformations, but not a group, since it is not closed. The transformation elements can be expressed as:

$$\begin{array}{llllll}
U(1)_V & \ni & U_{1V} = e^{-i\theta_V^0 T^0} & \text{with} & \theta_V^0/2 = \theta_R^0 = \theta_L^0, & \text{i.e.,} \quad U_{1V} = U_L = U_R, \\
SU(N_f)_V & \ni & U_V = e^{-i\theta_V^i T^i} & \text{with} & \theta_V^i/2 = \theta_R^i = \theta_L^i, & \text{i.e.,} \quad U_V = U_L = U_R, \\
U(1)_A & \ni & U_{1A} = e^{-i\theta_A^0 \gamma^5 T^0} & \text{with} & \theta_A^0/2 = \theta_R^0 = -\theta_L^0, & \text{i.e.,} \quad U_{1A} = U_R = U_L^\dagger\ , \\
SU(N_f)_A & \ni & U_A = e^{-i\theta_A^i \gamma^5 T^i} & \text{with} & \theta_A^i/2 = -\theta_R^0 = \theta_L^0, & \text{i.e.,} \quad U_A = U_L = U_R^\dagger\ ,
\end{array} \tag{1.31}$$

with $i = 1, 2, ..., N_f^2 - 1$. In the following we will give a short overview of the features and conditions of validity of these symmetry transformations concerning the QCD Lagrangian (1.1).

- **$U(1)_V$ symmetry**:
  This transformation corresponds to a multiplication of a phase for the quark field and it is clear that the QCD Lagrangian is symmetric under this transformation. According to the Noether theorem [16] the conserved current is

$$V_0^\mu = \bar{q}_f \gamma^\mu q_f \qquad \text{and} \qquad \partial_\mu V_0^\mu = 0, \tag{1.32}$$

  and the conserved charge obtained by the integration over the zero component,

$$Q = \int \mathrm{d}^3 \boldsymbol{x} \bar{q}_f \gamma^0 q_f, \tag{1.33}$$

  corresponds to the baryon number.

- **$SU(N_f)_V$ symmetry**:
  Transforming the quark fields q_f in the Dirac Lagrangian yields to first order in θ_V^i

$$\mathcal{L}_{\text{Dirac}} = \bar{q}_f \left(i\gamma^\mu \partial_\mu - m_f\right) q_f \xrightarrow{SU(N_f)_V} \cdots = \bar{q}_f \left(i\gamma^\mu \partial_\mu - m_f\right) q_f - i\theta_V^i \bar{q}_f \left[T^i, m_f\right] q_f. \tag{1.34}$$

  Hence the symmetry is only realized if the quark masses of all flavors are degenerate $m_1 = m_2 = ... = m_{N_f}$, because only then the mass matrix is proportional to the identity matrix. In nature quark masses are only roughly equal. This means that there is just an approximate $SU(2)_V$ isospin symmetry for $m_{up} \approx m_{down}$ or an approximate $SU(3)_V$ flavor symmetry for $m_{up} \approx m_{down} \approx m_{strange}$, although the breaking of the latter is sizeable in comparison to the first two. The corresponding conserved vector currents and their divergences (according to the Noether theorem [16]) are:

$$V^{\mu i} = \bar{q}_f \gamma^\mu T^i q_f \qquad \text{and} \qquad \partial_\mu V^{\mu i} = i\bar{q}_f \left[T^i, m_f\right] q_f. \tag{1.35}$$

  The divergences vanish only for degenerate quark masses, as expected.

- **$U(1)_A$ symmetry and anomaly**:
  In classical field theory this symmetry is fulfilled in the chiral limit, but is explicitly broken by quantum fluctuations [17]. Due to this fact it is a chiral $U(1)_A$ anomaly which has to be considered also in the construction of an effective chiral model. Furthermore, since no quark mass is zero, the axial symmetry is also broken at the classical level.

- **$SU(N_f)_A$ symmetry**:
  In the chiral representation for massless quarks, $SU(N_f)_A$ corresponds to an opposed rotation of left- and right-handed Weyl spinors, q_R and q_L, because of the $\gamma^5 = \mathrm{diag}(1, 1, -1, -1)$ matrix occurring in the transformation matrix. But a mass term mixes left- and right-handed components

$$m\bar{q}q = m(q_L^\dagger q_R + q_R^\dagger q_L), \tag{1.36}$$

  and thus breaks the $SU(N_f)_A$ symmetry. We can see this fact also by computing the transformed Dirac Lagrangian to first order in θ_A^i

$$\mathcal{L}_{\text{Dirac}} = \bar{q}_f \left(i\gamma^\mu \partial_\mu - m_f\right) q_f \xrightarrow{SU(N_f)_V} \cdots = \bar{q}_f \left(i\gamma^\mu \partial_\mu - m_f\right) q_f + i\theta_A^i \bar{q}_f \left\{T^i, m_f\right\} \gamma^5 q_f \tag{1.37}$$

or the corresponding axial-vector currents and their divergences

$$A^{\mu i} = \bar{q}_f \gamma^\mu \gamma^5 T^i q_f \qquad \text{and} \qquad \partial_\mu A^{\mu i} = -i\bar{q}_f \left\{T^i, m_f\right\} q_f. \tag{1.38}$$

We recognize that the axial-vector currents are only conserved if all quark masses are zero.

1.1.4. Explicit Symmetry Breaking

For non-vanishing quark masses, $m_f \neq 0$, i.e., the mass term in the QCD Lagrangian of the form

$$\mathcal{L}_{\text{mass}} = \sum_{f=1}^{N_f} m_f \bar{q}_f q_f \tag{1.39}$$

with $m_1 \neq m_2 \neq \cdots \neq m_{N_f} \neq 0$ explicitly breaks several of the above listed symmetries, as mentioned.

1.1.5. Spontaneous Breaking and Goldstone Theorem

We have seen that chiral symmetry is explicitly broken, if the quark masses are non-zero. However, even in the chiral limit $m_f \to 0$ the symmetry undergoes the phenomenon of spontaneous symmetry breaking. In general, a symmetry is called spontaneously broken if the ground state has a lower symmetry than the Lagrangian. Goldstone's theorem predicts that for each continuous global symmetry which is spontaneously broken a massless particle emerges. These particles are called (Nambu-)Goldstone bosons (see chapter 1.3 for an example). Namely, in QCD the vacuum is not invariant under $SU(N_f)_A$ transformations and therefore this symmetry is spontaneously broken,

$$SU(N_f)_L \times SU(N_f)_R = SU(N_f)_V \times SU(N_f)_A \longrightarrow SU(N_f)_V \qquad \text{in the ground state.} \tag{1.40}$$

To see why this is the case, we assume $SU(N_f)_A$ to be not spontaneously broken. Then we would obtain ground-state multiplets containing particles with opposed parity but with the same mass and same quantum numbers. Since such partners are not observed in nature, $SU(N_f)_A$ must be spontaneously broken. As a consequence massless Goldstone bosons arise: the pions. (Note that they are not exactly massless because of explicit symmetry breaking.) Furthermore, spontaneous symmetry breaking generates mass differences between the multiplets.

1.1.6. Dilatation Symmetry and Scale Anomaly

The dilation (or scale) transformation of a contravariant space-time vector is given by

$$x^\mu \longrightarrow \lambda^{-1} x^\mu, \tag{1.41}$$

where λ is the scale parameter. If the quark and gauge field in eq. (1.1) transform as

$$q_f \longrightarrow \lambda^{\frac{3}{2}} q_f \qquad \text{and} \qquad A^a_\mu(x) \longrightarrow \lambda A^a_\mu(x) \tag{1.42}$$

for massless quarks, $m_f = 0$, the QCD Lagrangian obtains a factor of λ^4:

$$\mathcal{L}_{\text{QCD},m_f=0} = \bar{q}_f i\gamma^\mu D_\mu q_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \quad \longrightarrow \quad \lambda^4 \mathcal{L}_{\text{QCD},m_f=0}. \tag{1.43}$$

Hence, the action which is the space-time integral over the Lagrangian is invariant under this transformation. This invariance is called dilatation or trace symmetry. The corresponding scale current is a product of the space-time vector and the energy-momentum tensor:

$$J^\mu = x_\nu T^{\mu\nu} \qquad \text{with} \quad T^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{QCD}}}{\partial(\partial_\mu A^a_\rho)} \partial^\nu A^a_\rho - g^{\mu\nu}\mathcal{L}_{\text{QCD}}. \tag{1.44}$$

Computing the divergence of this current,

$$\partial_\mu J^\mu = T_\mu{}^\mu, \tag{1.45}$$

we see that the trace of the energy-momentum tensor has to be zero in order to have a conserved current and therefore a symmetry of the system. That is why dilatation symmetry is also called trace symmetry. As we have seen in eq. (1.43) QCD is (classically) scale symmetric in the limit $m_f \to 0$, but for $m_f \neq 0$ the symmetry is explicitly broken, because

$$\mathcal{L}_{\text{mass}} = \sum_{f=1}^{N_f} m_f \bar{q}_f q_f \quad \longrightarrow \lambda^3 \mathcal{L}_{\text{mass}} \neq \lambda^4 \mathcal{L}_{\text{mass}} \qquad \text{and} \quad \partial_\mu J^\mu = 4\sum_{f=1}^{N_f} m_f \bar{q}_f q_f \neq 0. \tag{1.46}$$

Due to the light quark masses the breaking is small, but the symmetry is also broken on the quantum level when considering gluon loops (quantum fluctuations). Upon renormalisation of QCD this leads to a dependence of the strong coupling "constant" g on an energy scale μ (e.g. center-of-mass energy) [18], for more details see section 1.2.. This fact yields a non-vanishing divergence of the scale current. With a one-loop perturbative calculation one obtains:

$$\partial_\mu J^\mu = T_\mu{}^\mu = \frac{\beta(g)}{4g} G^a_{\mu\nu} G^{\mu\nu}_a \sum_{f=1}^{N_f} m_f \bar{q}_f q_f \neq 0 \qquad \text{with} \quad \beta(g) = \mu \frac{\partial g}{\partial \mu}. \tag{1.47}$$

Therefore, even for $m_f = 0$ the scale symmetry is broken at the quantum level and we have another anomaly: the so-called scale or trace anomaly. As shown in ref. [19], at the composite level one can parametrize this anomaly by introducing a scalar dilaton (glueball) field G which is described by the Lagrangian

$$\mathcal{L}_{\text{dil}} = \frac{1}{2}(\partial_\mu G)^2 - \frac{1}{4}\frac{m_G^2}{\Lambda^2}\left(\frac{1}{2}G^4 \ln \frac{G^2}{\Lambda^2} - \frac{G^2}{4}\right), \tag{1.48}$$

where the dilatation symmetry is explicitly broken by the scale factor Λ. For the construction of an effective QCD model this dilatation symmetry is an important point. Since the dilatation symmetry should be explicitly broken *only* in the potential of the dilaton field of eq. (1.48) all other terms in the Lagrangian have to be dilatation invariant. This means that only terms with dimensionless coupling constant are allowed, which is a very restrictive requirement that is fulfilled only by a finite number of terms in our effective model [20]. The parameter Λ of eq. (1.48) is proportional to the so-called Yang-Mills scale Λ_{YM}.

1.2. Running Coupling and Effective Theories

Studying QCD at the quantum level (i.e., including gluon loops) [18] implies that the coupling constant g of QCD is actually *not* constant - it is a function of the energy scale μ where the coupling is determined (e.g. the center-of-mass energy). For QCD with N_c colors and N_f approximately massless quarks one gets:

$$g^2(\mu) = \frac{1}{b\log(\mu/\Lambda_{\mathrm{YM}})} \qquad \text{with} \qquad b = \frac{11N_c - 2N_f}{48\pi^2}, \tag{1.49}$$

where $\Lambda_{\mathrm{YM}} \approx 200$ MeV is the Landau or the Yang-Mills scale. The coupling constant is called **running coupling constant**. For high energies (μ much larger than Λ_{YM}) the QCD coupling constant becomes small (*asymptotic freedom*) and perturbation theory is a good approximation. On the contrary, for low energies (μ is comparable or smaller Λ_{YM}) the QCD coupling constant is large and perturbation theory cannot be applied. Therefore, effective theories were developed, which contain no longer quarks and gluons but colorless hadrons as degrees of freedom. We thus use only appropriate degrees of freedom for our chosen energy scale (or length) scale, and ignore the substructure and degrees of freedom at shorter distances. However, effective theories have the same symmetries as QCD. One example of an effective theory is the linear sigma model which was suggested by Gell-Mann and Lévy in 1960 [1] and recently extended in ref. [2, 3, 4].

1.3. (Extended) Linear Sigma Model and Spontaneous Symmetry Breaking

In 1960 Gell-Mann and Lévy [1] constructed the linear sigma model (LSM) to study chiral symmetry and its spontaneous breaking in pion-nucleon interactions. Recently it was extended (eLSM) by the inclusion of vector and axial-vector mesons, e.g. [2, 3, 4, 11, 12]. Nuclear matter ground state has also been calculated as well as the chiral phase transition at nonzero density [3, 4] and the emergence of inhomogeneous chiral condensation [13].
The Lagrangian of (a simple version of) the linear sigma model is given by:

$$\mathcal{L}_{LSM} = \frac{1}{2}\left(\partial_\mu\sigma\right)^2 + \frac{1}{2}\left(\partial_\mu\boldsymbol{\pi}\right)^2 - \frac{\mu^2}{2}\left(\sigma^2 + \boldsymbol{\pi}^2\right) - \frac{\lambda}{4!}\left(\sigma^2 + \boldsymbol{\pi}^2\right)^2 + \bar{\Psi}i\gamma^\mu\partial_\mu\Psi + ig\bar{\Psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\Psi + g\bar{\Psi}\sigma\Psi, \tag{1.50}$$

where the scalar isosinglet field σ and the pseudoscalar isotriplet field $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ are present and interact with massless isodoublet nucleon fields Ψ through Yukawa couplings. The mass of the nucleons is generated by spontaneous symmetry breaking from $O(4) \to O(3)$ of the ground state. To see how it works, we should have a closer look at the potential of the Lagrangian. Setting $\boldsymbol{\Phi} \equiv (\Phi_1, \Phi_2, \Phi_3, \Phi_4) = (\sigma, \boldsymbol{\pi})$, we can rewrite the potential as[3]

$$V(\boldsymbol{\Phi}) = \frac{\mu^2}{2}\boldsymbol{\Phi}^2 + \frac{\lambda}{4!}\boldsymbol{\Phi}^4. \tag{1.51}$$

[3]This is equivalent to the potential of Φ^4 theory.

In order to find the ground state of the system, we need to find the minimum. If the mass term μ^2 is negative it is given by the relation

$$\mathbf{\Phi}^2 = -\frac{\mu^2}{\lambda} \equiv \varphi^2 \qquad \text{or} \qquad |\mathbf{\Phi}| = \sqrt{\frac{-\mu^2}{\lambda}} = \varphi. \tag{1.52}$$

This defines a 3-dimensional subspace in the 4-dimensional scalar space. Every point on this sphere is invariant under $O(3)$ rotations. Choosing only one point of the ground state (none of them is favoured most), the symmetry is spontaneously broken from $O(4)$ to $O(3)$. In Fig. 1.1 we show the potential along the Φ_1, Φ_2 directions by setting $\Phi_3 = \Phi_4 = 0$. It is the so-called Mexican-hat potential with the subspace being the brim of the hat.

In order to apply perturbation theory, we have to choose a minimum and expand around it. We pick

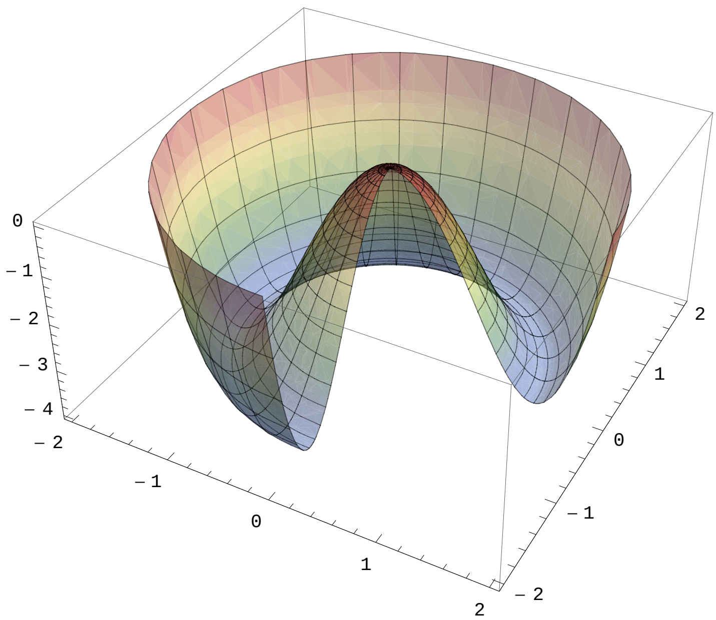


Figure 1.1.: The mexican-hat potential

for example the point $\boldsymbol{\varphi} = (\varphi, 0, 0, 0)$, i.e., the vacuum expectation values (VEVs) of σ and $\boldsymbol{\pi}$ are

$$\langle \sigma \rangle = \varphi \qquad \text{and} \qquad \langle \boldsymbol{\pi} \rangle = 0. \tag{1.53}$$

Carrying out the coordinate transformation $\sigma \to \sigma + \varphi$ the structure of the Lagrangian $\mathcal{L}_{\text{LSM}}$ changes as:

$$\begin{aligned} \mathcal{L}_{\text{LSM}} = \; & \frac{1}{2}\left(\partial_\mu \sigma\right)^2 + \frac{1}{2}\left(\partial_\mu \boldsymbol{\pi}\right)^2 + \mu^2\sigma^2 - \frac{\lambda\varphi}{6}\sigma\left(\sigma^2 + \boldsymbol{\pi}^2\right) - \frac{\lambda}{4!}\left(\sigma^2 + \boldsymbol{\pi}^2\right)^2 + \\ & + \bar{\Psi} i \gamma^\mu \partial_\mu \Psi + g\varphi\bar{\Psi}\Psi + ig\bar{\Psi}\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi}\Psi + g\bar{\Psi}\sigma\Psi, \end{aligned} \tag{1.54}$$

First we observe (as a consequence of the Goldstone theorem) that the σ meson has mass ($m_\sigma^2 = -2\mu^2 > 0$) while the pion $\boldsymbol{\pi}$ is a massless Goldstone boson. Another important point is that the

condensation of σ generates a mass term for the nucleon Ψ proportional to its vacuum expectation value.
Since in reality the pion is not massless we add a term to the Lagrangian (1.54), which explicitly breaks the chiral symmetry $SU(2)_R \times SU(2)_L$:

$$\mathcal{L}_{\text{LSM}} \quad \longrightarrow \quad \mathcal{L}_{\text{LSM}} + \epsilon\sigma. \tag{1.55}$$

This shifts the minimum of the potential (to first order in ϵ) to

$$\varphi = \sqrt{\frac{-6\mu}{\lambda}} + \frac{\epsilon}{2\mu^2}. \tag{1.56}$$

As a result the pion also acquires a mass of $m_{\boldsymbol{\pi}}^2 = \epsilon/\varphi \neq 0$.
In the following we will use a linear sigma model which contains, in addition to scalar and pseudoscalar mesons and baryons, also vector and axial-vector mesons.

2. A Short General Introduction to Perturbation Theory for Interacting Fields

The Lagrangians which we will apply do not describe free particles. There are interaction terms which include scattering and decay processes. In this chapter we show how to compute, in the framework of perturbation theory, the effects of interactions (such as decay widths) from an arbitrary Lagrangian which is made of a free part (indicated with "$_0$") and an interaction part (indicated with "$_{\text{int}}$"),

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}. \tag{2.1}$$

The decay widths resulting from such Lagrangians can subsequently be compared with experimental results. The presentation in this chapter is based on ref. [10].

2.1. Correlation Functions, Wick's Theorem, and Feynman Diagrams

A very important quantity to calculate transition amplitudes and therewith measurable quantities, such as decay rates or cross sections, is the n-point correlation function or n-point Green's function,

$$\langle\Omega|T\Big\{\phi(x_1)\phi(x_2)...\phi(x_n)\Big\}|\Omega\rangle\,, \tag{2.2}$$

where T is the time-ordering operator. We introduced the notation $|\Omega\rangle$ to denote the ground state of the interacting theory, which is different from the ground state of the free theory $|0\rangle$. Later we will see that the two-point correlation function, e.g. $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$, can be interpreted physically as the amplitude for a particle (resp. excitation) to propagate from y to x. The calculation of such vacuum expectation values in an interacting theory is not trivial, since we cannot use the creation and annihilation operators of the free theory straight-forwardly. That is because the field equation is non-linear and a general solution by linear superposition is not possible. Hence, in order to enable calculations we rewrite eq. (2.2) using only fields and states of the free theory[1]:

$$\langle\Omega|T\Big\{\phi(x_1)\phi(x_2)...\phi(x_n)\Big\}|\Omega\rangle = \frac{\langle 0|T\Big\{\phi_I(x_1)\phi_I(x_2)...\phi_I(x_n)\exp\Big\{-i\int_{-T}^{T}\mathrm{d}t\;H_{\text{int}}(t)\Big\}\Big\}|0\rangle}{\langle 0|\exp\Big\{-i\int_{-T}^{T}\mathrm{d}t\;H_{\text{int}}(t)\Big\}|0\rangle}, \tag{2.3}$$

where ϕ_I indicates a field in the interaction picture, which evolves like a free field and $H_{\text{int}} = -\mathcal{L}_{\text{int}}$ is the interacting part of the Hamiltonian of the corresponding theory. Before we evaluate this in more detail, we need to evaluate two-point correlation functions of the **free** theory.

[1]For the explicit calculation see e.g. [10].

2.1.1. Two-Point Correlation Function of the Free Theory: Feynman Propagators

We start with the two-point correlation function of the free theory for scalar fields. This quantity is also called Feynman propagator for scalar fields,

$$\Delta_F(x-y) = \langle 0|T\Big\{\Phi(x)\Phi(y)\Big\}|0\rangle\,, \tag{2.4}$$

and for $y^0 > x^0$ describes the amplitude for a free (Klein-Gordon) scalar field produced at the time y^0 at the place $\boldsymbol{y}$ to be found at the time x^0 at $\boldsymbol{x}$. If $x^0 > y^0$ it is the other way round. This aspect can be seen by simplifying the notation. We split the Klein-Gordon field Φ into two parts

$$\begin{aligned}\Phi = \Phi^+ + \Phi^- \qquad \text{with}\ \ \Phi^+(x) &:= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} a_{\boldsymbol{p}} e^{-ipx} \\ \text{and}\ \ \Phi^-(x) &:= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} a^\dagger_{\boldsymbol{p}} e^{ipx},\end{aligned} \tag{2.5}$$

such that $\Phi^+|0\rangle \sim a|0\rangle = 0$ and $\langle 0|\Phi^- \sim \langle 0|a^\dagger = 0$. The time-ordered product reads

$$T\Big\{\Phi(x)\Phi(y)\Big\} = \ :\Phi(x)\Phi(y): + \,\theta(x^0-y^0)\left[\Phi^+(x),\Phi^-(y)\right] + \theta(y^0-x^0)\left[\Phi^+(y),\Phi^-(x)\right] \tag{2.6}$$

with : ... : denoting the normal-ordered product (all creation operators $a^\dagger \sim \Phi^-$ at the left and all annihilation operators $a \sim \Phi^+$ at the right side). Using the commutator the non-trivial commutator of creation and annihilation operators for scalar fields,

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p'}}\right] = 2E_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'}), \tag{2.7}$$

and the fact that $\langle 0| : ... : |0\rangle = 0$ we obtain for the Feynman propagator:

$$\begin{aligned}\Delta_F(x-y) &= \langle 0|T\Big\{\Phi(x)\Phi(y)\Big\}|0\rangle = \\ &= \langle 0|\theta(x^0-y^0)\left[\Phi^+(x),\Phi^-(y)\right] + \theta(y^0-x^0)\left[\Phi^+(y),\Phi^-(x)\right]|0\rangle = \\ &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\Big(\theta(x^0-y^0)e^{-ip(x-y)} + \theta(y^0-x^0)e^{-ip(y-x)}\Big) = \\ &= \ldots \text{see eq. (2.9) and (2.10)} \cdots = \\ \Delta_F(x-y) &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip(x-y)} \frac{i}{p^2-m^2+i\epsilon},\end{aligned} \tag{2.8}$$

where ϵ is a infinitesimal real number. Here we can see why $\Delta_F(x-y)$ corresponds to the amplitude of the propagation of a scalar particle Φ from the space-time point x to y or the other way round depending on the time ordering. In the first line the term

$$\langle 0|\left[\Phi^+(x),\Phi^-(y)\right]|0\rangle = \langle 0|\Phi^+(x),\Phi^-(y)\,|0\rangle$$

(which exists only if $x^0 > y^0$) corresponds to the amplitude of a particle at x in the final bra-state and a particle at y in the initial ket-state and therefore the amplitude for the propagation of the particle Φ from y to x. The validity of the last step can be shown by complex integration:

$$\begin{aligned}\Delta_F(x-y) &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} e^{-ip^0(x^0-y^0)} \frac{i}{(p^0)^2 - (\boldsymbol{p}^2 + m^2 - i\epsilon)} = \\ &= -\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi i} \frac{e^{-ip^0(x^0-y^0)}}{\left(p^0 - \sqrt{E_{\boldsymbol{p}}^2 - i\epsilon}\right)\left(p^0 + \sqrt{E_{\boldsymbol{p}}^2 - i\epsilon}\right)} = \\ &= \ldots \end{aligned} \tag{2.9}$$

where we have used $\boldsymbol{p}^2 + m^2 = E_{\boldsymbol{p}}^2$. Extending to the complex plane (see fig. 2.1) and using the residue

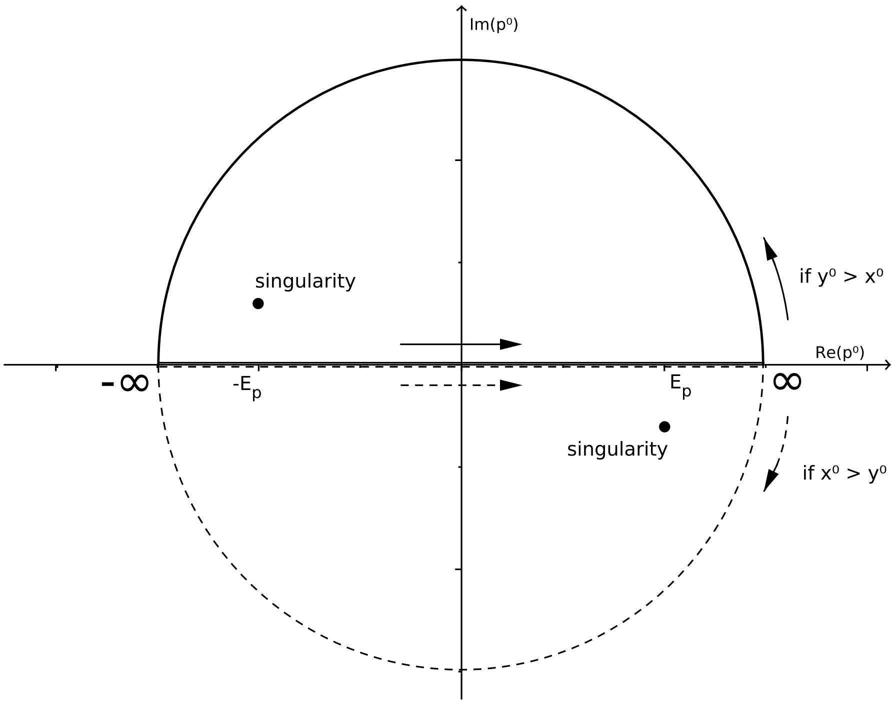


Figure 2.1.: Extending the integration to the complex plane. For $y^0 < x^0$ we have to close the integration path at infinity of the upper half plane, because only there the integrand vanishes for $|p^0| \to \infty$, and vice versa for $x^0 > y^0$.

theorem we obtain

$$\Delta_F(x-y) = \begin{cases} -\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \left(-\frac{\exp\left\{-iE_{\boldsymbol{p}}(x^0-y^0)\right\}}{2E_{\boldsymbol{p}}}\right) & \text{if } \; x^0 > y^0 \\ -\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} e^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \left(\frac{\exp\left\{iE_{\boldsymbol{p}}(x^0-y^0)\right\}}{-2E_{\boldsymbol{p}}}\right) & \text{if } \; y^0 > x^0 \end{cases} \tag{2.10}$$

which is exactly the same as eq. (2.8) - q.e.d.. Note that in the first line, we paid attention to the clockwise integration by adding a minus sign. In the second line we made use of the fact that the

integration of $\boldsymbol{p}$ goes over the whole $\mathbb{R}^3$ space and we can substitute $\boldsymbol{p}$ by $-\boldsymbol{p}$ under the integral.
In momentum space the **Feynman propagator for scalar fields** reads

$$\tilde{\Delta}_F(p) = \int \mathrm{d}^4x\; e^{ip(x-y)}\Delta_F(x-y) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{2.11}$$

In complete analogy the **Feynman propagator of fermions** in momentum space can be calculated as

$$\tilde{S}_F(p) = \int \mathrm{d}^4x\; e^{ip(x-y)}\langle 0|T\Big\{\Psi_A(x)\bar{\Psi}_b(y)\Big\}|0\rangle = \frac{i\left(\gamma^\mu p_\mu + m\right)}{p^2 - m^2 + i\epsilon}. \tag{2.12}$$

Having these results, it is easy to go one step further and evaluate ***n*-point** correlation functions of the free theory by using Wick's theorem.

2.1.2. Wick's Theorem and n-Point Correlation Functions of the Free Theory

Wick's Theorem tells us that a time-ordered product of a set of operators can be decomposed into the sum of all corresponding contracted normal-ordered products. Thereby all possible contractions of operators occur. **Wick's Theorem** reads as follows:

$$\begin{aligned} T\Big\{\phi(x_1)\phi(x_2)...\phi(x_n)\Big\} = \;& :\phi(x_1)\phi(x_2)...\phi(x_n): \; + \\ & + \text{ all combinations of normal-ordered elements and contractions.} \end{aligned} \tag{2.13}$$

Here, **contraction** means the replacement of two fields with the corresponding Feynman propagator, e.g. for scalar fields $\Phi(x_i)\Phi(x_k) \to \Delta_F(x_i - x_k)$. More precisely, a **contraction** of two fields $\phi(x_i)$ and $\phi(x_j)$ indicated by a (over- or under-)line joining them is defined by

$$\overbrace{\phi(x_i)\phi(x_j)} := \begin{cases} [\phi^+(x_i), \phi^-(x_j)] & \text{for } x^0 > y^0 \\ [\phi^+(x_j), \phi^-(x_i)] & \text{for } y^0 > x^0 \end{cases} \tag{2.14}$$

which corresponds exactly to the propagator of the field Φ. Hence Wick's theorem is nothing else than the generalization[2] of the results we obtained in the last section for the case of two scalar fields by using the splitting $\Phi = \Phi^+ + \Phi^-$. It is easy to compute (for example) the four-point correlation

[2] A proof of this theorem can e.g. be found in [21].

function for scalar fields. Using Wick's theorem,

$$\begin{aligned} T\Big\{\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\Big\} = \; & :\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4): \; + \\ & + :\Phi(x_1)\Phi(x_2):\Delta_F(x_3-x_4)+ :\Phi(x_1)\Phi(x_3):\Delta_F(x_2-x_4)+ \\ & + :\Phi(x_1)\Phi(x_4):\Delta_F(x_2-x_3)+ :\Phi(x_2)\Phi(x_3):\Delta_F(x_1-x_4)+ \\ & + :\Phi(x_2)\Phi(x_4):\Delta_F(x_1-x_3)+ :\Phi(x_3)\Phi(x_4):\Delta_F(x_1-x_2)+ \\ & + \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+ \\ & + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3), \end{aligned} \tag{2.15}$$

and the fact that $\langle 0| : ... : |0\rangle = 0$, the four-point correlation function for free scalar fields reads

$$\begin{aligned} \langle 0|T\Big\{\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\Big\}|0\rangle = \; & \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+ \\ & + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3). \end{aligned} \tag{2.16}$$

Consequently, in the most general sense Wick's theorem allows us to rewrite any n-point correlation function for free fields in terms of a sum of products of Feynman propagators. On that account it is possible to find a diagrammatic representation, called Feynman Diagrams.

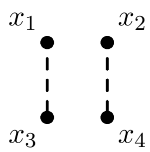
2.1.3. Feynman Diagrams for Free Fields

Considering eq. (2.16) we realize that $\langle 0|T\{\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\}|0\rangle$ represents a sum of different propagations of two Φ particles:

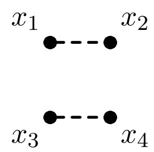
$$\begin{aligned} \langle 0|T\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\,|0\rangle = \; & \text{“}x_1 \longleftrightarrow x_2 \text{ and } x_3 \longleftrightarrow x_4\text{”}+ \\ & +\text{“}x_1 \longleftrightarrow x_3 \text{ and } x_2 \longleftrightarrow x_4\text{”}+ \\ & +\text{“}x_1 \longleftrightarrow x_4 \text{ and } x_2 \longleftrightarrow x_3\text{”}. \end{aligned} \tag{2.17}$$

This means that there are three possibilities for particles to be created at two spacetime points, each propagating to one of the other points, and then they are annihilated. If we now represent each point x_i with a dot indicated with x_i and each propagation respectively each factor $\Delta_F(x_i - x_j)$, by a dashed line joining x_i and x_j, we can express this in terms of diagrams. The VEV (2.17) of the four Φ fields is therefore a sum of three so-called **Feynman diagrams**:

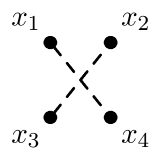
$\langle 0|T\{\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\}|0\rangle =$



$+$



$+$



Each (external) space-time point is denoted by a dot and each propagation between two space-time points by a line joining them. This line has different shapes depending on the type of particle which propagates. The most common ones are dashed lines for scalar particles, solid lines for fermions, wiggly lines for photons or bosons, and curly lines for gluons.

2.1.4. n-Point Correlation Function of Interaction Theory and Feynman Rules

Things get more interesting when we consider interactions. This means that the VEV contains more than one field at the same space-time point. The n-point correlation function is then given by eq. (2.3):

$$\boxed{\langle\Omega|T\Big\{\phi(x_1)\phi(x_2)...\phi(x_n)\Big\}|\Omega\rangle = \frac{\langle 0|T\Big\{\phi_I(x_1)\phi_I(x_2)...\phi_I(x_n)\exp\Big\{-i\int_{-T}^{T}\mathrm{d}t\ H_{\mathrm{int}}(t)\Big\}\Big\}|0\rangle}{\langle 0|\exp\Big\{-i\int_{-T}^{T}\mathrm{d}t\ H_{\mathrm{int}}(t)\Big\}|0\rangle}}$$

In order to evaluate this kind of calculations we follow ref. [10] and start with the evaluation of the two-point correlation function

$$\langle\Omega|T\big\{\Phi(x)\Phi(y)\big\}|\Omega\rangle = \frac{\langle 0|T\Big\{\Phi_I(x)\Phi_I(y)\exp\Big\{-i\int_{-T}^{T}\mathrm{d}t\ H_{\mathrm{int}}(t)\Big\}\Big\}|0\rangle}{\langle 0|\exp\Big\{-i\int_{-T}^{T}\mathrm{d}t\ H_{\mathrm{int}}(t)\Big\}|0\rangle} \tag{2.18}$$

of the Φ^4 theory, where the interaction part of the Hamiltonian reads

$$H_{\mathrm{int}} = \int \mathrm{d}^3z\ \frac{\lambda}{4!}\Phi^4. \tag{2.19}$$

Here we used eq. (2.3) to find the expression containing only free fields and states of the interaction-picture. From here on we will omit the subscript $_I$ since we work solely with these (free) interaction-picture fields. In particular, contractions contain always these fields.
Initially we will ignore the denominator. The numerator can be expanded as a power series,

$$\langle 0|T\Big\{\Phi(x)\Phi(y) + \Phi(x)\Phi(y)\left[-i\int \mathrm{d}t\ H_{\mathrm{int}}(t)\right] + ...\Big\}|0\rangle\,. \tag{2.20}$$

The first term is the free-field result as given in eq. (2.8). The second term (in Φ^4 theory) is equal to

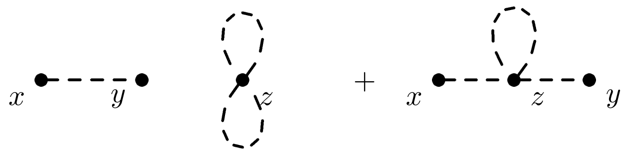
$$\begin{aligned}&\langle 0|T\Big\{\Phi(x)\Phi(y)\left[-i\int \mathrm{d}t\ \int \mathrm{d}^3z\ \frac{\lambda}{4!}\Phi^4(z)\right]\Big\}|0\rangle = \\ &= \langle 0|T\Big\{\Phi(x)\Phi(y)\left(\frac{-i\lambda}{4!}\right)\int \mathrm{d}^4z\ \Phi(z)\Phi(z)\Phi(z)\Phi(z)\Big\}|0\rangle\,.\end{aligned} \tag{2.21}$$

Making use of Wick's theorem and considering that all terms which are not fully contracted (meaning terms which are proportional to any normal-ordered product of field) vanish between $\langle 0|$ and $|0\rangle$, we can rewrite this as

$$\begin{aligned}\langle 0|T\Big\{\Phi(x)\Phi(y)&\left[-i\int \mathrm{d}t\ \int \mathrm{d}^3z\ \frac{\lambda}{4!}\Phi^4(z)\right]\Big\}|0\rangle = \\ &= 3\left(\frac{-i\lambda}{4!}\right)\Delta_F(x-y)\int \mathrm{d}^4z\ \Delta_F(z-z)\Delta_F(z-z) + \\ &+ 12\left(\frac{-i\lambda}{4!}\right)\int \mathrm{d}^4z\ \Delta_F(x-z)\Delta_F(y-z)\Delta_F(z-z).\end{aligned} \tag{2.22}$$

The factors 3 and 12 in front of the terms are of combinatorial nature. They arise because there are three ways to contract the $\Phi(z)$ fields among each other and for the second term four possibilities to contract the $\Phi(x)$ field with one $\Phi(z)$ times three possibilities for the contraction of $\Phi(y)$ and one $\Phi(z)$. The above result again can be expressed by Feynman diagrams. Again, each space-time point is represented by a dot and each Feynman propagator Δ_F by a dashed line. Ignoring all factors and the integration $\int \mathrm{d}^4 z$ we obtain:

$$\langle 0|T\Big\{\Phi(x)\Phi(y)\left[-i\int \mathrm{d}t \int \mathrm{d}^3 z\, \frac{\lambda}{4!}\Phi^4(z)\right]\Big\}|0\rangle =$$



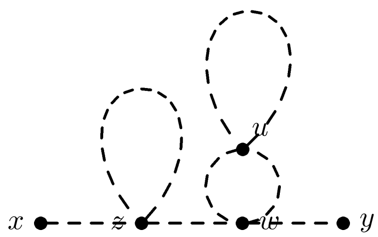
In these diagrams we now have to distinguish between external points x and y and an internal point z. Each internal point is associated with a factor and integration of $(-i\lambda)\int \mathrm{d}^4 z$ (see eq. (2.22)). We will keep this fact in mind until we discuss the so-called Feynman rules. Internal points where four lines meet are called **vertices**.
Of course it is possible to proceed to higher orders in λ. For instance the λ^3 term of the expansion of the correlation function with three internal points z, w, and u reads

$$\langle 0|T\Big\{\Phi(x)\Phi(y)\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int \mathrm{d}^4 z\ \Phi^4(z)\int \mathrm{d}^4 w\ \Phi^4(w)\int \mathrm{d}^4 u\ \Phi^4(u)\Big\}|0\rangle\,. \tag{2.23}$$

Using Wick's Theorem, we find again a sum of products of Feynman propagators. One possible fully contracted term is

$$\langle 0|\Phi(x)\Phi(y)\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int \mathrm{d}^4 z\ \Phi\Phi\Phi\Phi \int \mathrm{d}^4 w\ \Phi\Phi\Phi\Phi \int \mathrm{d}^4 u\ \Phi\Phi\Phi\Phi\,|0\rangle =$$
$$= \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int \mathrm{d}^4 z\ \mathrm{d}^4 w\ \mathrm{d}^4 u\ \Delta_F(x-z)\Delta_F(z-z)\Delta_F(z-w)\Delta_F(w-y)\Delta_F^2(w-u)\Delta_F(u-u), \tag{2.24}$$

which can be represented by the following Feynman diagram, sometimes called "cactus" diagram:



Actually this diagram represents not only one contraction of eq. (2.23) but 10,368 contractions. Again this number is a factor of combinatorial nature (as the 3 and the 12 in eq. (2.22)) and occurs because some contractions lead to the same form of diagrams. It arises from

$$\underbrace{3!}_{\text{interchange of vertices}} \times \underbrace{4\cdot 3}_{\text{placement of contraction into } z \text{ vertex}} \times \underbrace{4\cdot 3\cdot 2}_{\text{placement of contraction into } w \text{ vertex}} \times$$
$$\times \underbrace{4\cdot 3}_{\text{placement of contraction into } u \text{ vertex}} \times \underbrace{1/2}_{\text{interchange of } w\text{-}u \text{ contractions}} .$$

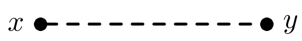
Finally, the calculation of the numerator of eq. (2.20) yields

$$\langle 0|T\Big\{\Phi(x)\Phi(y)\exp\Big\{-i\int_{-T}^{T}\mathrm{d}t\ H_{\text{int}}(t)\Big\}\Big\}|0\rangle = \begin{pmatrix}\text{sum of all possible Feynman diagrams}\\ \text{with two external points}\end{pmatrix}, \quad (2.25)$$

where each diagram is built from propagators, vertices, and external points. This corresponds in a way to the superposition principle of quantum mechanics: We add up all possible ways the propagation can take place. It can propagate directly $\hat{=}\langle 0|T\{\Phi(x)\Phi(y)\}|0\rangle$ or it can emit and absorb in the way some particles at any arbitrary ($\int \mathrm{d}^4z \dots$) vertex z, where $(-ig)$ then corresponds to the amplitude for the emission, respectively absorption, of a particle at a vertex.
In practice one starts not with the calculation of an n-point correlation function, but rather with drawing all possible diagrams (or most likely all diagrams up to the order in the coupling constant one is interested in) first. These diagrams are then mnemonic devices for writing down the corresponding analytic expressions. In order to do this in Φ^4 theory one makes use of the following Feynman rules.

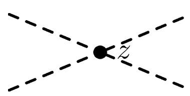
Position-Space Feynman Rules of Φ^4 Theory:

1. For each line between x and y insert a Feynman propagator.



$\hat{=}\ \Delta_F(x-y)$

2. Integrate over each vertex z.



$\hat{=}\ (-i\lambda)\int \mathrm{d}^4z,$

3. For each external line insert a factor 1.

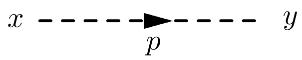
4. Multiply by the symmetry factor ($\hat{=}$ number of contractions which lead to the same diagram).

These are the space-time Feynman rules since they are written in terms of space-time points. Often it is useful to work in the momentum space. Therefore, in the Feynman diagrams we assign to each line a momentum by drawing arrows instead of lines. The direction of the arrow[3], respectively the momentum, is arbitrary for scalar fields since $\Delta_F(x-y) = \Delta(y-x)$. The momentum-space Feynman rules have then the following form:

[3] For fermions the arrows show the direction of charge flow.

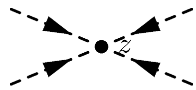
Momentum-Space Feynman Rules of Φ^4 Theory:

1. For each line, respectively arrow, insert the corresponding the Feynman propagator in momentum space.



$\hat{=}\ \tilde{\Delta}_F(p)$

2. For each vertex z



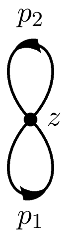
$\hat{=}\ (-i\lambda)$,

3. For each external point

$\hat{=}\ e^{-ipx}$,

4. Impose momentum conservation at each vertex: $\delta(\sum p_{\text{in}} - \sum p_{\text{out}})$
5. Integrate over each undetermined momentum: $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
6. Multiply by the symmetry factor ($\hat{=}$ number of contractions which lead to the same diagram).

It should now be clear now how to evaluate also the denominator of eq. (2.18), and our discussion about the two-point correlation function of Φ^4 theory is completed. In the same manner, it is possible to evaluate any n-point correlation function. However, the explicit computation of some diagrams or pieces of diagrams will cause troubles. Namely the pieces which are not connected to any external point yield infinities. For example take the following diagram evaluated with the Feynman rules in momentum and position space:



$\sim \delta^{(4)}(0)$ in momentum space
$\sim \int \mathrm{d}^4 z\ (\text{const}) \sim 2T \cdot V$ in position space

which yields infinity, because in momentum space it is proportional to the delta distribution of zero, $\delta^{(4)}(0)$, and in position space such a diagram will result in an integral of a constant over $\mathrm{d}^4 z$. Such diagrams are called "disconnected" diagrams or vacuum bubbles. In the next section we will have a closer look at their contribution to the n-point correlation function.

2.1.5. The Exponentiation of Disconnected Diagrams/Vacuum Bubbles

In this section we will consider the exponentiation of the disconnected diagrams to understand the contribution of vacuum bubbles to the n-point correlation function. We follow the explanation given in ref. [10]. We introduce a set of all various possible disconnected pieces and label the elements V_i.

Then an arbitrary Feynman diagram has (additionally to the connected diagram) n_i pieces of the form V_i for each i. If the *value* of the piece V_i is also denoted by V_i, the value of the diagram is

$$\text{(value of an arbitrary diagram)} = \text{(value of connected diagram)} \cdot \prod_i \frac{1}{n_i!}(V_i)^{n_i} \tag{2.26}$$

where $1/n_i!$ is coming from the interchange of the n_i copies of V_i. In the last section we found that the numerator of the two-point (the same holds for the n-point) correlation function is given by the sum of all possible diagrams with two (n) external points. Hence, factoring out the connected pieces, the value of the numerator reads

$$\text{(value of numerator)} = \left(\sum \text{connected}\right) \cdot \sum_{\text{all } \{n_i\}} \left(\prod_i \frac{1}{n_i!}(V_i)^{n_i}\right). \tag{2.27}$$

Here ($\sum$ connected) represents the sum of values of all connected pieces of diagrams and the sum of "all $\{n_i\}$" means that we have to sum over all the sets $\{n_1, n_2, ...\}$ for each diagram. This expression can be further rearranged to

$$\begin{aligned} \ldots &= \left(\sum \text{connected}\right) \cdot \prod_i \left(\sum_{n_i} \frac{1}{n_i!}(V_i)^{n_i}\right) = \\ &= \left(\sum \text{connected}\right) \cdot \prod_i \exp\left(V_i\right) = \\ &= \left(\sum \text{connected}\right) \cdot \exp\left(\sum_i V_i\right) \end{aligned} \tag{2.28}$$

This means that the value of the numerator of the n-point correlation function can be written as a product of the sum of all values of connected pieces times the exponential of the sum of all values of the disconnected pieces. This is the so-called "exponentiation of the disconnected diagrams" and the interesting point is getting clear when we take also the denominator into account: With the same arguments one can express the value of the denominator as

$$\text{(value of denominator)} = \exp\left(\sum_i V_i\right), \tag{2.29}$$

which cancels exactly with the exponential in the numerator. Therefore as a final result we conclude that the n-point correlation function is given by

$$\langle\Omega|T\Big\{\phi(x_1)\phi(x_2)...\phi(x_n)\Big\}|\Omega\rangle = \left(\begin{array}{c}\text{sum of all connected diagrams} \\ \text{with } n \text{ external points}\end{array}\right). \tag{2.30}$$

All disconnected pieces cancelled out and we got rid of the upper mentioned infinities. Note that with "disconnected" we denoted pieces of diagrams which are "disconnected to ALL external points", so-called vacuum bubbles.

2.2. S-Matrix and Decay Width in Dependence of the Invariant Matrix Element $\mathcal{M}$

In this section we will determine an expression to calculate the decay width of any particle decaying into an arbitrary number of particles. To this end, we require the overlap between the initial and final states, which can be expressed by use of the scattering or S-matrix,

$$ {}_{\text{out}}\langle \phi_B | \phi_A \rangle_{\text{in}} \equiv \langle \phi_B | S | \phi_A \rangle , \tag{2.31} $$

for an arbitrary initial state ϕ_A and final state ϕ_B. If the considered particles do not interact the S-matrix is simply the identity operator. If the theory contains interactions, then there also will be a chance that the particles do not interact. Therefore we can define the T matrix which isolates the interesting (interaction) part of the S-matrix[4]

$$ S = \mathbb{1} + iT. \tag{2.32} $$

Furthermore the S- or T.matrix should reflect the 4-momentum conservation by containing a factor $\delta^{(4)}(\sum p_{\text{initial}} - \sum p_{\text{final}})$. Extracting this factor we define the **invariant matrix element** $\mathcal{M}$ by

$$ \langle \phi_B | iT | \phi_A \rangle = (2\pi)^4 \delta^{(4)} \left(\sum p_i - \sum p_{\text{f}} \right) \cdot i\mathcal{M}(\{p_{\text{initial}}\} \to \{p_{\text{final}}\}). \tag{2.33} $$

Now we are ready to calculate the decay width of any particle A decaying into n particles $B_1, B_2, ..., B_n$. We assume that the particle A is at rest ($\boldsymbol{k}_A = 0$) and the momenta of the B-particles is given by $\boldsymbol{p}_f$ for B_f and $f \in \{1, 2, ..., n\}$. The probability of such a decay after a long time ($t_f - t_i \to \infty$) can be expressed with the T-matrix. We take the sum of the square of the absolute value of the transition amplitude over all final momenta of B-particles:

$$ \begin{aligned} P(A \to B_1 ... B_n) &= \left(\prod_f \sum_{\boldsymbol{p}_f} \right) \Big| {}_1\langle B(\boldsymbol{p}_1) ... B(\boldsymbol{p}_n) | iT | A(\boldsymbol{k}_A = 0) \rangle_1 \Big|^2 = \\ &= \left(\prod_f \frac{V}{(2\pi)^3} \int \mathrm{d}^3 \boldsymbol{p}_f \right) \Big| {}_1\langle B(\boldsymbol{p}_1) ... B(\boldsymbol{p}_n) | iT | A(\boldsymbol{k}_A = 0) \rangle_1 \Big|^2 \end{aligned} \tag{2.34} $$

The subscript $_1$ at the state indicates that it is very important to use normalised states to obtain the right probability. Furthermore, in the second line we replaced the sum with an integral in the following way

$$ \sum_{\boldsymbol{p}_f} = \frac{1}{\Delta p^3} \sum_{\boldsymbol{p}_f} \Delta p^3 \longrightarrow \frac{V}{(2\pi)^3} \int \mathrm{d}^3 \boldsymbol{p}_f \tag{2.35} $$

[4]One can get this result also by evaluating the so-called Born series.

with $\frac{(2\pi)^3}{V}$ being the smallest possible Δp^3 in a discrete box with volume V. If we use the general solutions of the Klein-Gordon and Dirac equation with the following choice of conventions,

$$\Phi(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ipx} + a_{\boldsymbol{p}}^\dagger e^{ipx}\right)$$
$$\Psi(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left(a_{\boldsymbol{p}}^s u^s(\boldsymbol{p}) e^{-ipx} + b_{\boldsymbol{p}}^{s\dagger} v^s(\boldsymbol{p}) e^{ipx}\right), \tag{2.36}$$

the states are not normalized:

for scalars:

$$\langle \boldsymbol{p}|\boldsymbol{k}\rangle = \langle 0|a_{\boldsymbol{p}} a_{\boldsymbol{k}}^\dagger |0\rangle = \langle 0| \left[a_{\boldsymbol{p}}, a_{\boldsymbol{k}}^\dagger\right] - a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{p}} |0\rangle = 2E_{\boldsymbol{p}} (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{k}) \tag{2.37}$$

or for fermions:

$$\langle \boldsymbol{p}|\boldsymbol{k}\rangle = \langle 0|a_{\boldsymbol{p}} b_{\boldsymbol{k}}^\dagger |0\rangle = \langle 0| \left\{a_{\boldsymbol{p}}, b_{\boldsymbol{k}}^\dagger\right\} - b_{\boldsymbol{k}}^\dagger a_{\boldsymbol{p}} |0\rangle = 2E_{\boldsymbol{p}} (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{k}) \tag{2.38}$$

where we have used the commutator (2.7) for scalar fields and the anti-commutator (1.8) for fermions. Since actually we want them to be normalized, we make a correction by multiplying the factor

$$\left[(2\pi)^3 2E_{\boldsymbol{p}} \delta(0)\right]^{-\frac{1}{2}} = \left[(2\pi)^3 2E_{\boldsymbol{p}} \frac{1}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{r}\ e^{i(\boldsymbol{p}=0)\cdot\boldsymbol{r}}\right]^{-\frac{1}{2}} = \frac{1}{\sqrt{2E_{\boldsymbol{p}} V}} \tag{2.39}$$

to each state/particle with momentum $\boldsymbol{p}$. Making use of this, the probability assumes the following form

$$P(A \to B_1 ... B_n) = \left(\prod_f \frac{V}{(2\pi)^3} \int \frac{\mathrm{d}^3\boldsymbol{p}_f}{2E_{\boldsymbol{p}_f} V}\right) \frac{1}{2m_A V} \left|\langle B(\boldsymbol{p}_1)...B(\boldsymbol{p}_n)| iT |A(\boldsymbol{k}_A = 0)\rangle\right|^2. \tag{2.40}$$

where $E_{\boldsymbol{k}_A} = m_A$ since A is at rest. With eq. (2.33) and using the so called Fermi trick to calculate the square of the delta distribution,

$$\begin{aligned}(2\pi)^8 \left[\delta^{(4)}\left(k_A - \sum p_f\right)\right]^2 &= (2\pi)^4 \delta^{(4)}\left(k_A - \sum p_f\right) \int \mathrm{d}^4x\ e^{i(k_A - \sum p_f)x} = \\ &= (2\pi)^4 \delta^{(4)}\left(k_A - \sum p_f\right) \int \mathrm{d}^3\boldsymbol{r} \int \mathrm{d}t = (2\pi)^4 \delta^{(4)}(k_A - \sum p_f) V t\end{aligned} \tag{2.41}$$

the probability reads

$$P(A \to B_1 ... B_n) = \left(\prod_f \frac{V}{(2\pi)^3} \int \frac{\mathrm{d}^3\boldsymbol{p}_f}{2E_{\boldsymbol{p}_f} V}\right) \frac{1}{2m_A V} (2\pi)^4 \delta^{(4)}\left(k_A - \sum p_f\right) V t \left|i\mathcal{M}\right|^2. \tag{2.42}$$

Since the decay width or decay rate is the probability per time, $\Gamma = \frac{P}{t}$ we obtain

$$\Gamma = \frac{1}{2m_A}\left(\prod_f \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\boldsymbol{p}_f}{2E_{\boldsymbol{p}_f}}\right)|i\mathcal{M}|^2\,(2\pi)^4\delta^{(4)}(k_A - \sum p_f). \tag{2.43}$$

Now we have a formula for the decay width in terms of the invariant matrix element $\mathcal{M}$. In the next section we will show a way of computing $\mathcal{M}$ for arbitrary processes in arbitrary interacting theories.

2.3. LSZ Reduction Formula and Computing S-Matrix Elements from Feynman Diagrams

Since we know how to compute n-point correlation functions, a general relation between S-matrix element and n-point correlation functions would be interesting. Exactly such a formula was first obtained by Lehmann, Symanzik, and Zimmermann [22] and is known as **LSZ reduction formula**. For the Φ^4-theory it reads

$$\begin{aligned}\prod_{i=1}^{n}\int \mathrm{d}^4x_i\; e^{ip_ix_i}\prod_{j=1}^{m}\int \mathrm{d}^4y_j e^{ik_jy_j}\langle\Omega|T\Big\{\Phi(x_1)\ldots\Phi(x_n)\Phi(y_1)\ldots\Phi(y_m)\Big\}|\Omega\rangle \\ \sim \left(\prod_{i=1}^{n}\frac{\sqrt{Z}i}{p_i^2 - m^2 + i\epsilon}\right)\left(\prod_{j=1}^{n}\frac{\sqrt{Z}i}{k_j^2 - m^2 + i\epsilon}\right)\langle\boldsymbol{p}_1\ldots\boldsymbol{p}_n|S\,|\boldsymbol{k}_1\ldots\boldsymbol{k}_m\rangle\end{aligned} \tag{2.44}$$

Here $|\boldsymbol{p}_1\ldots\boldsymbol{p}_n\rangle$ and $|\boldsymbol{k}_1\ldots\boldsymbol{k}_m\rangle$ are n-/m-particle states with definite momenta and for all $i \in \{1, 2, \ldots n\}$ and $j \in \{1, 2\ldots m\}$ one has $p_i \neq k_j$; meaning no particle is not interacting. We want to relate the transition amplitude $\langle\boldsymbol{p}_1\ldots\boldsymbol{p}_n|S\,|\boldsymbol{k}_1\ldots\boldsymbol{k}_m\rangle$ to the $(n{+}m)$-point correlation function, $\langle\Omega|T\{\Phi(x_1)\ldots\Phi(x_n)\Phi(y_1)$ $\ldots\;\Phi(y_m)\}|\Omega\rangle$. The factor Z is a re-normalisation factor (wave-function renormalisation) which is proportional to 1 for low orders in the coupling constant λ: $Z = 1 + \mathcal{O}(\lambda^2)$. Furthermore, the "$\sim$" instead of an equivalent sign means that the left- and right-hand side are only identical in the vicinity of the multi-poles $p_i^0 \to E_{\boldsymbol{p}_i}$ and $k_j^0 \to E_{\boldsymbol{k}_j}$. Away from the poles they differ by weakly divergent terms or finite terms.

Hence, in order to calculate the S- or T-matrix element we have to compute the Fourier transform of the $(n+m)$-point correlation function in the vicinity of the multi-poles and read off the corresponding coefficients.

A more handy way to calculate S-/T-matrix elements, or directly the invariant matrix element $\mathcal{M}$ respectively, is by using Feynman diagrams. In order find such a relation and corresponding Feynman rules we have to take a closer look at

$$\langle\boldsymbol{p}_1\ldots.\boldsymbol{p}_n|iT\,|\boldsymbol{k}_1\ldots\boldsymbol{k}_m\rangle = \left({}_0\langle\boldsymbol{p}_1\ldots.\boldsymbol{p}_n|T\Big\{\exp\Big[-i\int \mathrm{d}t\;H_I(t)\Big]\Big\}|\boldsymbol{k}_1\ldots\boldsymbol{k}_m\rangle_0\right)_{\text{connected and amputated}}, \tag{2.45}$$

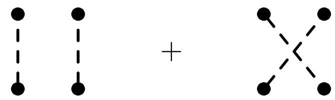
where the T on the left-hand side is the T-matrix element and must not be confused with the T on the right-hand side, which is the time-ordering operator! The states indexed with $_0$ are the free states of the unperturbed theory, which are eigenstates of the free Hamiltonian. The validity of this formula can be shown by the use of the LSZ reduction formula (for more details see e.g. ref. [10]). The phrase **fully connected** means that all external legs of the diagram are connected to each other and that the diagram contains no vacuum bubbles. Furthermore, we define a diagram to be **amputated** if all interactions of external legs, separated from the rest of the diagram only by a single propagator, are cut off. Evaluating this formula order by order yields

$$i\mathcal{M} \cdot (2\pi)^4 \delta^{(4)} \left(\sum p_i - \sum p_f\right) = \begin{pmatrix} \text{sum of all fully connected, amputated Feynman} \\ \text{diagrams with } p_i \text{ incoming and } p_f \text{ outgoing} \end{pmatrix} \tag{2.46}$$

with a kind of "new" Feynman rules listed later. The usage of Feynman diagrams is now a bit different from the above introduced diagrams. For two particles in the final and the initial state the 0th order contribution of eq. (2.45) reads

$$\begin{aligned} {}_0\langle \boldsymbol{p}_1 \boldsymbol{p}_2 | \boldsymbol{k}_1 \boldsymbol{k}_2 \rangle_0 &= \sqrt{2E_{\boldsymbol{p}_1} 2E_{\boldsymbol{p}_2} 2E_{\boldsymbol{k}_1} 2E_{\boldsymbol{k}_2}} \langle 0 | a_{\boldsymbol{p}_1} a_{\boldsymbol{p}_2} a^\dagger_{\boldsymbol{k}_1} a^\dagger_{\boldsymbol{k}_2} |0\rangle = \\ &= 2E_{\boldsymbol{k}_1} 2E_{\boldsymbol{k}_2} (2\pi)^6 \left[\delta^{(3)}\left(\boldsymbol{k}_1 - \boldsymbol{p}_1\right) \delta^{(3)}\left(\boldsymbol{k}_2 - \boldsymbol{p}_2\right) + \delta^{(3)}\left(\boldsymbol{k}_1 - \boldsymbol{p}_2\right) \delta^{(3)}\left(\boldsymbol{k}_2 - \boldsymbol{p}_1\right)\right]. \end{aligned} \tag{2.47}$$

The delta functions force the final state to be identical to the initial state and we can represent it diagrammatically as


.

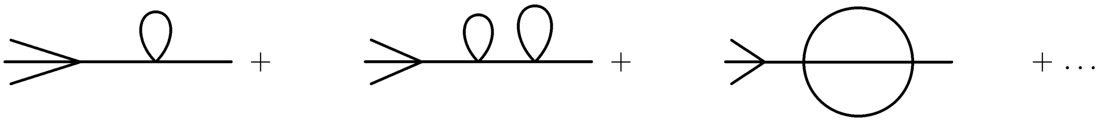
Now, in contrast to the evaluation of the VEV, for higher orders, not fully contracted terms do not necessarily vanish after using Wick's theorem . E.g.:

$$\begin{aligned} \Phi_I^+ |\boldsymbol{p}\rangle_0 &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{k}}}} a_{\boldsymbol{k}} e^{-ikx} \sqrt{2E_{\boldsymbol{p}}} a^\dagger_{\boldsymbol{p}} |0\rangle = \\ &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{k}}}} e^{-ikx} \sqrt{2E_{\boldsymbol{p}}} (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{p}) |0\rangle = e^{-ipx} |0\rangle, \end{aligned} \tag{2.48}$$

and therefore we introduce a contraction between fields and states as follows - which will be diagrammatically the external lines:

$$\overline{\Phi_I(x) |\boldsymbol{p}}\rangle = e^{-ipx} |0\rangle . \tag{2.49}$$

But since $i\mathcal{M} \cdot (2\pi)^4 \delta^{(4)}(\sum p_i - \sum p_f)$ is the part of the S-matrix, which contains only the interactions we are not interested in processes with identical final and initial states containing not more interactions than vacuum bubbles. Hence, to obtain interactions we have to contract all fields of the initial and final states with one field of the Hamiltonian. Therefore, the diagrams we are interested in have to be fully contracted. Moreover, corrections of the external legs by interactions like



represent the evolution of the free $|\boldsymbol{p}\rangle_0$ into the interacting $|\boldsymbol{p}\rangle$, in analogy to the difference between $|\Omega\rangle$ and $|0\rangle$. These corrections have nothing to do with the scattering and can therefore be neglected in the computation of $\mathcal{M}$. Cutting these interactions of external legs we obtain amputated diagrams as defined above. Finally, we have to represent the Feynman rules to compute invariant matrix elements $\mathcal{M}$ with eq. (2.46). In the following we will give the rules for fermions and scalars in a Yukawa Theory,

$$H = H_{\text{Dirac}} + H_{\text{Klein-Gordon}} + g \int \mathrm{d}^3\boldsymbol{r}\ \bar{\Psi}\Psi\Phi \tag{2.50}$$

in momentum space as they are also given in ref. [10].

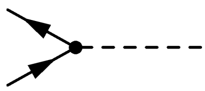
Momentum-Space Feynman Rules for Computing $i\mathcal{M}$ with eq. (2.46) for Fermions and Bosons in Yukawa Theory:

1. Propagators:

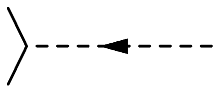
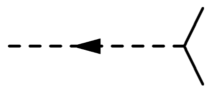
$\overbracket{\Phi(x)\Phi(y)} =$  $= \tilde{\Delta}_F(q)$

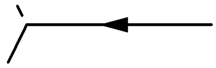
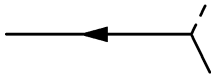
$\overbracket{\Psi(x)\bar{\Psi}(y)} =$  $= \tilde{S}_F(p)$

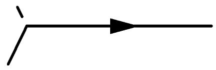
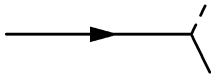
2. Vertices:

 $= -ig,$

3. External leg contractions:

$\overbracket{\Phi\ |\boldsymbol{q}}\rangle =$  $= 1,$ $\quad \langle\overbracket{\boldsymbol{q}|\ \Phi} =$  $= 1$

$\overbracket{\Psi\ |\boldsymbol{p}, s}\rangle =$  $= u^s(\boldsymbol{p}),$ $\quad \langle\overbracket{\boldsymbol{p}, s|\ \bar{\Psi}} =$  $= \bar{u}^s(\boldsymbol{p})$

$\overbracket{\bar{\Psi}\ |\boldsymbol{k}, s}\rangle =$  $= \bar{v}^s(\boldsymbol{k}),$ $\quad \langle\overbracket{\boldsymbol{k}, s|\ \Psi} =$  $= v^s(\boldsymbol{k})$

where $|\boldsymbol{p}, s\rangle$ denotes a fermion and $|\boldsymbol{k}, s\rangle$ denotes an antifermion. Note that in the diagrams with antifermions the vector points in the opposite direction than $\boldsymbol{k}$. For both, fermions and boson, hold: The direction of momentum is always ingoing for initial-state particles and outgoing for final-state particles.

4. Impose momentum conservation at each vertex.

5. Integrate over each undetermined loop momentum.

6. Figure out the overall sign of the diagram. (It can be demonstrated that the sign of a diagram involving fermions is equal to $(-1)^n$, where n is the number of times that fermion contraction lines intersect.)

Symmetry factors never occur in diagrams of Yukawa theory, since the interaction part of the Hamiltonian contains only three *different* fields which cannot substituted for one another in contractions and the $1/n!$ factor of the Taylor series always cancels with the factor $n!$ from interchanging vertices to obtain the same diagrams.

3. Mesonic Component of the Lagrangian for $N_f = 3$ Flavors

The aim of this work is to study baryons. However, since spontaneous symmetry breaking in the meson sector affects also the baryon part, we will first construct a three-flavor linear sigma model with vector and axial-vector mesons as it has been done in [2]. This effective model based on QCD should of course contain all properties of the QCD Lagrangian:

- Exact $SU(3)_c$ color symmetry.
- Exact $U(N_f)_R \times U(N_f)_L$ chiral symmetry in the limit of massless quarks.
- Spontaneous breaking of chiral symmetry.
- Chiral $U(1)_A$ anomaly.
- Explicit breaking of chiral symmetry.
- Poincaré and CP symmetry.

The meson Lagrangian which we will introduce is basically the same as in ref. [2] with the difference that the dilaton field is neglected here. In fact, the mesonic Lagrangian $\mathcal{L}_{\text{meson}}$ (3.12) we will construct, is actually valid for an arbitrary number of colors N_c and flavors N_f.

3.1. Construction of a Chiral Lagrangian for Mesons in the eLSM

Since we construct an *effective* model, our degrees of freedom are hadrons which are color neutral because of the confinement hypothesis. This means that the $SU(3)_c$ color symmetry will be implemented per construction. Furthermore, we have to fulfil chiral symmetry. To this end, we define a meson matrix Φ which has the quantum numbers and the transformation properties of an appropriately chosen quark-antiquark correlator:

$$\boxed{\Phi_{ij} \equiv \sqrt{2}\bar{q}_{j,R}q_{i,L} = \frac{1}{\sqrt{2}}\left(\bar{q}_j q_i + i\bar{q}_j i\gamma^5 q_i\right) \equiv S_{ij} + iP_{ij}, \qquad (3.1)}$$

In the last step we used the chiral projection operators (1.25) and recognised $S_{ij} \equiv \frac{1}{\sqrt{2}}\bar{q}_j q_i$ to be the scalar quark-antiquark current and $P_{ij} \equiv \frac{1}{\sqrt{2}}\bar{q}_j i\gamma^5 q_i$ the pseudoscalar quark-antiquark current[1].

[1]Namely, $S\left|0\right\rangle$ represents a scalar meson and $P\left|0\right\rangle$ a pseudoscalar meson.

With the chiral transformation behaviour of the left- and right-handed quarks, eq. (1.26), we find that the meson matrix transforms under chiral transformations as

$$\Phi_{ij} \longrightarrow \sqrt{2}\bar{q}_{k,R} U^{\dagger}_{kj,R} U_{il,L} q_{l,L} = U_{il,L} \Phi_{lk} U^{\dagger}_{kj,R}.$$

Ergo:

$$\boxed{\Phi \longrightarrow U_L \Phi U_R^{\dagger}.} \tag{3.2}$$

First we consider the chiral limit $m_i \to 0$ for all $i \in \{u, d, s\}$; i.e., when all quark masses vanish. Using only chirally invariant terms in powers of Φ yields:

$$\mathcal{L}_{\text{sym}} = \operatorname{Tr}\left\{(\partial^{\mu}\Phi)^{\dagger}\partial_{\mu}\Phi\right\} - m_0^2 \operatorname{Tr}\left\{\Phi^{\dagger}\Phi\right\} - \lambda_1 \left[\operatorname{Tr}\left\{\Phi^{\dagger}\Phi\right\}\right]^2 - \lambda_2 \operatorname{Tr}\left\{\Phi^{\dagger}\Phi\right\}^2. \tag{3.3}$$

This Lagrangian contains only scalar and pseudoscalar degrees of freedom and corresponds to the original form of the linear sigma model. Since an effective theory is not valid up to arbitrarily high energy scales, but only up to the energy of the heaviest resonance incorporated into the model, our Lagrangian does not have to preserve renormalisability. Therefore, one might think that higher-order chirally invariant terms as for instance $\alpha[\operatorname{Tr}\{\Phi^{\dagger}\Phi\}]^6$ should be possible. However, we make use of an additional criterion by taking the dilatation invariance into account. After we have included the dilaton field G, only dimensionless coupling constants are allowed. Hence, we see that for instance the coupling constant α of the term $\alpha[\operatorname{Tr}\{\Phi^{\dagger}\Phi\}]^6$ has dimension $[E^{-2}]$. Trying to render it dimensionless by $\frac{\alpha}{G^2}[\operatorname{Tr}\{\Phi^{\dagger}\Phi\}]^6$ will lead to a singularity for $G \to 0$, we consider terms up to the order four in Φ only. For details see e.g. ref. [2, 14].
As a next step we include vector and axial-vector degrees of freedom. To this end we define a right-handed matrix R^{μ} and a left-handed matrix L^{μ} as

$$\boxed{R^{\mu}_{ij} \equiv \sqrt{2}\bar{q}_{j,R}\gamma^{\mu}q_{i,R} = \frac{1}{\sqrt{2}}\left(\bar{q}_j\gamma^{\mu}q_i - \bar{q}_j\gamma^5\gamma^{\mu}q_i\right) \equiv V^{\mu}_{ij} + A^{\mu}_{ij},} \tag{3.4}$$

and

$$\boxed{L^{\mu}_{ij} \equiv \sqrt{2}\bar{q}_{j,L}\gamma^{\mu}q_{i,L} = \frac{1}{\sqrt{2}}\left(\bar{q}_j\gamma^{\mu}q_i + \bar{q}_j\gamma^5\gamma^{\mu}q_i\right) \equiv V^{\mu}_{ij} - A^{\mu}_{ij},} \tag{3.5}$$

with V^{μ} representing the vector and A^{μ} the axial-vector currents, eq. (1.35 and (1.38). With eq. (1.26) the right- and left-handed fields transform under chiral transformations as

$$\boxed{R^{\mu} \longrightarrow U_R R^{\mu} U_R^{\dagger} \qquad \text{and} \qquad L^{\mu} \longrightarrow U_L L^{\mu} U_L^{\dagger}.} \tag{3.6}$$

Finally, we define the right- left-handed field-strength tensor $R^{\mu\nu}$ and $L^{\mu\nu}$ as

$$R^{\mu\nu} = \partial^\mu R^\nu - \partial^\nu R^\mu \qquad \text{and} \qquad L^{\mu\nu} = \partial^\mu L^\nu - \partial^\nu L^\mu. \tag{3.7}$$

Considering the chiral transformations of Φ, R^μ, and L^μ we can construct further terms to include vector and axial-vector degrees of freedom in our Lagrangian:

$$\begin{aligned}
\mathcal{L}_{\text{sym,(ax)vec}} =& -\frac{1}{4}\operatorname{Tr}\{L_{\mu\nu}L^{\mu\nu} + R_{\mu\nu}R^{\mu\nu}\} + \operatorname{Tr}\left\{\frac{m_1^2}{2}\left(L_\mu L^\mu + R_\mu R^\mu\right)\right\} + \\
&+ i\frac{g_2}{2}\left(\operatorname{Tr}\{L_{\mu\nu}[L^\mu, L^\nu]\} + \operatorname{Tr}\{R_{\mu\nu}[R^\mu, R^\nu]\}\right) + \\
&+ \frac{h_1}{2}\operatorname{Tr}\left\{\Phi^\dagger\Phi\right\}\operatorname{Tr}\{L_\mu L^\mu + R_\mu R^\mu\} + \frac{h_2}{2}\operatorname{Tr}\left\{(L_\mu\Phi)^\dagger(L^\mu\Phi) + (R_\mu\Phi)^\dagger(R^\mu\Phi)\right\} + \\
&+ 2h_3\operatorname{Tr}\left\{\Phi R^\mu\Phi^\dagger L^\mu\right\} + g_3\left(\operatorname{Tr}\{L_\mu L_\nu L^\mu L^\nu\} + \operatorname{Tr}\{R_\mu R_\nu R^\mu R^\nu\}\right) + \\
&+ g_4\left(\operatorname{Tr}\{L_\mu L^\mu L_\nu L^\nu\} + \operatorname{Tr}\{R_\mu R^\mu R_\nu R^\nu\}\right) + g_5\operatorname{Tr}\{L_\mu L^\mu\}\operatorname{Tr}\{R_\nu R^\nu\} + \\
&+ g_6\left(\operatorname{Tr}\{L_\mu L^\mu\}\operatorname{Tr}\{L_\nu L^\nu\} + \operatorname{Tr}\{R_\mu R^\mu\}\operatorname{Tr}\{R_\nu R^\nu\}\right).
\end{aligned} \tag{3.8}$$

Another feature of QCD we have to implement is the explicit chiral symmetry breaking. We do this by adding in the (pseudo)scalar sector the term

$$\mathcal{L}_{\text{ESB,scalar}} = \operatorname{Tr}\left\{H\left(\Phi + \Phi^\dagger\right)\right\} \tag{3.9}$$

and in the (axial)vector sector

$$\mathcal{L}_{\text{ESB,vector}} = \operatorname{Tr}\{\Delta\left(L_\mu L^\mu + R_\mu R^\mu\right)\}, \tag{3.10}$$

where $H = \operatorname{diag}(h_0^u, h_0^d, h_0^s, ...) \sim \operatorname{diag}(m_u, m_d, m_s, ...)$ and $\Delta = \operatorname{diag}(\delta_u, \delta_d, \delta_s, ...) \sim \operatorname{diag}(m_u^2, m_d^2, m_s^2, ...)$. In order to describe the chiral anomaly in our model we add

$$\mathcal{L}_{\text{anomaly}} = c\left(\det\Phi - \det\Phi^\dagger\right)^2, \tag{3.11}$$

because the determinant is invariant under $SU(N_f)_L \times SU(N_f)_R$ but not under $U(1)_A$. This can be seen by using $\det(ABC) = \det A \det B \det C$ and the fact that the determinant of a $SU(N)$ matrix is equal to one.
Finally, we have the following mesonic Lagrangian:

$$\begin{aligned}
\mathcal{L}_{\text{meson}} =& \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{sym,(ax)vec}} + \mathcal{L}_{\text{ESB,scalar}} + \mathcal{L}_{\text{ESB,vector}} + \mathcal{L}_{\text{anomaly}} = \\
=& \operatorname{Tr}\left\{(D^\mu\Phi)^\dagger D_\mu\Phi)\right\} - m_0^2 \operatorname{Tr}\left\{\Phi^\dagger\Phi\right\} - \lambda_1\left[\operatorname{Tr}\left\{\Phi^\dagger\Phi\right\}\right]^2 - \lambda_2 \operatorname{Tr}\left\{\Phi^\dagger\Phi\right\}^2 + \\
& - \frac{1}{4}\operatorname{Tr}\left\{L_{\mu\nu}L^{\mu\nu} + R_{\mu\nu}R^{\mu\nu}\right\} + \operatorname{Tr}\left\{\left(\frac{m_1}{2} + \Delta\right)\left(L_\mu L^\mu + R_\mu R^\mu\right)\right\} + \\
& + \operatorname{Tr}\left\{H\left(\Phi + \Phi^\dagger\right)\right\} + c\left(\det\Phi - \det\Phi^\dagger\right)^2 + \\
& + i\frac{g_2}{2}\left(\operatorname{Tr}\left\{L_{\mu\nu}\left[L^\mu, L^\nu\right]\right\} + \operatorname{Tr}\left\{R_{\mu\nu}\left[R^\mu, R^\nu\right]\right\}\right) + \\
& + \frac{h_1}{2}\operatorname{Tr}\left\{\Phi^\dagger\Phi\right\}\operatorname{Tr}\left\{L_\mu L^\mu + R_\mu R^\mu\right\} + \frac{h_2}{2}\operatorname{Tr}\left\{(L_\mu\Phi)^\dagger(L^\mu\Phi) + (R_\mu\Phi)^\dagger(R^\mu\Phi)\right\} + \\
& + 2h_3 \operatorname{Tr}\left\{\Phi R^\mu \Phi^\dagger L^\mu\right\} + g_3\left(\operatorname{Tr}\left\{L_\mu L_\nu L^\mu L^\nu\right\} + \operatorname{Tr}\left\{R_\mu R_\nu R^\mu R^\nu\right\}\right) + \\
& + g_4\left(\operatorname{Tr}\left\{L_\mu L^\mu L_\nu L^\nu\right\} + \operatorname{Tr}\left\{R_\mu R^\mu R_\nu R^\nu\right\}\right) + g_5 \operatorname{Tr}\left\{L_\mu L^\mu\right\}\operatorname{Tr}\left\{R_\nu R^\nu\right\} + \\
& + g_6\left(\operatorname{Tr}\left\{L_\mu L^\mu\right\}\operatorname{Tr}\left\{L_\nu L^\nu\right\} + \operatorname{Tr}\left\{R_\mu R^\mu\right\}\operatorname{Tr}\left\{R_\nu R^\nu\right\}\right)
\end{aligned} \tag{3.12}$$

with

$$D^\mu\Phi = \partial^\mu\Phi - ig_1\left(L^\mu\Phi - \Phi R^\mu\right). \tag{3.13}$$

3.2. Discrete Symmetries of the eLSM

Last but not least we should check if the Lagrangian is invariant under charge conjugation, parity, and time reversal. Quarks being fermions transform as shown in eq. (1.15) and eq. (1.23):

$$\begin{aligned}
q(t,\boldsymbol{x}) &\xrightarrow{P} q^P(t,\boldsymbol{x}) = \gamma^0 q(t,-\boldsymbol{x}) \\
q &\xrightarrow{C} q^C = C\bar{q}^T = i\gamma^2 q^\star \qquad \text{with } C = i\gamma^2\gamma^0.
\end{aligned} \tag{3.14}$$

The transformation behaviour of Φ, R^μ, and L^μ can be computed by using proper features of the gamma matrices γ^μ, eq. (B.1) and (B.4), the charge conjugation matrix C, eq. (1.20) and (1.21), and the chiral projection operators, eq. (1.25). Furthermore, we always have to add a minus sign when we interchange two equal fermions. (For instance when we transpose an object with fermions in it.)

3.2.1. Parity Transformations:

For $\Phi_{ij} = \sqrt{2}\bar{q}_{j,R}q_{i,L} =$[2] $\sqrt{2}q_j^\dagger P_R\gamma^0 P_L q_i$ we get:

$$\Phi_{ij}(t,\boldsymbol{x}) \xrightarrow{P} \sqrt{2}\left[\gamma^0 q_j(t,-\boldsymbol{x})\right]^\dagger P_R\gamma^0 P_L\gamma^0\; q_i(t,-\boldsymbol{x}) = \sqrt{2}q_j^\dagger\gamma^0 P_R\gamma^0 P_L\gamma^0 q_i = \dots\,,$$

[2]Since $\gamma^{5\dagger} = \gamma^5$ and therefore $P^\dagger_{L/R} = P_{L/R}$.

since $(\gamma^0)^\dagger = \gamma^0$. Using $\gamma^{5\dagger} = \gamma^5$ we can identify $q_j^\dagger P_L$ with $q_{j,L}^\dagger$ and get:

$$\cdots = \sqrt{2} q_j^\dagger P_L \gamma^0 P_R q_i = \sqrt{2} q_j^\dagger \gamma^0 P_R q_i = \sqrt{2} \bar{q}_j P_R q_i = \dots ,$$

where we have used $\gamma^0, \gamma^5 = 0$ to exchange the projection operator with γ^0 and then applied that $P_R P_R = P_R$. Writing out the projection operator, eq. (1.25), yields

$$\cdots = \frac{1}{\sqrt{2}} \bar{q}_j q_i - \frac{1}{\sqrt{2}} \bar{q}_j \gamma^5 q_i = \frac{1}{\sqrt{2}} \bar{q}_j q_i - i \frac{1}{\sqrt{2}} \bar{q}_j i \gamma^5 q_i = S_{ij} - i P_{ij} \dots .$$

Since the scalar and pseudoscalar currents are hermitian that is identical to

$$\Phi_{ij}(t, \boldsymbol{x}) \xrightarrow{P} S_{ij}^\dagger - i P_{ij}^\dagger \equiv \Phi_{ij}^\dagger . \tag{3.15}$$

The calculations for the transformations of the fields $R_{ij}^\mu = \sqrt{2} \bar{q}_{j,R} \gamma^\mu q_{i,R} = \sqrt{2} q_{j,R}^\dagger \gamma^0 \gamma^\mu q_{i,R}$ and $L_{ij}^\mu = \sqrt{2} \bar{q}_{j,L} \gamma^\mu q_{i,L} = \sqrt{2} q_{j,L}^\dagger \gamma^0 \gamma^\mu q_{i,L}$ look quite similar. Using the same relations we find:

$$\begin{aligned}
R_{ij}^\mu(t, \boldsymbol{x}) &\xrightarrow{P} \sqrt{2} q_j^\dagger(t, -\boldsymbol{x}) \gamma^0 P_R \gamma^0 \gamma^\mu P_R \gamma^0 q_i(t, -\boldsymbol{x}) = \sqrt{2} q_{j,L}^\dagger(t, -\boldsymbol{x}) \gamma^\mu \gamma^0 q_{i,L}(t, -\boldsymbol{x}) = \\
&= \begin{cases} \sqrt{2} q_{j,L}^\dagger(t, -\boldsymbol{x}) \gamma^0 \gamma^\mu q_{i,L}(t, -\boldsymbol{x}) & \text{for } \mu = 0, \\ -\sqrt{2} q_{j,L}^\dagger(t, -\boldsymbol{x}) \gamma^0 \gamma^\mu q_{i,L}(t, -\boldsymbol{x}) & \text{for } \mu = 1,2,3, \end{cases} \\
&= \left\{ \begin{array}{ll} L_{ij}^\mu(t, -\boldsymbol{x}) & \text{for } \mu = 0 \\ -L_{ij}^\mu(t, -\boldsymbol{x}) & \text{for } \mu = 1,2,3 \end{array} \right\} = g^{\mu\nu} L_{\nu,ij}(t, -\boldsymbol{x}), \\
L_{ij}^\mu(t, \boldsymbol{x}) &\xrightarrow{P} \left\{ \begin{array}{ll} R_{ij}^\mu(t, -\boldsymbol{x}) & \text{for } \mu = 0 \\ -R_{ij}^\mu(t, -\boldsymbol{x}) & \text{for } \mu = 1,2,3 \end{array} \right\} = g^{\mu\nu} R_{\nu,ij}(t, -\boldsymbol{x}).
\end{aligned} \tag{3.16}$$

3.2.2. Charge Conjugations:

For $\Phi_{ij} = \sqrt{2}(P_R q_j)^\dagger \gamma^0 P_L q_i =^3 \sqrt{2} q_j^\dagger P_R \gamma^0 P_L q_i$ we get:

$$\Phi_{ij} \xrightarrow{C} \sqrt{2} \left(C \bar{q}_j^T \right)^\dagger P_R \gamma^0 P_L C \bar{q}_i^T = \sqrt{2} q_j^T \gamma^{0\star} C^\dagger P_R \gamma^0 P_L C \bar{q}_i^T = \dots .$$

Inserting ones by $\mathbb{1}_{4\times 4} = C^{-1} C$ we get:

$$\cdots = \sqrt{2} q_j^T \gamma^{0\star} (C^{-1} P_R C)(C^{-1} \gamma^0 C)(C^{-1} P_L C) \bar{q}_i^T = \sqrt{2} q_j^T \gamma^{0\star} P_R^T (-\gamma^0)^T P_L^T \bar{q}_i^T = \dots ,$$

where we used eq. (1.20) and (1.21) and the fact that the following relation holds:

$$C^{-1} \gamma^5 C = \gamma^{5T} \qquad \Rightarrow \qquad C^{-1} P_{R/L} C = P_{R/L}^T , \tag{3.17}$$

[3] Since $\gamma^{5\dagger} = \gamma^5$ and therefore $P_{L/R}^\dagger = P_{L/R}$.

which can be shown by writing out the γ^5 as $i\gamma^0\gamma^1\gamma^2\gamma^3$. The above expression can be further rearranged by using $\{\gamma^5, \gamma^0\} = 0$ and $\gamma^{0\star}\gamma^{0T} = (\gamma^0\gamma^\dagger)^T = (\gamma^0\gamma^0)^T = \mathbb{1}_{4\times 4}^T = \mathbb{1}_{4\times 4}$:

$$\cdots = -\sqrt{2} q_j^T \gamma^{0\star}\gamma^{0T} P_L^T P_L^T \bar{q}_i^T = -\sqrt{2} q_j^T P_L^T P_L^T \bar{q}_i^T = -\sqrt{2} q_{j,L}^T \gamma^{0T} (q_{i,R}^\dagger)^T = \dots \; .$$

As a next step we write our the spinor indices ($_\alpha$ and $_\beta$) and use the fact that for $\boldsymbol{u}$ being an $(1\times N)$-vector, M being an $(N \times N)$-matrix and $\boldsymbol{v}$ being a $(N \times 1)$-vector the equation

$$\boldsymbol{u} M \boldsymbol{v} = (\boldsymbol{u} M \boldsymbol{v})^T \,, \tag{3.18}$$

holds, since the product is a number. Therewith we get

$$\begin{aligned} \cdots &= -\sqrt{2}\Big[q_{j,L}^T\Big]_\alpha \left[\gamma^{0T}\right]_{\alpha\beta} \Big[(q_{i,R}^\dagger)^T\Big]_\beta = \sqrt{2}\Big\{\Big[q_{j,L}^T\Big]_\alpha \left[\gamma^{0T}\right]_{\alpha\beta} \Big[(q_{i,R}^\dagger)^T\Big]_\beta\Big\}^T \\ &= \sqrt{2}\Big[q_{j,L}^T \gamma^{0T} (q_{i,R}^\dagger)^T\Big]^T = \dots \; . \end{aligned} \tag{3.19}$$

Furthermore we regarded the fact that the transposition interchanges two fermions which yields a minus sign. Finally we perform the **transposition in spinor space**(!) and at the end include a **transposition in flavor space**(!). The result is:

$$\Phi_{ij} \xrightarrow{\;C\;} \sqrt{2} q_{i,R}^\dagger \gamma^0 q_{j,L} = \sqrt{2}\bar{q}_{i,R} q_{j,L} = \Phi_{ji} = \Phi_{ij}^T. \tag{3.20}$$

For $R_{ij}^\mu = \sqrt{2}\bar{q}_{j,R}\gamma^\mu q_{i,R} = \sqrt{2} q_{j,R}^\dagger \gamma^0\gamma^\mu q_{i,R}$ and $L_{ij}^\mu = \sqrt{2}\bar{q}_{j,L}\gamma^\mu q_{i,L} = \sqrt{2} q_{j,L}^\dagger \gamma^0 \gamma^\mu q_{i,L}$ we get:

$$\begin{aligned} R_{ij}^\mu &\xrightarrow{\;C\;} & & q_j^T \gamma^0 C^\dagger P_R \gamma^0\gamma^\mu P_R C \bar{q}_i^T = q_j^T \gamma^0 C^{-1} P_R C C^{-1}\gamma^0 C C^{-1}\gamma^\mu C C^{-1} P_R C \bar{q}_i^T = \\ & & &= q_j^T \gamma^0 P_R (-\gamma^0)(-\gamma^\mu)^T P_R \bar{q}_i^T \overset{3.18}{=} -\bar{q}_i P_R \gamma^\mu P_L q_j = -(L_{ij}^\mu)^T \\ L_{ij}^\mu &\xrightarrow{\;C\;} & & \sqrt{2}\left[\bar{q}_{i,R}\left(-\gamma^\mu\right) q_{j,R}\right]^T = -(R_{ij}^\mu)^T. \end{aligned} \tag{3.21}$$

For reasons of clarity and comprehensibility we will summarize the results again:

	Φ	R^μ	L^μ
P	$\Phi^\dagger$	$\begin{cases} L^\mu(t,-\boldsymbol{x}) & \text{for } \mu = 0 \\ -L^\mu(t,-\boldsymbol{x}) & \text{for } \mu = 1,2,3 \end{cases}$	$\begin{cases} R^\mu(t,-\boldsymbol{x}) & \text{for } \mu = 0 \\ -R^\mu(t,-\boldsymbol{x}) & \text{for } \mu = 1,2,3 \end{cases}$
C	Φ^T	$-\left(L^\mu\right)^T$	$-\left(R^\mu\right)^T$

(3.22)

It can be shown that the mesonic Lagrangian (3.12) is invariant under charge conjugation and parity transformations (using $\mathrm{Tr}\{M^T\} = \mathrm{Tr}\{M\}$).

3.2.3. Time Reversal:

As a consequence of the CPT theorem the symmetry under time reversal is then automatically fulfilled.

3.3. Explicit Form of the eLSM for $N_f = 3$

In the last section we already constructed a meson Lagrangian $\mathcal{L}_{\text{meson}}$ which is also globally chirally invariant for the special case of $N_f = 3$. The matrix Φ containing scalar and pseudoscalar degrees of freedom reads for three flavors

$$\Phi = S + iP = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{(\sigma_N + a_0^0) + i(\eta_N + \pi^0)}{\sqrt{2}} & a_0^+ + i\pi^+ & K_S^+ + iK^+ \\ a_0^- + i\pi^- & \frac{(\sigma_N - a_0^0) + i(\eta_N - \pi^0)}{\sqrt{2}} & K_S^0 + iK^0 \\ K_S^- + iK^- & \bar{K}_S^0 + i\bar{K}^0 & \sigma_S + i\eta_S \end{pmatrix}. \tag{3.23}$$

The left- and right-handed matrices L^μ and R^μ are combinations of vector V^μ and axial-vector A^μ degrees of freedom, eq. (3.24) and (3.25),

$$R^\mu = V^\mu - A^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu + \rho^{\mu 0}}{\sqrt{2}} - \frac{f_{1N}^\mu + a_1^{\mu 0}}{\sqrt{2}} & \rho^{\mu +} - a_1^{\mu +} & K^{*\mu +} - K_1^{\mu +} \\ \rho^{\mu -} - a_1^{\mu -} & \frac{\omega_N^\mu - \rho^{\mu 0}}{\sqrt{2}} - \frac{f_{1N}^\mu - a_1^{\mu 0}}{\sqrt{2}} & K^{*\mu 0} - K_1^{\mu 0} \\ K^{*\mu -} - K_1^{\mu -} & \bar{K}^{*\mu 0} - \bar{K}_1^{\mu 0} & \omega_S^\mu - f_{1S}^\mu \end{pmatrix} \tag{3.24}$$

and

$$L^\mu = V^\mu + A^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu + \rho^{\mu 0}}{\sqrt{2}} + \frac{f_{1N}^\mu + a_1^{\mu 0}}{\sqrt{2}} & \rho^{\mu +} + a_1^{\mu +} & K^{*\mu +} + K_1^{\mu +} \\ \rho^{\mu -} + a_1^{\mu -} & \frac{\omega_N^\mu - \rho^{\mu 0}}{\sqrt{2}} + \frac{f_{1N}^\mu - a_1^{\mu 0}}{\sqrt{2}} & K^{*\mu 0} + K_1^{\mu 0} \\ K^{*\mu -} + K_1^{\mu -} & \bar{K}^{*\mu 0} + \bar{K}_1^{\mu 0} & \omega_S^\mu + f_{1S}^\mu \end{pmatrix}. \tag{3.25}$$

Thus, our model describes four resonances for scalars $(\sigma_N, \sigma_S, \boldsymbol{a}_0, \boldsymbol{K}_S)$, pseudoscalars $(\eta_N, \eta_S, \boldsymbol{\pi}, \boldsymbol{K})$, vectors $(\omega_N^\mu, \omega_S^\mu, \boldsymbol{\rho}^\mu, \boldsymbol{K}^{*\mu})$, and axial-vectors $(f_{1N}^\mu, f_{1S}^\mu, \boldsymbol{a}_1^\mu, \boldsymbol{K}_1)$. We assign the fields in the following way:

$$\begin{aligned} \sigma_N &\;\hat{=}\; f_0(1370), \\ \sigma_S &\;\hat{=}\; f(1500), \\ \boldsymbol{a}_0 &\;\hat{=}\; a_0(980) \text{ or } a_0(1450), \\ \boldsymbol{K}_S &\;\hat{=}\; K_S^\star(1430), \end{aligned}$$

$$
\begin{aligned}
\eta_N &\hat{=} \text{ the } SU(2) \text{ counterpart of the } \eta \text{ meson} \equiv \tfrac{\bar{u}u+\bar{d}d}{\sqrt{2}}, \\
\eta_S &\hat{=} \text{ strange contribution to } \eta \text{ and } \eta' \equiv \bar{s}s, \\
\boldsymbol{\pi} &\hat{=} \text{ pion}, \\
\boldsymbol{K} &\hat{=} \text{ kaons}, \\
\omega_N &\hat{=} \omega(782), \\
\omega_S &\hat{=} \Phi(1020), \\
\boldsymbol{\rho} &\hat{=} \rho(770), \\
\boldsymbol{K}^* &\hat{=} K^*(892), \\
f_{1N} &\hat{=} f_1(1285), \\
f_{1S} &\hat{=} f_1(1420), \\
\boldsymbol{a}_1 &\hat{=} a_1(1260), \\
\boldsymbol{K}_1 &\hat{=} K_1(1270), \\
G &\hat{=} f_0(1710).
\end{aligned}
$$

Note that a mixing phenomena in the scalar-nonscalar sector occur, see ref. [14], but they will not be important for our work.
The matrices H and Δ in the explicit symmetry breaking terms are given by

$$
H = \begin{pmatrix} \frac{h_{0N}}{2} & 0 & 0 \\ 0 & \frac{h_{0N}}{2} & 0 \\ 0 & 0 & \frac{h_{0S}}{\sqrt{2}} \end{pmatrix} \quad , \quad \Delta = \begin{pmatrix} \delta_N & 0 & 0 \\ 0 & \delta_N & 0 \\ 0 & 0 & \delta_S \end{pmatrix} . \tag{3.26}
$$

If $m_0^2 < 0$, spontaneous symmetry breaking takes place and the two scalar-isosinglet fields σ_N and σ_S assume non-zero vacuum expectation values φ_N and φ_S. As a consequence, mixing terms between (axial-)vector and (pseudo)scalar fields occur. In order to discuss this issue we first consider the term $\mathrm{Tr}\{(D_\mu\Phi)^\dagger(D^\mu\Phi)\}$ in $\mathcal{L}_{\text{meson}}$ (3.12). If we write it out we get:

$$
\begin{aligned}
\mathrm{Tr}\left\{(D_\mu\Phi)^\dagger(D^\mu\Phi)\right\} &= \mathrm{Tr}\left\{[\partial_\mu\Phi - ig_1(L_\mu\Phi - \Phi R_\mu)]^\dagger\,[\partial^\mu\Phi - ig_1(L^\mu\Phi - \Phi R^\mu)]\right\} = \\
&= \mathrm{Tr}\left\{\left[\partial_\mu\Phi^\dagger + ig_1(\Phi^\dagger L_\mu^\dagger - R_\mu^\dagger\Phi^\dagger)\right][\partial^\mu\Phi - ig_1(L^\mu\Phi - \Phi R^\mu)]\right\} = \\
&= \ldots
\end{aligned}
$$

Simply plugging in the matrices Φ, eq. (3.23), R^μ, eq. (3.24) and L^μ, eq. (3.25) and expanding would lead to a lot of terms. At the moment we are interested only in some terms and we are able to simplify the calculation. We are looking for mixing terms of (pseudo)scalar and axial-vector states after condensation. Therefore, only the terms

$$
-ig_1(\partial_\mu\Phi^\dagger)(L^\mu\Phi - \Phi R^\mu) \qquad \text{and} \qquad + ig_1(\Phi^\dagger L_\mu^\dagger - R_\mu^\dagger\Phi^\dagger)(\partial^\mu\Phi) \tag{3.27}
$$

are interesting. We are searching for terms which are proportional to φ_N and φ_S after condensation. Hence, without loss of generality we can set all scalar and pseudoscalar fields in $L^\mu\Phi - \Phi R^\mu$ and $\Phi^\dagger L_\mu^\dagger - R_\mu^\dagger \Phi^\dagger$ to zero:

$$\begin{aligned} L^\mu\Phi - \Phi R^\mu &\longrightarrow L^\mu\Phi' - \Phi' R^\mu \\ \Phi^\dagger L_\mu^\dagger - R_\mu^\dagger\Phi^\dagger &\longrightarrow \Phi'^\dagger L_\mu^\dagger - R_\mu^\dagger \Phi'^\dagger \end{aligned} \tag{3.28}$$

$$\text{with} \qquad \Phi' = \frac{1}{2}\begin{pmatrix} \varphi_N & 0 & 0 \\ 0 & \varphi_N & 0 \\ 0 & 0 & \sqrt{2}\varphi_S \end{pmatrix} \equiv \Phi'^\dagger. \tag{3.29}$$

Using $L_\mu^\dagger = L_\mu$ and $R_\mu^\dagger = R_\mu$ the $\mathrm{Tr}\left\{(D_\mu\Phi)^\dagger(D^\mu\Phi)\right\}$ term yields us the following mixing terms:

$$\begin{aligned} &- g_1\varphi_N f_{1N}^\mu \partial_\mu \eta_N - \sqrt{2}g_1\varphi_S f_{1S}^\mu\partial_\mu\eta_S - g_1\varphi_N \boldsymbol{a}_1^\mu \cdot (\partial_\mu\boldsymbol{\pi}) + \\ &- ig_1\left(\frac{\varphi_N}{2} - \frac{\varphi_S}{\sqrt{2}}\right)\left[K_1^{-\mu}(\partial_\mu K^+) + \bar{K}_1^{0\mu}(\partial_\mu K^0) + K_1^{+\mu}(\partial_\mu K^-) + K_1^{0\mu}(\partial^\mu \bar{K}^0)\right] + \\ &- ig_1\left(\frac{\varphi_N}{2} - \frac{\varphi_S}{\sqrt{2}}\right)\left[\bar{K}_1^{0\mu}(\partial_\mu K_2^0) + K^{*-\mu}(\partial_\mu K_S^-) - K^{*+\mu}(\partial_\mu K_S^-) - K^{*0\mu}(\partial_\mu \bar{K}_S^0)\right]. \end{aligned} \tag{3.30}$$

These mixing terms indicate that the (axial-)vector and (pseudo)scalar fields do not yet correspond to physical resonances. We perform a shift of the (axial-)vectors as follows:

$$\boxed{\begin{array}{lllll} f_{1N}^\mu & \longrightarrow & f_{1N}^\mu & + & w_{f_{1N}}\partial^\mu\eta_N, \\ f_{1S}^\mu & \longrightarrow & f_{1S}^\mu & + & w_{f_{1S}}\partial^\mu\eta_S, \\ \boldsymbol{a}_1^\mu & \longrightarrow & \boldsymbol{a}_1^\mu & + & w_{a_1}\partial^\mu\boldsymbol{\pi}, \\ K_1^{\mu 0} & \longrightarrow & K_1^{\mu 0} & + & w_{K_1}\partial^\mu\bar{K}^0 \qquad \text{(and h.c.)}, \\ K_1^{\mu +} & \longrightarrow & K_1^{\mu +} & + & w_{K_1}\partial^\mu K^+ \qquad \text{(and h.c.)}, \\ K_1^{*\mu 0} & \longrightarrow & K_1^{*\mu 0} & + & w_{K^*}\partial^\mu K_S^0 \qquad \text{(and h.c.)}, \\ K_1^{*\mu +} & \longrightarrow & K_1^{*\mu +} & + & w_{K^*}\partial^\mu K_S^+ \qquad \text{(and h.c.)}. \end{array}} \tag{3.31}$$

The w parameters have to be determined by imposing that no mixing terms, eq. (3.30), remain. For instance to find $w_{f_{1N}}$ we have to extract all terms of the **full** Lagrangian (3.12) which describe transitions between (pseudo)scalar and (axial)vector fields after the condensation of the scalar isosinglets. First we have the mixing term in eq. (3.30):

$$-g_1\varphi_N f_{1N}^\mu \partial_\mu \eta_N. \tag{3.32}$$

But also terms which originally were proportional to $f_{1N\mu} f_{1N}^{\mu}$ yield a mixing between f_{1N}^{μ} and η_N after the shift (6.11).
For example, the $\mathrm{Tr}\{g_1^2 \left(L_\mu \Phi' - \Phi' R^\mu\right)^\dagger \left(L_\mu \Phi' - \Phi' R^\mu\right)\}$ part of $\mathrm{Tr}\{(D_\mu\Phi)^\dagger(D^\mu\Phi)\}$ contains the term

$$\frac{1}{2} g_1^2 \varphi_N^2 f_{1N\mu} f_{1N}^{\mu} \quad \xrightarrow{\text{shift}} \quad \frac{1}{2} g_1^2 \varphi_N^2 \left(f_{1N}^{\mu} + w_{f_{1N}} \partial^\mu \eta_N\right)^2. \tag{3.33}$$

Thus this yields the non-physical transition

$$g_1^2 \varphi_N^2 w_{f_{1N}} f_{1N}^{\mu} \partial_\mu \eta_N. \tag{3.34}$$

Going through the whole Lagrangian we finally find the vertex of the transition $f_{1N}^{\mu}(\partial_\mu \eta_N)$ to be

$$\left[-g_1\varphi_N + g_1^2 \varphi_N^2 w_{f_{1N}} + \frac{1}{2}\left(h_1 + h_2 - h_3\right) \varphi_N^2 w_{f_{1N}} + \frac{h_1}{2} \varphi_S^2 w_{f_{1N}} + \left(m_1^2 + 2\delta_N\right) w_{f_{1N}}\right] \stackrel{!}{=} 0. \tag{3.35}$$

which of course should be zero for all mixing terms to vanish. That results in

$$\boxed{w_{f_{1N}} = \frac{g_1 \varphi_N}{g_1^2 \varphi^2 + \frac{1}{2}(h_1 + h_2 - h_3)\varphi_N^2 + \frac{1}{2} h_1 \varphi_S^2 + m_1^2 + 2\delta_N} \equiv \frac{g_1 \varphi_N}{m_{f_{1N}}^2},} \tag{3.36}$$

where $m_{f_{1N}}$ is the mass of f_{1N} and could be read off the Lagrangian. It is the square root of the coefficient of the $\frac{1}{2} f_{\mu 1N} f_{1N}^{\mu}$-term:

$$m_{f_{1N}} = \sqrt{g_1^2 \varphi_N^2 + \frac{1}{2}(h_1 + h_2 - h_3)\varphi_N^2 + \frac{1}{2} h_1 \varphi_S^2 + m_1^2 + 2\delta_N}. \tag{3.37}$$

In analogy one finds all the other w parameters and (axial)vector masses:

$$\boxed{\begin{aligned}
w_{a_1} = w_{f_{1N}} &= \frac{g_1 \varphi_N}{g_1^2 \varphi^2 + \frac{1}{2}(h_1 + h_2 - h_3)\varphi_N^2 + \frac{1}{2} h_1 \varphi_S^2 + m_1^2 + 2\delta_N} \equiv \frac{g_1 \varphi_N}{m_{a_1}^2} \qquad \text{with } m_{a_1} = m_{f_{1N}}, \\
w_{f_{1S}} &= \frac{\sqrt{2} g_1 \varphi_S}{2 g_1^2 \varphi_S^2 + (\frac{h_1}{2} + h_2 - h_3)\varphi_S^2 + m_1^2 + 2\delta_S + \frac{h_1}{2}\varphi_N^2} \equiv \frac{\sqrt{2} g_1 \varphi_S}{m_{f_{1S}}^2}, \\
w_{K_1} &= \frac{1}{2} \frac{g_1(\varphi_N + \sqrt{2}\varphi_S)}{m_1^2 + \delta_N + \delta_S + \frac{1}{2}\varphi_N^2(\frac{g_1^1}{2} + h_1 + \frac{h_2}{2}) + \frac{g_1^2 - h_3}{\sqrt{2}}\varphi_N\varphi_S + \frac{\varphi_S^2}{2}(g_1^2 + h_1 + h_2)} \equiv \frac{g_1(\varphi_N + \sqrt{2}\varphi_S)}{2 m_{K_1}^2}, \\
w_{K^*} &= \frac{i\frac{1}{2} g_1(\varphi_N - \sqrt{2}\varphi_S)}{m_1^2 + \delta_N + \delta_S + \frac{1}{2}\varphi_N^2(\frac{g_1^2}{2} + h_1 + \frac{h_2}{2}) + \frac{h_3 - g_1^2}{\sqrt{2}}\varphi_N\varphi_S + \frac{\varphi_S^2}{2}(g_1^2 + h_1 + h_2)} \equiv i\frac{g_1(\varphi_N - \sqrt{2}\varphi_S)}{2 m_{K^*}^2}.
\end{aligned}} \tag{3.38}$$

All in all, with the shift we eliminated the non-physical mixing terms, but the kinetic terms of the (pseudo)scalar mesons are no longer properly normalized. Considering for instance the kinetic term of the η_N field: Because of the shift $f_{1N}^{\mu} \rightarrow f_{1N}^{\mu} + w_{f_{1N}} \partial^{\mu} \eta_N$ the term $\mathrm{Tr}\left\{(D_\mu \Phi)^\dagger (D^\mu \Phi)\right\}$ yields not only the contribution $\frac{1}{2}(\partial_\mu \eta_N)^2$, but additionally, from the former mixing terms $-g_1 \varphi_N b f_{1N} \partial_\mu \eta_N$ and from the term proportional to $f_{\mu 1N} f_{1N}^{\mu}$ the shift yields the terms $-g_1 \varphi_N w_{f_{1N}} (\partial_\mu \eta_N)^2$ and $\frac{1}{2} g_1^2 \varphi_N^2 w_{f_{1N}}^2 (\partial_\mu \eta_N)^2$. In the same way, from all the other terms proportional to $f_{\mu 1N} f_{1N}^{\mu}$ in the whole Lagrangian, after the shift we obtain terms which are proportional to $(\partial_\mu \eta_N)^2$. Altogether we find for the kinetic term of η_N:

$$\left(\frac{1}{2} - g_1 \varphi_N w_{f_{1N}} + \frac{1}{2} g_1^2 \varphi_N^2 w_{f_{1N}}^2 + \frac{m_1^2}{2} w_{f_{1N}}^2 + \delta_N w_{f_{1N}}^2 + \frac{1}{4} \varphi_N^2 w_{f_{1N}}^2 (h_1 + h_2 - h_3) + \frac{h_1}{2} \varphi_S^2 w_{f_{1N}}^2 \right) (\partial_\mu \eta)^2 \tag{3.39}$$

To renormalise this to $\frac{1}{2}(\partial_\mu \eta_N)^2$ we introduce the parameter Z_{η_N} so that the transformation

$$\boxed{\eta_N \longrightarrow Z_{\eta_N} \eta_N} \tag{3.40}$$

yields the correctly normalised kinetic term. In this way we find that Z_{η_N} is given by

$$Z_{\eta_N}^2 = \frac{m_{a_1^2}}{m_{a_1}^2 - g_1^2 \varphi_N^2}. \tag{3.41}$$

In an analogous way, we obtain the renormalisation of the following fields:

$$\boxed{\begin{aligned} \boldsymbol{\pi} &\longrightarrow Z_\pi \boldsymbol{\pi}, \\ \boldsymbol{K} &\longrightarrow Z_K \boldsymbol{K}, \\ \eta_S &\longrightarrow Z_{\eta_S} \eta_S, \\ \boldsymbol{K}_S &\longrightarrow Z_{K_S} \boldsymbol{K}_S, \end{aligned}} \tag{3.42}$$

with

$$\boxed{Z_\pi^2 = Z_{\eta_N}^2 = \frac{m_{a_1^2}}{m_{a_1}^2 - g_1^2 \varphi_N^2}.} \tag{3.43}$$

and

$$\begin{aligned} Z_K^2 &= \frac{4 m_{K_1}^2}{4 m_{K_1}^2 - g_1^2 (\varphi_N + \sqrt{2} \varphi_S)^2}, \\ Z_{\eta_S}^2 &= \frac{m_{f_1}^2}{m_{f_1}^2 - 2 g_1^2 \varphi_S^2}, \\ Z_{K_S}^2 &= \frac{4 m_{K^*}^2}{4 m_{K^*}^2 - g_1^2 (\varphi_N - \sqrt{2} \varphi_S)^2}. \end{aligned} \tag{3.44}$$

Summarizing, we have seen that spontaneous symmetry breaking in the meson sector leads to shifts of all axial-vector and some vector fields by the corresponding (pseudo)scalars, because the original ones were not physical. As a consequence, we have to renormalise the (pseudo)scalars.

4. Features Concerning Baryons and Diquarks

Before we construct a Lagrangian for baryons we will have a closer look at their properties and the inner structure. To this end, also diquarks will be discussed.

4.1. Baryons: qqq States

Baryons are strongly interacting fermions with baryon number $B = 1$. Since quarks have baryon number $B = 1/3$ and antiquarks have baryon number $B = -1/3$, baryons are composed (in the most general case) of three valence quarks (plus an arbitrary number of quark-antiquark and gluon pairs). As far as it is known [23], all established baryons can be understood as formed by three constituent quarks, where a constituent quark is a valence quark dressed a cloud of gluons and quark-antoquark pairs.

4.2. Baryon Flavor Multiplets with $N_f = 3$

In comparison with the typical hadronic mass scale, $M_h \sim 1$ GeV, the masses of up, down, and strange quarks are small[1] and approximately equal. In table 4.1 the properties of all quark flavors are listed.

flavor	spin	mass [MeV]	quark quantum numbers						
			Q	T_3	S	C	B	T	B
u	1/2	$2.3^{+0.1}_{-0.5}$	2/3	1/2	0	0	0	0	1/3
d	1/2	$4.8^{+0.5}_{-0.3}$	-1/3	-1/2	0	0	0	0	1/3
s	1/2	95 ± 5	-1/3	0	-1	0	0	0	1/3
c	1/2	$(1.275 \pm 0.025) \cdot 10^3$	2/3	0	0	1	0	0	1/3
b	1/2	$(4.18 \pm 0.03) \cdot 10^3$	-1/3	0	0	0	-1	0	1/3
t	1/2	$(173.07 \pm 1.14) \cdot 10^3$	2/3	0	0	0	0	1	1/3

Table 4.1.: Properties of the quark flavors. The quark quantum numbers are electrical charge Q, z-component of the isospin T_3, strangeness S, charm C, bottom charge B, top charge T, and baryon number B.

[1] This condition is well verified for light quarks u and d, but can also be extended to the s-quark, although in the latter case the breaking is larger.

Thus, if we consider only the lightest three flavors an approximate $SU(3)$ flavor symmetry exists. If we assume the $SU(3)$ flavor symmetry to be exact the three quarks form a $SU(3)$ triplet. It is the fundamental representation of the $SU(3)$ group from which all other multiplets can be built. Figure 4.1 illustrates this triplet and also the antiquark triplet in which the signs of additive quantum numbers are reversed. In order to distinguish the states of the mulitplets we use the two quantum numbers Y and T_3. Here $Y = 2/\sqrt{3}T_8 = B + S$ denotes the hypercharge, T_8 and T_3 are the two $SU(3)$ generators which form a Cartan subalgebra and B and S are the baryon number and strangeness, listed in table 4.1.

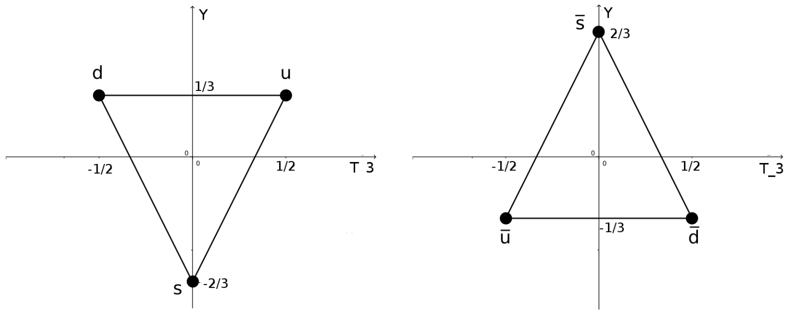


Figure 4.1.: $SU(3)$ quark and antiquark multiplets, $Y = 2/\sqrt{3}T_8 = B + S$.

As mentioned the baryon is built from three quarks. Thus, we have to combine three quark triplets to obtain all 27 possible combinations sorted into multiplets. We start with the combination of two quarks, which yields

$$[3] \otimes [3] = [6] \oplus [\bar{3}], \tag{4.1}$$

This means that the combination of the two quark triplets results in a (symmetric) sextet and an (anti-symmetric) antiquark triplet. The graphical form of this coupling is shown in Fig. 4.2, in which the quark flavor content is indicated, for the moment without taking care of the symmetry.

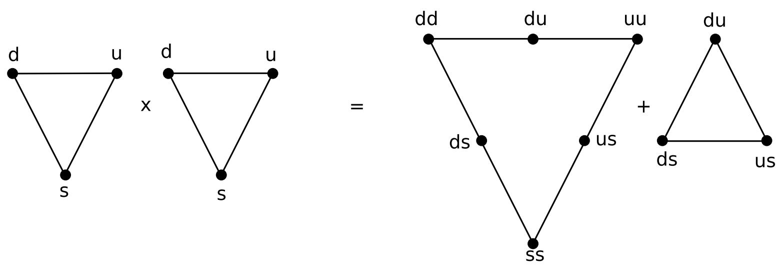


Figure 4.2.: The qq $SU(3)$ multiplets, $[3] \otimes [3] = [6] \oplus [\bar{3}]$.

In order to find the correct symmetry of the flavor content we consider separately the $SU(2)$ I-spin, $SU(2)$ U-spin and $SU(2)$ V-spin doublets, (u, d), (d, s), and (s, u). Figure 4.3 shows the three $SU(2)$ doublets.

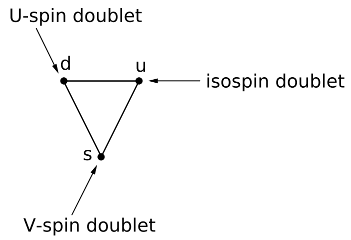


Figure 4.3.: $SU(2)$ I-spin, $SU(2)$ U-spin and $SU(2)$ V-spin doublets.

Following the rules for the addition of the angular momentum the possible combinations of two spin-1/2 states are constructed. Indeed, since $SU(2)$ underlies also the description of ordinary spin, the isospin is mathematically a carbon copy of the spin and we get the combination of two $SU(2)$-isospin doublets (u, d) in utter analogy. Fig. 4.4 displays the results.

with the Clebsch-Gordan coefficients (NOTE: A square-root sign is to be understood over *every* coefficient.):

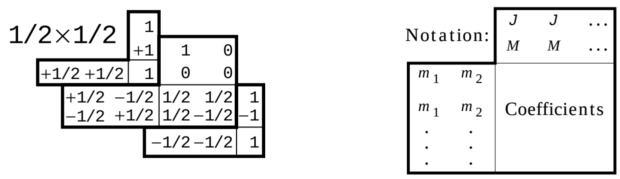


Figure 4.4.: The combination of two isospin doublets, $[2] \otimes [2] = [3] \oplus [1]$, and the respective Clebsch-Gordan coefficients.

Since the mathematics of I, U, and V spin are identical (all are based on the $SU(2)$ group) the quark content of the other states in figure 4.2 is found in the same way. In fact, we will see that almost all of the $SU(3)$ structure that we require can be obtained by successive application of $SU(2)$. Figure 4.5 outlines the resulting multiplets, now with the symmetry content of quark flavor.

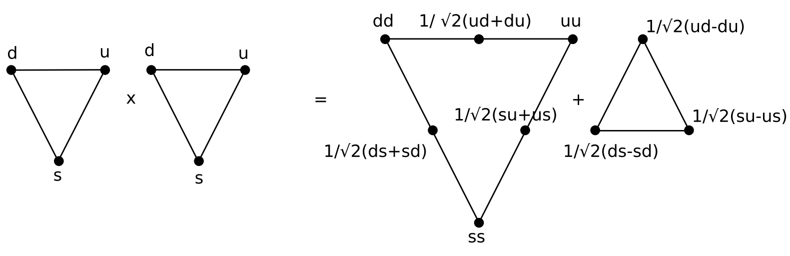


Figure 4.5.: The qq $SU(3)$ multiplets **with quark content**, $[3] \otimes [3] = [6] \oplus [\bar{3}]$.

In order to get the qqq baryon $SU(3)$ multiplets we have to add a further quark triplet. The final decomposition,

$$[3] \otimes [3] \otimes [3] = ([6] \otimes [3]) \oplus ([\bar{3}] \otimes [3]) = [10] \oplus [8] \oplus [8] \oplus [1], \tag{4.2}$$

is shown in figure 4.6. In order to get the quark contents we separately combine the I, U, and V

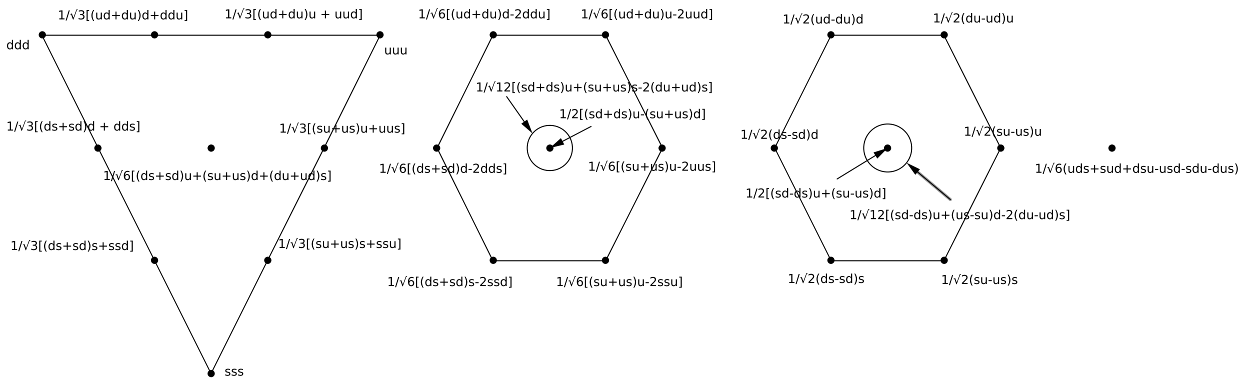


Figure 4.6.: The qqq $SU(3)$ multiplets, $[3] \otimes [3] \otimes [3] = [10] \oplus [8] \oplus [8] \oplus [1]$

multiplets. For example in the nonstrange sector we combine the triplet $(dd, 1/\sqrt{2}(ud+du), uu)$ with the doublet (d, u) and add the combination of the singlet $(1/\sqrt{2}(ud-du))$ and the doublet (d, u). Of course, we have to use the proper Clebsch-Gordan coefficients again. To get the quark contents of the "uds" states we should have a closer look at the symmetry of the multiplets. Close inspection reveals that the decuplet is totally symmetric under the exchange of two quarks. Both octets have a mixed symmetry; meaning one is symmetric and the other antisymmetric under exchange of the first two quarks only. Consequently the singlet is totally antisymmetric under the exchange of two arbitrary quarks. Regarding this and using orthogonality we constructed the "uds" quark contents denoted in figure 4.6.

4.3. Spin-Flavor Wave Function of Baryons in the Ground State

In the ground state it is quite easy to find the spins of the baryon. Since baryons contain three quarks, each carrying spin 1/2, we only have to add three spin 1/2 angular momenta. The combination gives

$$[2] \otimes [2] \otimes [2] = [4] \oplus [2] \oplus [2], \tag{4.3}$$

with the following spin state contents:

$$
\begin{aligned}
&[4]\begin{cases}
\left|S=\frac{3}{2}, m=-\frac{3}{2}\right\rangle &= \downarrow\downarrow\downarrow \\
\left|S=\frac{3}{2}, m=-\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}[\downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow + \downarrow\downarrow\uparrow] \\
\left|S=\frac{3}{2}, m=+\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}[\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow] \\
\left|S=\frac{3}{2}, m=+\frac{3}{2}\right\rangle &= \uparrow\uparrow\uparrow
\end{cases} \\
&[2]\begin{cases}
\left|S=\frac{1}{2}, m=-\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{6}}[\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow] \\
\left|S=\frac{1}{2}, m=+\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{6}}[\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow]
\end{cases} \\
&[2]\begin{cases}
\left|S=\frac{1}{2}, m=-\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{2}}[\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow] \\
\left|S=\frac{1}{2}, m=+\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{2}}[\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow].
\end{cases}
\end{aligned}
\tag{4.4}
$$

On closer examination one can recognize that the quartet states are total symmetric under exchange of two quarks, but the doublets have a mixed symmetry, meaning that they are symmetric under the interchange of the first two quarks only. For that reason we indicate the multiplets with S for symmetric, M_S for mixed symmetry with symmetry only in the first two quarks, and M_A for mixed symmetry with antisymmetry only in the first two quarks. Finally we have the following spin mulitplets for baryons indexed with the symmetry properties in the ground state:

$$[4]_S \oplus [2]_{M_S} \oplus [2]_{M_A}. \tag{4.5}$$

In order to obtain the spin-flavor wave function we have to combine the $SU(3)$-flavor[2] and the $SU(2)$-spin multiplets,

$$\overbrace{([10]_S \oplus [8]_{M_S} \oplus [8]_{M_A} \oplus [1]_A)}^{SU(3)\text{-flavor}}\ ,\quad \overbrace{([4]_S \oplus [2]_{M_S} \oplus [2]_{M_A})}^{SU(2)\text{-spin}}, \tag{4.6}$$

where we have indicated the flavor multiplets with the additional indices in the same manner as the spin multiplets. Then also the resulting spin-flavor state can have only one of the four symmetry types S, M_S, M_A, or A. With the notation "$(SU(3), SU(2))$" the following products appear sorted by symmetry:

$$
\begin{array}{rlclclcl}
\text{S:} & ([10]_S,[4]_S) & + & ([8],[2]) & & & & \\
\text{M}_S\text{:} & ([10]_S,[2]_{M_S}) & + & ([8]_{M_S},[4]_S) & + & ([8]_{M_S},[2]_{M_S}) & + & ([1]_A,[2]_{M_A}) \\
\text{M}_A\text{:} & ([10]_S,[2]_{M_A}) & + & ([8]_{M_A},[4]_S) & + & ([8]_{M_A},[2]_{M_A}) & + & ([1]_A,[2]_{M_S}) \\
\text{A:} & ([1]_A,[4]_S) & + & ([8],[2]). & & & &
\end{array}
\tag{4.7}
$$

The totally symmetric spin-flavor wave function of a (flavor) octet baryon is determined by the combination

$$S: \quad ([8],[2]) \equiv \frac{1}{\sqrt{2}}\left[([8]_{M_S},[2]_{M_S}) \oplus ([8]_{M_A},[2]_{M_A})\right] \tag{4.8}$$

[2]See last subsection.

and the totally antisymmetric spin-flavor wave function of the octet baryon is

$$A: \quad ([8],[2]) \;\equiv\; \frac{1}{\sqrt{2}}\left[([8]_{M_S},[2]_{M_A}) \oplus ([8]_{M_A},[2]_{M_S})\right], \tag{4.9}$$

which can easily be shown by an explicit calculation. As an example we will consider the proton spin-up wave function in the next section.

4.4. Complete Wave Function of Baryons and Particle Assignment to Multiplets

The ground state ($l = 0$) wave function of any baryon is the combination of space, color, flavor, and spin states. We may write it in the following way:

$$|\text{baryon}\rangle = |\text{space}\rangle_{l=0} \times |\text{flavor-spin}\rangle \times |\text{color}\rangle\,. \tag{4.10}$$

Since quarks are fermions, the state $|\text{baryon}\rangle$ has to be antisymmetric under the exchange of any two quarks. The confinement hypothesis makes sure that the color state is a singlet of $SU(3)_c$ and therefore completely antisymmetric[3]. In analogy to the $SU(3)_f$ singlet it has the following form

$$|\text{color}\rangle = \frac{1}{\sqrt{6}}\,(\text{RGB + BRG + GBR - RBG - BGR - GRB})\,. \tag{4.11}$$

For $l = 0$ (ground state) the space state is symmetric and we can write the state function of the baryon with the following symmetry properties (S: sym., A: antisym.) under the interchange of two arbitrary quarks as

$$|\text{baryon}\rangle_A = |\text{space}\rangle_{l=0,S} \times |\text{flavor-spin}\rangle_S \times |\text{color}\rangle_A\,. \tag{4.12}$$

Just, in order to achieve the antisymmetry of the $|\text{baryon}\rangle$-state we need a symmetric flavor-spin state. In the last section we have seen that the flavor decuplet with spin 3/2, eq. (4.7), and a combination of the flavor octets with spin 1/2, eq. (4.8), are totally symmetric. As a consequence the flavor singlet is forbidden by Fermi statistics in the ground state.

As an example we will give the exact flavor-spin state function for a proton with spin-up. The proton with $I(J^P) = \frac{1}{2}(\frac{1}{2})^+$ and the flavor content "uud" fit into the totally symmetric octet state of eq. (4.8).

$$\begin{aligned}
|\text{flavor-spin}\rangle_{\text{proton}\uparrow} &= \frac{1}{\sqrt{2}}\Big(|\text{uud}\rangle_{M_S}\,|\!\uparrow\rangle_{M_S} + |\text{uud}\rangle_{M_A}\,|\!\uparrow\rangle_{M_A}\Big) = \\
&= \frac{1}{\sqrt{2}}\Bigg[\frac{1}{\sqrt{6}}\,(\text{udu + duu - 2uud})\,\frac{1}{\sqrt{6}}\,(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow -2\uparrow\uparrow\downarrow) \\
&\qquad + \frac{1}{\sqrt{2}}\,(\text{udu - duu})\,\frac{1}{\sqrt{2}}\,(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\Bigg] = \\
&= \frac{1}{\sqrt{18}}\Big(\text{u}_\downarrow\text{d}^\uparrow\text{u}^\uparrow + \text{u}^\uparrow\text{d}^\uparrow\text{u}_\downarrow - 2\text{u}^\uparrow\text{d}_\downarrow\text{u}^\uparrow + \text{permutations}\Big)\,.
\end{aligned} \tag{4.13}$$

Finally, in the same manner we assign all symmetric states of the spin 3/2 decuplet $([10],[4])$ and the spin 1/2 octet with the lowest-mass ground state baryons as shown in fig. 4.7.

[3]Compare it to the singlet state of $SU(3)$ flavor in the section "Baryon flavor multiplets with $N_f = 3$".

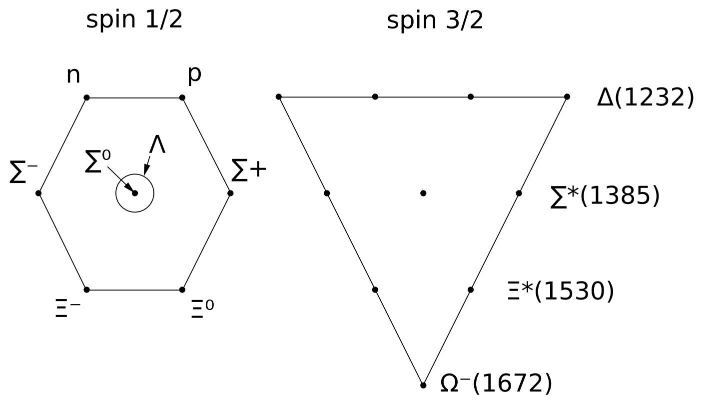


Figure 4.7.: Ground state baryons: $([8],[2]) + ([10],[4])$

4.5. Including Baryons - Naive and Mirror Assignment

Our goal is to add baryon interactions to the meson Lagrangian in the eLSM, introduced in chapter 3. Since we take three flavors into account and restrict ourselves to spin 1/2, the occurring baryons are the eight octet baryons, as shown in the last section. Since parity is not fixed there are baryons with $I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$ and the (more massive) chiral partners which have the same quantum numbers except for parity, i.e., $I(J^P) = \frac{1}{2}(\frac{1}{2}^-)$, and G-parity.
There are two ways to include baryons and their chiral partners in a chiral model: Either in the so-called naive assignment [6, 24] or in the so-called mirror assignment [3, 4, 6, 7, 8]. The main difference between these two possibilities is that in the naive assignment the left- and right-handed components of the baryon and its chiral partner transform identically, while in the mirror assignment they transform in a "mirror way", see below. For example, in a model with two flavors we have two spinors, one for the nucleon $\Psi_1 = \Psi_{1R} + \Psi_{1L}$ and another one representing the chiral partner $\Psi_2 = \Psi_{2R} + \Psi_{2L}$. In the naive assignment they transform as

$$\begin{aligned} &\Psi_{1R} \to U_R \Psi_{1R}, \qquad \Psi_{1L} \to U_L \Psi_{1L}, \\ &\Psi_{2R} \to U_R \Psi_{2R}, \qquad \Psi_{2L} \to U_L \Psi_{2L}. \end{aligned} \tag{4.14}$$

In comparison, in the mirror assignment they transform as

$$\begin{aligned} &\Psi_{1R} \to U_R \Psi_{1R}, \qquad \Psi_{1L} \to U_L \Psi_{1L}, \\ &\Psi_{2R} \to U_L \Psi_{2R}, \qquad \Psi_{2L} \to U_R \Psi_{2L}. \end{aligned} \tag{4.15}$$

In order to decide which assignment we choose, we have to know how the baryon fields transform in the three-flavor case. Therefore we consider the mathematical structure of baryons in the next section.

4.6. Diquarks

In order to describe baryons we use the so-called quark-diquark model (see e.g. [26]), in which baryons are bound states of a quark and a diquark, where a diquark is a hypothetical[4] bound state of two quarks. We can distinguish scalar diquarks which are antisymmetric in flavor- and color-space,

$$|qq\rangle_{l=s=0} = |\text{space}\rangle_{l=0}\,|\text{spin}\rangle_{s=0}\,|\text{color}\rangle_{N_c=3}\,|\text{flavor}\rangle_{N_f} \qquad \text{with} \qquad J^P = 0^+ \tag{4.16}$$

and pseudoscalar diquarks

$$|qq\rangle_{l=s=1} = |\text{space}\rangle_{l=1}\,|\text{spin}\rangle_{s=1}\,|\text{color}\rangle_{N_c=3}\,|\text{flavor}\rangle_{N_f} \qquad \text{with} \qquad J^P = 0^-, \tag{4.17}$$

where the color- and flavor-wave functions are also antisymmetric. In the following subsections we will examine their mathematical structure and behaviour under certain transformations.

4.6.1. Scalar Diquarks

For $N_f = 3$ there are exactly three such scalar diquarks $\mathcal{D}$,

$$\frac{1}{\sqrt{2}}\,[d,s]\,, \qquad -\frac{1}{\sqrt{2}}\,[u,s]\,, \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\,[u,d]\,, \tag{4.18}$$

where the anticommutator represents the antisymmetry in isospin space. They have the following mathematical structure:

$$\mathcal{D}_{ij} = \frac{1}{\sqrt{2}}\left(q_j^T C\gamma^5 q_i - q_i^T C\gamma^5 q_j\right) \equiv \sum_{k=1}^{\frac{1}{2}N_f(N_f-1)=3} D_k (A^k)_{ij}, \tag{4.19}$$

where the scalar diquark current D_k which occurs in the decomposition of $\mathcal{D}$ in the basis of antisymmetric $N_f \times N_f$ matrices $(A^k)_{ij}$ is

$$D_k = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^T C\gamma^5 q_l, \tag{4.20}$$

and in the case of $N_f = 3$ flavors $(A^k_{ij}) = \epsilon_{ijk}$. The color indices which are formally identical to the flavor indices are suppressed here. The charge-conjugation matrix C has to be included in order to get the required behaviour under charge conjugation. The γ^5 is needed to preserve the requested parity. With the parity transformation of the quark, eq. (3.14), we can compute the behaviour of the scalar diquark current:

$$\begin{aligned} D_i = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^T C\gamma^5 q_l \quad \xrightarrow{P} \quad & \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^T(t,-\boldsymbol{x})\gamma^0 C\gamma^5\gamma^0 q_l(t,-\boldsymbol{x}) = \\ & = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^T(t,-\boldsymbol{x}) C\gamma^5 q_l(t,-\boldsymbol{x}) \equiv D_i(t,-\boldsymbol{x}), \end{aligned} \tag{4.21}$$

[4]Since it is not colorless it is not a physical hadron.

where we have used $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$ and $(\gamma^0)^2 = \mathbb{1}$. With the charge conjugation of the quark, eq. (3.14), we find the charge conjugation of the scalar diquark current:

$$D_i = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^T C\gamma^5 q_l \quad \xrightarrow{\quad C \quad} \quad \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^\dagger \gamma^0 C^T C\gamma^5 C\gamma^0 q_l^\star = \dots \ .$$

With eq. (1.21) and $\{\gamma^0, \gamma^5\} = 0$ we get:

$$\dots = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^\dagger \gamma^0\gamma^5 C\gamma^0 q_l^\star = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^\dagger \left(-\gamma^5\right)\gamma^0 C\gamma^0 q_l^\star = \dots \ .$$

Including a one by $\mathbb{1}_{4\times4} = CC^{-1}$ between γ^5 and γ^0 and making use of eq. (1.20) and $\gamma^0\gamma^0 = \mathbb{1}_{4\times4}$ we find:

$$\dots = -\frac{1}{\sqrt{2}}\epsilon_{klm} q_m^\dagger \gamma^5 C\left(C^{-1}\gamma^0 C\right)\gamma^0 q_l^\star = -\frac{1}{\sqrt{2}}\epsilon_{klm} q_m^\dagger \gamma^5 C(-\gamma^0)\gamma^0 q_l^\star = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^\dagger \gamma^5 C q_l^\star = \dots \ .$$

We use eq. (3.18) and then interchange the indices $_l$ and $_m$, where we have to regard the antisymmetry of the ϵ-tensor, $\epsilon_{klm} = -\epsilon_{kml}$:

$$\dots = -\frac{1}{\sqrt{2}}\epsilon_{klm} q_l^\star C\gamma^5 q_m^\dagger = \frac{1}{\sqrt{2}}\epsilon_{kml} q_l^\star C\gamma^5 q_m^\dagger = \dots \ .$$

Finally, hermitian conjugation (note that D_i is a scalar in flavor space, $C^\dagger = C$, eq. (1.21), and $\gamma^{5\dagger} = \gamma^5$) yields:

$$D_i = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^T C\gamma^5 q_l \quad \xrightarrow{\quad C \quad} \quad D_i^\star \equiv D_i^\dagger. \tag{4.22}$$

Using the $SU(3)_V$ transformation of the quark, $q \to U_V q$, the calculation for the diquark transformation is straightforward.
Summarizing, the scalar diquark current has the following properties under parity, charge conjugation and $SU(N_f)_V$ transformations:

$$\begin{array}{lll} D_i & = \frac{1}{\sqrt{2}}\epsilon_{ijk} q_k^T C\gamma^5 q_j \quad \xrightarrow{\quad P \quad} & D_i^P = D_i(t, -\boldsymbol{x}), \\ D_i & = \frac{1}{\sqrt{2}}\epsilon_{ijk} q_k^T C\gamma^5 q_j \quad \xrightarrow{\quad C \quad} & D_i^C = D_i^\dagger, \\ D_i & = \frac{1}{\sqrt{2}}\epsilon_{ijk} q_k^T C\gamma^5 q_j \quad \xrightarrow{SU(N_f)_V} & D_i' = \frac{1}{\sqrt{2}}\epsilon_{ijk} U_V^{jj'} U_V^{kk'} q_{k'}^T C\gamma^5 q_{j'}. \end{array} \tag{4.23}$$

In our case, $N_f = 3$, the expression for the $SU(N_f = 3)_V$ transformed scalar diquark current can be rearranged further. Therefore, we consider the following expression with $U \in SU(3)_V$

$$\begin{aligned} \epsilon^{a'b'c'}\epsilon^{abc} U_{aa'} U_{bb'} U_{cc'} = \epsilon^{a'b'c'} & (U_{1a'}U_{2b'}U_{3c'} + U_{3a'}U_{1b'}U_{2c'} + U_{2a'}U_{3b'}U_{1c'} + \\ & - U_{1a'}U_{3b'}U_{2c'} - U_{2a'}U_{1b'}U_{3c'} - U_{3a'}U_{2b'}U_{1c'}) = \\ = \dots & \end{aligned}$$

using $\det U = \epsilon^{abc} U_{1a} U_{2b} U_{3c} \equiv 1$

$$\dots = 3\det U - (-3\det U) = 6\det U = 6.$$

With $\epsilon^{a'b'c'}\epsilon_{a'b'c'} = \sum_{b'c'}(\delta_{b'b'}\delta_{c'c'} - \delta_{b'c'}\delta_{c'b'}) = \sum_{b'c'}\delta_{b'b'}\delta_{c'c'} - \sum_{c'}\delta_{c'c'} = 6$ this is identical to

$$\epsilon^{a'b'c'}\epsilon^{abc}U_{aa'}U_{bb'}U_{cc'} = \epsilon^{a'b'c'}\epsilon_{a'b'c'}.$$

Therefore we find that the following relation holds:

$$\begin{aligned}\epsilon^{abc}U_{aa'}U_{bb'}U_{cc'} &= \epsilon_{a'b'c'} \\ \epsilon^{abc}U_{bb'}U_{cc'} &= \epsilon_{a'b'c'}U^{\dagger}_{a'a},\end{aligned} \tag{4.24}$$

where in the last step we have multiplied the equation from the right-hand side with $U^{\dagger}_{a'a}$. Using this relation the $SU(3)_V$ transformation of the D_i can be expressed as

$$D_i \quad \xrightarrow{SU(3)_V} \quad D'_i = D_k U^{\dagger}_{V,ki}. \tag{4.25}$$

Hence, the scalar diquark current transforms under $SU(3)_V$ exactly like an antiquark, $\bar{q}_i \to \bar{q}_k U^{\dagger}_{V,ki}$. Consequently, it is possible to find a correspondence between the scalar-diquark current and the antiquark:

$$\begin{aligned} D_1 &\sim \tfrac{1}{\sqrt{2}}[d,s] &\longleftrightarrow \quad &\bar{u}, \\ D_2 &\sim -\tfrac{1}{\sqrt{2}}[u,s] &\longleftrightarrow \quad &\bar{d}, \\ D_3 &\sim \tfrac{1}{\sqrt{2}}[u,d] &\longleftrightarrow \quad &\bar{s}, \end{aligned} \tag{4.26}$$

where"$\leftrightarrow$" refers to the same transformation under flavor transformations. Having this connection in mind one can understand the idea of the quark-diquark model: the construction of the baryon matrix conforms to the construction of the meson matrix in which a quark and an antiquark are combined $\Phi_{ij} \sim \bar{q}_{j,R}q_{i,L}$. Later on, we will say more about the matrix form of our baryon fields.

4.6.2. Pseudoscalar Diquarks

Furthermore, we also can define a pseudoscalar diquark matrix $\tilde{\mathcal{D}}$ by the mathematical expression

$$\tilde{\mathcal{D}}_{ij} = \frac{1}{\sqrt{2}}\left(q_j^T C q_i - q_i^T C q_j\right) \equiv \sum_{k=1}^{\frac{1}{2}N_f(N_f-1)=3} \tilde{D}_k (A^k)_{ij} \tag{4.27}$$

with

$$\tilde{D}_k = \frac{1}{\sqrt{2}}\epsilon_{klm} q_m^T C q_l. \tag{4.28}$$

The definitions of the previous subsection about scalar diquarks hold also in this case. Computing the transformations of the pseudoscalar diquark current in analogy to the scalar diquark current one finds

$$\begin{aligned} \tilde{D}_i &= \tfrac{1}{\sqrt{2}}\epsilon_{ijk}q_k^T C q_j \quad \xrightarrow{\quad P \quad} \quad \tilde{D}_i^P = -\tilde{D}_i(t,-\boldsymbol{x}), \\ \tilde{D}_i &= \tfrac{1}{\sqrt{2}}\epsilon_{ijk}q_k^T C q_j \quad \xrightarrow{\quad C \quad} \quad \tilde{D}_i^C = -\tilde{D}_i^{\dagger}, \\ \tilde{D}_i &= \tfrac{1}{\sqrt{2}}\epsilon_{ijk}q_k^T C q_j \quad \xrightarrow{SU(3)_V} \quad \tilde{D}'_i = \tilde{D}_k U^{\dagger}_{V,ki}. \end{aligned} \tag{4.29}$$

Note that the minus sign in the parity transformation and charge conjugation in contrast to the transformation of the scalar diquark current (eq. 4.21 and 4.22) occurs because of the missing γ^5 matrix.

4.6.3. Left- and Right-Handed Diquarks

Since our goal is the construction of a chirally invariant Lagrangian, we would like to have left- and right-handed objects with simple behaviour under chiral transformation. Therefore, we define two new matrices $\mathcal{D}_R$ and $\mathcal{D}_L$ as linear combinations of $\mathcal{D}$ and $\tilde{\mathcal{D}}$:

$$\boxed{\begin{aligned} \mathcal{D}_R &:= \frac{1}{\sqrt{2}}\left(\tilde{\mathcal{D}}+\mathcal{D}\right) = \sum_{i=1}^{3} D_i^R A^i \qquad \text{with} \qquad D_i^R \equiv \frac{1}{\sqrt{2}}\left(\tilde{D}_i + D_i\right), \\ \mathcal{D}_L &:= \frac{1}{\sqrt{2}}\left(\tilde{\mathcal{D}}-\mathcal{D}\right) = \sum_{i=1}^{3} D_i^L A^i \qquad \text{with} \qquad D_i^L \equiv \frac{1}{\sqrt{2}}\left(\tilde{D}_i - D_i\right). \end{aligned}} \tag{4.30}$$

In order to determine the transformation behaviour of the currents D_i^R and D_i^L under chiral transformations $U(3)_R \times U(3)_L$ we rearrange the currents using the properties of the chiral projection operators P_L and P_R, eq. (1.25), as follows

$$\begin{aligned} D_i^R = \frac{1}{\sqrt{2}}\left(\tilde{D}_i + D_i\right) &= \frac{1}{2}\epsilon_{ijk}\left(q_k^T C\gamma^5 q_j + q_k^T C q_j\right) = \\ &= \epsilon_{ijk} q_k^T C\left[\frac{1}{2}(\gamma^5+1)\right] q_j = \epsilon_{ijk} q_k^T C P_R q_j = \\ &= \epsilon_{ijk} q_k^T C P_R P_R q_j = \epsilon_{ijk} q_{R,k}^T C q_{R,j} \end{aligned}$$

and in analogy:

$$D_i^L = \qquad \ldots \qquad = \epsilon_{ijk} q_{L,k}^T C q_{L,j}. \tag{4.31}$$

We observe that the two diquark currents D_R and D_L behave under chiral transformations as (see eq. (4.24))

$$\begin{aligned} D_i^R &\xrightarrow{U(3)_R\times U(3)_L} D_i^{R\prime} = D_k^R U_{R,ki}^\dagger \qquad \text{with} \quad U_R \in U(3)_R, \\ D_i^L &\xrightarrow{U(3)_R\times U(3)_L} D_i^{L\prime} = D_k^L U_{L,ki}^\dagger \qquad \text{with} \quad U_L \in U(3)_L, \end{aligned} \tag{4.32}$$

and therefore D_i^R transforms like a right-handed and D_i^L like a left-handed antiquark. In order to compute the behaviour under parity transformation and charge conjugation we proceed as in eq. (4.21) and eq. (4.22). We solely have to pay attention to the additional chiral projection operators and the missing γ^5 matrix. Eventually, the mathematical properties of D^R and D^L are

$$
\begin{array}{lcl}
D_{R/L} & \xrightarrow{\quad P \quad} & D^P_{R/L} = -D_{L/R}(t, -\boldsymbol{x}), \\
D_{R/L} & \xrightarrow{\quad C \quad} & D^C_{R/L} = -D^\dagger_{L/R}, \\
D_{R/L} & \xrightarrow{SU(3)_V} & D'_{R/L} = D_{R/L} U^\dagger_V, \\
D_{R/L} & \xrightarrow{U(3)_R \times U(3)_L} & D'_{R/L} = D_{R/L} U^\dagger_{R/L}.
\end{array}
\tag{4.33}
$$

Thus, parity and charge conjugation exchange left- and right-handed diquark currents.

4.7. Baryons as Bound States of a Quark and a Diquark

With the results of the last subsections we are now able to construct baryons as quark-diquark states. Since two different diquarks, D_R and D_L, are available, we can construct at least two baryon fields. The simplest combinations are

$$
\begin{array}{lcl}
N^{(RR)} & \hat{=} & N^{(RR)}_{ij} = D_{R,j} q_i = \dfrac{1}{\sqrt{2}} \left(\tilde{D}_j + D_j \right) q_i \\
N^{(LL)} & \hat{=} & N^{(LL)}_{ij} = D_{L,j} q_i = \dfrac{1}{\sqrt{2}} \left(\tilde{D}_j - D_j \right) q_i
\end{array}
\tag{4.34}
$$

The label N is chosen because these two fields transform in a **n**aive way (as will be shown in the following). The indices $^{(RR)}$ and $^{(LL)}$ refer to the included left- or right-handed (quark content of the) diquark. Note that $N^{(RR)}$ is not right-handed and $N^{(LL)}$ is not left-handed - both are the full fields. Thus $N^{(RR)} = N^{(RR)}_R + N^{(RR)}_L$ and $N^{(LL)} = N^{(LL)}_R + N^{(LL)}_L$.
Two somewhat more complicated combinations are given by

$$
\begin{array}{lcl}
M^{(RR)} & \hat{=} & M^{(RR)}_{ij} = D^R_j \gamma^\mu \partial_\mu q_i = \dfrac{1}{\sqrt{2}} \left(\tilde{D}_j + D_j \right) \gamma^\mu \partial_\mu q_i \\
M^{(LL)} & \hat{=} & M^{(LL)}_{ij} = D^L_j \gamma^\mu \partial_\mu q_i = \dfrac{1}{\sqrt{2}} \left(\tilde{D}_j - D_j \right) \gamma^\mu \partial_\mu q_i,
\end{array}
\tag{4.35}
$$

where the letter M refers to the **m**irror assignment. In comparison to $N^{(RR)}$ and $N^{(LL)}$ we will see that $M^{(RR)}$ and $M^{(LL)}$ transform in a mirror way. The reason for this is the additional gamma matrix. Summarized, by looking at the microscopic decomposition we have found that four multiplets exist: positive-parity and negative-parity naive-transforming baryons and positive-parity and negative-parity mirror-transforming baryons. The presence of four multiplets was postulated in ref. [3] at the composite level of hadrons and can be justified with an study of the of the microscopic currents of the baryonic fields.

4.7.1. Behaviour under Chiral Transformations

Making use of the chiral projection operators P_L and P_R, eq. (1.25), we can determine the left- and right-handed components of $N^{(RR)}$ and $N^{(LL)}$:

$$\begin{aligned}
N^{(RR)}_{R,ij} &= P_R N^{(RR)}_{ij} = D^R_j P_R q_i &&= D^R_j q_{R,i} \\
N^{(RR)}_{L,ij} &= \dots &&= D^R_j q_{L,i} \\
N^{(LL)}_{R,ij} &= \dots &&= D^L_j q_{R,i} \\
N^{(LL)}_{L,ij} &= \dots &&= D^L_j q_{L,i}.
\end{aligned} \tag{4.36}$$

The chiral projectors act only on the quark q, because only the quark carries a spinor index. With the transformation behaviour of the diquarks, eq. (4.33), we immediately find the transformation of the two baryon fields under chiral transformation to be

$$\boxed{\begin{array}{llllll}
N^{(RR)}_R & \xrightarrow{U(3)_R\times U(3)_L} & U_R N^{(RR)}_R U^\dagger_R\,, & \bar{N}^{(RR)}_R & \xrightarrow{U(3)_R\times U(3)_L} & U_R \bar{N}^{(RR)}_R U^\dagger_R \\
N^{(RR)}_L & \xrightarrow{U(3)_R\times U(3)_L} & U_L N^{(RR)}_L U^\dagger_R\,, & \bar{N}^{(RR)}_L & \xrightarrow{U(3)_R\times U(3)_L} & U_R \bar{N}^{(RR)}_L U^\dagger_L \\
N^{(LL)}_R & \xrightarrow{U(3)_R\times U(3)_L} & U_R N^{(LL)}_R U^\dagger_L\,, & \bar{N}^{(LL)}_R & \xrightarrow{U(3)_R\times U(3)_L} & U_L \bar{N}^{(LL)}_R U^\dagger_R \\
N^{(LL)}_L & \xrightarrow{U(3)_R\times U(3)_L} & U_L N^{(LL)}_L U^\dagger_L\,, & \bar{N}^{(LL)}_L & \xrightarrow{U(3)_R\times U(3)_L} & U_L \bar{N}^{(LL)}_L U^\dagger_L.
\end{array}} \tag{4.37}$$

Considering only the chiral transformation matrix $U_{R/L}$ on the left of the right-hand sides of the equations (e.g. the two boxed parts), the left- and right-handed part of the two baryon fields transform in the same way, as it is characteristic for the naive assignment.
In the same manner we find for the fields $M^{(RR)}$ and $M^{(LL)}$ the left- and right-handed components to be

$$\begin{aligned}
M^{(RR)}_R &= P_R M^{(RR)} = D^R \gamma^\mu \partial_\mu P_L q &&= D^R \gamma^\mu \partial_\mu q_L, \\
M^{(RR)}_L &= \dots &&= D^R \gamma^\mu \partial_\mu q_R, \\
M^{(LL)}_R &= \dots &&= D^L \gamma^\mu \partial_\mu q_L, \\
M^{(LL)}_L &= \dots &&= D^L \gamma^\mu \partial_\mu q_R,
\end{aligned}$$

where we have used that the chiral operators switch when commuting them with a gamma matrix, since $\{\gamma^5, \gamma^\mu\} = 0$. The behaviour under chiral transformations is then

$$\boxed{\begin{array}{lll lll}
M_R^{(RR)} & \xrightarrow{U(3)_L\times U(3)_R} & U_L M_R^{(RR)} U_R^\dagger\,, & \bar{M}_R^{(RR)} & \xrightarrow{U(3)_L\times U(3)_R} & U_R \bar{M}_R^{(RR)} U_L^\dagger, \\
M_L^{(RR)} & \xrightarrow{U(3)_L\times U(3)_R} & U_R M_L^{(RR)} U_R^\dagger\,, & \bar{M}_L^{(RR)} & \xrightarrow{U(3)_L\times U(3)_R} & U_R \bar{M}_L^{(RR)} U_R^\dagger, \\
M_R^{(LL)} & \xrightarrow{U(3)_L\times U(3)_R} & U_L M_R^{(LL)} U_L^\dagger\,, & \bar{M}_R^{(LL)} & \xrightarrow{U(3)_L\times U(3)_R} & U_L \bar{M}_R^{(LL)} U_L^\dagger, \\
M_L^{(LL)} & \xrightarrow{U(3)_L\times U(3)_R} & U_R M_L^{(LL)} U_L^\dagger\,, & \bar{M}_L^{(LL)} & \xrightarrow{U(3)_L\times U(3)_R} & U_L \bar{M}_L^{(LL)} U_R^\dagger.
\end{array}} \tag{4.38}$$

Considering only the chiral transformation matrix $U_{R/L}$ on the left of the right-hand sides of the equations (e.g. the boxed parts), $M^{(RR)}$ and $M^{(LL)}$ transform for themselves also in a naive way. But compared to the $N^{(RR)}$ and $N^{(LL)}$ fields, eq. (4.37), they transform in a mirror way.

4.7.2. Parity Transformation

The parity transformation of the quark is given in eq. (3.14) and that of diquarks in eq. (4.33). Hence the baryon fields $N^{(RR)}$ and $N^{(LL)}$ behave under parity transformation as

$$\begin{array}{lllll}
N_R^{(RR)} & = D^R q_R & \xrightarrow{P} & -P_R D^L \gamma^0 q = -\gamma^0 D^L P_L q & = -\gamma^0 N_L^{(LL)}(t,-\boldsymbol{x}), \\
N_L^{(RR)} & = \dots & \xrightarrow{P} & \dots & = -\gamma^0 N_R^{(LL)}(t,-\boldsymbol{x}), \\
N_R^{(LL)} & = \dots & \xrightarrow{P} & \dots & = -\gamma^0 N_L^{(RR)}(t,-\boldsymbol{x}), \\
N_L^{(LL)} & = \dots & \xrightarrow{P} & \dots & = -\gamma^0 N_R^{(RR)}(t,-\boldsymbol{x}).
\end{array} \tag{4.39}$$

Paying attention to the fact that the sign of the spatial derivatives changes ($-\partial_i \to \partial_i$) under parity transformations, we find for $M^{(RR)}$ and $M^{(LL)}$

$$\begin{array}{lllll}
M_R^{(RR)} = & D^R \gamma^\mu \partial_\mu q_L & \xrightarrow{P} & -P_R D^L \left(\gamma^0 \partial_0 + \gamma^i \partial_i\right) \gamma^0 q = -\gamma^0 D^L \gamma^\mu \partial_\mu q_R & = -\gamma^0 M_L^{(LL)}(t,-\boldsymbol{x}), \\
M_L^{(RR)} = & \dots & \xrightarrow{P} & \dots & = -\gamma^0 M_R^{(LL)}(t,-\boldsymbol{x}), \\
M_R^{(LL)} = & \dots & \xrightarrow{P} & \dots & = -\gamma^0 M_L^{(RR)}(t,-\boldsymbol{x}), \\
M_L^{(LL)} = & \dots & \xrightarrow{P} & \dots & = -\gamma^0 M_R^{(RR)}(t,-\boldsymbol{x}).
\end{array} \tag{4.40}$$

Thus the parity transformation of $M^{(RR)}$ and $M^{(LL)}$ is identical to that of $N^{(RR)}$ and $N^{(LL)}$

4.7.3. Charge Conjugation

The charge conjugation of the quark is given in eq. (3.14) and that of diquarks in eq. (4.33). We compute the transformation behaviour of $N^{(RR)}$ and $N^{(LL)}$ under charge conjugation to be

$$\begin{array}{llcrl} N_R^{(RR)} & = D^R q_R & \xrightarrow{C} & -P_R(D^L)^\dagger C\bar{q}^T = -C\gamma^0(D^L)^\star q_L^\star & = -i\gamma^2\left(N_L^{(LL)}\right)^\star \\ N_L^{(RR)} & = \dots & \xrightarrow{C} & \dots & = -i\gamma^2\left(N_R^{(LL)}\right)^\star \\ N_R^{(LL)} & = \dots & \xrightarrow{C} & \dots & = -i\gamma^2\left(N_L^{(RR)}\right)^\star \\ N_L^{(LL)} & = \dots & \xrightarrow{C} & \dots & = -i\gamma^2\left(N_R^{(RR)}\right)^\star , \end{array} \tag{4.41}$$

where we used the fact that the diquark current is a scalar in spinor space and therefore commutes with the charge-conjugation matrix. In the last step we wrote the charge conjugation matrix in Dirac representation, $C = i\gamma^2\gamma^0$. For $M^{(RR)}$ and $M^{(LL)}$ we get in Dirac representation:

$$\begin{array}{llcrl} M_R^{(RR)} & = D^R\gamma^\mu\partial_\mu q_L & \xrightarrow{C} & -P_R(D^L)^\dagger\gamma^\mu\partial_\mu C\bar{q}^T & = \\ & & & = -i\gamma^2(D^L)^\star(-\gamma^0\partial_0+\gamma^1\partial_1-\gamma^2\partial_2+\gamma^3\partial_3)q_R^\star & = i\gamma^2\left(M_L^{(LL)}\right)^\star , \\ M_L^{(RR)} & = \dots & \xrightarrow{C} & \dots & = i\gamma^2\left(M_R^{(LL)}\right)^\star , \\ M_R^{(LL)} & = \dots & \xrightarrow{C} & \dots & = i\gamma^2\left(M_L^{(RR)}\right)^\star , \\ M_L^{(LL)} & = \dots & \xrightarrow{C} & \dots & = i\gamma^2\left(M_R^{(RR)}\right)^\star , \end{array} \tag{4.42}$$

where we paid attention to $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{4\times4}$ and the behaviour of the gamma matrices under complex conjugation: $(\gamma^{0,1,3,5})^* = \gamma^{0,1,3,5}$ and $(\gamma^2)^* = -\gamma^2$.
In the following we will compute the charge conjugation always in Dirac representation.

5. Baryon Lagrangian for $N_f = 3$

As mentioned earlier, when constructing a Lagrangian containing the baryon and its chiral partner we have two possibilities to introduce the chiral partner: the so-called naive and the mirror assignment. In the naive assignment alone a mass term of the baryon and its chiral parter is not chirally invariant and thus only the spontaneous symmetry breaking generates a mass. In contrast, in the mirror assignment the construction of a chirally invariant mass term is possible. Of course, the effective Lagrangian has to exhibit all properties of the QCD Lagrangian.
In our model we have four baryonic fields, $N^{(RR)}$ and $N^{(LL)}$, eq. (4.34), and $M^{(RR)}$ and $M^{(LL)}$, eq. (4.35). Each of the N and M fields transform in the naive way, but compared to each other they transform in a mirror way. This fact is clarified in table 5.1:

$N^{(RR)}$	$M^{(RR)}$	$\longleftarrow$ mirror
$N^{(LL)}$	$M^{(LL)}$	$\longleftarrow$ mirror
$\uparrow$ naive	$\uparrow$ naive	

Table 5.1.: Clarification of the naive and mirror transformation of the N and M fields of our model.

Therefore the Lagrangian that we will construct is a composite of two naive parts (one for the N and one for M fields) and a term which mixes N and M fields:

$$\boxed{\mathcal{L} = \mathcal{L}_{\text{naive},N} + \mathcal{L}_{\text{naive},M} + \mathcal{L}_{\text{mirror},NM}\,. \qquad (5.1)}$$

Since our effective model will contain colorless hadrons as degrees of freedom, color symmetry is fulfilled by default. In order to obtain the Lagrangian we work in the chiral limit, in which all quark masses are zero and chiral symmetry is exact. Explicit breaking of chiral symmetry and chiral $U(1)_A$ anomaly is included in the mesonic Lagrangian of chapter 3.

5.1. Baryonic Lagrangian for the Naive-Transforming Fields

The part of the Lagrangian (5.1) in the naive assignment will contain two baryonic fields which are combinations of the physical fields. The left- and the right-handed components of these two fields are assumed to behave in the same way under chiral transformations. Therefore, either the two fields $N^{(RR)}$ and $N^{(LL)}$, eq. (4.34), or $M^{(RR)}$ and $M^{(LL)}$, eq. (4.35), are suitable (but not all four together).

In this section we will present the naive parts $\mathcal{L}_{\text{naive},N}$ and $\mathcal{L}_{\text{naive},M}$. We start with $\mathcal{L}_{\text{naive},N}$ which contains the **two fields $\boldsymbol{N^{(RR)}}$ and $\boldsymbol{N^{(LL)}}$ only**.
Furthermore, in order to describe baryon-meson interactions, we use the meson field Φ, eq. (3.1) and the left- and right-handed fields R^μ and L^μ, eqs. (3.24), (3.25). The behaviour under chiral transformation of these fields is summarized (see eq. (3.2), (3.6) and (4.37)) in table 5.2:

current	chiral transformation
$N_R^{(RR)}$	$U_R N_R^{(RR)} U_R^\dagger$
$N_L^{(RR)}$	$U_L N_L^{(RR)} U_R^\dagger$
$N_R^{(LL)}$	$U_R N_R^{(LL)} U_L^\dagger$
$N_L^{(LL)}$	$U_L N_L^{(LL)} U_L^\dagger$
Φ	$U_L \Phi U_R^\dagger$
R^μ	$U_R R^\mu U_R^\dagger$
L^μ	$U_L L^\mu U_L^\dagger$

Table 5.2.: Behaviour of the fields under chiral transformations.

With these fields we are able to construct a chirally invariant hermitian Lagrangian for baryons interacting with (pseudo)scalar and (axial-)vector mesons:

$$\begin{aligned}\mathcal{L}_{\text{naive, }N} = \mathrm{Tr}\Big\{ & \bar{N}_R^{(RR)} i\gamma_\mu D_{1R}^\mu N_R^{(RR)} + \bar{N}_L^{(RR)} i\gamma_\mu D_{2L}^\mu N_L^{(RR)} + \\ & + \bar{N}_R^{(LL)} i\gamma_\mu D_{3R}^\mu N_R^{(LL)} + \bar{N}_L^{(LL)} i\gamma_\mu D_{4L}^\mu N_L^{(LL)} \Big\} + \\ & - g_1 \,\mathrm{Tr}\Big\{ \bar{N}_L^{(RR)} \Phi N_R^{(RR)} + \bar{N}_R^{(RR)} \Phi^\dagger N_L^{(RR)} \Big\} + \\ & - g_2 \,\mathrm{Tr}\Big\{ \bar{N}_L^{(LL)} \Phi N_R^{(LL)} + \bar{N}_R^{(LL)} \Phi^\dagger N_L^{(LL)} \Big\} + \mathcal{L}_{\text{meson}},\end{aligned} \tag{5.2}$$

with the covariant derivatives

$$D_{kR}^\mu = \partial^\mu - i c_k R^\mu \quad , \qquad D_{kL}^\mu = \partial^\mu - i c_k L^\mu, \qquad \text{with } k \in \{1,2,3,4\}, \tag{5.3}$$

where c_k, $k \in \{1,2,3,4\}$, are the baryon-(axial-)vector coupling constants and the coupling of baryons and (pseudo)scalar mesons is parametrized by g_1 and g_2. $\mathcal{L}_{\text{meson}}$ is given in eq. (3.12). As mentioned earlier it is not possible to construct a chirally invariant mass term in the naive assignment. For instance

$$\mathrm{Tr}(\bar{N}_R^{(RR)} N_R^{(RR)}) = \mathrm{Tr}(\bar{N}^{(RR)} P_L P_R N^{(RR)})$$

is admittedly chirally invariant, but since $P_R P_L = 0$, terms of this form vanish. Furthermore, one should note that it is not possible to construct mixing terms of $N^{(RR)}$ and $N^{(LL)}$, because they are

not chirally invariant or elicit a coupling constant which is not dimensionless[1] and therefore will break dilatation symmetry. This absence of mixing terms is in contrast to the case of two flavors studied in ref. [3] and, as we will see later, leads to degenerate masses.
The Lagrangian given in eq. (5.2) is hermitian and chirally invariant, but it is not yet invariant under parity transformations. We have to constrain the parameters to achieve this invariance. The parity transformation of the covariant derivatives is given by

$$\begin{aligned} D^\mu_{kR} &= \partial^\mu - ic_k R^\mu \quad \xrightarrow{\ P\ } \quad (D^\mu_{kR})' = (\partial^0, \partial^i)^T + ic_k(-L^0, L^i)^T, \\ D^\mu_{kL} &= \partial^\mu - ic_k L^\mu \quad \xrightarrow{\ P\ } \quad (D^\mu_{kL})' = (\partial^0, \partial^i)^T + ic_k(-R^0, R^i)^T, \end{aligned} \tag{5.4}$$

where we used $(-\partial_i) \xrightarrow{P} \partial_i$ and the transformation of the left- and right-handed fields L^μ and R^μ, given in eq. (3.22). An exemplary one of the kinetic terms with covariant derivative in the Lagrangian (5.2) has the following form after parity transformation:

$$\begin{aligned} \mathcal{L}^{\text{kin}}_{\text{naive, }N} &= \bar{N}^{(RR)}_L i\gamma_\mu D_{1L} N^{(RR)}_L \\ \xrightarrow{\ P\ } \quad \mathcal{L}^{\text{kin},P}_{\text{naive, }N} &= \mathrm{Tr}\Big\{-\bar{N}^{(LL)}_L \gamma^0 i\gamma_\mu \left[\left(\partial^0, \partial^i\right)^T - ic_1\left(L^0, -L^i\right)^T\right]\left(-\gamma^0 N^{(LL)}_L\right)\Big\} \\ &= \mathrm{Tr}\Big\{\bar{N}^{(LL)}_L i\gamma_\mu D_{1L} N^{(LL)}_L\Big\}, \end{aligned} \tag{5.5}$$

where we have used that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$ and $(\gamma^0)^2 = \mathbb{1}$. The remaining kinetic terms with covariant derivatives transform in the same manner. Comparing the transformed and original kinetic and baryon-(axial-)vector-interaction Lagrangian we realize that the constants c_1 and c_4 and the constants c_2 and c_3 have to be equal. The parity transformation of the remaining part of the Lagrangian (5.2), the baryon-(pseudo)vector interaction, is also easy to compute. For example the transformation of one of those terms reads

$$\begin{aligned} \mathcal{L}^{N\Phi N}_{\text{naive, }N} &= -g_1\,\mathrm{Tr}\Big\{\bar{N}^{(RR)}_R \Phi^\dagger N^{(RR)}_L\Big\} \\ \xrightarrow{\ P\ } \quad \mathcal{L}^{N\Phi N,P}_{\text{naive, }N} &= -\,g_1\,\mathrm{Tr}\Big\{-\bar{N}^{(LL)}_R \gamma^0 \Phi^\dagger\left(-\gamma^0 N^{(LL)}_L\right)\Big\} = \\ &= -\,g_1\,\mathrm{Tr}\Big\{\bar{N}^{(LL)}_R \Phi^\dagger N^{(LL)}_L\Big\}. \end{aligned} \tag{5.6}$$

In analogy we find the transformation of the remaining parts and finally determine that the constants g_1 and g_2 also have to be equal. Thus, we set

$$c_4 = c_1\ , \quad c_3 = c_2 \quad \text{and} \quad g_1 = g_2 \equiv g, \tag{5.7}$$

so that the Lagrangian (5.2) is invariant under parity transformations. As a consequence the La-

[1] See also the discussion of trace anomaly and of the construction of the meson Lagrangian.

grangian simplifies:

$$
\begin{aligned}
\mathcal{L}_{\text{naive, }N} = \quad & \mathrm{Tr}\Big\{ \bar{N}_R^{(RR)} i\gamma_\mu D_{1R}^\mu N_R^{(RR)} + + \bar{N}_L^{(RR)} i\gamma_\mu D_{2L}^\mu N_L^{(RR)} + \\
& + \bar{N}_R^{(LL)} i\gamma_\mu D_{2R}^\mu N_R^{(LL)} + \bar{N}_L^{(LL)} i\gamma_\mu D_{1L}^\mu N_L^{(LL)} \Big\} + \\
& - g\,\mathrm{Tr}\Big\{ \bar{N}_L^{(RR)} \Phi N_R^{(RR)} + \bar{N}_R^{(RR)} \Phi^\dagger N_L^{(RR)} + \bar{N}_L^{(LL)} \Phi N_R^{(LL)} + \bar{N}_R^{(LL)} \Phi^\dagger N_L^{(LL)} \Big\} + \mathcal{L}_{\text{meson}}.
\end{aligned} \tag{5.8}
$$

Hence we found a baryonic Lagrangian which is hermitian and invariant under chiral and parity transformations.
As a next step, we verify the invariance under charge conjugation. In order to show this we compute the charge conjugation of eq. (5.8) by plugging into it the charge conjugation of the fields $N^{(RR)}$ and $N^{(LL)}$, making use of the fact that the trace of a transposed matrix is the same as the trace of the matrix itself. Furthermore, one has to pay attention to the fact that interchanging two identical fermions via transposition yields a minus sign, because of Pauli-Dirac statistics. In the following we show as an example the transformation of one kinetic term:

$$
\begin{aligned}
\mathcal{L}^{\text{kin}}_{\text{naive, }N} &= \mathrm{Tr}\Big\{ \bar{N}_R^{(RR)} i\gamma_\mu D_{1R}^\mu N_R^{(RR)} \Big\} = \mathrm{Tr}\Big\{ \bar{N}_R^{(RR)} i\gamma_\mu \left[\partial_\mu - ic_1 R^\mu)\right] N_R^{(RR)} \Big\} \\
\xrightarrow{C} \mathcal{L}^{\text{kin},C}_{\text{naive, }N} &= \mathrm{Tr}\Big\{ -i(N_L^{(LL)})^T \gamma^2\gamma^0 i\gamma_\mu \left[\partial_\mu + ic_1 (L^\mu)^T\right] (-i\gamma^2)(N_L^{(LL)})^\star + \dots \Big\} = \\
&= \mathrm{Tr}\Big\{ -(N_L^{(LL)})^T iC\gamma_\mu C^{-1} \left[\partial_\mu + ic_1 (L^\mu)^T\right] \gamma^0 (N_L^{(LL)})^\star \Big\} = \\
&= \mathrm{Tr}\Big\{ -(N_L^{(LL)})^T i(-\gamma^\mu)^T \left[\partial_\mu + ic_1 (L^\mu)^T\right] \gamma^0 (N_L^{(LL)})^\star \Big\} = \\
&= \mathrm{Tr}\Big\{ (N_L^{(LL)})^T (i\gamma^\mu)^T \gamma^0 \partial_\mu (N_L^{(LL)})^\star + (N_L^{(LL)})^T (i\gamma^\mu)^T ic_1 (L^\mu)^T \gamma^0 (N_L^{(LL)})^\star \Big\} = \\
&= \mathrm{Tr}\Big\{ \left[(N_L^{(LL)})^T (i\gamma^\mu)^T \gamma^0 \partial_\mu (N_L^{(LL)})^\star\right]^T + \left[(N_L^{(LL)})^T (i\gamma^\mu)^T ic_1 (L^\mu)^T \gamma^0 (N_L^{(LL)})^\star\right]^T \Big\} = \\
&= \mathrm{Tr}\Big\{ -\left(\partial_\mu \bar{N}_L^{(LL)}\right) i\gamma^\mu N_L^{(LL)} - \bar{N}_L^{(LL)} i\gamma^\mu ic_1 L^\mu N_L^{(LL)} \Big\} = \\
&= \mathrm{Tr}\Big\{ \bar{N}_L^{(LL)} i\gamma^\mu \partial_\mu N_L^{(LL)} - \bar{N}_L^{(LL)} i\gamma^\mu ic_1 L^\mu N_L^{(LL)} \Big\} = \\
&= \mathrm{Tr}\Big\{ \bar{N}_L^{(LL)} i\gamma_\mu D_{1L}^\mu N_L^{(LL)} \Big\},
\end{aligned} \tag{5.9}
$$

where we performed an integration by parts in the second last line. One of the interaction terms of the Lagrangian transforms as

$$\begin{aligned}\mathcal{L}^{N\Phi N}_{\text{naive, }N} = -g\,\mathrm{Tr}\Big\{\bar{N}_L^{(RR)}\Phi N_R^{(RR)}\Big\}\\ \xrightarrow{C} \mathcal{L}^{N\Phi N,C}_{\text{naive, }N} &= -\,g\,\mathrm{Tr}\Big\{-i(N_R^{(LL)})^T\gamma^2\gamma^0\Phi^T(-i\gamma^2)(N_L^{(LL)})^\star + \dots\Big\} =\\ &= -\,g\,\mathrm{Tr}\Big\{-(N_R^{(LL)})^T(-\gamma^0)\Phi^T\gamma^2\gamma^2(N_L^{(LL)})^\star\Big\} =\\ &= -\,g\,\mathrm{Tr}\Big\{-(N_R^{(LL)})^T\Phi^T\gamma^0(N_L^{(LL)})^\star\Big\} =\\ &= -\,g\,\mathrm{Tr}\Big\{\Big[-(N_R^{(LL)})^T\Phi^T\gamma^0(N_L^{(LL)})^\star\Big]^T\Big\} =\\ &= -\,g\,\mathrm{Tr}\Big\{\bar{N}_L^{(LL)}\Phi N_R^{(LL)}\Big\}. \end{aligned} \tag{5.10}$$

Since the resulting terms exist exactly in this form in the original Lagrangian (5.8) and the same holds for the remaining terms, it is shown that the Lagrangian is invariant under charge conjugation also. For that reason we have found eq. (5.8) to be the final Lagrangian. Making use of the chiral projection operators, eq. (1.25), we rearrange it to

$$\begin{aligned}\mathcal{L}_{\text{naive, }N} =\, &\mathrm{Tr}\Big\{\bar{N}^{(RR)}i\gamma_\mu\frac{1}{2}\left[D^\mu_{1R}+D^\mu_{2L}+\gamma^5\left(D^\mu_{1R}-D^\mu_{2L}\right)\right]N^{(RR)}+\\ &+\bar{N}^{(LL)}i\gamma_\mu\frac{1}{2}\left[D^\mu_{2R}+D^\mu_{1L}+\gamma^5\left(D^\mu_{2R}-D^\mu_{1L}\right)\right]N^{(LL)}\Big\}+\\ &-g\,\mathrm{Tr}\Big\{\bar{N}^{(RR)}\frac{1}{2}\left[\Phi+\Phi^\dagger+\gamma^5\left(\Phi-\Phi^\dagger\right)\right]N^{(RR)}+\\ &+\bar{N}^{(LL)}\frac{1}{2}\left[\Phi+\Phi^\dagger+\gamma^5\left(\Phi-\Phi^\dagger\right)\right]N^{(LL)}\Big\}+\mathcal{L}_{\text{meson}}.\end{aligned} \tag{5.11}$$

This is the final Lagrangian including baryons in naive assignment expressed by the "full" fields (and not by the left- and right-handed components).
When assigning the included fields with the experimentally observed particles, we observe a problem. We take a closer look at the parity transformation, eq. (4.39), of the baryon fields $N^{(RR)}$ and $N^{(LL)}$: They transform into each other under parity and therefore they do not have a well-defined parity, so that we are not able to identify them with physical states. For that reason we will define new fields as linear combinations of the old ones, $N^{(RR)}$ and $N^{(LL)}$. We use the fact that Dq has positive parity (since the scalar diquark has positive parity and the quark has also (by convention) positive parity (eq. 3.14)), and that $\tilde{D}q$ has negative parity (since the pseudoscalar diquark has negative parity). Furthermore we know that $(N^{(RR)}-N^{(LL)}) \sim Dq$ and $(N^{(RR)}+N^{(LL)}) \sim \tilde{D}q$ and that adding a γ^5 to any term inverses its parity. Therefore we can construct a field with positive parity,

$$\boxed{B_N = \frac{N^{(RR)}-N^{(LL)}}{\sqrt{2}} \equiv Dq,} \tag{5.12}$$

which describes baryons with $J^P=\frac{1}{2}^+$, and a field with negative parity,

$$\boxed{B_{N\star} = \frac{N^{(RR)} + N^{(LL)}}{\sqrt{2}} \equiv \tilde{D}q,} \tag{5.13}$$

which describes the excited baryons with $J^P = \frac{1}{2}^-$. Note that one could also construct linear combinations with arbitrary mixing angle:

$$B_N = \cos\vartheta \overbrace{\frac{1}{\sqrt{2}}\left(N^{(RR)} - N^{(LL)}\right)}^{=Dq \text{ (pos. parity)}} + \sin\vartheta \overbrace{\gamma^5}^{\text{parity switch}} \overbrace{\frac{1}{\sqrt{2}}\left(N^{(RR)} + N^{(LL)}\right)}^{=\tilde{D}q \text{ (neg. parity)}}, \tag{5.14}$$

and

$$B_{N\star} = -\sin\vartheta \overbrace{\gamma^5}^{\text{parity switch}} \overbrace{\frac{1}{\sqrt{2}}\left(N^{(RR)} - N^{(LL)}\right)}^{=Dq \text{ (pos. parity)}} + \cos\vartheta \overbrace{\frac{1}{\sqrt{2}}\left(N^{(RR)} + N^{(LL)}\right)}^{=\tilde{D}q \text{ (neg. parity)}}, \tag{5.15}$$

but that would only complicate the following calculations. Remember that B_N and $B_{N\star}$ are still "bare fields" which will mix later on.
Inverting the two equations (5.12) and (5.13) yields

$$N^{(RR)} = \frac{B_{N\star} + B_N}{\sqrt{2}} \qquad \text{and} \qquad N^{(LL)} = \frac{B_{N\star} - B_N}{\sqrt{2}}. \tag{5.16}$$

Plugging this into eq. (5.11) gives the naive baryonic Lagrangian for B_N and $B_{N\star}$:

$$\boxed{\begin{aligned}\mathcal{L}_{\text{naive},N} =& \frac{1}{2}\operatorname{Tr}\Big\{\bar{B}_N i\gamma_\mu \left[D_R^\mu + D_L^\mu + \gamma^5\left(D_R^\mu - D_L^\mu\right)\right] B_N + \bar{B}_{N\star} i\gamma_\mu \left[D_R^\mu + D_L^\mu + \gamma^5\left(D_R^\mu - D_L^\mu\right)\right] B_{N\star}\Big\} + \\ &+ \frac{c_A}{2}\operatorname{Tr}\Big\{\bar{B}_N\gamma^\mu\left[R^\mu - L^\mu + \gamma^5\left(R^\mu + L^\mu\right)\right] B_{N\star} + \bar{B}_{N\star}\gamma^\mu\left[R^\mu - L^\mu + \gamma^5\left(R^\mu + L^\mu\right)\right] B_N\Big\} + \\ &- \frac{g}{2}\operatorname{Tr}\Big\{\bar{B}_N\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_N + \bar{B}_{N\star}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_{N\star}\Big\},\end{aligned}} \tag{5.17}$$

with the covariant derivatives

$$D_R^\mu = \partial^\mu - icR^\mu \quad \text{and} \quad D_L^\mu = \partial^\mu - icL^\mu \qquad , c := c_1 + c_2, \tag{5.18}$$

and

$$c_A := c_1 - c_2. \tag{5.19}$$

With eqs. (5.12) and (5.13) we can find a matrix structure of the two baryon fields in analogy to the construction of the meson matrix (3.23). We can make use of the fact that diquarks correspond to antiquarks (as mentioned in eq. (4.26)):

$$D \text{ and } \tilde{D} \sim ([d,s], -[u,s], [u,d]) \leftrightarrow (\bar{u}, \bar{d}, \bar{s}). \tag{5.20}$$

Therefore the (not symmetrized) quark flavor content of the baryon matrices reads

$$B_N \sim \begin{pmatrix} [d,s] \\ -[u,s] \\ [u,d] \end{pmatrix} (u,d,s) \equiv \begin{pmatrix} u[d,s] & -u[u,s] & u[u,d] \\ d[d,s] & -d[u,s] & d[u,d] \\ s[d,s] & -s[u,s] & s[u,d] \end{pmatrix} \hat{=} \begin{pmatrix} uds & uus & uud \\ dds & uds & udd \\ dss & uss & uds \end{pmatrix} . \tag{5.21}$$

For example the (1 3)-element corresponds to the proton and the (2 3)-element to the neutron. So the whole matrix can be assigned as follows:

$$B_N \equiv \begin{pmatrix} \frac{\Lambda}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\ \Sigma^- & \frac{\Lambda}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix} . \tag{5.22}$$

5.2. Baryonic Lagrangian for the Mirror-Transforming Fields

We repeat the same steps to reveal the Lagrangian for the fields $M^{(RR)}$ and $M^{(LL)}$. Fulfilling chiral symmetry, parity, and charge conjugation (and therefore because of the CPT-theorem also time reversal) it reads:

$$\begin{aligned}
\mathcal{L}_{\text{naive, } M} = \mathrm{Tr}\Big\{ & \bar{M}_R^{(RR)} i\gamma_\mu D^\mu_{1_M L} M_R^{(RR)} + + \bar{M}_L^{(RR)} i\gamma_\mu D^\mu_{2_M R} M_L^{(RR)} + \\
& + \bar{M}_R^{(LL)} i\gamma_\mu D^\mu_{2_M L} M_R^{(LL)} + \bar{M}_L^{(LL)} i\gamma_\mu D^\mu_{1_M R} M_L^{(LL)} \Big\} + \\
- g_M \, \mathrm{Tr}\Big\{ & \bar{M}_L^{(RR)} \Phi^\dagger M_R^{(RR)} + \bar{M}_R^{(RR)} \Phi M_L^{(RR)} + \bar{M}_L^{(LL)} \Phi^\dagger M_R^{(LL)} + \bar{M}_R^{(LL)} \Phi M_L^{(LL)} \Big\} \\
= \mathrm{Tr}\Big\{ & \bar{M}^{(RR)} i\gamma_\mu \frac{1}{2} \left[D^\mu_{1R} + D^\mu_{2L} - \gamma^5 \left(D^\mu_{1R} - D^\mu_{2L} \right) \right] M^{(RR)} + \\
& + \bar{M}^{(LL)} i\gamma_\mu \frac{1}{2} \left[D^\mu_{2R} + D^\mu_{1L} - \gamma^5 \left(D^\mu_{2R} - D^\mu_{1L} \right) \right] M^{(LL)} \Big\} + \\
- g_M \, \mathrm{Tr}\Big\{ & \bar{M}^{(RR)} \frac{1}{2} \left[\Phi + \Phi^\dagger - \gamma^5 \left(\Phi - \Phi^\dagger \right) \right] M^{(RR)} + \\
& + \bar{M}^{(LL)} \frac{1}{2} \left[\Phi + \Phi^\dagger - \gamma^5 \left(\Phi - \Phi^\dagger \right) \right] M^{(LL)} \Big\} .
\end{aligned} \tag{5.23}$$

The covariant derivatives are $D^\mu_{k_M R} = \partial^\mu - i c_{k_M} R^\mu$ and $D^\mu_{k_M L} = \partial^\mu - i c_{k_M} L^\mu$ for $k_M \in \{1,2\}$. c_{k_M} are the baryon-(axial-)vector coupling constants and the coupling of baryons and (pseudo)scalar mesons is parametrized by g_M. The only difference to the Lagrangian containing $N^{(RR)}$ and $N^{(LL)}$, eq. (5.11), is a minus sign in front of the γ^5 terms which occurs to achieve chiral invariance for the combinations of the (now mirror-transforming) M fields. Also in this case we introduce fields with well-defined parity:

$$B_M := \frac{M^{(RR)} - M^{(LL)}}{\sqrt{2}} \qquad \text{and} \qquad B_{M,\star} := \frac{M^{(RR)} + M^{(LL)}}{\sqrt{2}}. \tag{5.24}$$

Again, these fields are not yet physical, because of the remaining term $\mathcal{L}_{\text{mirror},NM}$ in eq. (5.1), which generates a mixing.
Therefore the final expression for the Lagrangian for the mirror-transforming fields, which contains now the two physical fields B_M and $B_{M\star}$ reads:

$$\begin{aligned}
\mathcal{L}_{\text{naive},M} =& \frac{1}{2}\,\mathrm{Tr}\Big\{\bar{B}_M i\gamma_\mu \left[D^\mu_{MR} + D^\mu_{ML} - \gamma^5\left(D^\mu_{MR} - D^\mu_{ML}\right)\right] B_M + \\
&\qquad + \bar{B}_{M\star} i\gamma_\mu \left[D^\mu_{MR} + D^\mu_{ML} - \gamma^5\left(D^\mu_{MR} - D^\mu_{ML}\right)\right] B_{M\star}\Big\} + \\
&+ \frac{c_{A_M}}{2}\,\mathrm{Tr}\Big\{\bar{B}_M \gamma^\mu \left[R^\mu - L^\mu - \gamma^5\left(R^\mu + L^\mu\right)\right] B_{M\star} + \\
&\qquad + \bar{B}_{M\star} \gamma^\mu \left[R^\mu - L^\mu - \gamma^5\left(R^\mu + L^\mu\right)\right] B_M\Big\} + \\
&- \frac{g}{2}\,\mathrm{Tr}\Big\{\bar{B}_M \left[\Phi + \Phi^\dagger - \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_M + \bar{B}_{M\star}\left[\Phi + \Phi^\dagger - \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_{M\star}\Big\}.
\end{aligned} \tag{5.25}$$

with the covariant derivatives

$$D^\mu_{MR} = \partial^\mu - ic_M R^\mu \quad \text{and} \quad D^\mu_{ML} = \partial^\mu - ic_M L^\mu, \tag{5.26}$$

where

$$c_M := c_{1_M} + c_{2_M} \qquad \text{and} \qquad c_{A_M} := c_{1_M} - c_{2_M}. \tag{5.27}$$

Finally we have a closer look at the resulting baryonic Lagrangians, eqs. (5.17) and (5.25). As it is characteristic for the naive assignment, the mass of the particles is solely generated by spontaneous symmetry breaking (SSB). This means that the mass terms arise after the condensation of the sigma fields ($\sigma_N \to \sigma_N + \varphi_N$ and $\sigma_S \to \sigma_S + \varphi_S$) included in the (pseudo)scalar meson matrix Φ.
Comparing now the two terms in the Lagrangian (5.17), which contribute to the mass of B_N and $B_{N\star}$ after SSB,

$$\frac{g}{2}\,\mathrm{Tr}\Big\{\bar{B}_N\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_N\Big\} \qquad \text{and} \qquad \frac{g}{2}\,\mathrm{Tr}\Big\{\bar{B}_{N\star}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_{N\star}\Big\}, \tag{5.28}$$

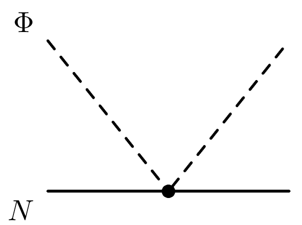
we realize that both terms are identical. As a consequence, the masses of B_N and $B_{N\star}$ will be degenerate. This is a problem, because experiments show that the exited $J^P = (1/2)^-$ baryons have a larger mass than the $J^P = (1/2)^+$ baryons. Of course our Lagrangian should cope with this fact. As a consequence, the Lagrangian of the above form in naive assignment cannot be correct on its own. We need further terms which generate a mass shift. In $\mathcal{L}_{\text{mirror},NM}$ of eq. (5.1) terms are included, which mix N and M fields. But, as we will see in chapter 5.4, these terms will not be sufficient to generate a mass splitting.

5.3. Additional λ-Terms

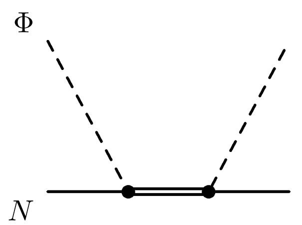
In order to generate a mass shift, we add the following chirally invariant terms to our naive Lagrangian (5.17):

$$\begin{aligned}\mathcal{L}^{\text{mix}}_{\text{naive},N} = &-\lambda_1 \operatorname{Tr}\left\{\bar{N}_L^{(LL)}\Phi N_R^{(RR)}\Phi^\dagger + \bar{N}_R^{(RR)}\Phi^\dagger N_L^{(LL)}\Phi\right\} + \\ &- \lambda_2 \operatorname{Tr}\left\{\bar{N}_R^{(LL)}\Phi^\dagger N_L^{(RR)}\Phi^\dagger + \bar{N}_L^{(RR)}\Phi N_R^{(LL)}\Phi\right\} + \\ &- \tilde{\lambda_1}\left(\operatorname{Tr}\left\{\bar{N}_L^{(LL)}\Phi\right\}\operatorname{Tr}\left\{N_R^{(RR)}\Phi^\dagger\right\} + \operatorname{Tr}\left\{\bar{N}_R^{(RR)}\Phi^\dagger\right\}\operatorname{Tr}\left\{N_L^{(LL)}\Phi\right\}\right) + \\ &- \tilde{\lambda_2}\left(\operatorname{Tr}\left\{\bar{N}_R^{(LL)}\Phi^\dagger\right\}\operatorname{Tr}\left\{N_L^{(RR)}\Phi^\dagger\right\} + \operatorname{Tr}\left\{\bar{N}_L^{(RR)}\Phi\right\}\operatorname{Tr}\left\{N_R^{(LL)}\Phi\right\}\right). \end{aligned} \tag{5.29}$$

They produce a mixing of $N^{(RR)}$ and $N^{(LL)}$ after spontaneous symmetry breaking. Besides chiral invariance, these terms are naturally Lorentz scalars and CP-invariant. However, the constants λ_1, λ_2, $\tilde{\lambda_1}$ and $\tilde{\lambda_2}$ have dimension $[E^{-1}]$ and therefore break dilatation symmetry. $\mathcal{L}_{\text{mix}}$ represents a coupling of the form



which is not dilatation invariant. But such terms (may) arise due to the fact that we have not included heavier baryons (as for instance baryons with spin $J = 3/2$). Namely, the upper diagram might be a "squashed" version of the following diagram



which is dilatation invariant. (The field denoted by the double line refers to a heavier baryonic field which is not included in the present version of the model.)
Thus, we will use $\mathcal{L}^{\text{mix}}_{\text{naive},N}$ as a mixing term for the Lagrangian in the naive assignment. After replacing $N^{(RR)}$ and $N^{(LL)}$ with the field B_N and $B_{N\star}$, eqs. (5.12) and (5.13), and adding these term to eq. (5.17), we find

$$
\begin{aligned}
\mathcal{L}_{\text{naive},N+\lambda} &= \mathcal{L}_{\text{naive},N} + \mathcal{L}^{\text{mix}}_{\text{naive},N} = \\
&= \frac{1}{2}\operatorname{Tr}\Big\{\bar{B}_N i\gamma_\mu\left[D^\mu_R + D^\mu_L + \gamma^5\left(D^\mu_R - D^\mu_L\right)\right]B_N + \bar{B}_{N\star} i\gamma_\mu\left[D^\mu_R + D^\mu_L + \gamma^5\left(D^\mu_R - D^\mu_L\right)\right]B_{N\star}\Big\} + \\
&+ \frac{c_{A_N}}{2}\operatorname{Tr}\Big\{\bar{B}_N i\gamma^\mu\left[R^\mu - L^\mu + \gamma^5\left(R^\mu + L^\mu\right)\right]B_{N\star} + \bar{B}_{N\star} i\gamma^\mu\left[R^\mu - L^\mu + \gamma^5\left(R^\mu + L^\mu\right)\right]B_N\Big\} + \\
&- \frac{g}{2}\operatorname{Tr}\Big\{\bar{B}_N\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right]B_N + \bar{B}_{N\star}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right]B_{N\star}\Big\} + \\
&- \frac{\lambda_1}{2}\operatorname{Tr}\Big\{\bar{B}_{N\star}\Phi P_R B_{N\star}\Phi^\dagger + \bar{B}_{N\star}\Phi P_R B_N\Phi^\dagger - \bar{B}_N\Phi P_R B_{N\star}\Phi^\dagger - \bar{B}_N\Phi P_R B_N\Phi^\dagger + \\
&\qquad + \bar{B}_{N\star}\Phi^\dagger P_L B_{N\star}\Phi - \bar{B}_{N\star}\Phi^\dagger P_L B_N\Phi + \bar{B}_N\Phi^\dagger P_L B_{N\star}\Phi - \bar{B}_N\Phi^\dagger P_L B_N\Phi\Big\} + \\
&- \frac{\lambda_2}{2}\operatorname{Tr}\Big\{\bar{B}_{N\star}\Phi^\dagger P_L B_{N\star}\Phi^\dagger + \bar{B}_{N\star}\Phi^\dagger P_L B_N\Phi^\dagger - \bar{B}_N\Phi^\dagger P_L B_{N\star}\Phi^\dagger - \bar{B}_N\Phi^\dagger P_L B_N\Phi^\dagger + \\
&\qquad + \bar{B}_{N\star}\Phi P_R B_{N\star}\Phi - \bar{B}_{N\star}\Phi P_R B_N\Phi + \bar{B}_N\Phi P_R B_{N\star}\Phi - \bar{B}_N\Phi P_R B_N\Phi\Big\} + \\
&+ \mathcal{L}_{\tilde{\lambda}_{1,2}}(B_N, B_{N\star}).
\end{aligned} \tag{5.30}
$$

where we have shortened the Lagrangian (5.30) by abbreviating the terms proportional to $\tilde{\lambda_1}$ and $\tilde{\lambda_2}$ with

$$
\begin{aligned}
\mathcal{L}_{\tilde{\lambda}_{1,2}}(B_N, B_{N\star}) := -\frac{\tilde{\lambda_1}}{2}\Big(&\operatorname{Tr}\{\bar{B}_{N\star}\Phi\}P_R\operatorname{Tr}\{B_{N\star}\Phi^\dagger\} + \operatorname{Tr}\{\bar{B}_{N\star}\Phi\}P_R\operatorname{Tr}\{B_N\Phi^\dagger\} + \\
&- \operatorname{Tr}\{\bar{B}_N\Phi\}P_R\operatorname{Tr}\{B_{N\star}\Phi^\dagger\} - \operatorname{Tr}\{\bar{B}_N\Phi\}P_R\operatorname{Tr}\{B_N\Phi^\dagger\} + \\
&+ \operatorname{Tr}\{\bar{B}_{N\star}\Phi^\dagger\}P_L\operatorname{Tr}\{B_{N\star}\Phi\} - \operatorname{Tr}\{\bar{B}_{N\star}\Phi^\dagger\}P_L\operatorname{Tr}\{B_N\Phi\} + \\
&+ \operatorname{Tr}\{\bar{B}_N\Phi^\dagger\}P_L\operatorname{Tr}\{B_{N\star}\Phi\} - \operatorname{Tr}\{\bar{B}_N\Phi^\dagger\}P_L\operatorname{Tr}\{B_N\Phi\}\Big) + \\
-\frac{\tilde{\lambda_2}}{2}\Big(&\operatorname{Tr}\{\bar{B}_{N\star}\Phi^\dagger\}P_L\operatorname{Tr}\{B_{N\star}\Phi^\dagger\} + \operatorname{Tr}\{\bar{B}_{N\star}\Phi^\dagger\}P_L\operatorname{Tr}\{B_N\Phi^\dagger\} + \\
&- \operatorname{Tr}\{\bar{B}_N\Phi^\dagger\}P_L\operatorname{Tr}\{B_{N\star}\Phi^\dagger\} - \operatorname{Tr}\{\bar{B}_N\Phi^\dagger\}P_L\operatorname{Tr}\{B_N\Phi^\dagger\} + \\
&+ \operatorname{Tr}\{\bar{B}_{N\star}\Phi\}P_R\operatorname{Tr}\{B_{N\star}\Phi\} - \operatorname{Tr}\{\bar{B}_{N\star}\Phi\}P_R\operatorname{Tr}\{B_N\Phi\} + \\
&+ \operatorname{Tr}\{\bar{B}_N\Phi\}P_R\operatorname{Tr}\{B_{N\star}\Phi\} - \operatorname{Tr}\{\bar{B}_N\Phi\}P_R\operatorname{Tr}\{B_N\Phi\}\Big),
\end{aligned} \tag{5.31}
$$

since the terms are easily reconstructed by comparing it with the terms proportional to λ_1 and λ_2 and in the limit of two flavors these terms will vanish anyway. P_L and P_R are the chiral projection operators, eq. (1.25).
After condensation, $\Phi \to \Phi + \phi$, where $\phi = 1/2\ \mathrm{diag}(\varphi_N, \varphi_N, \sqrt{2}\varphi_S)$, the mass terms are

$$
\begin{aligned}
\mathcal{L}^{\text{mass}}_{\text{naive},N+\lambda} = &-g_N\operatorname{Tr}\Big\{\bar{B}_N\phi B_N + \bar{B}_{N\star}\phi B_{N\star}\Big\} + \\
&- \lambda_1\operatorname{Tr}\left\{\bar{B}_{N\star}\phi B_{N\star}\phi - \bar{B}_N\phi B_N\phi + \bar{B}_{N\star}\phi\gamma^5 B_N\phi - \bar{B}_N\phi\gamma^5 B_{N\star}\phi\right\} +
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_2 \operatorname{Tr}\left\{\bar{B}_{N\star}\phi B_{N\star}\phi - \bar{B}_N\phi B_N\phi - \bar{B}_{N\star}\phi\gamma^5 B_N\phi + \bar{B}_N\phi\gamma^5 B_{N\star}\phi\right\} + \\
&- \tilde{\lambda_1}\Big(\operatorname{Tr}\big\{\bar{B}_{N\star}\phi\big\}\operatorname{Tr}\big\{B_{N\star}\phi\big\} - \operatorname{Tr}\big\{\bar{B}_N\phi\big\}\operatorname{Tr}\big\{B_N\phi\big\} + \operatorname{Tr}\big\{\bar{B}_{N\star}\phi\big\}\gamma^5\operatorname{Tr}\big\{B_N\phi\big\} + \\
&\qquad - \operatorname{Tr}\big\{\bar{B}_N\phi\big\}\gamma^5\operatorname{Tr}\big\{B_{N\star}\phi\big\}\Big) + \\
&- \tilde{\lambda_2}\Big(\operatorname{Tr}\big\{\bar{B}_{N\star}\phi\big\}\operatorname{Tr}\big\{B_{N\star}\phi\big\} - \operatorname{Tr}\big\{\bar{B}_N\phi\big\}\operatorname{Tr}\big\{B_N\phi\big\} - \operatorname{Tr}\big\{\bar{B}_{N\star}\phi\big\}\gamma^5\operatorname{Tr}\big\{B_N\phi\big\} + \\
&\qquad + \operatorname{Tr}\big\{\bar{B}_N\phi\big\}\gamma^5\operatorname{Tr}\big\{B_{N\star}\phi\big\}\Big) \approx \\
\approx &- \operatorname{Tr}\Big\{\left[g_N\phi - (\lambda_1+\lambda_2)\phi^2\right]\bar{B}_N B_N + \left[g_N\phi + (\lambda_1+\lambda_2)\phi^2\right]\bar{B}_{N\star}B_{N\star} + \\
&\qquad + \left[(\lambda_1-\lambda_2)\phi^2\right]\left(-\bar{B}_{N\star}\gamma^5 B_N + \bar{B}_N\gamma^5 B_{N\star}\right)\Big\} + \\
&- \Big[(\tilde{\lambda_1}+\tilde{\lambda_2})\phi^2\Big]\Big(-\operatorname{Tr}\big\{\bar{B}_N\big\}\operatorname{Tr}\big\{B_N\big\} + \operatorname{Tr}\big\{\bar{B}_{N\star}\big\}\operatorname{Tr}\big\{B_{N\star}\big\}\Big) + \\
&- \Big[(\tilde{\lambda_1}-\tilde{\lambda_2})\phi^2\Big]\Big(-\operatorname{Tr}\big\{\bar{B}_{N\star}\big\}\gamma^5\operatorname{Tr}\big\{B_N\big\} + \operatorname{Tr}\big\{\bar{B}_N\big\}\gamma^5\operatorname{Tr}\big\{B_{N\star}\big\}\Big),
\end{aligned} \tag{5.32}
$$

where we assumed at the approximately-equal sign, that $\phi \approx \varphi\mathbb{1}_{3\times 3}$, so that we can get an impression of the elements of the mass matrix. Now off-diagonal terms appear, which correspond to the mixture of the fields and therefore ensure the splitting of masses in the naive assignment.

The construction of the Lagrangian for the M fields with isomorphic λ_i terms proceeds in complete analogy.

5.4. The Full Baryonic Lagrangian

As mentioned in eq. (5.1), the "full" baryonic Lagrangian containing all four fields will be a sum of the naive Lagrangians and a mixing term:

$$
\mathcal{L} = \mathcal{L}_{\text{naive, } N(+\lambda)} + \mathcal{L}_{\text{naive, } M(+\lambda)} + \mathcal{L}_{\text{mirror},NM}. \tag{5.33}
$$

In this section we include the last term. In order to be hermitian and invariant under parity and charge conjugation it must have the form:

$$
\begin{aligned}
\mathcal{L}_{\text{mirror},NM} &= -m_{0,1}\operatorname{Tr}\Big\{\bar{M}_R^{(RR)}N_L^{(RR)} + \bar{M}_L^{(LL)}N_R^{(LL)} + \text{h.c.}\Big\} + \\
&\quad - m_{0,2}\operatorname{Tr}\Big\{\bar{M}_L^{(RR)}N_R^{(RR)} + \bar{M}_R^{(LL)}N_L^{(LL)} + \text{h.c.}\Big\} = \\
&= -\frac{m_{0,1}+m_{0,2}}{2}\operatorname{Tr}\Big\{\bar{B}_M B_N + \bar{B}_{M\star}B_{N\star} + \bar{B}_N B_M + \bar{B}_{N\star}B_{M\star}\Big\} + \\
&\quad - \frac{m_{0,2}-m_{0,1}}{2}\operatorname{Tr}\Big\{\bar{B}_M\gamma^5 B_{N\star} + \bar{B}_{M\star}\gamma^5 B_N - \bar{B}_{N\star}\gamma^5 B_M - \bar{B}_N\gamma^5 B_{M\star}\Big\}.
\end{aligned} \tag{5.34}
$$

where we replaced $N^{(RR)}$, $N^{(LL)}$, $M^{(RR)}$, and $M^{(LL)}$ with the fields B_N, $B_{N\star}$, B_M, and $B_{M\star}$, defined in eqs. (5.12), (5.13) and (5.24).

In the last section we introduced two types of naive Lagrangians: one without the λ_i-terms , eqs. (5.17 and (5.25) and one with them, eq. (5.30). We study them separately in the following sections and chapters in order to understand, why the additional λ_i-terms are necessary:

1) Without the λ_i-terms:

We plug the naive Lagrangians without λ_i-terms, which we have found in the last section for the two N and M fields, eqs. (5.8)and (5.23), into the Lagrangian (5.33). That yields in terms of the fields with definite parity:

$$\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{naive, }N} + \mathcal{L}_{\text{naive, }M} + \mathcal{L}_{\text{mirror},NM} \\
=& \operatorname{Tr}\Big\{ \bar{B}_N i\gamma_\mu \frac{1}{2} \left[D^\mu_{NR} + D^\mu_{NL} + \gamma^5 \left(D^\mu_{NR} - D^\mu_{NL} \right) \right] B_N + \\
&\qquad + \bar{B}_{N\star} i\gamma_\mu \frac{1}{2} \left[D^\mu_{NR} + D^\mu_{NL} + \gamma^5 \left(D^\mu_{NR} - D^\mu_{NL} \right) \right] B_{N\star} \Big\} + \\
&+ \operatorname{Tr}\Big\{ \bar{B}_M i\gamma_\mu \frac{1}{2} \left[D^\mu_{MR} + D^\mu_{ML} - \gamma^5 \left(D^\mu_{MR} - D^\mu_{ML} \right) \right] B_M + \\
&\qquad + \bar{B}_{M,\star} i\gamma_\mu \frac{1}{2} \left[D^\mu_{MR} + D^\mu_{ML} - \gamma^5 \left(D^\mu_{MR} - D^\mu_{ML} \right) \right] B_{M,\star} \Big\} + \\
&+ \frac{c_{A_N}}{2} \operatorname{Tr}\Big\{ \bar{B}_N \gamma^\mu \left[R^\mu - L^\mu + \gamma^5 \left(R^\mu + L^\mu \right) \right] B_{N\star} + \\
&\qquad + \bar{B}_{N\star} \gamma^\mu \left[R^\mu - L^\mu + \gamma^5 \left(R^\mu + L^\mu \right) \right] B_N \Big\} + \\
&+ \frac{c_{A_M}}{2} \operatorname{Tr}\Big\{ \bar{B}_M \gamma^\mu \left[R^\mu - L^\mu - \gamma^5 \left(R^\mu + L^\mu \right) \right] B_{M\star} + \\
&\qquad + \bar{B}_{M\star} \gamma^\mu \left[R^\mu - L^\mu - \gamma^5 \left(R^\mu + L^\mu \right) \right] B_M \Big\} + \\
&- g_N \operatorname{Tr}\Big\{ \bar{B}_N \frac{1}{2} \left[\Phi + \Phi^\dagger + \gamma^5 \left(\Phi - \Phi^\dagger \right) \right] B_N + \\
&\qquad + \bar{B}_{N\star} \frac{1}{2} \left[\Phi + \Phi^\dagger + \gamma^5 \left(\Phi - \Phi^\dagger \right) \right] B_{N\star} \Big\} + \\
&- g_M \operatorname{Tr}\Big\{ \bar{B}_M \frac{1}{2} \left[\Phi + \Phi^\dagger - \gamma^5 \left(\Phi - \Phi^\dagger \right) \right] B_M + \\
&\qquad + \bar{B}_{M,\star} \frac{1}{2} \left[\Phi + \Phi^\dagger - \gamma^5 \left(\Phi - \Phi^\dagger \right) \right] B_{M,\star} \Big\} + \\
&- \frac{m_{0,1} + m_{0,2}}{2} \operatorname{Tr}\Big\{ \bar{B}_M B_N + \bar{B}_{M\star} B_{N\star} + \bar{B}_N B_M + \bar{B}_{N\star} B_{M\star} \Big\} + \\
&- \frac{m_{0,2} - m_{0,1}}{2} \operatorname{Tr}\Big\{ \bar{B}_M \gamma^5 B_{N\star} + \bar{B}_{M\star} \gamma^5 B_N - \bar{B}_{N\star} \gamma^5 B_M - \bar{B}_N \gamma^5 B_{M\star} \Big\}. \qquad (5.35)
\end{aligned}$$

In the last subsection about the naive assignment we found that the masses of the $B_\star$ are identical to the masses of the particles B (without λ_i-terms). In order to avoid these problem, a first try was the construction of the above Lagrangian in mirror assignment, which now also contains mass terms. These additional terms mix the four fields and might cause a splitting of the masses. In order to check if the model can indeed reveals the splitting, we have to evaluate the masses by diagonalising the Lagrangian, i.e., we have to calculate the eigenvalues of the so-called mass matrix. But instead of analysing the (36×36)-mass matrix of the $N_f = 3$ case, we will reduce the above Lagrangian to $N_f = 2$ in the next chapter and analyse the (4×4)-mass matrix in this case.

2) With the λ_i-terms:

When including the λ_i-terms the "full" baryonic Lagrangian takes the form:

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{naive, } N+\lambda} + \mathcal{L}_{\text{naive, } M+\lambda} + \mathcal{L}_{\text{mirror},NM} \\
=& \operatorname{Tr}\Big\{ \bar{B}_N i\gamma_\mu \frac{1}{2}\left[D^\mu_{NR} + D^\mu_{NL} + \gamma^5\left(D^\mu_{NR} - D^\mu_{NL}\right)\right] B_N + \\
&\qquad + \bar{B}_{N\star} i\gamma_\mu \frac{1}{2}\left[D^\mu_{NR} + D^\mu_{NL} + \gamma^5\left(D^\mu_{NR} - D^\mu_{NL}\right)\right] B_{N\star}\Big\} + \\
&+ \operatorname{Tr}\Big\{ \bar{B}_M i\gamma_\mu \frac{1}{2}\left[D^\mu_{MR} + D^\mu_{ML} - \gamma^5\left(D^\mu_{MR} - D^\mu_{ML}\right)\right] B_M + \\
&\qquad + \bar{B}_{M,\star} i\gamma_\mu \frac{1}{2}\left[D^\mu_{MR} + D^\mu_{ML} - \gamma^5\left(D^\mu_{MR} - D^\mu_{ML}\right)\right] B_{M,\star}\Big\} + \\
&+ \frac{c_{A_N}}{2} \operatorname{Tr}\Big\{ \bar{B}_N i\gamma^\mu \left[R^\mu - L^\mu + \gamma^5\left(R^\mu + L^\mu\right)\right] B_{N\star} + \\
&\qquad + \bar{B}_{N\star} i\gamma^\mu \left[R^\mu - L^\mu + \gamma^5\left(R^\mu + L^\mu\right)\right] B_N\Big\} + \\
&+ \frac{c_{A_M}}{2} \operatorname{Tr}\Big\{ \bar{B}_M i\gamma^\mu \left[R^\mu - L^\mu - \gamma^5\left(R^\mu + L^\mu\right)\right] B_{M\star} + \\
&\qquad + \bar{B}_{M\star} i\gamma^\mu \left[R^\mu - L^\mu - \gamma^5\left(R^\mu + L^\mu\right)\right] B_M\Big\} + \\
&- g_N \operatorname{Tr}\Big\{ \bar{B}_N \frac{1}{2}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_N + \bar{B}_{N\star}\frac{1}{2}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_{N\star}\Big\} + \\
&- g_M \operatorname{Tr}\Big\{ \bar{B}_M \frac{1}{2}\left[\Phi + \Phi^\dagger - \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_M + \bar{B}_{M,\star}\frac{1}{2}\left[\Phi + \Phi^\dagger - \gamma^5\left(\Phi - \Phi^\dagger\right)\right] B_{M,\star}\Big\} + \\
&- \frac{\lambda_1}{2} \operatorname{Tr}\Big\{ \bar{B}_{N\star}\Phi P_R B_{N\star}\Phi^\dagger + \bar{B}_{N\star}\Phi P_R B_N \Phi^\dagger - \bar{B}_N \Phi P_R B_{N\star}\Phi^\dagger - \bar{B}_N \Phi P_R B_N \Phi^\dagger + \\
&\qquad + \bar{B}_{N\star}\Phi^\dagger P_L B_{N\star}\Phi - \bar{B}_{N\star}\Phi^\dagger P_L B_N \Phi + \bar{B}_N \Phi^\dagger P_L B_{N\star}\Phi - \bar{B}_N\Phi^\dagger P_L B_N \Phi\Big\} + \\
&- \frac{\lambda_2}{2} \operatorname{Tr}\Big\{ \bar{B}_{N\star}\Phi^\dagger P_L B_{N\star}\Phi^\dagger + \bar{B}_{N\star}\Phi^\dagger P_L B_N \Phi^\dagger - \bar{B}_N \Phi^\dagger P_L B_{N\star}\Phi^\dagger - \bar{B}_N \Phi^\dagger P_L B_N \Phi^\dagger + \\
&\qquad + \bar{B}_{N\star}\Phi P_R B_{N\star}\Phi - \bar{B}_{N\star}\Phi P_R B_N \Phi + \bar{B}_N \Phi P_R B_{N\star}\Phi - \bar{B}_N\Phi P_R B_N \Phi\Big\} + \\
&- \frac{\lambda_3}{2} \operatorname{Tr}\Big\{ \bar{B}_{M\star}\Phi^\dagger P_R B_{M\star}\Phi^\dagger + \bar{B}_{M\star}\Phi^\dagger P_R B_M \Phi^\dagger - \bar{B}_M \Phi^\dagger P_R B_{M\star}\Phi^\dagger - \bar{B}_M \Phi^\dagger P_R B_M \Phi^\dagger + \\
&\qquad + \bar{B}_{M\star}\Phi P_L B_{M\star}\Phi - \bar{B}_{M\star}\Phi P_L B_M \Phi + \bar{B}_M \Phi P_L B_{M\star}\Phi - \bar{B}_M\Phi P_L B_M \Phi\Big\} + \\
&- \frac{\lambda_4}{2} \operatorname{Tr}\Big\{ \bar{B}_{M\star}\Phi^\dagger P_L B_{M\star}\Phi + \bar{B}_{M\star}\Phi^\dagger P_L B_M \Phi - \bar{B}_M \Phi^\dagger P_L B_{M\star}\Phi - \bar{B}_M \Phi^\dagger P_L B_M \Phi + \\
&\qquad + \bar{B}_{M\star}\Phi P_R B_{M\star}\Phi^\dagger - \bar{B}_{M\star}\Phi P_R B_M \Phi^\dagger + \bar{B}_M \Phi P_R B_{M\star}\Phi^\dagger - \bar{B}_M\Phi P_R B_M \Phi^\dagger\Big\} + \\
&- \frac{m_{0,1}+m_{0,2}}{2} \operatorname{Tr}\Big\{ \bar{B}_M B_N + \bar{B}_{M\star}B_{N\star} + \bar{B}_N B_M + \bar{B}_{N\star}B_{M\star}\Big\} + \\
&- \frac{m_{0,2}-m_{0,1}}{2} \operatorname{Tr}\Big\{ \bar{B}_M \gamma^5 B_{N\star} + \bar{B}_{M\star}\gamma^5 B_N - \bar{B}_{N\star}\gamma^5 B_M - \bar{B}_N\gamma^5 B_{M\star}\Big\} + \\
&+ \mathcal{L}_{\tilde{\lambda}_{1,2}}(B_N, B_{N\star}) + \mathcal{L}_{\tilde{\lambda}_{3,4}}(B_M, B_{M\star}).
\end{aligned}
\tag{5.36}
$$

This is indeed the most general Lagrangian of our work which contains all other cases as subcases. As we will see in the next chapter, with this Lagrangian it will be possible to describe the masses of the particles correctly.

6. The Baryonic Lagrangian for Two Flavors

In this chapter and for the following calculations of this work, we reduce our model eqs. (5.35) and (5.36), to two flavors.

6.1. Reduction of the Baryon Lagrangian to the $N_f = 2$ Case

In order to achieve the reduction, we set all quark fields with strangeness to zero. With eq. (5.21) we realize that only the (1 3)- and (2 3)-elements of the baryon matrices are nonzero,

$$B \xrightarrow{s=0} \begin{pmatrix} 0 & 0 & \Psi_{1,1} \\ 0 & 0 & \Psi_{1,2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_\star \xrightarrow{s=0} \begin{pmatrix} 0 & 0 & \Psi_{2,1} \\ 0 & 0 & \Psi_{2,2} \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.1}$$

$$B_M \xrightarrow{s=0} \begin{pmatrix} 0 & 0 & \Psi_{3,1} \\ 0 & 0 & \Psi_{3,2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_{M,\star} \xrightarrow{s=0} \begin{pmatrix} 0 & 0 & \Psi_{4,1} \\ 0 & 0 & \Psi_{4,2} \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.2}$$

where Ψ_{ij} are fields with quark content $\Psi_{i1} \hat{=} uud$ and $\Psi_{i2} \hat{=} udd$. Applying the same to the meson matrix eq. (3.23) and to the left- and right-handed (axial-)vector fields eqs. (3.24) and (3.25) we obtain

$$\Phi \xrightarrow{S=0} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{(\sigma_N+\varphi_N+a_0^0)+i(\eta_N+\pi^0)}{\sqrt{2}} & a_0^+ + i\pi^+ & 0 \\ a_0^- + i\pi^- & \frac{(\sigma_N+\varphi_N-a_0^0)+i(\eta_N-\pi^0)}{\sqrt{2}} & 0 \\ 0 & 0 & \sqrt{2}\varphi_S \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \Phi_{N_f=2} \end{pmatrix} & & 0 \\ & & 0 \\ 0 & 0 & \sqrt{2}\varphi_S \end{pmatrix}, \tag{6.3}$$

$$R^\mu \xrightarrow{S=0} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu+\rho^{\mu 0}}{\sqrt{2}} - \frac{f_{1N}^\mu+a_1^{\mu 0}}{\sqrt{2}} & \rho^{\mu +} - a_1^{\mu +} & 0 \\ \rho^{\mu -} - a_1^{\mu -} & \frac{\omega_N^\mu-\rho^{\mu 0}}{\sqrt{2}} - \frac{f_{1N}^\mu-a_1^{\mu 0}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} R^\mu_{N_f=2} \end{pmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.4}$$

$$L^\mu \xrightarrow{S=0} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu+\rho^{\mu 0}}{\sqrt{2}} + \frac{f_{1N}^\mu+a_1^{\mu 0}}{\sqrt{2}} & \rho^{\mu +} + a_1^{\mu +} & 0 \\ \rho^{\mu -} + a_1^{\mu -} & \frac{\omega_N^\mu-\rho^{\mu 0}}{\sqrt{2}} + \frac{f_{1N}^\mu-a_1^{\mu 0}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} L^\mu_{N_f=2} \end{pmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.5}$$

Note that it is crucial to first include the condensation of the sigma mesons in $\Phi_{N_f=3}$ and only then set the mesons with s quarks to zero, because otherwise one would loose the vacuum expectation value φ_S of σ_S. In this chapter strange mesons will not contribute to the Lagrangian, but in later chapters it will be very important to take them into account.
For the $N_f = 2$ case it is common to write the (2×2) meson matrix $\Phi_{N_f=2}$ in the basis of the three $SU(2)_f$ generators $\boldsymbol{T} = \boldsymbol{\tau}/2$, where $\boldsymbol{\tau}$ are the Pauli matrices, and $T^0 = \mathbb{1}_{2x2}/2$:

$$\Phi_{N_f=2} = \sum_{a=0}^{3} \Phi_a T^a = (\sigma_N + \varphi_N + i\eta_N)\, T^0 + (\boldsymbol{a}_0 + i\boldsymbol{\pi}) \cdot \boldsymbol{T}. \tag{6.6}$$

Similarly the left- and right-handed fields can be expressed as

$$L^\mu_{N_f=2} = (\omega^\mu + f_1^\mu)\, T^0 + (\boldsymbol{\rho}^\mu + \boldsymbol{a}_1^\mu) \cdot \boldsymbol{T}, \tag{6.7}$$
$$R^\mu_{N_f=2} = (\omega^\mu - f_1^\mu)\, T^0 + (\boldsymbol{\rho}^\mu - \boldsymbol{a}_1^\mu) \cdot \boldsymbol{T}. \tag{6.8}$$

As in the last chapter, we separate the following study into the two parts: first the case without λ_i terms and after that the case with the λ_i terms included.

1) Without λ_i-terms:

When we plug these non-strange fields into the mirror Lagrangian (5.35), it reduces to

$$\begin{aligned}
\mathcal{L}_{N_f=2} = {} & \bar{\Psi}_1 i\gamma_\mu \frac{1}{2}\left[D^\mu_{N,R} + D^\mu_{N,L} + \gamma^5\left(D^\mu_{N,R} - D^\mu_{N,L}\right)\right]\Psi_1 + \\
& + \bar{\Psi}_2 i\gamma_\mu \frac{1}{2}\left[D^\mu_{N,R} + D^\mu_{N,L} + \gamma^5\left(D^\mu_{N,R} - D^\mu_{N,L}\right)\right]\Psi_2 + \\
& + \bar{\Psi}_3 i\gamma_\mu \frac{1}{2}\left[D^\mu_{M,R} + D^\mu_{M,L} - \gamma^5\left(D^\mu_{M,R} - D^\mu_{M,L}\right)\right]\Psi_3 + \\
& + \bar{\Psi}_4 i\gamma_\mu \frac{1}{2}\left[D^\mu_{M,R} + D^\mu_{M,L} - \gamma^5\left(D^\mu_{M,R} - D^\mu_{M,L}\right)\right]\Psi_4 + \\
& + \frac{c_{A_N}}{2}\left\{\bar{\Psi}_1\gamma_\mu\left[R^\mu - L^\mu + \gamma^5(R^\mu + L^\mu)\right]\Psi_2 + \bar{\Psi}_2\gamma_\mu\left[R^\mu - L^\mu + \gamma^5(R^\mu + L^\mu)\right]\Psi_1\right\} + \\
& + \frac{c_{A_M}}{2}\left\{\bar{\Psi}_3\gamma_\mu\left[R^\mu - L^\mu - \gamma^5(R^\mu + L^\mu)\right]\Psi_4 + \bar{\Psi}_4\gamma_\mu\left[R^\mu - L^\mu - \gamma^5(R^\mu + L^\mu)\right]\Psi_3\right\} + \\
& - g_N\left(\bar{\Psi}_1\frac{1}{2}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right]\Psi_1 + \bar{\Psi}_2\frac{1}{2}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right]\Psi_2\right) + \\
& - g_M\left(\bar{\Psi}_3\frac{1}{2}\left[\Phi + \Phi^\dagger - \gamma^5\left(\Phi - \Phi^\dagger\right)\right]\Psi_3 + \bar{\Psi}_4\frac{1}{2}\left[\Phi + \Phi^\dagger - \gamma^5\left(\Phi - \Phi^\dagger\right)\right]\Psi_4\right) + \\
& - \frac{m_{0,1} + m_{0,2}}{2}\left(\bar{\Psi}_4\Psi_2 + \bar{\Psi}_3\Psi_1 + \bar{\Psi}_2\Psi_4 + \bar{\Psi}_1\Psi_3\right) +
\end{aligned}$$

$$
\begin{aligned}
& - \frac{m_{0,2}-m_{0,1}}{2}\left(\bar{\Psi}_4\gamma^5\Psi_1 + \bar{\Psi}_3\gamma^5\Psi_2 - \bar{\Psi}_1\gamma^5\Psi_4 - \bar{\Psi}_2\gamma^5\Psi_3\right) + \mathcal{L}_{\text{meson}}^{N_F=2} = \\
= & \bar{\Psi}_{1R} i\gamma_\mu D^\mu_{NR}\Psi_{1R} + \bar{\Psi}_{1L} i\gamma_\mu D^\mu_{NL}\Psi_{1L} + \bar{\Psi}_{2R} i\gamma_\mu D^\mu_{NR}\Psi_{2R} + \bar{\Psi}_{2L} i\gamma_\mu D^\mu_{NL}\Psi_{2L} + \\
& + \bar{\Psi}_{3R} i\gamma_\mu D^\mu_{ML}\Psi_{3R} + \bar{\Psi}_{3L} i\gamma_\mu D^\mu_{MR}\Psi_{3L} + \bar{\Psi}_{4R} i\gamma_\mu D^\mu_{ML}\Psi_{4R} + \bar{\Psi}_{4L} i\gamma_\mu D^\mu_{MR}\Psi_{4L} + \\
& + c_{A_N}\left(\bar{\Psi}_{1R} i\gamma_\mu R^\mu\Psi_{2R} - \bar{\Psi}_{1L}\gamma_\mu L^\mu\Psi_{2L} + \bar{\Psi}_{2R} i\gamma_\mu R^\mu\Psi_{1R} - \bar{\Psi}_{2L}\gamma_\mu L^\mu\Psi_{1L}\right) + \\
& + c_{A_M}\left(\bar{\Psi}_{3L} i\gamma_\mu R^\mu\Psi_{4L} - \bar{\Psi}_{3R}\gamma_\mu L^\mu\Psi_{4R} + \bar{\Psi}_{4L} i\gamma_\mu R^\mu\Psi_{3L} - \bar{\Psi}_{4R}\gamma_\mu L^\mu\Psi_{3R}\right) + \\
& - g_N\left(\bar{\Psi}_{1L}\Phi\Psi_{1R} + \bar{\Psi}_{1R}\Phi^\dagger\Psi_{1L} + \bar{\Psi}_{2L}\Phi\Psi_{2R} + \bar{\Psi}_{2R}\Phi^\dagger\Psi_{2L}\right) + \\
& - g_M\left(\bar{\Psi}_{3L}\Phi^\dagger\Psi_{3R} + \bar{\Psi}_{3R}\Phi\Psi_{3L} + \bar{\Psi}_{4L}\Phi^\dagger\Psi_{4R} + \bar{\Psi}_{4R}\Phi\Psi_{4L}\right) + \\
& - \frac{m_{0,1}+m_{0,2}}{2}\left(\bar{\Psi}_{4R}\Psi_{2L} + \bar{\Psi}_{4L}\Psi_{2R} + \bar{\Psi}_{3R}\Psi_{1L} + \bar{\Psi}_{3L}\Psi_{1R} + \text{ h.c.}\right) + \\
& - \frac{m_{0,1}-m_{0,2}}{2}\left(\bar{\Psi}_{4R}\Psi_{1L} - \bar{\Psi}_{4L}\Psi_{1R} + \bar{\Psi}_{3R}\Psi_{2L} - \bar{\Psi}_{3L}\Psi_{2R} - \text{ h.c.}\right) + \mathcal{L}_{\text{meson}}^{N_F=2},
\end{aligned}
\tag{6.9}
$$

where we dropped the indices "${}_{N_f=2}$" and some of "${}_N$," since φ_N does no longer appear. Furthermore, we have defined $\Psi_i := (\Psi_{i,1}, \Psi_{i,2})$.

2) With λ_i terms:

Analogously, with the Lagrangian (5.36) we find :

$$
\begin{aligned}
\mathcal{L}_{N_f=2} = & \bar{\Psi}_1 i\gamma_\mu \frac{1}{2}\left[D^\mu_{NR} + D^\mu_{NL} + \gamma^5\left(D^\mu_{NR} - D^\mu_{NL}\right)\right]\Psi_1 + \\
& + \bar{\Psi}_2 i\gamma_\mu \frac{1}{2}\left[D^\mu_{NR} + D^\mu_{NL} + \gamma^5\left(D^\mu_{NR} - D^\mu_{NL}\right)\right]\Psi_2 + \\
& + \bar{\Psi}_3 i\gamma_\mu \frac{1}{2}\left[D^\mu_{MR} + D^\mu_{ML} - \gamma^5\left(D^\mu_{MR} - D^\mu_{ML}\right)\right]\Psi_3 + \\
& + \bar{\Psi}_4 i\gamma_\mu \frac{1}{2}\left[D^\mu_{MR} + D^\mu_{ML} - \gamma^5\left(D^\mu_{MR} - D^\mu_{ML}\right)\right]\Psi_4 + \\
& + \frac{c_{A_N}}{2}\left\{\bar{\Psi}_1\gamma^\mu\left[R^\mu - L^\mu + \gamma^5\left(R^\mu + L^\mu\right)\right]\Psi_2 + \bar{\Psi}_2\gamma^\mu\left[R^\mu - L^\mu + \gamma^5\left(R^\mu + L^\mu\right)\right]\Psi_1\right\} + \\
& + \frac{c_{A_M}}{2}\left\{\bar{\Psi}_3\gamma^\mu\left[R^\mu - L^\mu - \gamma^5\left(R^\mu + L^\mu\right)\right]\Psi_4 + \bar{\Psi}_4\gamma^\mu\left[R^\mu - L^\mu - \gamma^5\left(R^\mu + L^\mu\right)\right]\Psi_3\right\} + \\
& - g_N\left\{\bar{\Psi}_1\frac{1}{2}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right]\Psi_1 + \bar{\Psi}_2\frac{1}{2}\left[\Phi + \Phi^\dagger + \gamma^5\left(\Phi - \Phi^\dagger\right)\right]\Psi_2\right\} + \\
& - g_M\left\{\bar{\Psi}_3\frac{1}{2}\left[\Phi + \Phi^\dagger - \gamma^5\left(\Phi - \Phi^\dagger\right)\right]\Psi_3 + \bar{\Psi}_4\frac{1}{2}\left[\Phi + \Phi^\dagger - \gamma^5\left(\Phi - \Phi^\dagger\right)\right]\Psi_4\right\} + \\
& - \frac{\lambda_1}{\sqrt{2}}\varphi_S\Big(\bar{\Psi}_2\Phi P_R\Psi_2 + \bar{\Psi}_2\Phi P_R\Psi_1 - \bar{\Psi}_1\Phi P_R\Psi_2 - \bar{\Psi}_1\Phi P_R\Psi_1 + \bar{\Psi}_2\Phi^\dagger P_L\Psi_2 + \\
& \qquad - \bar{\Psi}_2\Phi^\dagger P_L\Psi_1 + \bar{\Psi}_1\Phi^\dagger P_L\Psi_2 - \bar{\Psi}_1\Phi^\dagger P_L\Psi_1\Big) + \\
& - \frac{\lambda_2}{\sqrt{2}}\varphi_S\Big(\bar{\Psi}_2\Phi^\dagger P_L\Psi_2 + \bar{\Psi}_2\Phi^\dagger P_L\Psi_1 - \bar{\Psi}_1\Phi^\dagger P_L\Psi_2 - \bar{\Psi}_1\Phi^\dagger P_L\Psi_1 + \bar{\Psi}_2\Phi P_R\Psi_2 + \\
& \qquad - \bar{\Psi}_2\Phi P_R\Psi_1 + \bar{\Psi}_1\Phi P_R\Psi_2 - \bar{\Psi}_1\Phi P_R\Psi_1\Big) +
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\lambda_3}{\sqrt{2}}\varphi_S\Big(\bar{\Psi}_4\Phi^\dagger P_R\Psi_4+\bar{\Psi}_4\Phi^\dagger P_R\Psi_3-\bar{\Psi}_3\Phi^\dagger P_R\Psi_4-\bar{\Psi}_3\Phi^\dagger P_R\Psi_3+\bar{\Psi}_4\Phi P_L\Psi_4+\\
&\qquad\qquad -\bar{\Psi}_4\Phi P_L\Psi_3+\bar{\Psi}_3\Phi P_L\Psi_4-\bar{\Psi}_3\Phi P_L\Psi_3\Big)+\\
&-\frac{\lambda_4}{\sqrt{2}}\varphi_S\Big(\bar{\Psi}_3\Phi^\dagger P_L\Psi_4+\bar{\Psi}_4\Phi^\dagger P_L\Psi_3-\bar{\Psi}_3\Phi^\dagger P_L\Psi_4-\bar{\Psi}_3\Phi^\dagger P_L\Psi_3+\bar{\Psi}_4\Phi P_R\Psi_4+\\
&\qquad\qquad -\bar{\Psi}_4\Phi P_R\Psi_3+\bar{\Psi}_3\Phi P_R\Psi_4-\bar{\Psi}_3\Phi P_R\Psi_3\Big)+\\
&-\frac{m_{0,1}+m_{0,2}}{2}\Big(\bar{\Psi}_3\Psi_1+\bar{\Psi}_4\Psi_2+\bar{\Psi}_1\Psi_3+\bar{\Psi}_2\Psi_4\Big)+\\
&-\frac{m_{0,2}-m_{0,1}}{2}\Big(\bar{\Psi}_3\gamma^5\Psi_2+\bar{\Psi}_4\gamma^5\Psi_1-\bar{\Psi}_2\gamma^5\Psi_3-\bar{\Psi}_1\gamma^5\Psi_4\Big)+\mathcal{L}_{\text{meson}}.
\end{aligned}
\tag{6.10}
$$

The second $\Phi^{(\dagger)}$ matrix in the λ_i terms contributes only with the vacuum expectation value φ_S of σ_S and the terms proportional to $\tilde{\lambda}_i$ vanish.

6.2. Effects of the Spontaneous Symmetry Breaking in the Meson Sector on the Baryon Sector

As explained in section 3.3 spontaneous symmetry breaking in the mesonic sector leads to the condensation of the sigma meson $\sigma \rightarrow \sigma + \varphi$ and a shift of axial-vector fields. For the two-flavor case one has

$$
f_1^\mu \quad\longrightarrow\quad f_1^\mu + Zw\partial^\mu\eta_N \qquad\text{and}\qquad \boldsymbol{a}_1^\mu \quad\longrightarrow\quad \boldsymbol{a}_1^\mu + Zw\partial^\mu\boldsymbol{\pi}, \tag{6.11}
$$

with

$$
w = g_1\varphi/m_{f_1}^2, \tag{6.12}
$$

where m_{f_1} corresponding to $m_{f_{1N}}$ given in eq. (3.37) and

$$
Z^2 = {m_{f_1}}^2/({m_{f_1}}^2 - {g_1}^2\varphi^2). \tag{6.13}
$$

In order to simplify the following expressions we renamed:

$$
\varphi_N \equiv \varphi. \tag{6.14}
$$

1) Without λ_i-terms:

With eq. (6.11) the Lagrangian (6.9) reads after the condensation $\sigma \rightarrow \sigma + \varphi$

$$
\begin{aligned}
\mathcal{L}_{\text{mirror}}^{N_f=2} =&\bar{\Psi}_1 i\gamma_\mu\partial^\mu\Psi_1+\bar{\Psi}_2 i\gamma_\mu\partial^\mu\Psi_2+\bar{\Psi}_3 i\gamma_\mu\partial^\mu\Psi_3+\bar{\Psi}_4 i\gamma_\mu\partial^\mu\Psi_4+\\
&+c_N\bar{\Psi}_1\gamma_\mu\left\{\left[\omega^\mu-\gamma^5\left(f_1^\mu+Zw\partial^\mu\eta_N\right)\right]T^0+\left[\boldsymbol{\rho}^\mu-\gamma^5\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}\right)\right]\cdot\boldsymbol{T}\right\}\Psi_1+\\
&+c_N\bar{\Psi}_2\gamma_\mu\left\{\left[\omega^\mu-\gamma^5\left(f_1^\mu+Zw\partial^\mu\eta_N\right)\right]T^0+\left[\boldsymbol{\rho}^\mu-\gamma^5\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}\right)\right]\cdot\boldsymbol{T}\right\}\Psi_2+\\
&+c_M\bar{\Psi}_3\gamma_\mu\left\{\left[\omega^\mu+\gamma^5\left(f_1^\mu+Zw\partial^\mu\eta_N\right)\right]T^0+\left[\boldsymbol{\rho}^\mu+\gamma^5\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}\right)\right]\cdot\boldsymbol{T}\right\}\Psi_3+\\
&+c_M\bar{\Psi}_4\gamma_\mu\left\{\left[\omega^\mu+\gamma^5\left(f_1^\mu+Zw\partial^\mu\eta_N\right)\right]T^0+\left[\boldsymbol{\rho}^\mu+\gamma^5\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}\right)\right]\cdot\boldsymbol{T}\right\}\Psi_4+
\end{aligned}
$$

$$
\begin{aligned}
&- c_{A_N} \bar{\Psi}_1 \gamma_\mu \left[\left(f_1^\mu + Zw\partial^\mu \eta_N - \gamma^5 \omega^\mu \right) T^0 + \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} - \gamma^5 \boldsymbol{\rho}^\mu \right) \cdot \boldsymbol{T} \right] \Psi_2 + \\
&- c_{A_N} \bar{\Psi}_2 \gamma_\mu \left[\left(f_1^\mu + Zw\partial^\mu \eta_N - \gamma^5 \omega^\mu \right) T^0 + \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} - \gamma^5 \boldsymbol{\rho}^\mu \right) \cdot \boldsymbol{T} \right] \Psi_1 + \\
&- c_{A_M} \bar{\Psi}_3 \gamma_\mu \left[\left(f_1^\mu + Zw\partial^\mu \eta_N + \gamma^5 \omega^\mu \right) T^0 + \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} + \gamma^5 \boldsymbol{\rho}^\mu \right) \cdot \boldsymbol{T} \right] \Psi_4 + \\
&- c_{A_M} \bar{\Psi}_4 \gamma_\mu \left[\left(f_1^\mu + Zw\partial^\mu \eta_N + \gamma^5 \omega^\mu \right) T^0 + \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} + \gamma^5 \boldsymbol{\rho}^\mu \right) \cdot \boldsymbol{T} \right] \Psi_3 + \\
&- g_N \bar{\Psi}_1 \left[\left(\sigma + \varphi + i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 + i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_1 + \\
&- g_N \bar{\Psi}_2 \left[\left(\sigma + \varphi + i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 + i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_2 + \\
&- g_M \bar{\Psi}_3 \left[\left(\sigma + \varphi - i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 - i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_3 + \\
&- g_M \bar{\Psi}_4 \left[\left(\sigma + \varphi - i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 - i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_4 + \\
&- \frac{m_{0,1} + m_{0,2}}{2} \left(\bar{\Psi}_4 \Psi_2 + \bar{\Psi}_3 \Psi_1 + \bar{\Psi}_2 \Psi_4 + \bar{\Psi}_1 \Psi_3 \right) + \\
&- \frac{m_{0,2} - m_{0,1}}{2} \left(\bar{\Psi}_4 \gamma^5 \Psi_1 + \bar{\Psi}_3 \gamma^5 \Psi_2 - \bar{\Psi}_1 \gamma^5 \Psi_4 - \bar{\Psi}_2 \gamma^5 \Psi_3 \right) + \mathcal{L}_{\text{meson}}^{N_F=2} .
\end{aligned} \tag{6.15}
$$

2) With λ_i-terms:

When including the λ_i-terms, one gets:

$$
\begin{aligned}
\mathcal{L}_{N_f=2} = &\bar{\Psi}_1 i\gamma^\mu \partial_\mu \Psi_1 + \bar{\Psi}_2 i\gamma^\mu \partial_\mu \Psi_2 + \bar{\Psi}_3 i\gamma^\mu \partial_\mu \Psi_3 + \bar{\Psi}_4 i\gamma^\mu \partial_\mu \Psi_4 + \\
&+ c_N \bar{\Psi}_1 \gamma_\mu \left\{ \left[\omega^\mu - \gamma^5 \left(f_1^\mu + Zw\partial^\mu \eta_N \right) \right] T^0 + \left[\boldsymbol{\rho}^\mu - \gamma^5 \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} \right) \right] \cdot \boldsymbol{T} \right\} \Psi_1 + \\
&+ c_N \bar{\Psi}_2 \gamma_\mu \left\{ \left[\omega^\mu - \gamma^5 \left(f_1^\mu + Zw\partial^\mu \eta_N \right) \right] T^0 + \left[\boldsymbol{\rho}^\mu - \gamma^5 \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} \right) \right] \cdot \boldsymbol{T} \right\} \Psi_2 + \\
&+ c_M \bar{\Psi}_3 \gamma_\mu \left\{ \left[\omega^\mu + \gamma^5 \left(f_1^\mu + Zw\partial^\mu \eta_N \right) \right] T^0 + \left[\boldsymbol{\rho}^\mu + \gamma^5 \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} \right) \right] \cdot \boldsymbol{T} \right\} \Psi_3 + \\
&+ c_M \bar{\Psi}_4 \gamma_\mu \left\{ \left[\omega^\mu + \gamma^5 \left(f_1^\mu + Zw\partial^\mu \eta_N \right) \right] T^0 + \left[\boldsymbol{\rho}^\mu + \gamma^5 \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} \right) \right] \cdot \boldsymbol{T} \right\} \Psi_4 + \\
&- c_{A_N} \bar{\Psi}_1 \gamma_\mu \left[\left(f_1^\mu + Zw\partial^\mu \eta_N - \gamma^5 \omega^\mu \right) T^0 + \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} - \gamma^5 \boldsymbol{\rho}^\mu \right) \cdot \boldsymbol{T} \right] \Psi_2 + \\
&- c_{A_N} \bar{\Psi}_2 \gamma_\mu \left[\left(f_1^\mu + Zw\partial^\mu \eta_N - \gamma^5 \omega^\mu \right) T^0 + \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} - \gamma^5 \boldsymbol{\rho}^\mu \right) \cdot \boldsymbol{T} \right] \Psi_1 + \\
&- c_{A_M} \bar{\Psi}_3 \gamma_\mu \left[\left(f_1^\mu + Zw\partial^\mu \eta_N + \gamma^5 \omega^\mu \right) T^0 + \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} + \gamma^5 \boldsymbol{\rho}^\mu \right) \cdot \boldsymbol{T} \right] \Psi_4 + \\
&- c_{A_M} \bar{\Psi}_4 \gamma_\mu \left[\left(f_1^\mu + Zw\partial^\mu \eta_N + \gamma^5 \omega^\mu \right) T^0 + \left(\boldsymbol{a}_1^\mu + Zw\partial^\mu \boldsymbol{\pi} + \gamma^5 \boldsymbol{\rho}^\mu \right) \cdot \boldsymbol{T} \right] \Psi_3 + \\
&- g_N \bar{\Psi}_1 \left[\left(\sigma + \varphi_N + i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 + i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_1 + \\
&- g_N \bar{\Psi}_2 \left[\left(\sigma + \varphi_N + i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 + i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_2 + \\
&- g_M \bar{\Psi}_3 \left[\left(\sigma + \varphi_N - i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 - i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_3 + \\
&- g_M \bar{\Psi}_4 \left[\left(\sigma + \varphi_N - i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 - i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_4 + \\
&- \lambda_1' \varphi_S \bar{\Psi}_2 \left[\left(\sigma + \varphi_N + i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 + i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_2 + \\
&- \lambda_1' \varphi_S \bar{\Psi}_2 \left[\left(\gamma^5 \sigma + \gamma^5 \varphi_N + iZ\eta_N \right) T^0 + \left(\gamma^5 \boldsymbol{a}_0 + iZ\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_1 + \\
&+ \lambda_1' \varphi_S \bar{\Psi}_1 \left[\left(\gamma^5 \sigma + \gamma^5 \varphi_N + iZ\eta_N \right) T^0 + \left(\gamma^5 \boldsymbol{a}_0 + iZ\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_2 + \\
&+ \lambda_1' \varphi_S \bar{\Psi}_1 \left[\left(\sigma + \varphi_N + i\gamma^5 Z\eta_N \right) T^0 + \left(\boldsymbol{a}_0 + i\gamma^5 Z\boldsymbol{\pi} \right) \cdot \boldsymbol{T} \right] \Psi_1 + \; ...
\end{aligned}
$$

$$
\begin{aligned}
\ldots\ & -\lambda_2'\varphi_S\bar{\Psi}_2\left[\left(\sigma+\varphi_N+i\gamma^5 Z\eta_N\right)T^0+\left(\boldsymbol{a}_0+i\gamma^5 Z\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_2+\\
&+\lambda_2'\varphi_S\bar{\Psi}_2\left[\left(\gamma^5\sigma+\gamma^5\varphi_N+iZ\eta_N\right)T^0+\left(\gamma^5\boldsymbol{a}_0+iZ\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_1+\\
&-\lambda_2'\varphi_S\bar{\Psi}_1\left[\left(\gamma^5\sigma+\gamma^5\varphi_N+iZ\eta_N\right)T^0+\left(\gamma^5\boldsymbol{a}_0+iZ\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_2+\\
&+\lambda_2'\varphi_S\bar{\Psi}_1\left[\left(\sigma+\varphi_N+i\gamma^5 Z\eta_N\right)T^0+\left(\boldsymbol{a}_0+i\gamma^5 Z\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_1+\\
&-\lambda_3'\varphi_S\bar{\Psi}_4\left[\left(\sigma+\varphi_N-i\gamma^5 Z\eta_N\right)T^0+\left(\boldsymbol{a}_0-i\gamma^5 Z\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_4+\\
&-\lambda_3'\varphi_S\bar{\Psi}_4\left[\left(\gamma^5\sigma+\gamma^5\varphi_N-iZ\eta_N\right)T^0+\left(\gamma^5\boldsymbol{a}_0-iZ\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_3+\\
&+\lambda_3'\varphi_S\bar{\Psi}_3\left[\left(\gamma^5\sigma+\gamma^5\varphi_N-iZ\eta_N\right)T^0+\left(\gamma^5\boldsymbol{a}_0-iZ\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_4+\\
&+\lambda_3'\varphi_S\bar{\Psi}_3\left[\left(\sigma+\varphi_N-i\gamma^5 Z\eta_N\right)T^0+\left(\boldsymbol{a}_0-i\gamma^5 Z\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_3+\\
&-\lambda_4'\varphi_S\bar{\Psi}_4\left[\left(\sigma+\varphi_N+i\gamma^5 Z\eta_N\right)T^0+\left(\boldsymbol{a}_0+i\gamma^5 Z\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_4+\\
&+\lambda_4'\varphi_S\bar{\Psi}_4\left[\left(\gamma^5\sigma+\gamma^5\varphi_N+iZ\eta_N\right)T^0+\left(\gamma^5\boldsymbol{a}_0+iZ\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_3+\\
&-\lambda_4'\varphi_S\bar{\Psi}_3\left[\left(\gamma^5\sigma+\gamma^5\varphi_N+iZ\eta_N\right)T^0+\left(\gamma^5\boldsymbol{a}_0+iZ\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_4+\\
&+\lambda_4'\varphi_S\bar{\Psi}_3\left[\left(\sigma+\varphi_N+i\gamma^5 Z\eta_N\right)T^0+\left(\boldsymbol{a}_0+i\gamma^5 Z\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\Psi_3+\\
&-\frac{m_{0,1}+m_{0,2}}{2}\left(\bar{\Psi}_4\Psi_2+\bar{\Psi}_3\Psi_1+\bar{\Psi}_2\Psi_4+\bar{\Psi}_1\Psi_3\right)+\\
&-\frac{m_{0,2}-m_{0,1}}{2}\left(\bar{\Psi}_4\gamma^5\Psi_1+\bar{\Psi}_3\gamma^5\Psi_2-\bar{\Psi}_1\gamma^5\Psi_4-\bar{\Psi}_2\gamma^5\Psi_3\right)+\mathcal{L}_{\text{meson}}^{N_F=2}.
\end{aligned}
\tag{6.16}
$$

Here we have defined

$$\lambda_i' := \lambda_i/\sqrt{2} \tag{6.17}$$

and dropped the index $_N$ of the sigma meson.

7. Mixing and Fit of the Parameters $m_{0,1}$, $m_{0,2}$, g_N, g_M, and $\lambda_1, \ldots, \lambda_4$

In this chapter we determine the masses of the baryons included in $\mathcal{L}_{\text{mirror}}$, eqs. (5.35) and (5.36). Coincidently we fit the parameters $m_{0,1}, m_{0,2}, g_N, g_M$, and $\lambda_1, \ldots, \lambda_4$, which are included in the mass matrix.

7.1. Mixing of the Four Nucleon Fields and Diagonalisation of the Lagrangian without λ_i terms

After spontaneous symmetry breaking (σ in $\Phi_{N_f=2}$ becomes $\sigma + \varphi$) the Lagrangian (6.15) shows the following mixing terms of the four fields Ψ_1, ..., Ψ_4:

$$\begin{aligned}\mathcal{L}_{\text{mass}} = &-\frac{g_N \varphi}{2}\left(\bar{\Psi}_1\Psi_1 + \bar{\Psi}_2\Psi_2\right) - \frac{g_M \varphi}{2}\left(\bar{\Psi}_3\Psi_3 + \bar{\Psi}_4\Psi_4\right) + \\ &- \frac{m_{0,1}+m_{0,2}}{2}\left(\bar{\Psi}_4\Psi_2 + \bar{\Psi}_3\Psi_1 + \bar{\Psi}_2\Psi_4 + \bar{\Psi}_1\Psi_3\right) + \\ &- \frac{m_{0,2}-m_{0,1}}{2}\left(\bar{\Psi}_4\gamma^5\Psi_1 + \bar{\Psi}_3\gamma^5\Psi_2 + \bar{\Psi}_1\gamma^5\Psi_4 + \bar{\Psi}_2\gamma^5\Psi_3\right). \end{aligned} \tag{7.1}$$

Because of these mixing terms, the four nucleon field in the Lagrangian are not yet the physical ones. In order to determine them we have to diagonalise the Lagrangian. We define the vector $\Psi = (\Psi_1, \gamma^5\Psi_2, \Psi_3, \gamma^5\Psi_4)^T$ and rewrite the mass terms in matrix form. Then we introduce a so-called mass matrix M through the definition:

$$\mathcal{L}_{\text{mass}} \overset{!}{=} -\left(\bar{\Psi}_1, -\bar{\Psi}_2\gamma^5, \bar{\Psi}_3, -\bar{\Psi}_4\gamma^5\right) M \begin{pmatrix} \Psi_1 \\ \gamma^5\Psi_2 \\ \Psi_3 \\ \gamma^5\Psi_4 \end{pmatrix} = -\bar{\Psi} M \Psi,$$

$$\text{with } M = \frac{1}{2}\begin{pmatrix} g_N\varphi & 0 & m_{0,1}+m_{0,2} & -m_{0,1}+m_{0,2} \\ 0 & -g_N\varphi & m_{0,1}-m_{0,2} & -m_{0,1}-m_{0,2} \\ m_{0,1}+m_{0,2} & m_{0,1}-m_{0,2} & g_M\varphi & 0 \\ -m_{0,1}+m_{0,2} & -m_{0,1}-m_{0,2} & 0 & -g_M\varphi \end{pmatrix}. \tag{7.2}$$

In order to avoid γ^5s in the mass matrix all four components of the vectors to the left and the right should have the same parity. For that reason we have redefined the fields which are proportional to

the pseudoscalar diquark (i.e., with negative parity), by inserting a γ^5: $\Psi_2 \to \gamma^5\Psi_2$ and $\Psi_4 \to \gamma^5\Psi_4$. Furthermore, we made use of $\overline{(\gamma^5\Psi_i)} = -\bar{\Psi}_i\gamma^5$.
In order to diagonalise $\mathcal{L}_{\text{mass}}$ we have to solve the eigenvalue problem

$$\begin{aligned} M\boldsymbol{u}^{(i)} &= m_i\boldsymbol{u}^{(i)}, \\ M_{ij}u_j^{(k)} &= m_k u_i^{(k)}, \end{aligned} \tag{7.3}$$

where $\boldsymbol{u}^{(i)}$ ($i \in \{1, ..., \dim(M)\}$) are the so-called eigenvectors and m_i are the four eigenvalues of the mass matrix M. Notice that there is no Einstein sum over i, but there are four equations of this type. If we multiply eq. (7.3) with $\boldsymbol{u}^{(l)}$ from the left hand side, we find

$$u_i^{(l)}M_{ij}u_j^{(k)} = m_k u_i^{(l)}u_i^{(k)} = m_k\delta^{kl}, \tag{7.4}$$

since the eigenvectors are orthogonal, $\boldsymbol{u}^{(l)} \cdot \boldsymbol{u}^{(k)} = \delta^{lk}$. Hence the matrix

$$\boxed{U_{ij} = u_i^{(j)}} \tag{7.5}$$

diagonalises M:

$$U^\dagger M U = \text{diag}(m_1, m_2, m_3, m_4). \tag{7.6}$$

Going back to the Lagrangian $\mathcal{L}_{\text{mass}}$ we realize that it is diagonalized by

$$\mathcal{L}_{\text{mass}} = -\bar{\Psi}UU^\dagger MUU^\dagger\Psi = -\bar{\Psi}^{\text{phys}}\text{diag}(m_1, m_2, m_3, m_4)\Psi^{\text{phys}}, \tag{7.7}$$

with the physical fields

$$\Psi^{\text{phys}} = U^\dagger\Psi. \tag{7.8}$$

Thus, we have to determine the eigenvalues of M, which correspond to the masses of the physical fields. We evaluate

$$\begin{aligned} M\boldsymbol{u}^{(i)} &= m_i\boldsymbol{u}^{(i)} \\ (M - m_i\mathbb{1}_{4\times4})\,\boldsymbol{u}^{(i)} &= 0 \\ \Rightarrow \qquad \det\Big[M - m_i\mathbb{1}_{4\times4}\Big] &\overset{!}{=} 0 \end{aligned} \tag{7.9}$$

and find

$$\begin{aligned} m_1 &= +\sqrt{2}\sqrt{\Omega_1 + \Omega_2}, \\ -m_2 &= -\sqrt{2}\sqrt{\Omega_1 + \Omega_2}, \\ m_3 &= +\sqrt{2}\sqrt{\Omega_1 - \Omega_2}, \\ -m_4 &= -\sqrt{2}\sqrt{\Omega_1 - \Omega_2}, \end{aligned} \tag{7.10}$$

with

$$\begin{aligned}
\Omega_1 &:= g_N^2\varphi^2 + g_M^2\varphi^2 + 4(m_{0,1}+m_{0,2})^2, \\
\Omega_2 &:= \sqrt{\Big[(g_N\varphi - g_M\varphi)^2 + 4(m_{0,1}+m_{0,2})^2\Big]\Big[(g_N\varphi + g_M\varphi)^2 + 4(m_{0,1}-m_{0,2})^2\Big]}.
\end{aligned} \tag{7.11}$$

Thus we realize that the masses of the nucleon and its chiral partner are still degenerate. Note that the masses of the baryons with negative parity (Ψ_2 and Ψ_4) have a minus sign because of the definitions $\Psi_2 \to \gamma^5\Psi_2$ resp. $\bar{\Psi}_2 \to -\bar{\Psi}_2\gamma^5$ and $\Psi_4 \to \gamma^5\Psi_4$ resp. $\bar{\Psi}_4 \to -\bar{\Psi}_2\gamma^5$. Therefore we relabelled $m_2 \to -m_2$ and $m_4 \to -m_4$.
There are at least two possibilities why our model does not yield four different masses. The first point might be the fact that our model does not pay attention to the inner structure of the baryons. Thus, the mass difference between the scalar and the pseudoscalar diquark is not included. Another point might be the fact that we work only on tree level and heavier baryons or baryons with larger momenta are not included. (With respect to the latter point we added the λ_i-terms to the Lagrangian in the last chapter.)
The first idea can easily be implemented in a phenomenological (but not rigorous) way by adding the mass difference Δm between the pseudoscalar and scalar diquark to the diagonal elements of the mass matrix M, which contain the pseudoscalar diquark ($\hat{=}$ have negative parity). Thus, the mass matrix would read

$$M = \frac{1}{2}\begin{pmatrix} g_N\varphi & 0 & m_{0,1}+m_{0,2} & -m_{0,1}+m_{0,2} \\ 0 & -g_N\varphi - 2\Delta m & m_{0,1}-m_{0,2} & -m_{0,1}-m_{0,2} \\ m_{0,1}+m_{0,2} & m_{0,1}-m_{0,2} & g_M\varphi & 0 \\ -m_{0,1}+m_{0,2} & -m_{0,1}-m_{0,2} & 0 & -g_M\varphi - 2\Delta m \end{pmatrix}. \tag{7.12}$$

The following masses are obtained:

$$\begin{aligned}
m_1 &= -\frac{\Delta m}{2} - \frac{\sqrt{\Omega_1 - \Omega_2}}{4}, \\
-m_2 &= -\frac{\Delta m}{2} + \frac{\sqrt{\Omega_1 - \Omega_2}}{4}, \\
m_3 &= -\frac{\Delta m}{2} - \frac{\sqrt{\Omega_1 + \Omega_2}}{4}, \\
-m_4 &= -\frac{\Delta m}{2} + \frac{\sqrt{\Omega_1 + \Omega_2}}{4},
\end{aligned} \tag{7.13}$$

with

$$\begin{aligned}
\Omega_1 &:= 2g_N^2\varphi^2 + 2g_M^2\varphi^2 + 2(g_N\varphi + g_M\varphi)\Delta m + \Delta m^2 + 8(m_{0,1}^2 + m_{0,2}^2), \\
\Omega_2 &:= 2\sqrt{(g_N\varphi + g_M\varphi + \Delta m)^2 + 4(m_{0,1}-m_{0,2})^2}\sqrt{(g_N\varphi - g_M\varphi)^2 + 4(m_{0,1}+m_{0,2})^2}.
\end{aligned} \tag{7.14}$$

As expected the masses are no longer degenerate. However, there is still a problem: When computing the difference between the masses of the nucleons (parity = +) and the chiral partners (parity = -),

$$\begin{aligned} m_2 - m_1 &= \Delta m, \\ m_4 - m_3 &= \Delta m, \end{aligned} \tag{7.15}$$

we see that the mass differences are given by the mass difference between pseudoscalar and scalar diquark. Moreover, the mass differences of both partners are identical. Unfortunately that is in contrast to experiment. Namely, the resonances listed in ref. [23] are the particles $N(939) \equiv N$ and $N(1440)$ with positive parity and the resonances $N(1535)$ and $N(1650)$ with negative parity. The masses of these particles are given in ref. [23] as

$$\begin{aligned} m^{\text{exp}}_{N(939)} &\simeq 939 \text{ MeV}, \\ m^{\text{exp}}_{N(1440)} &\simeq 1440 \text{ MeV}, \\ m^{\text{exp}}_{N(1535)} &\simeq 1535 \text{ MeV}, \\ m^{\text{exp}}_{N(1650)} &\simeq 1655 \text{ MeV}. \end{aligned} \tag{7.16}$$

If we assume $N(1535)$ to be the chiral partner of $N(939)$ and the $N(1650)$ the chiral partner of $N(1440)$, the approximate mass differences are

$$\begin{aligned} m_{N(1535)} - m_{N(939)} &= 596 \text{ MeV}, \\ m_{N(1650)} - m_{N(1440)} &= 210 \text{ MeV}, \end{aligned} \tag{7.17}$$

or when $N(1650)$ is the chiral partner of $N(939)$ and $N(1535)$ the chiral partner of $N(1440)$:

$$\begin{aligned} m_{N(1650)} - m_{N(939)} &= 711 \text{ MeV}, \\ m_{N(1535)} - m_{N(1440)} &= 95 \text{ MeV}. \end{aligned} \tag{7.18}$$

Hence the simple ansatz of including “ad hoc” the missing mass difference between pseudoscalar and scalar diquark Δm in the mass matrix failed. This is indeed the reason why we have included the λ_i-terms in sec. 5.3. In the next section we will see that these terms generate a splitting of the masses.

7.2. Diagonalisation of the Lagrangian with λ_i terms

From the Lagrangian (6.16) which includes the additional λ_i-terms we extract the following mixing terms:

$$\begin{aligned}
\mathcal{L}_{\text{mass}} = &-\left(\frac{g_N\varphi_N}{2} - \frac{\lambda_1+\lambda_2}{2\sqrt{2}}\varphi_N\varphi_S\right)\bar{\Psi}_1\Psi_1 - \left(\frac{g_N\varphi}{2} + \frac{\lambda_1+\lambda_2}{2\sqrt{2}}\varphi_N\varphi_S\right)\bar{\Psi}_2\Psi_2 + \\
&-\left(\frac{g_M\varphi_N}{2} - \frac{\lambda_3+\lambda_4}{2\sqrt{2}}\varphi_N\varphi_S\right)\bar{\Psi}_3\Psi_3 - \left(\frac{g_M\varphi}{2} + \frac{\lambda_3+\lambda_4}{2\sqrt{2}}\varphi_N\varphi_S\right)\bar{\Psi}_4\Psi_4 + \\
&-\left(\frac{\lambda_1-\lambda_2}{2\sqrt{2}}\varphi_N\varphi_S\right)\left(-\bar{\Psi}_2\gamma^5\Psi_1 + \bar{\Psi}_1\gamma^5\Psi_2\right) + \\
&-\left(\frac{\lambda_3-\lambda_4}{2\sqrt{2}}\varphi_N\varphi_S\right)\left(-\bar{\Psi}_4\gamma^5\Psi_3 + \bar{\Psi}_3\gamma^5\Psi_4\right) + \\
&-\left(\frac{m_{0,1}+m_{0,2}}{2}\right)\left(\bar{\Psi}_4\Psi_2 + \bar{\Psi}_3\Psi_1 + \bar{\Psi}_2\Psi_4 + \bar{\Psi}_1\Psi_3\right) + \\
&-\left(\frac{m_{0,2}-m_{0,1}}{2}\right)\left(\bar{\Psi}_4\gamma^5\Psi_1 + \bar{\Psi}_3\gamma^5\Psi_2 - \bar{\Psi}_1\gamma^5\Psi_4 - \bar{\Psi}_2\gamma^5\Psi_3\right).
\end{aligned} \tag{7.19}$$

Again, we can rewrite this as a vector-matrix-vector product:

$$\mathcal{L}_{\text{mass}} = -\left(\bar{\Psi}_1, -\bar{\Psi}_2\gamma^5, \bar{\Psi}_3, -\bar{\Psi}_4\gamma^5\right) M \begin{pmatrix} \Psi_1 \\ \gamma^5\Psi_2 \\ \Psi_3 \\ \gamma^5\Psi_4 \end{pmatrix}, \tag{7.20}$$

with the mass matrix M now being:

$$\frac{1}{2}\begin{pmatrix}
g_N\varphi_N - \frac{(\lambda_1+\lambda_2)\varphi_N\varphi_S}{\sqrt{2}} & \frac{(\lambda_1-\lambda_2)\varphi_N\varphi_S}{\sqrt{2}} & m_{0,1}+m_{0,2} & m_{0,1}-m_{0,2} \\
\frac{(\lambda_1-\lambda_2)\varphi_N\varphi_S}{\sqrt{2}} & -g_N\varphi_N - \frac{(\lambda_1+\lambda_2)\varphi_N\varphi_S}{\sqrt{2}} & m_{0,2}-m_{0,1} & -m_{0,1}-m_{0,2} \\
m_{0,1}+m_{0,2} & m_{0,2}-m_{0,1} & g_M\varphi_N - \frac{(\lambda_3+\lambda_4)\varphi_N\varphi_S}{\sqrt{2}} & \frac{(\lambda_3-\lambda_4)\varphi_N\varphi_S}{\sqrt{2}} \\
m_{0,1}-m_{0,2} & -m_{0,1}-m_{0,2} & \frac{(\lambda_3-\lambda_4)\varphi_N\varphi_S}{\sqrt{2}} & -g_M\varphi_N - \frac{(\lambda_3+\lambda_4)\varphi_N\varphi_S}{\sqrt{2}}
\end{pmatrix}. \tag{7.21}$$

When computing the eigenvalues of this matrix, we find that the results are very long expressions and therefore we cannot give an analytic expression. Thus, from now on we are forced to work **numerically**.

We will try to find numerical values for the eight parameters $g_N, g_M, m_{0,1}, m_{0,2}, \lambda_1, \ldots, \lambda_4$, which are included in the mass matrix (7.21), in such a way that the eigenvalues match to the experimental masses (7.16). Since we have only **four** eigenvalue equations to fit these **eight** parameters, it is clear that we will find more than one combination of values that yields perfect eigenvalues. However, note that the correct description of masses is non-trivial, because of the particular form of the equations.

At this point we should note that the appropriate numerical approach would be a larger fit, which includes first and foremost sufficiently many experimental quantities or results from lattice calculations and also all twelve parameters (the above ones plus $c_N, c_M, c_{A_N}, c_{A_M}$) of our model. As additional input from experiment or lattice calculations we can find five decay widths and four axial coupling constants (see chapters 8.3 and 9). In this work, for the sake of simplicity, we will not perform such a fit, but show a step-by-step procedure. First we take only the mass matrix and the included parameters into account and perform a fit. This method will not lead to perfect results, but will be a first attempt towards a solution and yields hints in which part of the parameter space one should search for solutions.

With the vacuum expectation values $\varphi_N = (164.6 \pm 0.1)$ MeV and $\varphi_S = (126.2 \pm 0.1)$ MeV [2] the choice (for more details, see Appendix C)

$$\begin{aligned} g_N &= 15.13 \quad , & \lambda_1 &= 0.02260 \text{ MeV}^{-1}, \\ g_M &= 17.80 \quad , & \lambda_2 &= 0.02060 \text{ MeV}^{-1}, \\ m_{0,1} &= -166.4\text{MeV} \quad , & \lambda_3 &= 0.00201 \text{ MeV}^{-1}, \\ m_{0,2} &= 294.5\text{MeV} \quad , & \lambda_4 &= 0.01000 \text{ MeV}^{-1}, \end{aligned} \tag{7.22}$$

yields the correct masses. But there are further interesting solutions: Namely it is also possible to determine a solution in which $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$:

g_N	15.37
g_M	17.80
$m_{0,1}$	-260.4 MeV
$m_{0,2}$	332.6 MeV
$\lambda_1 = \lambda_2$ [MeV^{-1}]	0.02410
$\lambda_3 = \lambda_4$ [MeV^{-1}]	0.003507

Table 7.1.: The choice of values for the parameters $g_N, g_M, m_{0,1}, m_{0,2}, \lambda_1, \ldots, \lambda_4$, we will work with in the following.

With this choice we reduce the number of parameters in our Lagrangian by two, since now

$$\begin{aligned} \lambda_1 - \lambda_2 &= 0, \\ \lambda_3 - \lambda_4 &= 0. \end{aligned} \tag{7.23}$$

This means that the Lagrangian simplifies even more with this special choice. From now on we will do all calculations with the values given in table 7.1.

The next step is to determine the physical fields by using eq. (7.8):

$$\Psi^{\text{phys}} = U^\dagger \Psi,$$
$$\Psi_i^{\text{phys}} = U_{ij}^\dagger \Psi_j = U_{ji}\Psi_j = u_j^{(i)}\Psi_j,$$

$$\Rightarrow \quad \begin{pmatrix} \Psi_1^{\text{phys}} \\ \gamma^5\Psi_2^{\text{phys}} \\ \Psi_3^{\text{phys}} \\ \gamma^5\Psi_4^{\text{phys}} \end{pmatrix} \equiv \begin{pmatrix} N_{939} \\ \gamma^5 N_{1535} \\ N_{1440} \\ \gamma^5 N_{1650} \end{pmatrix} = \begin{pmatrix} u_1^{(1)} & u_1^{(2)} & u_1^{(3)} & u_1^{(4)} \\ u_2^{(1)} & u_2^{(2)} & u_2^{(3)} & u_2^{(4)} \\ u_3^{(1)} & u_3^{(2)} & u_3^{(3)} & u_3^{(4)} \\ u_4^{(1)} & u_4^{(2)} & u_4^{(3)} & u_4^{(4)} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \gamma^5\Psi_2 \\ \Psi_3 \\ \gamma^5\Psi_4 \end{pmatrix}, \tag{7.24}$$

where $\boldsymbol{u}^{(i)}, i \in \{1,2,3,4\}$ are the eigenvectors of M and we relabelled Ψ_1^{phys} to N_{939}, Ψ_2^{phys} to N_{1535}, Ψ_3^{phys} to N_{1440} and Ψ_4^{phys} to N_{1650} in accordance to the eigenvalues resp. masses. We have to determine the eigenvectors to find the transformation matrix which transforms the fields included in the Lagrangian into the physical fields. With the parameters given in table 7.1 the eigenvectors are

$$\begin{aligned} \boldsymbol{u}^{(1)} &= (-0.9902, 0.006661, 0.07194, 0.1197)^T, \\ \boldsymbol{u}^{(2)} &= (0.1139, 0.3158, 0.03041, 0.9415)^T, \\ \boldsymbol{u}^{(3)} &= (0.0718, 0.09646, 0.9927, 0.008389)^T, \\ \boldsymbol{u}^{(4)} &= (-0.03776, 0.9439, 0.09179, 0.3150)^T. \end{aligned} \tag{7.25}$$

The fields included in the Lagrangian Ψ_i are related to the physical fields $N_{939}, N_{1535}, N_{1440}, N_{1650}$ by

$$\boxed{\begin{aligned} \begin{pmatrix} \Psi_1 \\ \gamma^5\Psi_2 \\ \Psi_3 \\ \gamma^5\Psi_4 \end{pmatrix} &= \begin{pmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} & u_4^{(1)} \\ u_1^{(2)} & u_2^{(2)} & u_3^{(2)} & u_4^{(2)} \\ u_1^{(3)} & u_2^{(3)} & u_3^{(3)} & u_4^{(3)} \\ u_1^{(4)} & u_2^{(4)} & u_3^{(4)} & u_4^{(4)} \end{pmatrix} \begin{pmatrix} N_{939} \\ \gamma^5 N_{1535} \\ N_{1440} \\ \gamma^5 N_{1650} \end{pmatrix} = \\ &= \begin{pmatrix} \mathbf{-0.9902} & 0.006661 & 0.07194 & 0.1197 \\ 0.1139 & 0.3158 & 0.03041 & \mathbf{0.9415} \\ 0.0718 & 0.09646 & \mathbf{0.9927} & 0.008389 \\ -0.03776 & \mathbf{0.9439} & 0.09179 & 0.3150 \end{pmatrix} \begin{pmatrix} N_{939} \\ \gamma^5 N_{1535} \\ N_{1440} \\ \gamma^5 N_{1650} \end{pmatrix}. \end{aligned}} \tag{7.26}$$

An interesting point is that, since N_{1650} has the largest amount in Ψ_2, $N(1650)$ is predominantly the chiral partner of the nucleon $N(939)$. $N(1535)$ is then the chiral partner of $N(1440)$. This is in agreement with the results of ref. [3, 4].

In order to obtain the Lagrangian in dependence of the physical fields (without mixing terms) we have to plug the relations (7.26) into the Lagrangian (6.16). The result is an extremely long expression. Therefore, we will not write down the full Lagrangian, but give only the relevant parts.
As expected, the mass terms are given by:

$$\mathcal{L}_{\text{mass}} = m_{939}\bar{N}_{939}N_{939} + m_{1535}\bar{N}_{1535}N_{1535} + m_{1440}\bar{N}_{1440}N_{1440} + m_{1650}\bar{N}_{1650}N_{1650} \tag{7.27}$$

where the masses m_i are given in eq. (7.16).
Since we will need it in the following chapter 8.3, we exemplarily also give the coupling terms which are relevant for the calculation of the decay width of a chiral partner (N_{1535} or N_{1650}) into the nucleon (N_{939}) and a pseudoscalar P ($\boldsymbol{P}\cdot\boldsymbol{\tau}$ or $\partial_\mu \boldsymbol{P}\cdot\boldsymbol{\tau}$) which can be π or η,

$$\boxed{\begin{aligned}\mathcal{L}_{N_\star\to NP} =& -ig^{N_{1535}\to N_{939}P}\ \ \bar{N}_{939}\boldsymbol{P}\cdot\boldsymbol{\tau}N_{1535} + g^{N_{1535}\to N_{939}\partial P}\ \ \bar{N}_{939}\gamma^\mu\partial_\mu\boldsymbol{P}\cdot\boldsymbol{\tau}N_{1535} + \\ & -ig^{N_{1650}\to N_{939}P}\ \ \bar{N}_{939}\boldsymbol{P}\cdot\boldsymbol{\tau}N_{1650} + g^{N_{1650}\to N_{939}\partial P}\ \ \bar{N}_{939}\gamma^\mu\partial_\mu\boldsymbol{Pi}\cdot\boldsymbol{\tau}N_{1650},\end{aligned}} \tag{7.28}$$

and the coupling terms which are relevant for the calculation of the decay width of N_{1440} decaying into the nucleon N_{939} and a pseudoscalar P:

$$\boxed{\mathcal{L}_{N_{1440}\to N_{939}P} = -i\gamma^5 g^{N_{1440}\to N_{939}P}\ \ \bar{N}_{939}\boldsymbol{P}\cdot\boldsymbol{\tau}N_{1440} + g^{N_{1440}\to N_{939}\partial P}\ \ \bar{N}_{939}\gamma^5\gamma^\mu\partial_\mu\boldsymbol{P}\cdot\boldsymbol{\tau}N_{1440}.} \tag{7.29}$$

The coupling constants $g^{\cdots\to\cdots}$ can be read off eq. (6.16) regarding eq. (7.26). With

$$\begin{aligned} N_{939} &=: N_1, \\ N_{1535} &=: N_2, \\ N_{1440} &=: N_3, \\ N_{1650} &=: N_4, \end{aligned} \tag{7.30}$$

we find **for $\boldsymbol{i} \in \{\mathbf{1,2,3,4}\}$ and $\boldsymbol{j} \in \{\mathbf{1,2,3,4}\}$**:

$$\boxed{\begin{aligned} g^{N_i\to N_jP} = \frac{1}{2}Z\Big[& g_N\left(u_j^{(1)}u_i^{(1)} - u_j^{(2)}u_i^{(2)}\right) + g_M\left(-u_j^{(3)}u_i^{(3)} + u_j^{(4)}u_i^{(4)}\right) + \\ & + \lambda_1'\left(-u_j^{(1)}u_i^{(1)} - u_j^{(1)}u_i^{(2)} - u_j^{(2)}u_i^{(1)} - u_2^{(j)}u_2^{(i)}\right) + \\ & + \lambda_2'\left(-u_j^{(1)}u_i^{(1)} + u_j^{(1)}u_i^{(2)} + u_j^{(2)}u_i^{(1)} - u_j^{(2)}u_i^{(2)}\right) + \\ & + \lambda_3'\left(u_j^{(3)}u_i^{(3)} + u_j^{(3)}u_i^{(4)} + u_j^{(4)}u_i^{(3)} + u_j^{(4)}u_i^{(4)}\right) + \\ & + \lambda_4'\left(-u_j^{(3)}u_i^{(3)} + u_j^{(3)}u_i^{(4)} + u_j^{(4)}u_i^{(3)} - u_j^{(4)}u_i^{(4)}\right)\Big].\end{aligned}} \tag{7.31}$$

For $k \in \{2,4\}$ and $l \in \{1,3\}$:

$$\boxed{\begin{aligned} g^{N_i \to N_j \partial P} = \frac{Zw}{2}\Big[& c_N \left(-u_l^{(1)}u_k^{(1)} + u_l^{(2)}u_k^{(2)}\right) + c_M \left(u_l^{(3)}u_k^{(3)} - u_l^{(4)}u_k^{(4)}\right) + \\ & - c_{A_N}\left(u_l^{(1)}u_k^{(2)} - u_l^{(2)}u_k^{(1)}\right) - c_{A_M}\left(u_l^{(3)}u_k^{(4)} - u_l^{(4)}u_k^{(3)}\right)\Big] \end{aligned}} \tag{7.32}$$

and

$$\boxed{\begin{aligned} g^{N_3 \to N_1 \partial P} = \frac{Zw}{2}\Big[& c_N \left(u_1^{(1)}u_3^{(1)} + u_1^{(2)}u_3^{(2)}\right) - c_M \left(u_1^{(3)}u_3^{(3)} + u_1^{(4)}u_3^{(4)}\right) + \\ & + c_{A_N}\left(u_1^{(1)}u_3^{(2)} + u_1^{(2)}u_3^{(1)}\right) + c_{A_M}\left(u_1^{(3)}u_3^{(4)} + u_1^{(4)}u_3^{(3)}\right)\Big]. \end{aligned}} \tag{7.33}$$

8. Decays of Baryonic Resonances

The baryonic Lagrangian describes various interactions. A glance at the experimental results [23], however, reveals that there are only five decays which are kinematically allowed. These are

$$\begin{aligned}
N(1535) &\longrightarrow N(939)\pi, \\
N(1535) &\longrightarrow N(939)\eta, \\
N(1650) &\longrightarrow N(939)\pi, \\
N(1650) &\longrightarrow N(939)\eta, \\
N(1440) &\longrightarrow N(939)\pi.
\end{aligned} \tag{8.1}$$

For this reason, in this chapter we will compute the decay width of $N_\star \to NP$, where $N_\star$ corresponds to the resonances $N(1535)$ or $N(1650)$, N to the nucleon $N(939)$ or $N(1440)$ and P to a pseudoscalar, which can be π or η. In the end, this result will be extended to the calculation of the decay $N(1440) \to N(939)P$.

8.1. Decay of the type $N_\star \longrightarrow NP$

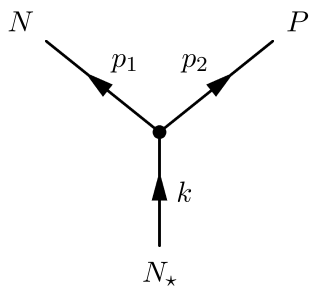
If we look at the interaction terms (7.28), we notice that derivative interactions are present. However, as a simple start for this kind of calculations we will evaluate the decay width considering only the simpler couplings $N_\star NP$ without derivatives. Later we can easily include the derivative couplings.

8.1.1. Decay Channel Without Derivatives

An interaction Lagrangian which describes the coupling $N_\star NP$ and contains no derivative couplings is given by

$$\mathcal{L}_{N_\star NP} = -\, ig^{N\to N_\star P}\bar{N}_\star \boldsymbol{P}\cdot\boldsymbol{\tau}N - ig^{N_\star\to NP}\bar{N}\boldsymbol{P}\cdot\boldsymbol{\tau}N_\star, \tag{8.2}$$

where $g^{\cdots\to\cdots}$ are coupling constants. The product of the pseudoscalar triplet field $\boldsymbol{P}$ with the vector of Pauli matrices $\boldsymbol{\tau}$ is the hermitian pseudoscalar particle matrix P. The second term of this Lagrangian is the one which is interesting for us since we want to calculate $\langle f|S_{\text{int}}|i\rangle = \langle NP|S_{\text{int}}|N_\star\rangle$ with $S_{\text{int}} = i\int \mathrm{d}^4x\mathcal{L}_{\text{int}}$. It corresponds to the Feynman diagram:



where we have assigned the 4-momenta p_1, p_2, and k to the fields. The decay is evaluated in the rest frame of the decaying particle, i.e.,

$$\boldsymbol{k} = 0. \tag{8.3}$$

Using conservation of momentum yields also a relation between the outgoing momenta:

$$\boldsymbol{p}_1 = -\boldsymbol{p}_2. \tag{8.4}$$

In order to calculate the decay width we make use of eq. (2.43) which reads for this case:

$$\Gamma_{N_\star \to NP} = \frac{1}{2m_{N_\star}} \frac{1}{(2\pi)^6} \int \frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_N(\boldsymbol{p}_1)} \frac{\mathrm{d}^3\boldsymbol{p}_2}{2E_P(\boldsymbol{p}_2)} \left| i\mathcal{M} \right|^2 (2\pi)^4 \delta^{(4)}(k - p_1 - p_2). \tag{8.5}$$

We rewrite the Dirac delta distribution by splitting it into a product of a space- and time-component:

$$\begin{aligned} \delta^{(4)}(k - p_1 - p_2) &= \delta(m_{N_\star} - E_N(\boldsymbol{p}_1) - E_P(\boldsymbol{p}_2))\delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = \\ &= \delta(m_{N_\star} - \sqrt{\boldsymbol{p}_1^2 + m_N^2} - \sqrt{\boldsymbol{p}_1^2 + m_P^2})\delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = \dots \end{aligned}$$

where, in the last line, we have used the last delta distribution, from which follows that $\boldsymbol{p}_2^2 = \boldsymbol{p}_1^2$. The first delta distribution can be rewritten further by using the following formula for delta distributions with a function as an argument:

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \tag{8.6}$$

where x_i are the zeros of the function ($f(x_i) = 0$) and $f'(x_i)$ is the first derivative of f with respect to x at x_i, which must not be zero. Hence we have to find the zero of the function $f(|\boldsymbol{p}_1|) = m_{N_\star} - \sqrt{\boldsymbol{p}_1^2 + m_N^2} - \sqrt{\boldsymbol{p}_1^2 + m_P^2}$, which we label p_f. It is given by

$$\boxed{p_f = \frac{1}{2m_{N_\star}} \sqrt{(m_{N_\star}^2 - m_N^2 - m_P^2)^2 - 4m_N^2 m_P^2}.} \tag{8.7}$$

Furthermore we need the derivative of $f(|\boldsymbol{p}_1|) = m_{N_\star} - \sqrt{\boldsymbol{p}_1^2 + m_N^2} - \sqrt{\boldsymbol{p}_1^2 + m_P^2}$ at the zero p_f, which reads

$$\frac{\partial f(|\boldsymbol{p}_1|)}{\partial |\boldsymbol{p}_1|}\Big|_{|\boldsymbol{p}_1| = p_f} = \frac{p_f m_{N_\star}}{\sqrt{p_f^2 + m_N^2}\sqrt{p_f^2 + m_P^2}}. \tag{8.8}$$

Then, the delta distribution (8.6) can be expressed as

$$\begin{aligned}\delta^{(4)}(k - p_1 - p_2) &= \left| \left(\frac{\partial f(|\boldsymbol{p}_1|)}{\partial |\boldsymbol{p}_1|}\Big|_{|\boldsymbol{p}_1| = p_f} \right)^{-1} \right| \delta(|\boldsymbol{p}_1| - p_f)\delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = \\ &= \frac{\sqrt{p_f^2 + m_N^2}\sqrt{p_f^2 + m_P^2}}{p_f m_{N_\star}} \delta(|\boldsymbol{p}_1| - p_f)\delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2).\end{aligned} \tag{8.9}$$

Plugging this result back into the decay width (8.5) yields:

$$\begin{aligned}\Gamma_{N_\star \to NP} &= \frac{1}{2m_{N_\star}} \frac{1}{(2\pi)^6} \int \frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_N(\boldsymbol{p}_1)} \frac{\mathrm{d}^3\boldsymbol{p}_2}{2E_P(\boldsymbol{p}_2)} \left| i\mathcal{M} \right|^2 (2\pi)^4 \frac{\sqrt{p_f^2 + m_N^2}\sqrt{p_f^2 + m_P^2}}{p_f m_{N_\star}} \times \\ &\qquad \times \delta(|\boldsymbol{p}_1| - p_f)\delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = \\ &= \frac{1}{2m_{N_\star}} \frac{1}{(2\pi)^2} \int \frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_N(\boldsymbol{p}_1) 2E_P(\boldsymbol{p}_1)} \left| i\mathcal{M} \right|^2 \frac{\sqrt{p_f^2 + m_N^2}\sqrt{p_f^2 + m_P^2}}{p_f m_{N_\star}} \delta(|\boldsymbol{p}_1| - p_f)\delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2),\end{aligned} \tag{8.10}$$

where in the last line we used the last delta distribution to replace $\boldsymbol{p}_2$ with $-\boldsymbol{p}_1$ and substituted the integration variable $-\boldsymbol{p}_1$ with $\boldsymbol{p}_1$. This is possible, because the integration runs over the whole $\mathbb{R}^3$ space. When we convert to spherical coordinates $\mathrm{d}^3\boldsymbol{p}_1 \to \mathrm{d}p_1\ \mathrm{d}\Omega\ p_1^2$ and regard that $\int \mathrm{d}\Omega = 4\pi$, since we have no angular dependences, the decay width (8.10) reads

$$\boxed{\Gamma_{N_\star \to NP} = \frac{p_f}{8\pi m_{N_\star}^2} \left| i\mathcal{M} \right|^2 \delta(|\boldsymbol{p}_1| - p_f)\delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2).} \tag{8.11}$$

Finally, in order to evaluate the matrix element $\mathcal{M}$ of the second term of the interaction Lagrangian (8.2), we use Feynman rules (see sec. 2.3) and get

$$i\mathcal{M}^{ss'} = -ig^{N_\star \to NP}\ \bar{u}^{s'}(\boldsymbol{p}_1) u^s(\boldsymbol{k}), \tag{8.12}$$

where s and s' are the spins of the incoming and outgoing particle. Since we do not know the initial and final spins of the particles of the decay, we have to average over initial spins and to sum over the final ones. (The decaying particle has one certain spin and the final particle can have arbitrary spin.)

Mathematically we have

$$
\begin{aligned}
|i\mathcal{M}|^2 \equiv \overline{|i\mathcal{M}|^2} &= \frac{1}{2}\sum_{s,s'}\left|i\mathcal{M}^{ss'}\right|^2 = \\
&= \frac{\left|g^{N_\star\to NP}\right|^2}{2}\sum_{s,s'}\left|\bar{u}^{s'}(\boldsymbol{p}_1)u^s(\boldsymbol{k})\right|^2 = \\
&= \frac{\left|g^{N_\star\to NP}\right|^2}{2}\sum_{s,s'}\bar{u}^{s'}_\alpha(\boldsymbol{p}_1)u^s_\alpha(\boldsymbol{k})\bar{u}^s_\beta(\boldsymbol{k})u^{s'}_\beta(\boldsymbol{p}_1) = \ldots,
\end{aligned}
\tag{8.13}
$$

where α and β are spinor indices. For the spin sum over the 4-spinors $u^s(\boldsymbol{p})$ holds the relation

$$
\sum_s u^s_\alpha(\boldsymbol{p})\bar{u}^s_\beta(\boldsymbol{p}) = (\gamma^\mu p_\mu + m)_{\alpha\beta}\,. \tag{8.14}
$$

Therefore, the spin averaged square amplitude (8.13) reads

$$
\begin{aligned}
\overline{|i\mathcal{M}|^2} &= \frac{\left|g^{N_\star\to NP}\right|^2}{2}(\gamma^\mu k_\mu + m_{N_\star})_{\alpha\beta}(\gamma^\nu p_{1\nu} + m_N)_{\beta\alpha} = \\
&= \frac{\left|g^{N_\star\to NP}\right|^2}{2}\,\mathrm{Tr}\left\{(\gamma^\mu k_\mu + m_{N_\star})(\gamma^\nu p_{1\nu} + m_N)\right\} = \\
&= \frac{\left|g^{N_\star\to NP}\right|^2}{2}\left(\mathrm{Tr}\left\{\gamma^\mu\gamma^\nu\right\}k_\mu p_{1\nu} + \mathrm{Tr}\left\{\mathbb{1}_{4\times4}\right\}m_{N_\star}m_N\right) = \\
&= \frac{\left|g^{N_\star\to NP}\right|^2}{2}\left(4k^\mu p_{1\mu} + 4m_{N_\star}m_N\right) = 2\left|g_{N_\star NP}\right|\left(m_{N_\star}E_N(\boldsymbol{p}_1) + m_{N_\star}m_N\right) = \\
&= 2\left|g^{N_\star\to NP}\right|^2 m_{N_\star}m_N\left(\frac{E_N(\boldsymbol{p}_1)}{m_N} + 1\right),
\end{aligned}
\tag{8.15}
$$

where in the third line we have used that the gamma matrices are traceless ($\mathrm{Tr}\,\gamma^\mu = 0\ \forall\ \mu = 0,1,2,3$). In the fourth line we applied the relation $\mathrm{Tr}\left\{\gamma^\sigma\gamma^\mu\right\} = 4g^{\sigma\mu}$. Furthermore we regard that $\boldsymbol{k} = 0$, eq. (8.3), from which follows that $k^\mu p_{1\mu} = k^0 p_1^0 = m_{N_\star}E_N(\boldsymbol{p}_1)$. Plugging this into the decay width (8.11) and regarding the delta distributions yields

$$
\Gamma_{N_\star\to NP} = \frac{p_f m_N}{2\pi m_{N_\star}}\frac{\left|g^{N_\star\to NP}\right|^2}{2}\left(\frac{E_N}{m_N} + 1\right), \tag{8.16}
$$

where

$$
E_N = E_N(p_f) = \sqrt{p_f^2 + m_N^2}. \tag{8.17}
$$

This result is now easy to extend to the decay width of a Lagrangian with derivative couplings.

8.1.2. Including Derivative Couplings

Since the Lagrangian of our model contains also derivative interactions, we have to extend the previous result (8.16). Therefore, we consider the following Lagrangian, which is based on the interaction terms (7.28):

$$\begin{aligned}\mathcal{L}_{N_\star NP} =& - ig^{N\to N_\star P}\bar{N}_\star \boldsymbol{P}\cdot\boldsymbol{\tau}N + g^{N\to N_\star \partial P}\bar{N}_\star\gamma^\mu\partial_\mu\boldsymbol{P}\cdot\boldsymbol{\tau}N +\\ &- ig^{N_\star\to NP}\bar{N}\boldsymbol{P}\cdot\boldsymbol{\tau}N_\star + g_{N_\star\to N\partial P}\bar{N}\gamma^\mu\partial_\mu\boldsymbol{P}\cdot\boldsymbol{\tau}N_\star.\end{aligned} \tag{8.18}$$

Thus, due to the last term there is an additional Feynman diagram which contributes to the $N_\star \to NP$ decay. It includes a coupling to the derivative of the meson field.

This new diagram now has a different coupling constant. Making use of the correspondence $i\partial_\mu = p_\mu$ the momentum p^μ of the meson enters in the expression. The relation (8.11) for the decay width is still valid - only the matrix element $|i\mathcal{M}|$ is affected. Using the Feynman rules (see sec. 2.3) we find for the amplitude

$$i\mathcal{M}^{ss'} = -i\bar{u}^{s'}(\boldsymbol{p}_1)Cu^s(\boldsymbol{k}=0) \qquad \text{with} \qquad C = -ig^{N_\star\to NP} + ig^{N_\star\to N\partial P}\gamma^\rho p_{2\rho}. \tag{8.19}$$

Again, we have to average over initial spins and sum over final spins to find the averaged squared amplitude

$$\overline{|i\mathcal{M}^{ss'}|^2} = \frac{1}{2}\sum_{s,s'}\left|i\mathcal{M}^{ss'}\right|^2 = \frac{1}{2}\sum_{s,s'}\bar{u}^{s'}(\boldsymbol{p}_1)Cu^s(\boldsymbol{k})\bar{u}^s(\boldsymbol{k})\tilde{C}u^{s'}(\boldsymbol{p}_1) = \ldots, \tag{8.20}$$

where we used that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ and defined $\tilde{C} = ig^{N_\star\to NP} - ig^{N_\star\to N\partial P}\gamma^\rho p_{2\rho} = \gamma^0 C^\dagger\gamma^0$. Applying the relation for the spin sums over the 4-spinors $u^s(\boldsymbol{p})$ (8.14) yields

$$\begin{aligned}\overline{|i\mathcal{M}^{ss'}|^2} =& \frac{1}{2}\,\mathrm{Tr}\left\{C(\gamma^\mu k_\mu + m_{N_\star})\tilde{C}(\gamma^\nu p_{1\nu} + m_N)\right\} =\\ =& \frac{\left(g^{N_\star\to NP}\right)^2}{2}\,\mathrm{Tr}\left\{(\gamma^\mu k_\mu + m_{N_\star})(\gamma^\nu p_{1\nu} + m_N)\right\} +\\ &+ \frac{\left(g^{N_\star\to N\partial P}\right)^2}{2}\,\mathrm{Tr}\left\{\gamma^\sigma p_{2\sigma}(\gamma^\mu k_\mu + m_{N_\star})\gamma^\rho p_{2\rho}(\gamma^\nu p_{1\nu} + m_N)\right\} +\\ &- \frac{g^{N_\star\to NP}g^{N_\star\to N\partial P}}{2}\,\mathrm{Tr}\left\{\gamma^\sigma p_{2\sigma}(\gamma^\mu k_\mu + m_{N_\star})(\gamma^\nu p_{1\nu} + m_N) +\right.\\ &\left.+(\gamma^\mu k_\mu + m_{N_\star})\gamma^\sigma p_{2\sigma}(\gamma^\nu p_{1\nu} + m_N)\right\} = \ldots\end{aligned} \tag{8.21}$$

For reasons of clarity and comprehensibility we will evaluate the three traces separately.

- The **first** trace is the same as we have evaluated in eq. (8.15):

$$\frac{\left(g^{N_\star\to NP}\right)^2}{2}\,\mathrm{Tr}\left\{(\gamma^\mu k_\mu + m_{N_\star})(\gamma^\mu p_{1\mu}+m_N)\right\} = 2\left(g^{N_\star\to NP}\right)^2 m_{N_\star} m_N\left(\frac{E_N(\boldsymbol{p}_1)}{m_N}+1\right), \quad (8.22)$$

- The **second** trace can be rearranged in the following way:

$$\begin{aligned}
&\frac{\left(g^{N_\star\to N\partial P}\right)^2}{2}\,\mathrm{Tr}\left\{\gamma^\mu p_{2\mu}(\gamma^\nu k_\nu + m_{N_\star})\gamma^\rho p_{2\rho}(\gamma^\sigma p_{1\sigma}+m_N)\right\} = \\
=&\frac{\left(g^{N_\star\to N\partial P}\right)^2}{2}\left[\mathrm{Tr}\left\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right\}p_{2\mu}p_\nu p_{2\rho}p_{1\sigma} + \mathrm{Tr}\left\{\gamma^\mu\gamma^\rho\right\}p_{2\mu}m_{N_\star}p_{2\rho}m_N\right] = \\
=&\frac{\left(g^{N_\star\to N\partial P}\right)^2}{2}\left[4(p_{2\mu}k^\mu p_{2\nu}p_1^\nu - p_{2\mu}p_2^\mu k_\nu p_1^\nu + p_{2\mu}p_1^\mu k_\nu p_2^\nu) + 4m_{N_\star}m_N p_{2\mu}p_{2\mu}\right] = \ldots,
\end{aligned}$$

where we made use of $\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = 4g^{\mu\nu}$ and $\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\sigma})$ and regarded that $\mathrm{Tr}\{\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n}\} = 0$ for odd n. With $\boldsymbol{k} = 0$,, eq. (8.3), and $p_{2\mu}p_2^\mu = m_P^2$ we get:

$$\ldots = \frac{\left(g^{N_\star\to NP}\right)^2}{2}\left[4(E_P(\boldsymbol{p}_2)m_{N_\star}p_{2\nu}p_1^\nu - m_P^2 m_{N_\star}E_N(\boldsymbol{p}_1) + p_{2\mu}p_1^\mu m_{N_\star}E_P(\boldsymbol{p}_1)) + 4m_{N_\star}m_N m_P^2\right] = \ldots$$

With the four-momentum conservation $k = p_1 + p_2$ we can find an expression for the product $p_{1\mu}p_2^\mu$:

$$\begin{aligned}
k^\mu &= p_1^\mu + p_2^\mu \\
k^2 &= (p_1+p_2)^2 = p_1^2 + 2p_{1\mu}p_2^\mu + p_2^2 \\
m_{N_\star}^2 &= m_N^2 + 2p_{1\mu}p_2^\mu + m_P^2 \qquad \Rightarrow \qquad p_{1\mu}p_2^\mu = \frac{1}{2}(m_{N_\star}^2 - m_N^2 - m_P^2).
\end{aligned}$$

If we plug this into eq. (8.23), we find a final expression for the second trace:

$$\ldots = 2\left(g^{N_\star\to N\partial P}\right)^2\left[(m_{N_\star}^2 - m_N^2 - m_P^2)E_P(\boldsymbol{p}_2)m_{N_\star} - m_P^2 m_{N_\star}E_N(\boldsymbol{p}_1) + m_{N_\star}m_N m_P^2\right]. \quad (8.23)$$

- Analogously the **third** trace can be rearranged to

$$\begin{aligned}
&-\frac{g^{N_\star\to NP}g^{N_\star\to N\partial P}}{2}\,\mathrm{Tr}\left\{\gamma^\rho p_{2\rho}(\gamma^\mu k_\mu + m_{N_\star})(\gamma^\mu p_{1\mu}+m_N) + \right. \\
&\qquad\qquad \left. +(\gamma^\mu k_\mu + m_{N_\star})\gamma^\rho p_{2\rho}(\gamma^\mu p_{1\mu}+m_N)\right\} = \\
&= -\frac{g^{N_\star\to NP}g^{N_\star\to N\partial P}}{2}\left[4(m_{N_\star}^2 - m_N^2 - m_P^2)m_{N_\star} + 8E_P(\boldsymbol{p}_2)m_{N\star}m_N\right]. \qquad (8.24)
\end{aligned}$$

Hence the final expression for the averaged squared matrix element (8.21) is

$$\begin{aligned}\overline{|i\mathcal{M}^{ss'}|^2} &= 2\left(g^{N_\star\to NP}\right)^2 m_{N_\star} m_N \left(\frac{E_N(\boldsymbol{p}_1)}{m_N}+1\right)+\\ &+2\left(g^{N_\star\to N\partial P}\right)^2\Big[(m_{N_\star}^2-m_N^2-m_P^2)E_P(\boldsymbol{p}_2)m_{N_\star}-m_P^2 m_{N_\star}E_N(\boldsymbol{p}_1)+m_{N_\star}m_N m_P^2\Big]+\\ &-2g^{N_\star\to NP}g^{N_\star\to N\partial P}\Big[(m_{N_\star}^2-m_N^2-m_P^2)m_{N_\star}+2E_P(\boldsymbol{p}_2)m_{N\star}m_N\Big]. \end{aligned} \tag{8.25}$$

Plugging the result (8.25) into eq. (8.11) yields us the decay width of the decay $N_\star \to NP$ including derivative couplings:

$$\boxed{\begin{aligned}\Gamma_{N_\star\to NP} &= \lambda_P \frac{p_f}{8\pi m_{N_\star}^2}\,|i\mathcal{M}|^2\\ &= \frac{p_f m_N}{4\pi m_{N_\star}}\Big\{\left(g^{N_\star\to NP}\right)^2\left(\frac{E_N}{m_N}+1\right)+\\ &\quad+\left(g^{N_\star\to N\partial P}\right)^2\Big[(m_{N_\star}^2-m_N^2-m_P^2)\frac{E_P}{m_N}-m_P^2\frac{E_N}{m_N}+m_P^2\Big]+\\ &\quad-g^{N_\star\to NP}g^{N_\star\to N\partial P}\Big[(m_{N_\star}^2-m_N^2-m_P^2)\frac{1}{m_N}+2E_P\Big]\Big\}, \qquad (8.26)\\ &\text{with}\\ p_f &= \frac{1}{2m_{N_\star}}\sqrt{(m_{N_\star}^2-m_N^2-m_P^2)^2-4m_N^2m_P^2}, \qquad (8.27)\end{aligned}}$$

where $E_N = E_N(p_f) = \sqrt{p_f^2+m_N^2}$ and $E_P = E_P(p_f) = \sqrt{p_f^2+m_P^2}$, because of the delta distribution in eq. (8.11). Furthermore we added a factor λ_P by hand. This factor should

- for $P=\pi$ pay attention to the three possible isospin states of the pion, i.e.,

$$\boxed{\lambda_\pi = 3.} \tag{8.28}$$

- and for $P=\eta$ take into account that

$$\eta = \eta_N \cos\phi_P + \eta_S \sin\phi_P, \tag{8.29}$$

where $\eta_N \equiv (\bar{u}u+\bar{d}d)/\sqrt{2}$ and $\eta_S \equiv \bar{s}s$ and ϕ_P is the mixing angle. Its value lies between $-32°$ and $-45°$ [27]. For the following results we will choose $\phi_P = -38.7° \pm 6°$.
It is assumed that the amplitude of the decay $N_\star \to N\eta_S$ is massively suppressed. This means that to good approximation

$$\Gamma_{N_\star\to N\eta} \simeq \cos^2\phi_P \Gamma_{N_\star\to N\eta_N}.$$

Thus:

$$\boxed{\lambda_\eta = \cos^2 \phi_P. \qquad (8.30)}$$

8.2. Decay Width of $N_3 \longrightarrow N_1 P$

An interaction Lagrangian which is based on the interaction terms (7.29) and describes the coupling of $N_3 N_1 P$ is given by

$$\begin{aligned}\mathcal{L}_{N_3 N_1 P} = &- i g^{N_1 \to N_3 P} \bar{N}_3 \gamma^5 \boldsymbol{P} \cdot \boldsymbol{\tau} N_1 + g^{N_1 \to N_3 \partial P} \bar{N}_3 \gamma^5 \gamma^\mu \partial_\mu \boldsymbol{P} \cdot \boldsymbol{\tau} N_1 + \\ &- i g^{N_3 \to N_1 P} \bar{N}_1 \gamma^5 \boldsymbol{P} \cdot \boldsymbol{\tau} N_3 + g^{N_3 \to N_1 \partial P} \bar{N}_1 \gamma^5 \gamma^\mu \partial_\mu \boldsymbol{P} \cdot \boldsymbol{\tau} N_3,\end{aligned} \tag{8.31}$$

where the indices are chosen according to eq. (7.30). If we assign the 4-momenta in analogy to sec. 8.1.2:

- The decaying particle N_3 has 4-momentum k,
- the outgoing nucleon N_1 has 4-momentum p_1, and
- the pseudoscalar P has 4-momentum p_2,

we find that the decay width for $N_3 \longrightarrow N_1 P$ has the same form as the relation given in eq. (8.11):

$$\Gamma_{N_3 \to N_1 P} = \frac{p_f}{8\pi m_{N_3}^2} \left| i\mathcal{M} \right|^2 \delta(|\boldsymbol{p}_1| - p_f) \delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2), \tag{8.32}$$

with

$$p_f = \frac{1}{2 m_{N_3}} \sqrt{(m_{N_3}^2 - m_{N_1}^2 - m_P^2)^2 - 4 m_{N_1}^2 m_P^2}. \tag{8.33}$$

The only difference to the calculations of sec. 8.1.2 is the matrix element $|i\mathcal{M}|$, because the coupling constants in the Lagrangian (8.31) include additional γ^5 matrices. In comparison to eq. (8.19) this leads to an additional minus sign:

$$i\mathcal{M}^{ss'} = -i \bar{u}^{s'}(\boldsymbol{p}_1) C' u^s(\boldsymbol{k} = 0) \qquad \text{with} \qquad C' = -i\gamma^5 g^{N_3 \to N_1 P} + i \gamma^5 g^{N_3 \to N_1 \partial P} \gamma^\rho p_{2\rho}. \tag{8.34}$$

Again, we have to average over initial spins and sum over final spins to find the averaged squared amplitude

$$\overline{|i\mathcal{M}^{ss'}|^2} = \frac{1}{2} \sum_{s,s'} \left| i\mathcal{M}^{ss'} \right|^2 = \frac{1}{2} \sum_{s,s'} \bar{u}^{s'}(\boldsymbol{p}_1) C' u^s(\boldsymbol{k}) \bar{u}^s(\boldsymbol{k}) \tilde{C}' u^{s'}(\boldsymbol{p}_1) = \ldots, \tag{8.35}$$

where $\tilde{C}' = -i\gamma^5 g^{N_3\to N_1 P} - i\gamma^5 g^{N_3\to N_1 \partial P}\gamma^\rho p_{2\rho} = \gamma^0 C'^\dagger \gamma^0$. Applying the relation for the spin sums over the 4-spinors $u^s(\boldsymbol{p})$ (8.14) yields

$$\begin{aligned}
\overline{|i\mathcal{M}^{ss'}|^2} &= \frac{1}{2}\,\mathrm{Tr}\left\{C'(\gamma^\mu k_\mu + m_{N_3})\tilde{C}'(\gamma^\nu p_{1\nu} + m_{N_1})\right\} = \\
&= -\frac{\left(g^{N_3\to N_1 P}\right)^2}{2}\,\mathrm{Tr}\left\{\gamma^5(\gamma^\mu k_\mu + m_{N_3})\gamma^5(\gamma^\nu p_{1\nu} + m_{N_1})\right\} + \\
&\quad + \frac{\left(g^{N_3\to N_1 \partial P}\right)^2}{2}\,\mathrm{Tr}\left\{\gamma^5\gamma^\sigma p_{2\sigma}(\gamma^\mu k_\mu + m_{N_3})\gamma^5\gamma^\rho p_{2\rho}(\gamma^\nu p_{1\nu} + m_{N_1})\right\} + \\
&\quad - \frac{g^{N_3\to N_1 P} g^{N_3\to N_1 \partial P}}{2}\,\mathrm{Tr}\left\{-\gamma^5\gamma^\sigma p_{2\sigma}(\gamma^\mu k_\mu + m_{N_3})\gamma^5(\gamma^\nu p_{1\nu} + m_{N_1}) + \right.\\
&\qquad\qquad \left. +\gamma^5(\gamma^\mu k_\mu + m_{N_3})\gamma^5\gamma^\sigma p_{2\sigma}(\gamma^\nu p_{1\nu} + m_{N_1})\right\} = \\
&= -\frac{\left(g^{N_3\to N_1 P}\right)^2}{2}\,\mathrm{Tr}\left\{(-\gamma^\mu k_\mu + m_{N_3})(\gamma^\nu p_{1\nu} + m_{N_1})\right\} + \\
&\quad + \frac{\left(g^{N_3\to N_1 \partial P}\right)^2}{2}\,\mathrm{Tr}\left\{-\gamma^\sigma p_{2\sigma}(-\gamma^\mu k_\mu + m_{N_3})\gamma^\rho p_{2\rho}(\gamma^\nu p_{1\nu} + m_{N_1})\right\} + \\
&\quad - \frac{g^{N_3\to N_1 P} g^{N_3\to N_1 \partial P}}{2}\,\mathrm{Tr}\left\{\gamma^\sigma p_{2\sigma}(-\gamma^\mu k_\mu + m_{N_3})(\gamma^\nu p_{1\nu} + m_{N_1}) + \right.\\
&\qquad\qquad \left. +(-\gamma^\mu k_\mu + m_{N_3})\gamma^\sigma p_{2\sigma}(\gamma^\nu p_{1\nu} + m_{N_1})\right\} = \ldots ,
\end{aligned} \tag{8.36}$$

where we have used that $\{\gamma^5, \gamma^\mu\} = 0$ and $\gamma^5\gamma^5 = 1$. The two traces can be rearranged further:

- The **first** trace is apart from one sign the same as we have evaluated in (8.15). It is given by:

$$-\frac{\left(g^{N_3\to N_1 P}\right)^2}{2}\,\mathrm{Tr}\left\{(-\gamma^\mu k_\mu + m_{N_3})(\gamma^\mu p_{1\mu} + m_{N_1})\right\} = 2\left(g^{N_3\to N_1 P}\right)^2 m_{N_3} m_{N_1}\left(\frac{E_{N_1}(\boldsymbol{p}_1)}{m_{N_1}} - 1\right). \tag{8.37}$$

- The **second** trace can be rearranged to:

$$\begin{aligned}
&\frac{\left(g^{N_3\to N_1 \partial P}\right)^2}{2}\,\mathrm{Tr}\left\{-\gamma^\sigma p_{2\sigma}(-\gamma^\mu k_\mu + m_{N_3})\gamma^\rho p_{2\rho}(\gamma^\nu p_{1\nu} + m_{N_1})\right\} = \\
&= 2\left(g^{N_3\to N_1 \partial P}\right)^2\left[(m_{N_3}^2 - m_{N_1}^2 - m_P^2)E_P(\boldsymbol{p}_2)m_{N_3} - m_P^2 m_{N_3} E_{N_1}(\boldsymbol{p}_1) - m_{N_3} m_{N_1} m_P^2\right].
\end{aligned} \tag{8.38}$$

- The **third** trace is given by:

$$\begin{aligned}
&-\frac{g^{N_3\to N_1 P} g^{N_3\to N_1 \partial P}}{2}\,\mathrm{Tr}\left\{\gamma^\sigma p_{2\sigma}(-\gamma^\mu k_\mu + m_{N_3})(\gamma^\nu p_{1\nu} + m_{N_1}) + \right.\\
&\qquad\qquad \left. +(-\gamma^\mu k_\mu + m_{N_3})\gamma^\sigma p_{2\sigma}(\gamma^\nu p_{1\nu} + m_{N_1})\right\} = \\
&= -\frac{g^{N_3\to N_1 P} g^{N_3\to N_1 \partial P}}{2}\left[4(m_{N_3}^2 - m_{N_1}^2 - m_P^2)m_{N_3} - 8E_P(\boldsymbol{p}_2)m_{N_3} m_{N_1}\right].
\end{aligned} \tag{8.39}$$

Hence the final expression for the averaged squared matrix element (8.36) is

$$\begin{aligned}
\overline{|i\mathcal{M}^{ss'}|^2} = 2\left(g^{N_3\to N_1 P}\right)^2 m_{N_3} m_{N_1}\left(\frac{E_{N_1}(\boldsymbol{p}_1)}{m_{N_1}} - 1\right) + \\
+ 2\left(g^{N_3\to N_1\partial P}\right)^2\left[(m_{N_3}^2 - m_{N_1}^2 - m_P^2)E_P(\boldsymbol{p}_2)m_{N_3} - m_P^2 m_{N_3} E_{N_1}(\boldsymbol{p}_1) - m_{N_3} m_{N_1} m_P^2\right] + \\
- \frac{g^{N_3\to N_1 P} g^{N_3\to N_1\partial P}}{2}\left[4(m_{N_3}^2 - m_{N_1}^2 - m_P^2)m_{N_3} - 8E_P(\boldsymbol{p}_2)m_{N_3}m_{N_1}\right]. \qquad (8.40)
\end{aligned}$$

Plugging the result (8.40) into eq. (8.32) yields us the decay width of the decay $N_3 \to N_1 P$:

$$\begin{aligned}
\Gamma_{N_3\to N_1 P} &= \lambda_P \frac{p_f}{8\pi m_{N_\star}^2}\,|i\mathcal{M}|^2 \\
&= \frac{p_f m_{N_1}}{4\pi m_{N_3}}\Big\{\left(g^{N_3\to N_1 P}\right)^2\left(\frac{E_{N_1}}{m_{N_1}} - 1\right) + \\
&\quad + \left(g^{N_3\to N_1\partial P}\right)^2\left[(m_{N_3}^2 - m_{N_1}^2 - m_P^2)\frac{E_P}{m_{N_1}} - m_P^2\frac{E_{N_1}}{m_{N_1}} - m_P^2\right] + \\
&\quad - g^{N_3\to N_1 P} g^{N_3\to N_1\partial P}\left[(m_{N_3}^2 - m_{N_1}^2 - m_P^2)\frac{1}{m_{N_1}} - 2E_P\right]\Big\} \qquad (8.41)
\end{aligned}$$

with

$$p_f = \frac{1}{2m_{N_3}}\sqrt{(m_{N_3}^2 - m_{N_1}^2 - m_P^2)^2 - 4m_{N_1}^2 m_P^2}\,, \qquad (8.42)$$

where $E_{N_1} = E_{N_1}(p_f) = \sqrt{p_f^2 + m_{N_1}^2}$ and $E_P = E_P(p_f) = \sqrt{p_f^2 + m_P^2}$. The factor λ_P is defined in eq. (8.28) and (8.30).

8.3. Fit of c_N, c_M, c_{A_N} and c_{A_M} to Decay Widths

The remaining undetermined parameters c_N, c_M, c_{A_N}, and c_{A_M} in the Lagrangian (6.16) can be fixed by using the experimental values of the decay widths which belong to the kinematically allowed decays (8.1). According to ref. [23] the observed values are

$$\begin{aligned}
\Gamma^{\text{exp}}_{N(1535)\to N(939)\pi} &= (67.5 \pm 15)\text{ MeV} \\
\Gamma^{\text{exp}}_{N(1535)\to N(939)\eta} &= (63.0 \pm 15)\text{ MeV} \\
\Gamma^{\text{exp}}_{N(1650)\to N(939)\pi} &= (105 \pm 30)\text{ MeV} \\
\Gamma^{\text{exp}}_{N(1650)\to N(939)\eta} &= (15.0 \pm 7.5)\text{ MeV} \\
\Gamma^{\text{exp}}_{N(1440)\to N(939)\pi} &= (195 \pm 30)\text{ MeV}. \qquad (8.43)
\end{aligned}$$

In order to calculate the decay widths which result from our model we use the expression (8.26), the values for the coupling constants given in eqs. (7.31) and (7.32), and the values for the parameters given in table 7.1. Furthermore, we use the following values, for the masses of a_1, the pseudoscalars π and η [23], and the coupling constant g_1 of the meson sector [2] :

$$\begin{aligned} m_{a_1} &= (1230 \pm 61.5)\ \text{MeV} \\ m_\pi &= (138 \pm 6.9)\ \text{MeV} \\ m_\eta &= (547.9 \pm 27.4)\ \text{MeV} \\ g_1 &= 5.8433 \pm 0.0176 \end{aligned} \tag{8.44}$$

Then we impose that they should be equal to the experimental quantities (8.43).
Therefore we define a function

$$\begin{aligned} \chi^2(c_N, c_M, c_{A_N}, c_{A_M}) := \; & \left[\Gamma_{N(1535)\to N(939)\pi}(c_N, c_M, c_{A_N}, c_{A_M}) - \Gamma^{\text{exp}}_{N(1535)\to N(939)\pi}\right]^2 + \\ & + \left[\Gamma_{N(1535)\to N(939)\eta}(c_N, c_M, c_{A_N}, c_{A_M}) - \Gamma^{\text{exp}}_{N(1535)\to N(939)\eta}\right]^2 + \\ & + \left[\Gamma_{N(1650)\to N(939)\pi}(c_N, c_M, c_{A_N}, c_{A_M}) - \Gamma^{\text{exp}}_{N(1650)\to N(939)\pi}\right]^2 + \\ & + \left[\Gamma_{N(1650)\to N(939)\eta}(c_N, c_M, c_{A_N}, c_{A_M}) - \Gamma^{\text{exp}}_{N(1650)\to N(939)\eta}\right]^2 + \\ & + \left[\Gamma_{N(1440)\to N(939)\pi}(c_N, c_M, c_{A_N}, c_{A_M}) - \Gamma^{\text{exp}}_{N(1440)\to N(939)\pi}\right]^2 , \end{aligned} \tag{8.45}$$

which should be minimized with a preferably small minimum, in order to determine the parameters c_N, c_M, c_{A_N}, and c_{A_M}. Doing this we find that the minimum is not narrow and the results of the fit depend very strongly in the starting point we choose. Table 8.1 shows the results with two different starting points for each parameter.

We see that the values can be completely different just because of choosing different starting points of the fit. Nevertheless, the results for the decay widths are always equal, because all minima have the same depth.

Considering the decays into nucleon and pion the results are very good. Only the decay widths of the decay into the nucleon and η differ from the experimental results. Here the decay width of the decay of the resonance $N(1650)$ into $N\eta$ is much closer to the imposed values. This is a result which is in accordance to the results of ref. [3].

	starting point (SP) and result of the fit				experimental values [23]
	SP 1	result	SP 2	result	
c_N	-15	-21.18	-1	-49.02	-
c_M	10	-21.11	100	2.537	-
c_{A_N}	0	-0.006340	-3	-2.040	-
c_{A_M}	0	15.91	-1	8.149	-
$\Gamma_{N(1535)\to N(939)\pi}$ [MeV]	72.75				67.5 ± 15
$\Gamma_{N(1535)\to N(939)\eta}$ [MeV]	5.567				63.0 ± 15
$\Gamma_{N(1650)\to N(939)\pi}$ [MeV]	105.0				105 ± 30
$\Gamma_{N(1650)\to N(939)\eta}$ [MeV]	9.775				15.0 ± 7.5
$\Gamma_{N(1440)\to N(939)\pi}$ [MeV]	195.0				195 ± 30

Table 8.1.: Results of the fit with two different starting points for each parameter.

9. The Axial Coupling Constants

Other interesting quantities which yield information of the chiral structure of the baryon sector are the axial coupling constants. These are the constants in front of the axial currents (1.38) which have the following form:

$$A^{a,\mu} = g_A \bar{\Psi}\gamma^\mu\gamma^5 \frac{\tau^a}{2}\Psi. \tag{9.1}$$

In general, the current of a Lagrangian (resulting from the conservation under an arbitrary transformation) is given as [16]

$$J^\mu \sum_k \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi^k\right)}\delta\Phi^k \qquad \text{with} \qquad \Phi^k \xrightarrow{\text{infinitesimal}} \Phi^k + \delta\Phi^k. \tag{9.2}$$

Using that in our case $\Phi^k = \pi^k$ and $\delta\Phi^k = \delta\pi^k$ and that (for infinitesimal axial transformations) $U_A = U_L = U_R^\dagger = e^{-i\theta_A^i\gamma^5 T^i} \simeq 1 - i\theta_A^i\gamma^5 T^i$, the pion field transforms as

$$\begin{aligned}
\pi^k \hat{=} i\bar{q}\gamma^5\tau^k q \quad &\xrightarrow{U_A} \quad iq^\dagger(1+i\theta_A^i\gamma^5T^i)\gamma^0\gamma^5\tau^k(1-i\theta_A^i\gamma^5T^i)q = \\
&\qquad i\bar{q}(1-i\theta_A^i\gamma^5T^i)\gamma^5\tau^k(1-i\theta_A^i\gamma^5T^i)q = \\
&\qquad = i\bar{q}\gamma^5\tau^k q - \bar{q}\theta_A^i\gamma^5\gamma^5\frac{1}{2}\left(\tau^i\tau^k + \tau^k\tau^i\right)q + \mathcal{O}(\theta_A^2) = \\
&\qquad \simeq \pi^k - \bar{q}\theta_A^i\frac{1}{2}2\delta_{ik}q = \pi^k - \theta_A^k\bar{q}q = \pi^k - \theta_A^k\sigma_N \\
\boldsymbol{\pi} \quad &\xrightarrow{U_A} \quad \boldsymbol{\pi} - \boldsymbol{\theta}_A\sigma_N,
\end{aligned} \tag{9.3}$$

where we have used the anti-commutator for the Pauli matrices $\{\tau^i, \tau^k\} = 2\delta_{ik}$ and $\sigma_N \hat{=} \bar{q}q$. Thus $\delta\pi^k = -\sigma_N$. Additionally, have to pay attention to the fact that we had to renormalise $\boldsymbol{\pi}$ after the condensation of the sigma meson: $\boldsymbol{\pi} \to Z\boldsymbol{\pi}$. Hence the infinitesimal variation (in first order) of the pion under axial transformation after the condensation of the sigma meson, $\sigma_N \to \sigma_N + \varphi_N$, is given by

$$\delta\pi^k = -\frac{\varphi_N}{Z} + \ldots, \tag{9.4}$$

where only the relevant term is listed. In order to find the axial coupling constants of our baryon Lagrangian (6.16) we first extract the terms generating the axial currents. These are kinetic terms and the pseudo-vectorial couplings of nucleon and pion, which occur after the shift of the $\boldsymbol{a}_1^\mu$ axial field from the terms describing the interaction of nucleons with (axial-) vector fields. First, omitting the kinetic terms the relevant terms of the Lagrangian (6.16) are

$$\begin{aligned}\mathcal{L}_A = &-c_N Z w \sum_{i=1}^{2} \left(\bar{\Psi}_i \gamma^\mu \gamma^5 \frac{\boldsymbol{\tau}}{2} \cdot \partial_\mu \boldsymbol{\pi} \Psi_i\right) + c_M Z w \sum_{i=3}^{4} \left(\bar{\Psi}_i \gamma^\mu \gamma^5 \frac{\boldsymbol{\tau}}{2} \cdot \partial_\mu \boldsymbol{\pi} \Psi_i\right) + \\ &- c_{A_N} Z w \sum_{\substack{i,j=1,\\ i\neq j}}^{2} \left(\bar{\Psi}_i \gamma^\mu \frac{\boldsymbol{\tau}}{2} \cdot \partial_\mu \boldsymbol{\pi} \Psi_j\right) - c_{A_M} Z w \sum_{\substack{i,j=3,\\ i\neq j}}^{4} \left(\bar{\Psi}_i \gamma^\mu \frac{\boldsymbol{\tau}}{2} \cdot \partial_\mu \boldsymbol{\pi} \Psi_j\right). \end{aligned} \tag{9.5}$$

Using eqs. (9.2) and (9.4) we can compute the axial currents resulting from these terms:

$$\begin{aligned}A_c^{a,\mu} = \frac{\partial \mathcal{L}_A}{\partial(\partial_\mu \pi^a)} \left(-\frac{\varphi_N}{Z}\right) = &\, c_N w \varphi_N \sum_{i=1}^{2} \bar{\Psi}_i \gamma^\mu \gamma^5 \frac{\tau^a}{2} \Psi_i - c_M w \varphi_N \sum_{i=3}^{4} \bar{\Psi}_i \gamma^\mu \gamma^5 \frac{\tau^a}{2} \Psi_i + \\ &+ c_{A_N} w \varphi_N \sum_{\substack{i,j=1,\\ i\neq j}}^{2} \bar{\Psi}_i \gamma^\mu \frac{\tau^a}{2} \Psi_j + c_{A_M} w \varphi_N \sum_{\substack{i,j=3,\\ i\neq j}}^{4} \bar{\Psi}_i \gamma^\mu \frac{\tau^a}{2} \cdot \Psi_j \end{aligned} \tag{9.6}$$

Considering also the transformation of Ψ_i under axial transformations shows that the kinetic term yields an axial current, too. In order to calculate them we need the infinitesimal axial transformation ($U_A = U_L = U_R^\dagger$) of a spinor. First we consider one whose left- and right-handed components transform under $U(3)_R \times U(3)_L$ as $\Psi_R \to U_R \Psi_R$ and $\Psi_R \to U_L \Psi_L$. Then we find:

$$\begin{aligned}\Psi = \Psi_R + \Psi_L \quad \xrightarrow{U_A} \quad & U_A^\dagger \Psi_R + U_A \Psi_L = (1 + i\theta_A^i T^i \gamma^5)\Psi_R + (1 - i\theta_A^i T^i \gamma^5)\Psi_L = \\ &= \Psi_R + \Psi_L + i\theta_A^i T^i \gamma^5 \Psi_R - i\theta_A^i T^i \gamma^5 \Psi_L = \\ &= \Psi + i\theta_A^i T^i \gamma^5 \frac{1+\gamma^5}{2} \Psi - i\theta_A^i T^i \gamma^5 \frac{1-\gamma^5}{2} \Psi = \\ &= \Psi + i\theta_A^i T^i \gamma^5 \Psi. \end{aligned} \tag{9.7}$$

Thus, the infinitesimal change of this spinor under axial transformations is given by

$$\delta \Psi^a = i T^a \gamma^5 \Psi. \tag{9.8}$$

In analogy we find for a spinor whose left- and right-handed components transform under $U(3)_R \times U(3)_L$ as $\Psi_R \to U_L \Psi_R$ and $\Psi_R \to U_R \Psi_L$:

$$\delta \Psi^a = -i T^a \gamma^5 \Psi. \tag{9.9}$$

Then, from the kinetic term for Ψ_i results the following part of axial current:

$$A_{\Psi_i,\mathrm{kin}}^{a,\mu} = \frac{\partial \mathcal{L}_{\mathrm{kin},\Psi_i}}{\partial(\partial_\mu \Psi_i)} \delta \Psi_i^a = \begin{cases} \bar{\Psi}_i i \gamma^\mu i T^a \gamma^5 \Psi_i = -\bar{\Psi}_i \gamma^\mu \gamma^5 \frac{\tau^a}{2} \Psi_i & \text{for } i = 1,2, \\ -\bar{\Psi}_i i \gamma^\mu i T^a \gamma^5 \Psi_i = \bar{\Psi}_i \gamma^\mu \gamma^5 \frac{\tau^a}{2} \Psi_i & \text{for } i = 3,4. \end{cases} \tag{9.10}$$

Adding up these currents for $i = 1,2,3,4$ and the upper current $A_c^{a,\mu}$ we find the total current

$$A^{a,\mu} = g_A^{(1)} \bar{\Psi}_1 \gamma^\mu \gamma^5 \frac{\tau^a}{2} \Psi_1 + g_A^{(1)} \bar{\Psi}_2 \gamma^\mu \gamma^5 \frac{\tau^a}{2} \Psi_2 + g_A^{(2)} \bar{\Psi}_3 \gamma^\mu \gamma^5 \frac{\tau^a}{2} \Psi_3 + g_A^{(2)} \bar{\Psi}_4 \gamma^\mu \gamma^5 \frac{\tau^a}{2} \Psi_4 + A_{\mathrm{mix}}^{a,\mu},$$

with

$$A^{a,\mu}_{\text{mix}} = g_A^{(12)} \sum_{\substack{i,j=1,\\ i\neq j}}^{2} \bar{\Psi}_i \gamma^\mu \frac{\tau^a}{2} \Psi_j + g_A^{(34)} \sum_{\substack{i,j=3,\\ i\neq j}}^{4} \bar{\Psi}_i \gamma^\mu \frac{\tau^a}{2} \Psi_j, \tag{9.11}$$

and the following abbreviations:

$$\begin{aligned} g_A^{(1)} &= -1 + c_N w \varphi_N = -1 + \frac{c_N}{g_1}\left(1 - \frac{1}{Z^2}\right), \\ g_A^{(2)} &= 1 - c_M w \varphi_N = 1 - \frac{c_M}{g_1}\left(1 - \frac{1}{Z^2}\right), \\ g_A^{(12)} &= c_{A_N} w \varphi_N = \frac{c_{A_N}}{g_1}\left(1 - \frac{1}{Z^2}\right), \\ \text{and} \qquad g_A^{(34)} &= c_{A_M} w \varphi_N = \frac{c_{A_M}}{g_1}\left(1 - \frac{1}{Z^2}\right), \end{aligned} \tag{9.12}$$

where we have used

$$Z^2 = \frac{m_{a_1}^2}{m_{a_1}^2 - g_1^2 \varphi_N^2} \qquad \text{and} \qquad w = \frac{g_1 \varphi_N}{m_{a_1}^2} \qquad \Rightarrow \qquad w\varphi_N = \frac{1}{g_1}\left(1 - \frac{1}{Z^2}\right) \tag{9.13}$$

from eqs. (3.38) and (3.43).
If we now plug in the physical fields, given in eq. (7.26), and extract the axial current terms of the form $\bar{N}_i \gamma^\mu \gamma^5 \frac{\tau^a}{2} N_i$ with $i \in \{939, 1535, 1440, 1650\}$ we find the axial coupling constant for N_{939}

$$\boxed{\begin{aligned} g_A^{N_{939}} = g_A^{(1)} \left(u_1^{(1)} u_1^{(1)} + u_1^{(2)} u_1^{(2)}\right) + g_A^{(2)} \left(u_1^{(3)} u_1^{(3)} + u_1^{(4)} u_1^{(4)}\right) + \\ + 2 g_A^{(12)} u_1^{(1)} u_1^{(2)} + 2 g_A^{(34)} u_1^{(3)} u_1^{(4)}, \end{aligned}} \tag{9.14}$$

for N_{1535}:

$$\boxed{\begin{aligned} g_A^{N_{1535}} = g_A^{(1)} \left(u_2^{(1)} u_2^{(1)} + u_2^{(2)} u_2^{(2)}\right) + g_A^{(2)} \left(u_2^{(3)} u_2^{(3)} + u_2^{(4)} u_2^{(4)}\right) + \\ + 2 g_A^{(12)} u_2^{(1)} u_2^{(2)} + 2 g_A^{(34)} u_2^{(3)} u_2^{(4)}, \end{aligned}} \tag{9.15}$$

for N_{1440}:

$$\boxed{\begin{aligned} g_A^{N_{1440}} = g_A^{(1)} \left(u_3^{(1)} u_3^{(1)} + u_3^{(2)} u_3^{(2)}\right) + g_A^{(2)} \left(u_3^{(3)} u_3^{(3)} + u_3^{(4)} u_3^{(4)}\right) + \\ + 2 g_A^{(12)} u_3^{(1)} u_3^{(2)} + 2 g_A^{(34)} u_3^{(3)} u_3^{(4)}, \end{aligned}} \tag{9.16}$$

and for N_{1650}:

$$g_A^{N_{1650}} = g_A^{(1)}\left(u_4^{(1)}u_4^{(1)} + u_4^{(2)}u_4^{(2)}\right) + g_A^{(2)}\left(u_4^{(3)}u_4^{(3)} + u_4^{(4)}u_4^{(4)}\right) + \\ + 2g_A^{(12)}u_4^{(1)}u_4^{(2)} + 2g_A^{(34)}u_4^{(3)}u_4^{(4)}. \tag{9.17}$$

Using eq. (7.26) and the results listed in table 8.1 we can compute the axial coupling constants. The results are listed in table 9.1.

	with SP 1	with SP 2	experimental, lattice or RQCM result
g_A^{939}	3.142	6.080	1.267 ± 0.004 [28]
g_A^{1535}	-2.841	-0.1924	0.2 ± 0.3 [29]
g_A^{1440}	-2.853	-0.5410	1.16 [29]
g_A^{1650}	2.559	5.356	0.55 ± 0.2 [30]

Table 9.1.: Result for the axial coupling constants for the set of parameters c_N, c_M, c_{A_N}, and c_{A_M} with the starting point (SP) 1 and 2, given in table 8.1. The axial coupling constant for the nucleon $N(939)$ is known from experiment, the constants for $N(1535)$ and $N(1650)$ are known from lattice calculations and studies in the framework of the relativistic constituent quark model (RCQM) by employing in the first instance the extended Goldstone-boson exchange (EGBE) yields the value for the axial coupling constant of $N(1440)$.

Surely, with the present set of parameters the coupling constants cannot be well described.
Moreover, we realize that the results for the axial coupling constants depend very strongly on the choice of the parameters $c_N, c_M, c_{A_N}, c_{A_M}$. Therefore it is necessary to perform a fit including all parameters and experimental quantities to gain better results.

10. Conclusions

The way to describe the nucleon N and its chiral partner $N^{\star}$ in the extended linear sigma model was first studied in ref. [3]. There the resonance $N(1535)$ was considered to be the chiral partner of the nucleon $N(939)$. This is the most natural assignment, because it is the lightest resonance with the correct quantum numbers. Among other things, the decay width $N^{\star} \to N\eta$ was computed to be (10.9 ± 3.8) MeV. The experimental width, given in [23], is (78.7 ± 24.3) MeV, hence the result was not satisfactory. On the contrary with $N(1650)$ as the chiral partner of the nucleon the decay width $N^{\star} \to N\eta$ is calculated to be (18.3 ± 8.5) MeV and the experimental width in [23] is (10.7 ± 6.7) MeV. Thus, with this assignment the result is much better.

A possible improvement has already been outlined in the outlook of ref. [3] and implies the inclusion of both resonances, $N(1535)$ and $N(1650)$, as well as the nucleon fields $N(940)$ and $N(1440)$, in one unique Lagrangian. For two flavors the constructed chiral Lagrangian in the mirror assignment would contain 14 free parameters. These are four (axial-)vector coupling constants, six constants which parametrize the coupling to (pseudo)scalar mesons, and four parameters of mixing terms. In order to fix these free parameters we have only 13 experimental quantities: four axial couplings, five kinematicaly allowed decay widths of $N^{\star} \to N\pi$, $N^{\star} \to N\eta$, and $N(1440) \to N\pi$, and four masses.

The idea of this work is to start with three flavors, hence with baryon matrices instead of spinors (baryon doublets). We have constructed the inner structure of the baryon fields in the diquark-quark model which assumes that baryons are bound states of a diquark and a quark. By this study of the microscopic baryonic currents, we are automatically led to four baryonic multiplets, as postulated in ref. [3]. Two of them transforming according to the naive assignment and two others according to the mirror assignment. Then, we have built a chirally invariant[1] Lagrangian which has a smaller number of free parameters than the Lagrangian constructed for only two flavors. Namely, there are twelve parameters: two coupling constants which parametrize the coupling to (pseudo)scalar mesons, two parameters of mixing terms, four (axial-)vector coupling constants, and four parameters which have been included to obtain a correct description of masses.

In order to fix these parameters we performed a step-by-step fit. We first took only the mass matrix and the eight corresponding parameters into account and have done a fit. Then, we used decay widths to fix the remaining parameters. This method is a first attempt to study the model and leads to a quantitative agreement with data (but not yet fully satisfactory).

From the fit with the mass matrix, we found that $N(1650)$ is predominantly the chiral partner of the nucleon as anticipated in ref. [5]. However, we realized that the decay width of a resonance into the nucleon and a pseudoscalar cannot be described very well even in this enlarged mixing. The decay width of $N(1650) \to N\eta$ is close to the experimental result, but the decay width of $N(1535) \to N\eta$ is not.

The calculation of the axial coupling constants does not yield good results yet, but a scan of the the

[1] Of course the Lagrangian is also a Lorentz scalar and invariant under CPT.

whole parameter space was not yet performed.
Hence, an important and necessary outlook of this work is to perform a fit which includes all parameters and experimental quantities at one time.
Moreover the three-flavor case should be studied in more detail. In this way various decay widths can be investigated in a unique framework.

A. Units, Conventions, and Notations

Units:

Throughout the whole thesis we will work in **natural units**:

$$\hbar = c = \epsilon_0 = 1.$$

Conventions:

We use the **West Coast metric**, i.e., the metric tensor in Cartesian coordinates in Minkowski space is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Notations:

In order to distinguish between 4-vectors in Minkowski space and three-dimensional vectors in Euclidean space, we choose the convention to write **3-vectors in bold letters** and we indicate the components by Roman indices (which run over $\{1, 2, 3\}$). We always use (and not only to indicate components) Greek indices (which run over $\{0, 1, 2, 3\}$) for 4-vectors.
A **contravariant** 4-vector carries a raised and a **covariant** a lowered Greek index. E.g. the contra- and covariant space-time vector is then written as

$$x^\mu = (t, \boldsymbol{x})^T \qquad \text{and} \qquad x_\mu = g_{\mu\nu} x^\nu = (t, -\boldsymbol{x}),$$

where the transposed symbol indicates that x^μ actually is a column vector. The covariant 4-vector is obtained by pulling the indices down using the metric tensor.
The **scalar product** of two 4-vectors is defined with the metric tensor. E.g. the scalar product of two space-time vectors reads as

$$x^\mu g_{\mu\nu} x^\nu \equiv x_\nu x^\nu = t^2 - \boldsymbol{x}^2.$$

The co - and contravariant **4-gradients** are given by

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \qquad \text{and} \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right)^T .$$

We use the Dirac matrices in **Dirac notation**, see sec. B.3.1, unless otherwise indicated.

B. Summery of Properties of Dirac Matrices

B.1. Definition

The gamma or Dirac matrices γ^0, γ^1, γ^2, and γ^3 are defined by fulfilling the following anticommutator relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{4\times 4}. \tag{B.1}$$

The covariant gamma matrices are given by

$$\gamma_\mu = g_{\mu\nu}\gamma^\nu = \left(\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3\right). \tag{B.2}$$

The γ^5 matrix is defined by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma_0\gamma^1\gamma^2\gamma^3 = \gamma_5. \tag{B.3}$$

B.2. Properties

The products of gamma matrices and the identity matrix form a group with 32 elements. Every representation of a finite group can be chosen unitary (by a proper choice of an basis). Together with the Dirac-algebra (B.1) we obtain that γ^0 is hermitian and the all the other are antihermitian:

$$\gamma^{0\dagger} = \gamma^0 \qquad \text{and} \qquad \gamma^{i\dagger} = -\gamma^i \ , \quad i = 1, 2, 3. \tag{B.4}$$

These relations can be combined in

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{B.5}$$

Therewith γ^5 is also hermitian

$$\gamma^{5\dagger} = \gamma^5. \tag{B.6}$$

Furthermore it is its own inverse,

$$\gamma^5\gamma^5 = \mathbb{1}_{4\times 4}, \tag{B.7}$$

and anticommutates with the gamma matrices,

$$\left\{\gamma^5, \gamma^\mu\right\} = 0. \tag{B.8}$$

As a consequence of the fundamental anticommutation relation (B.1) the following identities hold:

$$
\begin{aligned}
\gamma^\mu\gamma_\mu &= 4\mathbb{1}_{4\times4}, \\
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu, \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho}\mathbb{1}_{4\times4}, \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu, \\
\gamma^\mu\gamma^\nu\gamma^\lambda &= g^{\mu\nu}\gamma^\lambda + g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu - i\epsilon^{\sigma\mu\nu\lambda}\gamma_\sigma\gamma^5,
\end{aligned} \tag{B.9}
$$

and for traces:

$$
\begin{aligned}
\mathrm{Tr}\left(\gamma^\mu\right) &= 0, \\
\mathrm{Tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\ldots\gamma^{\mu_n}\right) &= 0 \qquad \text{for an odd number } n, \\
\mathrm{Tr}\left(\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\ldots\gamma^{\mu_n}\right) &= 0 \qquad \text{for an odd number } n, \\
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) &= 4g^{\mu\nu}, \\
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right), \\
\mathrm{Tr}\left(\gamma^5\right) = \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) &= 0, \\
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) &= -4i\epsilon^{\mu\nu\rho\sigma}, \\
\mathrm{Tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\ldots\gamma^{\mu_n}\right) &= \mathrm{Tr}\left(\gamma^{\mu_n}\ldots\gamma^{\mu_3}\gamma^{\mu_2}\gamma^{\mu_1}\right).
\end{aligned} \tag{B.10}
$$

B.3. Common Representations

The following bases are common in physics.

B.3.1. Dirac Representation

In a proper basis the gamma matrices have the following form which harks back to Dirac:

$$
\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \tag{B.11}
$$

where

$$
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{B.12}
$$

are the **Pauli matrices**.Therewith the γ^5 matrix reads

$$
\gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}. \tag{B.13}
$$

B.3.2. Weyl Representation

The representation which is named after Herman Weyl is sometimes also called chiral representation. The gamma matrices have the following form.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \tag{B.14}$$

The γ^5 matrix is diagonal:

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix} \tag{B.15}$$

B.3.3. Majorana Representation

In the Majorana Representation all gamma matrices are imaginary[1],

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} \\ \gamma^2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & -i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix}, \end{aligned} \tag{B.16}$$

and γ^5 reads

$$\gamma^5 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{B.17}$$

[1]Therewith the Dirac equations are a real set of differential equations.

C. The QCD Lagrangian

In order to obtain the QCD Lagrangian we start with the Dirac Lagrangian for N_f quarks

$$\mathcal{L}_{\mathrm{Dirac}} = \sum_{f}^{u,d,s,c,b,t} \bar{q}_f \left(i\gamma^\mu \partial_\mu - m\right) q_f, \tag{C.1}$$

where the (six) quarks q_f are triplets in $SU(3)$ color space,

$$q_f = \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix} \qquad\qquad f = u, d, s, c, b, t, \tag{C.2}$$

and behave under local $SU(3)_C$ transformations as

$$q_f \longrightarrow \exp\left\{ -i \sum_{a=1}^{N_c^2-1} \theta_a(x) T^a \right\} q_f \equiv U_c(x) q_f, \tag{C.3}$$

with T^a being the eight generators of $SU(3)$, which are equal to half the Gell-Mann matrices $T^a = \lambda^a/2$. The next step is to claim (additionally to Poincare and CPT theorem) a local $SU(3)$ symmetry. As a consequence we have to replace the partial derivative in $\mathcal{L}_{\mathrm{Dirac}}$ by a covariant derivative

$$D_\mu = \partial_\mu + ig\mathcal{A}_\mu, \tag{C.4}$$

containing the coupling constant g of the strong interaction and the gauge field $\mathcal{A}_\mu = \sum_{a=1}^{N_c^2-1} A_\mu^a T_a$ which transforms under local $SU(3)$ transformations as

$$\mathcal{A}_\mu \longrightarrow \mathcal{A}'_\mu = U_c(x) \left(\mathcal{A}_\mu - \frac{i}{g}\partial_\mu \right) U_c^\dagger(x). \tag{C.5}$$

These eight fields describe the exchange bosons, named gluons, of the strong interaction. Hence they need a kinetic term, respectively a self-interaction term, in the Lagrangian:

$$-\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}, \tag{C.6}$$

with the field strength tensor

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_{b,\mu} A_{c,\nu}, \tag{C.7}$$

where f^{abc} are the structure constants of $SU(3)$. The field strength tensor transforms in the following way under local $SU(3)$ transformations:

$$\left(G^a_{\mu\nu}T_a\right) \longrightarrow \left(G^a_{\mu\nu}T_a\right)' = U_c(x)\left(G^a_{\mu\nu}T_a\right)U_c^\dagger(x). \tag{C.8}$$

Finally, the gauge invariant QCD Lagrangian is given by

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=1}^{N_f} \bar{q}_f \left(i\gamma^\mu D_\mu - m\right) q_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a, \tag{C.9}$$

where the first term describes a coupling of a gluon with quarks and the last term contains the kinetic term and self interaction of gluons.

D. How We Determined the Eight Parameters of the Mass Matrix

In order to determine the eight parameters of the mass matrix in section 7.2, we used the program Mathematica.

First we define the mass matrix `M` and set the VEV of the sigma mesons to the given values. Then we calculate the eigenvalues and since the expressions are very long we define four eigenvalue functions `eigvalue`i ($i \in \{1,2,3,4\}$):

Given mass matrix :

```
M := ( g1 ϕN / 2 - (λ1 + λ2) ϕN ϕS / (2 Sqrt[2])      (λ1 - λ2) ϕN ϕS / (2 Sqrt[2])
       (λ1 - λ2) ϕN ϕS / (2 Sqrt[2])       -g1 ϕN / 2 - (λ1 + λ2) ϕN ϕS / (2 Sqrt[2])
       (m01 + m02) / 2                      (m02 - m01) / 2                      g2 ϕN / 2
       - (m02 - m01) / 2                    - (m01 + m02) / 2                    (λ3

ϕN := 164.6 (*\pm 0.1 MeV*)
ϕS := 126.2 (*\pm 0.1 MeV*)
```

Eigenvalue-functions:

```
eigvalue1[g1_, g2_, m01_, m02_, λ1_, λ2_, λ3_, λ4_] :=
 Root[16 m01^2 m02^2 - 8 g1 g2 m01 m02 ϕN^2 + g1^2 g2^2 ϕN^4 - 8 m01^2 λ2 λ3 ϕN^2 ϕS^2 -
   8 m02^2 λ1 λ4 ϕN^2 ϕS^2 - 2 g2^2 λ1 λ2 ϕN^4 ϕS^2 - 2 g1^2 λ3 λ4 ϕN^4 ϕS^2 + 4 λ1 λ2 λ3 λ4 ϕN^4 ϕS^4 +
   (-8 Sqrt[2] m02^2 λ1 ϕN ϕS - 8 Sqrt[2] m01^2 λ2 ϕN ϕS - 8 Sqrt[2] m01^2 λ3 ϕN ϕS - 8 Sqrt[2] m02^2 λ4 ϕN ϕS -
     2 Sqrt[2] g2^2 λ1 ϕN^3 ϕS - 2 Sqrt[2] g2^2 λ2 ϕN^3 ϕS - 2 Sqrt[2] g1^2 λ3 ϕN^3 ϕS - 2 Sqrt[2] g1^2 λ4 ϕN^3 ϕS + 4 Sqrt[2] λ1
      λ2 λ3 ϕN^3 ϕS^3 + 4 Sqrt[2] λ1 λ2 λ4 ϕN^3 ϕS^3 + 4 Sqrt[2] λ1 λ3 λ4 ϕN^3 ϕS^3 + 4 Sqrt[2] λ2 λ3 λ4 ϕN^3 ϕS^3) #1 +
   (-16 m01^2 - 16 m02^2 - 4 g1^2 ϕN^2 - 4 g2^2 ϕN^2 + 8 λ1 λ2 ϕN^2 ϕS^2 + 8 λ1 λ3 ϕN^2 ϕS^2 +
     8 λ2 λ3 ϕN^2 ϕS^2 + 8 λ1 λ4 ϕN^2 ϕS^2 + 8 λ2 λ4 ϕN^2 ϕS^2 + 8 λ3 λ4 ϕN^2 ϕS^2) #1^2 +
   (8 Sqrt[2] λ1 ϕN ϕS + 8 Sqrt[2] λ2 ϕN ϕS + 8 Sqrt[2] λ3 ϕN ϕS + 8 Sqrt[2] λ4 ϕN ϕS) #1^3 + 16 #1^4 &, 3]

eigvalue2[g1_, g2_, m01_, m02_, λ1_, λ2_, λ3_, λ4_] :=
```

Figure D.1.: Mass matrix and eigenvalue functions.

As a next step we checked if the order of the eigenvalue functions is correct. Therefore we chose any arbitrary values for the eight parameters and set the masses $m_{0,1}$ and $m_{0,2}$ to zero, so that there is no longer any mixing. With we calculated the values of the `eigvalue`i functions and compared them to the eigenvalues we can read off the mass matrix immediately:

Assigne eigenvalue functions to the right eigenvalues:

```
λ1pr := 0.003;
λ2pr := 0.003;
λ3pr := 0.001;
λ4pr := 0.001;
g1pr := 9;
g2pr := 10;
m01pr := 300;
m02pr := 400;

eigvalue1[g1pr, g2pr, m01pr * 0, m02pr * 0, λ1pr, λ2pr, λ3pr, λ4pr]
eigvalue2[g1pr, g2pr, m01pr * 0, m02pr * 0, λ1pr, λ2pr, λ3pr, λ4pr]
eigvalue3[g1pr, g2pr, m01pr * 0, m02pr * 0, λ1pr, λ2pr, λ3pr, λ4pr]
eigvalue4[g1pr, g2pr, m01pr * 0, m02pr * 0, λ1pr, λ2pr, λ3pr, λ4pr]

696.635

-784.765

808.312

-837.688
```

Figure D.2.: Check the order of the eigenvalue functions.

```
g1pr ϕN / 2 - (λ1pr + λ2pr) ϕN ϕS / (2 Sqrt[2])
-g1pr ϕN / 2 - (λ1pr + λ2pr) ϕN ϕS / (2 Sqrt[2])
g2pr ϕN / 2 - (λ3pr + λ4pr) ϕN ϕS / (2 Sqrt[2])
-g2pr ϕN / 2 - (λ3pr + λ4pr) ϕN ϕS / (2 Sqrt[2])

696.635

-784.765

808.312

-837.688
```

Figure D.3.: Check the order of the eigenvalue functions.

In order to fix the parameters included in the mass matrix we use that the eigenvalues (`eigvalue`i) should be equal to the four masses of the two nucleons and their chiral partners $m_{939} = 939$ MeV, $m_{1535} = 1535$ MeV, $m_{1440} = 1440$ MeV, $m_{1650} = 1650$ MeV. Therefore we defined a function which is similar to the χ^2 function and thus we label it with `chi2`. It is the sum of squares of the conditions. Now we have four conditions to fix eight parameters and so we should have some free choices. In order to realize this we use the `Manipulate`[.] function of Mathematica to vary the values of λ_1 to λ_4. After that we calculated the minimum of `chi2` with `FindMinimum` to find the values of the remaining parameters at the point where the minimum is deepest:

Find minimum of “chi^2”:

```
In[24]:= chi2[g1_, g2_, m01_, m02_, λ1_, λ2_, λ3_, λ4_] :=
          (eigvalue1[g1, g2, m01, m02, λ1, λ2, λ3, λ4] - 939) ^2 +
           (eigvalue2[g1, g2, m01, m02, λ1, λ2, λ3, λ4] + 1535) ^2 +
           (eigvalue3[g1, g2, m01, m02, λ1, λ2, λ3, λ4] - 1440) ^2 +
           (eigvalue4[g1, g2, m01, m02, λ1, λ2, λ3, λ4] + 1655) ^2

In[25]:= Clear[g1, g2, λ1, λ2, λ3, λ4, m01, m02]

In[26]:= Manipulate[FindMinimum[chi2[g1, g2, m01, m02, λ1, λ2, λ3, λ4], {g1, 0, 20},
           {g2, 0, 20}, {m01, -500, 0}, {m02, 400, 800}], {λ1, 0.01, 0.05, 0.0001},
          {λ2, 0.001, 0.05, 0.0001}, {λ3, 0.001, 0.05, 0.0001}, {λ4, 0.009, 0.05, 0.0001}]
```

Out[26]=

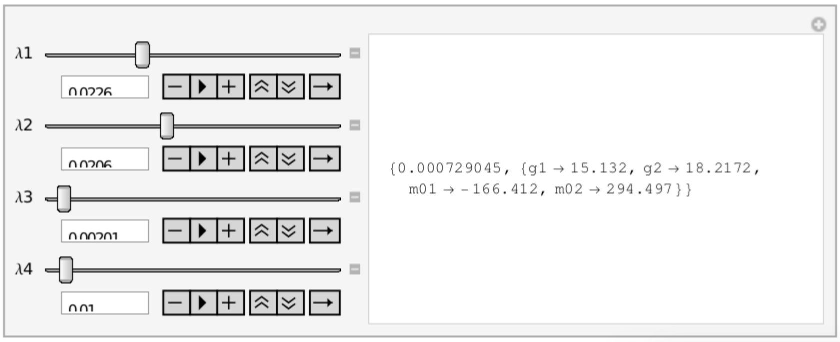


Figure D.4.: Determine the parameters through varying the lambda parameters and subsequently find the minimum of a defined `chi2` function.

And last but not least we can compute the eigenvectors:

```
Eigenvectors[M]

{{0.0673438, 0.615001, -0.0326089, 0.784968}, {0.060123, -0.784733, 0.0722817, 0.612662},
 {-0.139782, -0.0770814, -0.986678, 0.031395},
 {-0.986058, 0.00508133, 0.14205, 0.0865155}}
```

Which Vector and Values belong together :

```
{vals, vecs} = Eigensystem[M]

{{-1654.99, -1534.99, 1440.01, 939.014},
 {{0.0673438, 0.615001, -0.0326089, 0.784968}, {0.060123, -0.784733, 0.0722817, 0.612662},
  {-0.139782, -0.0770814, -0.986678, 0.031395},
  {-0.986058, 0.00508133, 0.14205, 0.0865155}}}

M.vecs[[1]] == vals[[1]] vecs[[1]]
M.vecs[[2]] == vals[[2]] vecs[[2]]
M.vecs[[3]] == vals[[3]] vecs[[3]]
M.vecs[[4]] == vals[[4]] vecs[[4]]

True

True

True

True
```

Figure D.5.: Compute the eigenvectors.

Of course the same procedure can be done for $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. In the following figure the results of the parameters can be seen:

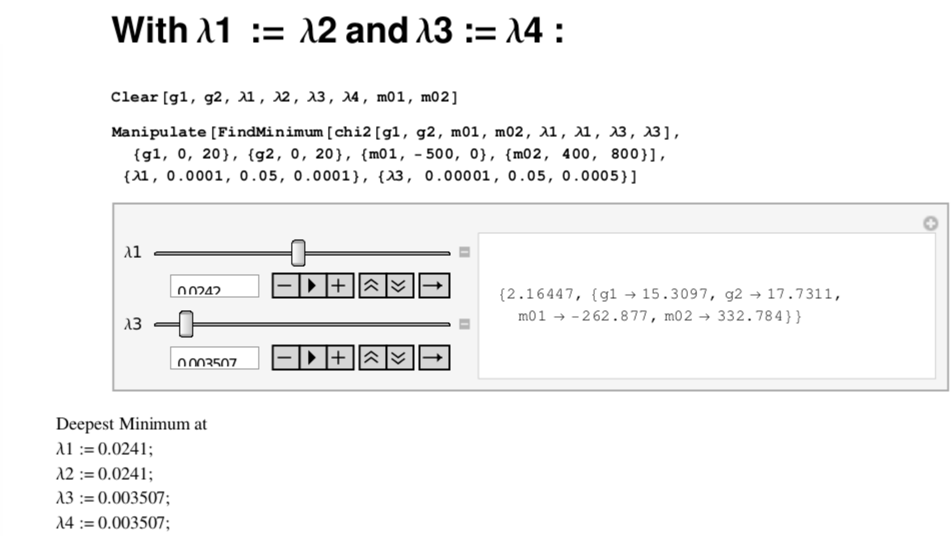


Figure D.6.: Same procedure for $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$.

Bibliography

[1] M. Gell-Mann and M. Levy, *The Axial Vector Current In Beta Decay*, Nuovo Cim. 16 (1960) 705.

[2] D. Parganlija, P. Kovacs, G. Wolf, F. Giacosa and D. H. Rischke, Phys. Rev. D87, 014011 (2013) [arXiv:1208.0585 [hep-ph]]. D. Parganlija, F. Giacosa, D. H. Rischke, Phys. Rev. D **82**, 054024 (2010) [arXiv:1003.4934 [hep-ph]].

[3] S. Gallas, F. Giacosa and D. H. Rischke, Phys. Rev. D **82**, 014004 (2010). Susanna Gallas, *Das Nukleon und sein chiraler Partner im Vakuum und in dichter Kernmaterie*; Dissertation in Frankfurt am Main (2010)

[4] S. Gallas and F. Giacosa, Int. J. Mod. Phys. A **29** (2014) 17, 1450098 [arXiv:1308.4817 [hep-ph]]

[5] S. Gallas, F. Giacosa and G. Pagliara, Nucl. Phys. A **872** (2011) 13 [arXiv:1105.5003 [hep-ph]].

[6] B. W. Lee, Chiral Dynamics (Gordon and Breach, New York, 1972).

[7] C. E. DeTar and T. Kunihiro, Phys. Rev. D **39** (1989) 2805.

[8] D. Jido, M. Oka and A Hosaka, *Chiral symmetry of baryons*, Prog. Theor. Phys. **106** (2001) 873. D. Jido, Y. Nemoto, M. Oka and A. Hosaka, *Chiral symetry for positive and negative parity nucleons*, Nucl. Phys. A **671** (2000) 471.

[9] D. Bailin and A. Love, *Introduction to Gauge Field Theory*; 1993 by Taylor and Francis Group, LLC, New York.

[10] Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory*; 1995 by Westview Press.

[11] S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. **41** (1969) 531.

[12] P. Ko and S. Rudaz, Phys. Rev. D **50** (1994) 6877. M. Urban, M. Buballa and J. Wambach, Nucl. Phys. A **697** (2002) 338 [hep-ph/0102260].

[13] A. Heinz, F. Giacosa and D. H. Rischke, arXiv:1312.3244 [nucl-th].

[14] S. Janowski, F. Giacosa and D. H. Rischke, arXiv:1408.4921 [hep-ph]. S. Janowski, D. Parganlija, F. Giacosa and D. H. Rischke, Phys. Rev. D **84** (2011) 054007 [arXiv:1103.3238 [hep-ph]].

[15] Raymond F. Streater und Arthur S. Wightman, *PCT, Spin and Statistics, and All That* (Princeton Landmarks in Mathematics & Physics) Princeton University Press (16. Oktober 2000)

[16] E. Noether, Gott. Nachr. **1918**, 235 (1918) [Transp. Theory Statist. Phys **1**, 186 (1971)] [physics/0503066].

[17] J. S. Bell and R. Jackiw, Nuovo Cim. A **60**, 47 (1969); K. Fujikawa, Phys. Rev. Lett. **42** 1195 (1979).

[18] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); D. J. Gross and F. Wilczek, Phys. Rev. D **8**, 3633 (1973); H. D. Politzer, Phys. Rev. D **9**, 2174 (1974); H. D. Politzer Phys. Rept. **14**, 129 (1974).

[19] C. Rosenzweig, A. Salomone and J. Schechter, Phys. Rev. D **24**, 2545 (1981); A. Salomone, J. Schechter and T. Tudron, Phys. Rev. D **23**, 1143 (1981); C. Rosenzweig, A. Salomone and J. Schechter, Nucl. Phys. B **206**, 12 (1982) [Erratum-ibid. B **207**, 546 (1982)]; A. A. Migdal and M. A. Shifman, Phys. Lett. B **114**, 445 (1982); H. Gomm and J. Schechter, Phys. Lett. B **158**, 449 (1985); R. Gomm, P. Jain, R. Johnson and J. Schechter, Phys. Rev. D **33**, 801 (1986).

[20] F. Giacosa, Phys. Rev. D **80**, 074028 (2009) [arXiv:0903.4481 [hep-ph]].

[21] L. Olbrich *Phänomenologie der Pseudovektormesonen und Mischung mit Axialvektormesonen im kaonischen Sektor* (Bachelorarbeit), Fachbereich Theoretische Physik der Johann Wolfgang von Goethe-Universität Frankfurt am Main (2012)

[22] H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cimento* **1**, 1425 (1955)

[23] Particle Data Group, *Review of Partical Physics*, Volume 37, Number 7A, July 2010, Article 075021, Journal of Physics G - Nuclear and Particle Physics, IOP Publishing (http://pdg.lbl.gov).

[24] V. Koch, nucl-th/9512029.

[25] V. Dmitrasinovic and F. Myhrer, Phys. Rev. C **61** (2000) 025202 [hep-ph/9911320].

[26] D. B. Lichtenberg, W. Namgung, E. Predazzi, J. G. Wills (1982). "Baryon Masses In A Relativistic Quark-Diquark Model". Physical Review Letters 48 (24): 1653–1656. Bibcode:1982PhRvL..48.1653L. doi:10.1103/PhysRevLett.48.1653.

[27] E.P. Venugopal and B. R. Holstein, “Chiral anomaly and eta eta’ mixing”, Phys. Rev. D **57** (1998) 4397 [arXiv:hep-ph/9710382]; T. Feldmann, P. Kroll and B. Stech, “Mixing and decay constants of pseudoscalar mesons”, Phys. Rev. D **58** (1998) 114006 [arXiv:hep-ph/9802409]

[28] J. Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012)

[29] T. T. Takahashi and T Kunihiro, Phys. Rev. D **78** (2008) 011503 [arXiv:0801.4707 [hep-lat]]. T. T. Takahashi and T. Kunihiro, eConf **C070910** (2007) [Mod. Phys. Lett. A **23** (2008) 2340] [arXiv:0711.1961 [hep-lat]].

[30] K. S. Choi, W. Plessas and R. F. Wagenbrunn, Phys. Rev. C **81** (2010) 028201 [arXiv:0908.3959 [hep-ph]].

[31] G. 't Hooft, Phys. Rev. D 14, 3432 (1976) [Erratum-ibid. D **18**, 2199 (1978)]; G. 't Hooft, Phys. Rept. 142, 357 (1986)

Acknowledgements

I would like to express my gratitude to Professor Dr. Dirk H. Rischke, who gave me the opportunity to do my master thesis in his group. I thank him for his useful comments and remarks and the very careful correction of the thesis.

Moreover, I would like to express my special appreciation and thanks to my supervisor Priv. Doz. Dr. Francesco Giacosa; he has been a tremendous mentor for me. I would like to thank him for supporting me, having excellent explanations for everything, revealing tips and tricks, being perpetually ready to answer my questions and encouraging my research. Furthermore, I thank him very much for his detailed correction of the thesis.

I also thank the whole “chiral group” for instructive discussions after my presentation of the work. Also many of them willingly shared their precious time to answer my questions and gave very useful remarks. Thanks for that.

Furthermore I thank Christian Wesp who always helped me with Mathematica. I am also grateful to his dog Nadi who so often came into my office to say hello and make me smile.

Last but not least I would like to thank my life partner Ingo Endemann and my family who supported me throughout the entire process.

Statutory declaration

I herewith declare that I have completed the present thesis independently, without making use of other than the specified literature and aids. Sentences or parts of sentences quoted literally are marked as quotations; identification of other references with regard to the statement and scope of the work is quoted. The thesis in this form or in any other form has not been submitted to an examination body and has not been published. This thesis has not been used, either in whole or part, for another examination achievement.

Frankfurt am Main, September 2014