Up to now we have only discussed only "periodic" systems.

They are not "really" unstable states. In order to study "truly unstable state" we have to couple the system to an ∞ of d.o.f.

Indeed, we have seen this property already.

$$H = H_0 + H_{int}$$

$$H_0 = M_0 |S\rangle\langle S| + \sum_{k = \frac{2\pi n}{L}} \omega(k) |k\rangle\langle k|$$

For instance, we can take for $\omega(k)$ the positive defined function $\omega(k) = 2\sqrt{k^2 + m^2}$

Intuitively:

$|S\rangle$ represents one state

$|k\rangle$ represents "two states", each of those with momentum "k"

$$\overset{2}{\longleftarrow} \underset{s}{O} \overset{1}{\longrightarrow}$$

$$\boxed{|k\rangle \equiv |1_k, 2_{-k}\rangle}$$

of course, at the present level we cannot "see it".

$$H_{int} = \sum_{k=\frac{2\pi n}{L}} g \frac{f(k)}{\sqrt{L}} \left(|k\rangle\langle S| + |S\rangle\langle k| \right)$$

$|S\rangle$ can transform into "$|k\rangle$ ...

Obviously, we have that

$$\langle k_1 | k_2 \rangle = \delta_{n_1 n_2}$$

$$\langle k | S \rangle = 0 \;\; \forall k = \frac{2\pi n}{L} .$$

The continuous limit is obtained in the following way: $L \mapsto \infty$

$$\sum_k \mapsto L \int_{-\infty}^{\infty} \frac{dk}{2\pi}$$

$$|k\rangle \mapsto \sqrt{\frac{2\pi}{L}} \; |k\rangle$$

This is indeed a subtle point that we encounter also in QFT.

$$|k_D\rangle = \sqrt{\frac{2\pi}{L}} \; |k_C\rangle$$

In this way:

$$\langle K_{1,c} | K_{2,c} \rangle = \frac{L}{2\pi} \langle K_{1,D} | K_{2,D} \rangle = \begin{cases} 0 & \text{for } m_1 \neq m_2 \\ \frac{L}{2\pi} & \text{for } m_1 = m_2 . \end{cases}$$

Therefore:

$$\langle K_{1,c} | K_{2,c} \rangle = \int_{-L/2}^{L/2} \frac{dx}{2\pi} \, e^{i(K_{1,c} - K_{2,c})x} = \begin{cases} 0 & K_{1,c} \neq K_{2,c} \\ \frac{L}{2\pi} & K_{1,c} = K_{2,c} \end{cases}$$

Ergo, for $L \mapsto \infty$ we find:

$$\langle K_{1,c} | K_{2,c} \rangle = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \, e^{i(K_{1c} - K_{2c})x} = \delta(K_{1c} - K_{2c})$$

Then, we "obviously" omit the subscript "c" and write simply

$$\langle K_1 | K_2 \rangle = \delta(K_1 - K_2) .$$

But the link is not "that" trivial...

The time-evolution operator is

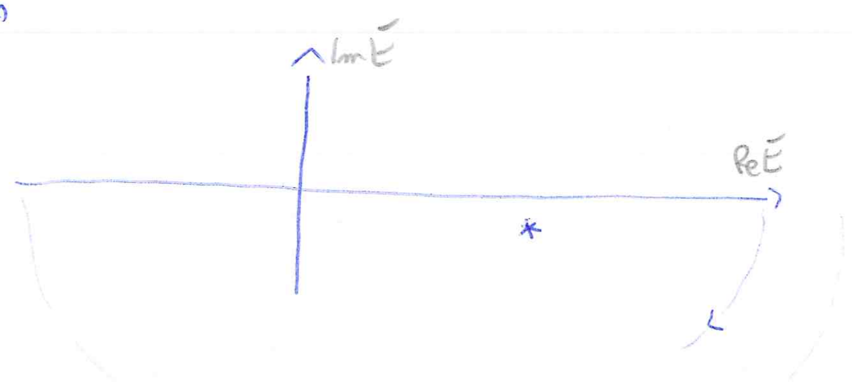
$$U(t) = e^{-iHt} = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-H+i\varepsilon}\, e^{-iEt}$$

propagator...

if H were a "number", this is obvious. In fact for $H = E_0 \in \mathbb{R}^+$

$$e^{-iE_0 t} = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-E_0+i\varepsilon}\, e^{-iEt}$$

Pole for $E = E_0 - i\varepsilon$



Jordan-lemma:

$$\frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-E_0+i\varepsilon}\, e^{-iEt} = \frac{i}{2\pi}(-2\pi i)\, e^{-iE_0 t} = e^{-iE_0 t}$$

But H is an operator and <u>not</u> a number! But it has eigenvalues:

$$H|E\rangle = E|E\rangle$$

$$e^{-iHt}|E_0\rangle = e^{-iE_0 t}|E_0\rangle = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-H+i\varepsilon}\, e^{-iEt}|E_0\rangle$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-E_0+i\varepsilon}\, e^{-iEt} \; ;$$

The operator $G(E)$ is defined as

$$G(E) = \frac{1}{E - H + i\varepsilon}$$

where H is the full Hamiltonian.

The propagator of the "particle $|S\rangle$" is given by

$$G_S(E) = \langle S | \frac{1}{E - H + i\varepsilon} | S \rangle .$$

$$a(t) = \langle S | e^{-iHt} | S \rangle = \langle S | \int_{-\infty}^{\infty} dE \, \frac{e^{-iEt}}{E - H + i\varepsilon} | S \rangle$$

survival probability amplitude

$$= \int_{-\infty}^{+\infty} dE \left(\langle S | \frac{1}{E - H + i\varepsilon} | S \rangle \right) e^{-iEt} =$$

$$= \int_{-\infty}^{+\infty} dE \, G_S(E) \, e^{-iEt}$$

The survival probability amplitude is "simply" the Fourier transform of the propagator. ("$a(t)$" is so-to-say the propagator in "position space", while $G_S(E)$ is the propagator in momentum space).