Up to now we have only discussed only "periodic" systems. They are not "really" unstable states. In order to study "truly unstable state" we have to couple the system to an ∞ of d.o.f.

Indeed, we have seen this property already.

$$H = H_0 + H_{int}$$

$$H_0 = M_0 |S\rangle\langle S| + \sum_{k=\frac{2\pi n}{L}} \omega(k)\, |k\rangle\langle k|$$

For instance, we can take for $\omega(k)$ the positive defined function $\omega(k) = 2\sqrt{k^2+m^2}$

Intuitively:

$|S\rangle$ represents one state

$|k\rangle$ represents "two states", each of those with momentum "k"

$$\xleftarrow{2} \underset{s}{o} \xrightarrow{1}$$

$$\boxed{|k\rangle \equiv |1_k, 2_{-k}\rangle}$$

Of course, at the present level we cannot "see it".

$$H_{int} = \sum_{k=\frac{2\pi m}{L}} g \frac{f(k)}{\sqrt{L}} \left(|k\rangle\langle S| + |S\rangle\langle k| \right)$$

$|S\rangle$ can transform into "$|k\rangle$" ...

Obviously, we have that

$$\langle k_1 | k_2 \rangle = \delta_{m_1 m_2}$$

$$\langle k | S \rangle = 0 \quad \forall k = \frac{2\pi m}{L} .$$

The continuous limit is obtained in the following way: $L \mapsto \infty$

$$\sum_k \mapsto L \int_{-\infty}^{\infty} \frac{dk}{2\pi}$$

$$|k\rangle \mapsto \sqrt{\frac{2\pi}{L}} \; |k\rangle$$

This is indeed a subtle point that we encounter also in QFT.

$$|k_D\rangle = \sqrt{\frac{2\pi}{L}} \; |k_C\rangle$$

In this way:

$$\langle K_{1,c} | K_{2,c} \rangle = \frac{L}{2\pi} \langle K_{1,D} | K_{2,D} \rangle = \begin{cases} 0 & \text{for } m_1 \neq m_2 \\ \frac{L}{2\pi} & \text{for } m_1 = m_2 . \end{cases}$$

Therefore:

$$\langle K_{1,c} | K_{2,c} \rangle = \int_{-L/2}^{L/2} \frac{dx}{2\pi} e^{i(K_{1,c} - K_{2,c})x} = \begin{cases} 0 & K_{1,c} \neq K_{2,c} \\ \frac{L}{2\pi} & K_{1,c} = K_{2,c} \end{cases}$$

Ergo, for $L \mapsto \infty$ we find:

$$\langle K_{1,c} | K_{2,c} \rangle = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{i(K_{1c} - K_{2c})x} = \delta(K_{1c} - K_{2c})$$

Then, we "obviously" omit the subscript "c" and write simply

$$\langle K_1 | K_2 \rangle = \delta(K_1 - K_2) .$$

But the link is not "that" trivial ...

The time-evolution operator is

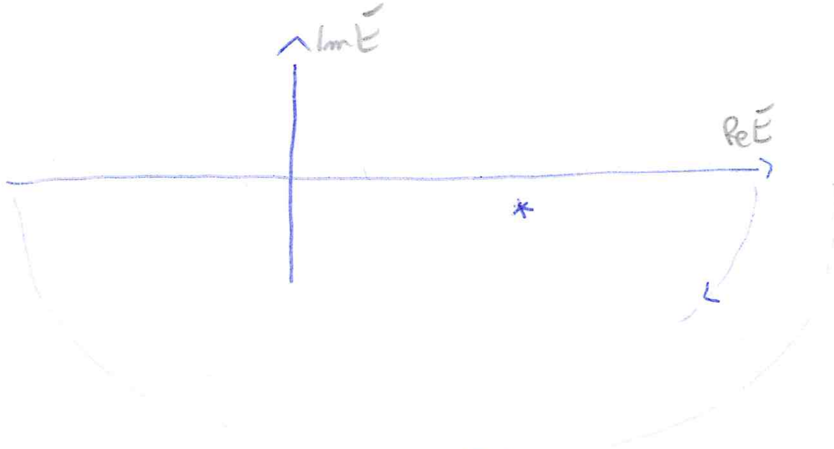
$$U(t) = e^{-iHt} = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-H+i\varepsilon}\, e^{-iEt}$$

$\rightarrow$ propagator...

[if H were a "number", this is obvious. In fact for $H = E_0 \in \mathbb{R}^+$

$$e^{-iE_0 t} = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-E_0+i\varepsilon}\, e^{-iEt}$$

[Pole for $E = E_0 - i\varepsilon$]



Jordan-lemma:

$$\frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-E_0+i\varepsilon}\, e^{-iEt} = \frac{i}{2\pi}(-2\pi i)\, e^{-iE_0 t} = e^{-iE_0 t}$$

But H is an operator and <u>not</u> a number! But it has eigenvalues:

$$H|E\rangle = E|E\rangle$$

$$e^{-iHt}|E_0\rangle = e^{-iE_0 t}|E_0\rangle \stackrel{?}{=} \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-H+i\varepsilon}\, e^{-iEt}|E_0\rangle$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{1}{E-E_0+i\varepsilon}\, e^{-iEt} \, ;$$
]

The operator $G(E)$ is defined as

$$G(E) = \frac{1}{E - H + i\varepsilon}$$

where H is the full Hamiltonian.

The propagator of the "particle $|s\rangle$" is given by

$$G_s(E) = \langle s | \frac{1}{E - H + i\varepsilon} | s \rangle .$$

$$a(t) = \langle s | e^{-iHt} | s \rangle = \frac{i}{2\pi} \langle s | \int_{-\infty}^{\infty} dE \, \frac{e^{-iEt}}{E - H + i\varepsilon} | s \rangle$$

survival probability amplitude

$$= \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \left(\langle s | \frac{1}{E - H + i\varepsilon} | s \rangle \right) e^{-iEt} =$$

$$= \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \, G_s(E) \, e^{-iEt}$$

The survival probability amplitude is "simply" the fourier transform of the propagator. ("$a(t)$" is so-to-say the propagator in "position space", while $G_s(E)$ is the propagator in momentum spa

We now need to calculate $G_s(E)$.

First, lets rewrite the operator $G(E)$ as:

$$G(E) = \frac{1}{E - H + i\varepsilon} = \frac{1}{E - H_0 - H_1 + i\varepsilon} = \frac{1}{(E - H_0 + i\varepsilon)\left(1 - \frac{1}{E - H_0 + i\varepsilon} H_1\right)} =$$

$$= \frac{1}{E - H_0 + i\varepsilon} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{E - H_0 + i\varepsilon} \cdot H_1 \right)^n =$$

Ergo:

$$G_s(E) = \langle s | G(E) | E \rangle = \sum_{n=0}^{\infty} \langle s | \frac{1}{E - H_0 + i\varepsilon} \left(\frac{1}{E - H_0 + i\varepsilon} H_1 \right)^n | s \rangle$$

$$n = 0 \rightarrow \langle s | \frac{1}{E - H_0 + i\varepsilon} | s \rangle = \frac{1}{E - M_0 + i\varepsilon}$$

"free propagator"

$n = 1$

$$\langle s | \frac{1}{E - H_0 + i\varepsilon} \frac{1}{E - H_0 + i\varepsilon} H_1 | s \rangle = \frac{1}{(E - M_0 + i\varepsilon)^2} \langle s | H_1 | s \rangle = 0$$

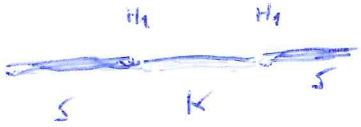
First order... s can go to k

, but there are no such diagrams for $|s\rangle \mapsto |s\rangle$. This is why it vanishes.

$n = 2$

$$\langle s| \frac{1}{E-H_0+i\varepsilon}\,\frac{1}{E-H_0+i\varepsilon}\, H_1 \,\frac{1}{E-H_0+i\varepsilon}\, H_1 |s\rangle =$$

$$= \frac{1}{(E-M_0+i\varepsilon)^2}\,\langle s| H_1 \,\frac{1}{E-H_0+i\varepsilon}\, H_1 |s\rangle$$



(this corresponds to )

$$\langle s| \left[\sum_k \frac{g\, f(k)}{\sqrt{L}} \big(|k\rangle\langle s| + |s\rangle\langle k|\big)\right] \frac{1}{E-H_0+i\varepsilon} \left[\sum_p \frac{g\, f(p)}{\sqrt{L}} \big(|p\rangle\langle s| + |s\rangle\langle p|\big)\right]$$

$$= \sum_k \frac{g\, f(k)}{\sqrt{L}} \underbrace{\langle k| \frac{1}{E-H_0+i\varepsilon} |p\rangle}_{\frac{1}{E-\omega_k+i\varepsilon}\,\delta_{kp}} \sum_p \frac{g\, f(p)}{\sqrt{L}}$$

$$= g^2 \sum_k \frac{f^2(k)}{L}\, \frac{1}{E-\omega_k+i\varepsilon} \longmapsto g^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{f^2(k)}{E-\omega(k)+i\varepsilon}$$

(cont. limit $\sum_k \mapsto L \int_{-\infty}^{\infty} \frac{dk}{2\pi}$)

Ergo, we obtain:

$$n=2 \qquad \frac{1}{(E-M_0+i\varepsilon)^2}\left(g^2\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{f^2(k)}{E-\omega(k)+i\varepsilon}\right)$$

Introducing $\Sigma(E) := -\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{f^2(k)}{E-\omega(k)+i\varepsilon}$

we find:

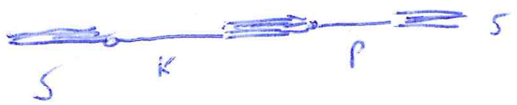
$$n=2 \qquad \frac{1}{(E-M_0+i\varepsilon)^2}\left(-g^2\,\Sigma(E)\right)$$

↪ we see the Feynman rules...

For a generic even n we find a "recursive appearance" of $\Sigma(E)$.

($n=3 \mapsto$ "0")

$n=4$



$$\frac{1}{(E-M_0+i\varepsilon)^3}\left(-g^2\,\Sigma(E)\right)^2$$

Ergo, the full propagator $G_S(E)$ is given by

$$G_S(E) = \frac{1}{E-M_0+i\varepsilon}\sum_{n=0}^{\infty}\left(\frac{-g^2\Sigma(E)}{E-M_0+i\varepsilon}\right)^n =$$

$$= \frac{1}{E-M_0+i\varepsilon}\,\frac{1}{1+\frac{g^2\Sigma(E)}{E-M_0+i\varepsilon}} = \frac{1}{E-M_0+\underbrace{g^2\Sigma(E)}_{\text{loop contribution...}}+i\varepsilon}$$

In general, $\Sigma(E)$ is a "complex number".

$$\Sigma(E) = \mathrm{Re}\,\Sigma(E) + i\,\mathrm{Im}\,\Sigma(E).$$

We see that the propagator is modified in the following way:

* mass shift.

$$\frac{1}{\underbrace{E - M_0 + g^2 \mathrm{Re}\,\Sigma(E)}\ + i g^2 \mathrm{Im}\,\Sigma(E) + i\varepsilon}$$

The new mass M is found as

$$E - M_0 + g^2 \mathrm{Re}\,\Sigma(E) = 0 \longmapsto E = M$$

Indeed, this is just one of the possible def of mass. In principle, $|S\rangle$ has no "mass" anyhow

* $\mathrm{Im}\,\Sigma(E) \longmapsto$ imaginary part! This indeed generates a imaginary mass → as we shall see, this is responsible of the exp. decay law for short times.

* one also see where complex functions pop in... still, we have

$$\Sigma(E) : \mathbb{R} \longmapsto \mathbb{C}$$

The next step is to "shift" $E \longmapsto z \in \mathbb{C}$

$$\Sigma(z) : \mathbb{C} \longmapsto \mathbb{C}$$

Σ is in general a complex function living in a "Riemann surface" and with a 'cut' along the real axis.

A usual simplification consists in neglecting further dependence on E and writing

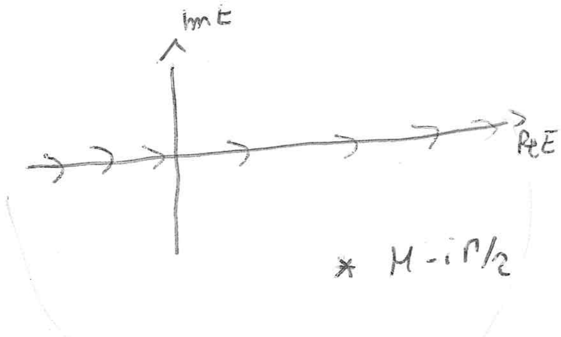
$$G_S^{BW}(E) = \frac{1}{E - M + i g^2 \,\mathrm{Im}[\Sigma(M)]} = \frac{1}{E - M + i\Gamma/2}$$

where

$$\Gamma = 2 g^2 \,\mathrm{Im}\,\Sigma(M)$$

In fact.

$$a^{BW}(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE \, \frac{1}{E - M + i\Gamma/2} \, e^{-iEt}$$



$$= \frac{i}{2\pi} \cdot (-i 2\pi) \, e^{-i(M - i\Gamma/2)t} = e^{-iMt - \Gamma/2\, t} = e^{-iMt} \cdot e^{-\Gamma/2\, t}$$

$$p^{BW}(t) = |a^{BW}(t)|^2 = e^{-\Gamma t} \;!!!$$

Spectral function

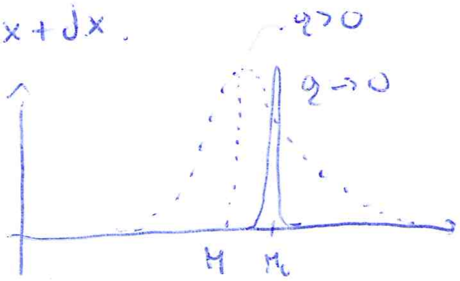
The spectral function $d_s(x)$ is defined as

$$G_s(E) = \int_0^\infty \frac{d_s(x)}{E - x + i\varepsilon}\, dx$$

Before going into mathematical details, let us see what it means:

- when $g = 0$, $G_s(E) = \dfrac{1}{E - M_0 + i\varepsilon} \longrightarrow d_s(x) = \delta(x - M_0)$
- when $g \neq 0$ we are making the "sum" of many free propagators.

Thus, $d_s(x)\,dx$ is the probability that the state $|s\rangle$ has a "mass" between x and $x + dx$.



$M < M_0$ (in QFT it is in general so ... problem of renormalisation!)

- Obviously, being a probability distribution, we 'expect that'

$$\int_0^\infty d_s(x)\, dx = 1 \qquad \text{(For } g \mapsto 0 \text{ this is indeed the case)}$$

$$G_s(E) = \mathrm{Re}\, G_s(E) + i\, \mathrm{Im}\, G_s(E) \; ; \quad G_s(E) = \frac{1}{E - M_0 + g^2 \Sigma(E) + i\varepsilon}$$

$$\begin{cases} \mathrm{Re}\, G_s(E) = \dfrac{(E - M_0 + g^2 \mathrm{Re}\Sigma)}{(E - M_0 + g^2 \mathrm{Re}\Sigma)^2 + (g^2 \mathrm{Im}\Sigma + \varepsilon)^2} ; \\[2em] \mathrm{Im}\, G_s(E) = \dfrac{g^2 \Sigma(E) + i\varepsilon}{(E - M_0 + g^2 \mathrm{Re}\Sigma)^2 + (g^2 \mathrm{Im}\Sigma + \varepsilon)^2} ; \end{cases}$$

$$G_s(E) = \int_0^\infty \frac{d_s(x)\,dx}{E - x + i\varepsilon} = \underbrace{\int_0^\infty d_s(x)\,dx \cdot \frac{(E - x)}{(E - x)^2 + \varepsilon^2}}_{\mathrm{Re} \int} + i \underbrace{\int_0^\infty d_s(x)\,dx \cdot \frac{\varepsilon}{(E - x)^2 + \varepsilon^2}}_{\mathrm{Im} \int}$$

OK, let us now perform the limit $\varepsilon \mapsto 0$:

$$\lim_{\varepsilon \mapsto 0} \int_0^\infty dx\, d_s(x) \frac{\varepsilon}{(E - x)^2 + \varepsilon^2} = \int_0^\infty dx\, d_s(x)\, \pi \delta(E - x) = \pi\, d_s(E)$$

$$\lim_{\varepsilon \mapsto 0} \frac{1}{\pi} \frac{\varepsilon}{(E - x)^2 + \varepsilon^2} = \delta(E - x)$$

Ergo: $$d_s(E) = \frac{1}{\pi} \mathrm{Im}\, \Sigma(E) = \lim_{\varepsilon \mapsto 0} \frac{1}{\pi} \frac{g^2 \Sigma(E) + \varepsilon}{(E - M_0 + g^2 \mathrm{Re}\Sigma)^2 + (g^2 \mathrm{Im}\Sigma + \varepsilon)^2}$$

Note, as long as g is finite in many cases we can simply perform the limit and get

$$d_s(E) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} G_s(E) = \frac{1}{\pi} \frac{g^2 \operatorname{Im}\Sigma(E)}{(E - M_0 + g^2 \operatorname{Re}\Sigma(E))^2 + (g^2 \operatorname{Im}\Sigma(E))^2}$$

[Note, for $g \mapsto 0$ we really have $\delta(E - M_0)$ as stated before]

This expression is valid if

$$M - M_0 + g^2 \operatorname{Re}\Sigma(M) = 0$$

is such that

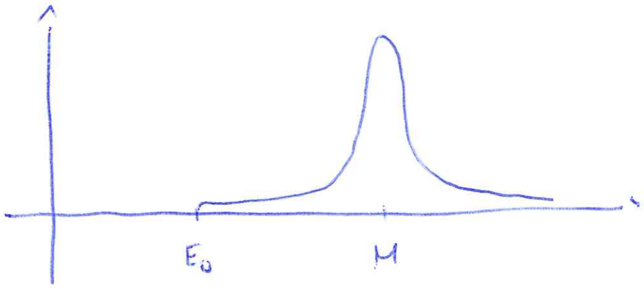
$$\operatorname{Im}\Sigma(M) = 0.$$

$$\operatorname{Re}\Sigma(E) = PP \int_0^\infty \frac{d_s(x)\,dx}{E - x} = \lim_{\delta \to 0} \left\{ \int_0^{E_0 - \delta} \frac{d_s(x)\,dx}{E - x} + \int_{E_0 + \delta}^\infty \frac{d_s(x)\,dx}{E - x} \right\}$$

<u>Principal part</u>

In general there is a threshold.

$\mathrm{Im}\,\Sigma(E) = 0$ for $E < E_0$



$E \geq \omega(k)$

$E \geq 2\sqrt{k^2+m^2} \rightarrow E \geq 2m \rightarrow E_0 = 2m$ is the threshold

(in general, $E_0 = \min \omega(k) = \omega(0)$)

Approximation of $\underline{d_S(E)}$:

$(M - M_0 + g^2\Sigma(M)) = 0$

$M =$ "nominal mass" or "Breit

$$d_S(E) = \frac{1}{\pi} \frac{g^2\,\mathrm{Im}\,\Sigma(E)}{(E - M_0 + g^2\Sigma(E)) + (g^2\,\mathrm{Im}\,\Sigma(E))^2}$$

$$\simeq \frac{1}{\pi} \frac{g^2\,\mathrm{Im}\,\Sigma(M)}{(E-M)^2 + (g^2\,\mathrm{Im}\,\Sigma)^2}\,\theta(E-E_0) \simeq \frac{1}{\pi} \frac{g^2\,\mathrm{Im}\,\Sigma(M)}{(E-M)^2 + (g^2\,\mathrm{Im}\,\Sigma)^2}$$

Let us define $\Gamma = 2g^2\,\mathrm{Im}\,\Sigma(M) \rightarrow \Gamma$ is indeed the decay width

$$d_S^{BW}(E) = \frac{1}{\pi} \frac{\Gamma/2}{(E-M)^2 + (\Gamma/2)^2}$$

$$a(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE\, G_s(E)\, e^{-iEt} = \langle S| e^{-iHt} |S\rangle = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE\, \underbrace{\langle S| \frac{1}{E-H+i\varepsilon} |S\rangle}_{G_s(E)} e^{-iEt}$$

but

$$G_s(E) = \int_{-\infty}^{\infty} \frac{d_s(x)}{E-x+i\varepsilon} e^{-ixt}\, dx$$

$d_s(x) = 0$ for $x < E_0$, $E_0 > 0$.

$$a(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} \frac{d_s(x)}{E-x+i\varepsilon} e^{-ixt}\, dx = \frac{i}{2\pi}\int_{-\infty}^{\infty} dx\, d_s(x) \int_{-\infty}^{\infty} \frac{dE\, e^{-ixt}}{E-x+i\varepsilon} =$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} dx\, d_s(x) \left(-2\pi i\, e^{-ixt}\right) = \int_{-\infty}^{\infty} dx\, d_s(x)\, e^{-ixt}$$

This is indeed intuitive.

$$a(t) = \int_{-\infty}^{\infty} d_s(x) e^{-ixt}$$

is "intuitive".

$$|S\rangle = \int_0^{\infty} f(x)|x\rangle dx \qquad \text{where } H|x\rangle = x|x\rangle$$

Then: $|\langle x|S\rangle|^2 = d_s(x) = |f(x)|^2$

$$\langle S| e^{-iHt} |S\rangle = \int_0^{\infty} |f(x)|^2 dx \, e^{-ixt} = \int_0^{\infty} d_s(x) dx \, e^{-ixt}$$