Determine $|s(t)\rangle$ $\forall t$ (initial condition $|s(0)\rangle = |S\rangle$).

$$|s(t)\rangle = e^{-iHt}|S\rangle = e^{-iHt}\left(c_2|E_2\rangle + c_1|E_1\rangle\right)$$

whereas:

$c_2 = \cos(\theta) = c$

$c_1 = \sin(\theta) = s$

are the coeff.

Then:

$$|s(t)\rangle = \cos\theta\, e^{-iE_2t}|E_2\rangle + \sin\theta\, e^{-iE_1t}|E_1\rangle$$

But:

$$\begin{pmatrix}|W\rangle\\|S\rangle\end{pmatrix} = \begin{pmatrix}c & -s\\ +s & c\end{pmatrix}\begin{pmatrix}|E_1\rangle\\|E_2\rangle\end{pmatrix} \quad ; \quad \begin{pmatrix}|E_1\rangle\\|E_2\rangle\end{pmatrix} = \begin{pmatrix}c & s\\ -s & c\end{pmatrix}\begin{pmatrix}|W\rangle\\|S\rangle\end{pmatrix}$$

Then:

$$|s(t)\rangle = c\, e^{-iE_2t}(c|S\rangle - s|W\rangle) + s\, e^{-iE_1t}(c|W\rangle + s|S\rangle)$$

$$= |S\rangle \cdot \left(c^2 e^{-iE_2t} + s^2 e^{-iE_1t}\right) + |W\rangle\left(-s\cdot c\, e^{-iE_2t} + sc\, e^{-iE_1t}\right)$$

$$|s(t)\rangle = a(t)|S\rangle + r(t)|W\rangle$$

2

$$|\psi(t)\rangle = a(t)|S\rangle + r(t)|W\rangle$$

$$a(t) = \cos^2\alpha\, e^{-iE_2 t} + s^2 e^{-iE_1 t}$$

$$r(t) = sc\left(e^{-iE_1 t} - e^{-iE_2 t}\right)$$

$$\begin{cases} P(t) = \left|c^2 e^{-iE_2 t} + s^2 e^{-iE_1 t}\right|^2 = c^4 + s^4 + 2sc\cos\left((E_2 - E_1)t\right) \\ \bar{P}(t) = r^2 = s^2c^2\left(1 + 1 - e^{-i(E_1 - E_2)t} - e^{-i(E_2 - E_1)t}\right) = \end{cases}$$

$$= 2s^2c^2\left(1 - \cos\left((E_2 - E_1)t\right)\right)$$

$$P(t) + \bar{P}(t) = c^4 + s^4 + 2s^2c^2\cos(\Delta E t) + 2s^2c^2 - 2s^2c^2\cos(\Delta E t) =$$

$$= 1\ !!!$$

$$\begin{cases} P(t) = \cos^2\alpha \\ \bar{P}(t) = \sin^2\alpha \end{cases}$$

$$\cos^2\alpha + \sin^2\alpha = 1$$

The case $M_0 = \omega$.

It implies that $\theta = \frac{\pi}{4}$.

We then get.

$$\begin{cases} P(t) = \frac{1}{4} + \frac{1}{4} + 2 \cdot \frac{1}{4} \cos(\Delta E t) = \frac{1}{2} + \frac{1}{2} \cos(\Delta E t) = \cos^2(2 \Delta E t) \\ \bar{P}(t) = \frac{1}{2} - \frac{1}{2} \cos(\Delta E t) = \sin^2(2 \Delta E t) \end{cases}$$

Ergo:

$$\begin{cases} a(t) = \cos(2 \Delta E t)\, e^{i \varphi(t)} \\ \bar{a}(t) = \sin(2 \Delta E t)\, e^{i \bar{\varphi}(t)} \end{cases}$$

(Achtung... complex numbers are ongoing).

Exercise of diagonalisation:

$$H_1 = z\,|s\rangle\langle w| + z^*\,|w\rangle\langle s|$$

$\rightsquigarrow$ Evaluate the mixing in this case

First case: $z = i\alpha$, α real

Then for the full case $z = \varrho + i\alpha$!

$$H = H_0 + H_1$$

$$= M_0 |s\rangle\langle s| + w|w\rangle\langle w| + z|s\rangle\langle w| + z^*|w\rangle\langle s|$$

$$z = a + ib = \rho e^{i\varphi}$$

$$= M_0 |s\rangle\langle s| + w|w\rangle\langle w| + \rho e^{i\varphi} |s\rangle\langle w| + \rho e^{-i\varphi} |w\rangle\langle s|$$

Define:

$$|\tilde{s}\rangle = e^{i\varphi}|s\rangle$$

$$\langle\tilde{s}| = \langle s| e^{-i\tilde{\varphi}}$$

Ergo:

$$|s\rangle\langle s| = |\tilde{s}\rangle\langle\tilde{s}|$$

$$H = M_0 |\tilde{s}\rangle\langle\tilde{s}| + w|w\rangle\langle w| + \rho|\tilde{s}\rangle\langle w| + \rho|w\rangle\langle\tilde{s}|$$

ρ is real, therefore we can diag. as usual:

$$\begin{pmatrix} |E_1\rangle \\ |E_2\rangle \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} |w\rangle \\ |\tilde{s}\rangle \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} |w\rangle \\ e^{i\varphi}|s\rangle \end{pmatrix} =$$

$$= \begin{pmatrix} c & se^{i\varphi} \\ -s & ce^{i\varphi} \end{pmatrix}\begin{pmatrix} |w\rangle \\ |s\rangle \end{pmatrix}$$

$$B = \begin{pmatrix} c & s e^{i\varphi} \\ -s & c e^{i\varphi} \end{pmatrix}$$

Is it unitary?

$$B^{+} B = \begin{pmatrix} c & -s \\ s e^{-i\varphi} & c e^{-i\varphi} \end{pmatrix} \begin{pmatrix} c & s e^{i\varphi} \\ -s & c e^{i\varphi} \end{pmatrix} =$$

$$= \begin{pmatrix} c^2 + s^2 & cs e^{+i\varphi} - sc e^{i\varphi} = 0 \\ sc e^{-i\varphi} - sc e^{-i\varphi} & s^2 + c^2 = 1 \end{pmatrix}$$

How to express it in term of

$$e^{i n^a \sigma^a} \; ?$$

$\varphi = \pi/2$

$$B = \begin{pmatrix} c & is \\ -s & ic \end{pmatrix}$$

$$\begin{cases} H_0 = M_0 |S\rangle\langle S| + \omega |\omega\rangle\langle\omega| \\ H_1 = g\cos(\alpha t)\left(|S\rangle\langle\omega| + |\omega\rangle\langle S|\right)\end{cases}$$

Define also $\beta = M_0 - \omega$

Start with the Ansatz

$$|s(t)\rangle = a(t)e^{-iM_0 t}|S\rangle + r(t)e^{-i\omega t}|\omega\rangle \qquad (*)$$

(note, it is a bit different from sheet "1" because we factorise out the phases $e^{-iM_0 t}$ and $e^{-i\omega t}$).

Put the Ansatz $(*)$ into the equation of Schrödinger:

$$i\frac{d|s(t)\rangle}{dt} = H|s(t)\rangle$$

... ergo ... :

$$i\frac{d|s(t)\rangle}{dt} = i\dot{a}(t)e^{-iM_0 t}|S\rangle + M_0 a(t)e^{-iM_0 t}|S\rangle + i\dot{r}(t)e^{-i\omega t}|\omega\rangle + \omega r(t)e^{-i\omega t}|\omega\rangle$$

$$H|s(t)\rangle = M_0\, a(t)e^{-iM_0 t}|S\rangle + g\cos(\alpha t)a(t)e^{-iM_0 t}|\omega\rangle + \omega r(t)e^{-i\omega t}|\omega\rangle + g\cos(\alpha t)\, r(t)e^{-i\omega t}|S\rangle$$

We therefore get the eqs:

$$\begin{cases} i\dot{a}(t)e^{-iM_0 t} = g\cos(\alpha t)r(t)e^{-i\omega t} \\ i\dot{r}(t)e^{-i\omega t} = g\cos(\alpha t)\,a(t)e^{-iM_0 t} \end{cases}$$

The first eq. reads

$$\dot{a}(t) = -ig\cos(\alpha t)\,e^{i\beta t}r(t)$$

$$= -ig\frac{1}{2}\left(e^{i\alpha t} + e^{-i\alpha t}\right)e^{i\beta t}r(t)$$

Ergo, we get a piece as

$$e^{i(\alpha+\beta)t}$$

and a piece as

$$e^{i(\alpha+\beta)t}$$

Let us consider $\beta = M_0 - \omega > 0$.

$e^{i(\alpha+\beta)t}$ can be neglected...

$$\boxed{\dot{a}(t) = \frac{-ig}{2}e^{i(\beta-\alpha)t}r(t)} \quad \left(\frac{-ig}{2}e^{i(\beta+\alpha)t}r(t)\right)$$

This approx. makes sense if $\beta \simeq \alpha$! In fact, in that case second term shows variation for $t \propto 1/(\beta+\alpha) \ll \frac{1}{\beta-\alpha}$ if $\beta \sim \alpha$!

Let us do the same for $r(t)$:

$$\dot{r}(t) = -iq \cos(\alpha t)\, a(t)\, e^{-i\beta t}$$

$$= \frac{-iq}{2}\left(e^{i\alpha t} + e^{-i\alpha t}\right) a(t)\, e^{-i\beta t}$$

$$\boxed{\dot{r}(t) \simeq \frac{-iq}{2}\, e^{-i(\beta-\alpha)t}\, a(t)}$$

We can eliminate $a(t)$ and get:

$$\ddot{a}(t) = \frac{-iq}{2}\, i(\beta-\alpha)\, e^{i(\beta-\alpha)} r(t) \quad - \frac{iq}{2}\, \dot{r}(t)\, e^{i(\beta-\alpha)t}$$

$$= i(\beta-\alpha)\,\dot{a}(t) \;\; \frac{-iq}{2}\, e^{i(\beta-\alpha)t}\left(\frac{-iq}{2}\, e^{-i(\beta-\alpha)}\, a(t)\right)$$

$$= i(\beta-\alpha)\,\dot{a}(t) - \frac{q^2}{4}\, a(t)$$

$$\boxed{\ddot{a}(t) + i(\alpha-\beta)\,\dot{a}(t) + \frac{q^2}{4}\, a(t)} = 0$$

This equation can be solved ...

Ansatz:

$$a = e^{\lambda t}$$

$$\lambda^2 + i(\alpha-\beta)\lambda + \frac{g^2}{4} = 0$$

$$\lambda = i\,\frac{-(\alpha-\beta) \pm \sqrt{(\alpha-\beta)^2 - 4\frac{g^2}{4}}}{2} = \frac{i}{2}\left[(\beta-\alpha) \pm \sqrt{(\beta-\alpha)^2 + g^2}\right]$$

$$= \frac{i}{2}\left[\Delta \pm \sqrt{\Delta^2 + g^2}\right]. \qquad \boxed{\Delta = \beta - \alpha}$$

$$=$$

$$a(t) = a_+ e^{\frac{i}{2}\left[\Delta + \sqrt{\Delta^2+g^2}\right]t} + a_- e^{\frac{i}{2}\left[\Delta - \sqrt{\Delta^2+g^2}\right]t}$$

if we impose that $a(0) = 1$ we get

$$a(0) = a_+ + a_- = 1$$

Let us look also at $r(t)$:

$$\dot r(t) = \frac{-ig}{2} e^{-i\Delta t} a(t) = \frac{-ig}{2}\left(a_+ e^{\frac{i}{2}\left[-\Delta + \sqrt{\Delta^2+g^2}\right]t} + a_- e^{-\frac{i}{2}\left[\Delta + \sqrt{\Delta^2+g^2}\right]}\right.$$

Ergo:

$$r(t) = \frac{-ig}{2}\left(\frac{a_+}{\frac{i}{2}\left[-\Delta + \sqrt{\ }\right]} e^{\frac{i}{2}\left[-\Delta+\sqrt{\ }\right]t} - \frac{a_-}{\frac{i}{2}\left[\Delta + \sqrt{\ }\right]} e^{-\frac{i}{2}\left[\Delta + \sqrt{\ }\right]t}\right)$$

$r(0) = 0$ implies:

$$\boxed{\frac{a_+}{-\Delta + \sqrt{\;}} = \frac{a_-}{\Delta + \sqrt{\;}}}$$

for $g = 0$ .. $\frac{a_+}{g} = \frac{a_-}{g} \rightarrow a_+ = a$

$\Delta = 0$

$a_- = a_+ \frac{g}{\;}$

$$a_- = a_+ \frac{\Delta + \sqrt{\;}}{\sqrt{\;} - \Delta}$$

Ergo $a(t)$ can be rewritten as:

$$a(t) = e^{\frac{i}{2}\Delta t}\left[a_+ e^{\frac{i}{2}\sqrt{\;}\, t} + a_+ \frac{\Delta + \sqrt{\;}}{\sqrt{\;} - \Delta} e^{-\frac{i}{2}\sqrt{\;}\, t}\right]$$

$$a(0) = a_+ + a_+ \frac{\Delta + \sqrt{\;}}{\Delta - \sqrt{\;}} = 1$$

$$a_+\left(1 + \frac{\Delta + \sqrt{\;}}{\sqrt{\;} - \Delta}\right) = a_+ \frac{2\sqrt{\;}}{\sqrt{\;} - \Delta} = 1$$

$$a_+ = \frac{\sqrt{\Delta^2 + g^2} - \Delta}{2\sqrt{\Delta^2 + g^2}}$$

$$a(t) = e^{\frac{i}{2}\Delta t}\left[\frac{\sqrt{\;} - \Delta}{2\sqrt{\;}} e^{\frac{i}{2}\sqrt{\;}\, t} + \frac{\Delta + \sqrt{\;}}{2\sqrt{\;}} e^{-\frac{i}{2}\sqrt{\;}\, t}\right]$$

There is still one mistake.

but it is ok for our case.

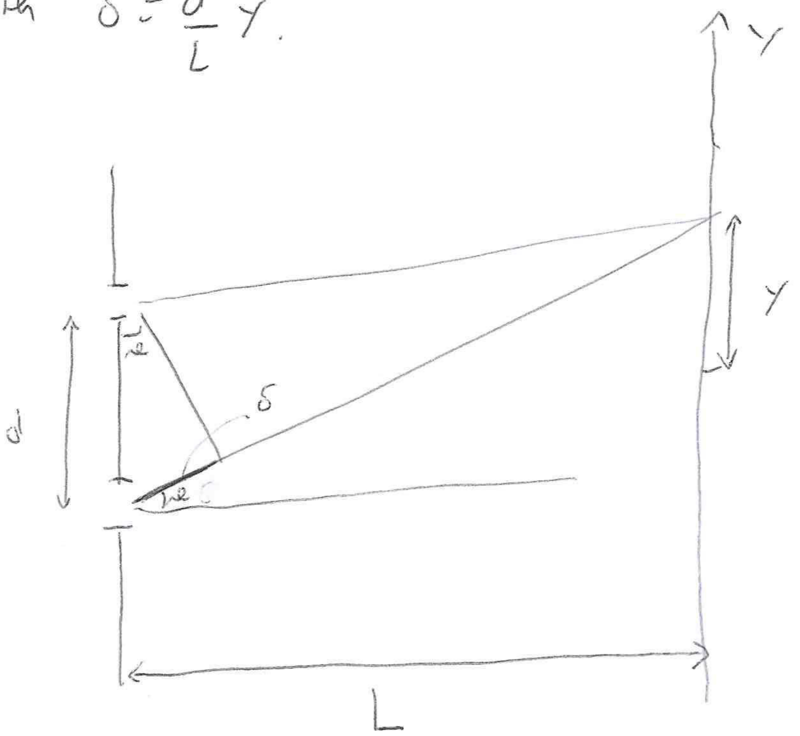
$$\boxed{a(t) = e^{\frac{i}{2}\Delta t}\left[\cos\left(\frac{\sqrt{\Delta^2 + g^2}}{2} t\right) - \frac{i\Delta}{2\sqrt{\Delta^2 + g^2}} \sin\left(\frac{\sqrt{\;}}{2} t\right)\right]}$$

Plot that function...

Ex. 2

$$\psi = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2 = \frac{1}{\sqrt{2}} e^{-i\omega y}\left(e^{-i\omega\delta/2} + e^{+i\omega\delta/2}\right)$$

with $\delta = \frac{d}{L} y$.



$\delta = d \sin\theta \cong d \cdot \theta$, but $\theta = \frac{Y}{L}$

$\delta = \frac{d}{L} Y \qquad (L >> d)$

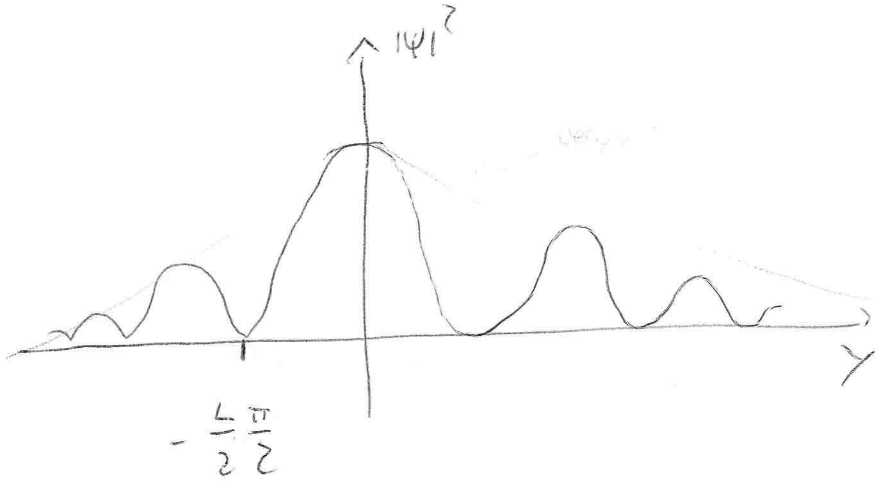
Then:

$$\left|\frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2\right|^2 = \frac{1}{2}\left(1+1+e^{i\omega\delta}+e^{-i\omega\delta}\right) \cdot f(y^2)$$

$$= f(y^2)\left[1+\cos(\omega\delta)\right] = f(y)\left(1+\cos\left(\frac{d}{L}y\omega\right)\right)$$

Ergo, for $y = \frac{L}{d}\cdot\frac{\pi}{\cancel{2}}\frac{1}{\omega}$ → we get the first minimum.

Then:



Now, if we include the detector

$$\Psi_{tot} = \frac{1}{\sqrt{2}} \psi_1 \phi_1(\alpha_i) + \frac{1}{\sqrt{2}} \psi_2 \phi_2(\alpha_i)$$

$\Rightarrow$

$$|\Psi_{tot}|^2 = \frac{1}{2} f(y^2) \left[|\phi_1(\alpha_i)|^2 + |\phi_2(\alpha_i)|^2 + \phi_1 \phi_2^* e^{i\omega\delta} + \phi_1^* \phi_2 e^{-i\omega\delta} \right]$$

But $\phi_1 \cdot \phi_2^* \simeq 0$!

Infact, if we make the assumption

$$\phi_1 = \prod_{K=1}^{N_A} \varphi_K^{(1)}(y)$$

$$\phi_2 = \prod_{K=2}^{N_A} \varphi_K^{(2)}(y)$$

$\to$ Avogadro,

$$\phi_1 \cdot \phi_2 = \prod_{K=1}^{N_A} \varphi_K^{(1)}(y) \varphi_K^{(2)}(y)$$

Assuming also that $\varphi_K^{(1)} \cdot \varphi_K^{(2)} \Big|_{y=y_0} \simeq 0.99$ (very small numbers ...)

$$\phi_1 \phi_2 \Big|_{y=y_0} = (0.99)^{N_A} =$$

Ergo, we get no interference

$$|\Psi_{tot}| \simeq \frac{1}{2} f(y^2) \left[|\phi_1(\alpha_i)|^2 + |\phi_2(\alpha_i)|^2 \right]$$

Integrating over α_i ...

$$\int D\alpha_i \, |\Psi_{tot}|^2 = f(y^2).$$

$$H = H_0 + H_1 = M_0 |S\rangle\langle S| + \sum_{k=\frac{2\pi n}{L}} \omega_k |k\rangle\langle k| + H_1$$

$|\psi(t)\rangle \;/\; |\psi(0)\rangle = |S\rangle$

$$|\psi(t)\rangle = a(t)\, e^{-iM_0 t} |S\rangle + \sum_k r_k(t)\, e^{-i\omega_k t} |k\rangle$$

$$i \frac{d}{dt} |\psi(t)\rangle = i\dot{a}\, e^{-iM_0 t} |S\rangle + M_0\, a\, e^{-iM_0 t} + \sum_k \left(\dot{r}_k(t)\, e^{-i\omega_k t} |k\rangle + \omega_k r_k e^{-i\omega_k t} |\right.$$

$$= a(t) M_0 e^{-iM_0 t} |S\rangle + \sum_k r_k(t)\, \omega_k e^{-i\omega_k t} |k\rangle$$

$$+ a(t)\, e^{-iM_0 t} H_1 |S\rangle + \sum_k r_k(t)\, e^{-i\omega_k t} H_1 |k\rangle$$

Ergo:

$$i\dot{a}\, e^{-iM_0 t} |S\rangle + \sum_k i\dot{r}_k e^{-i\omega_k t} |k\rangle = a(t)\, e^{-iM_0 t} H_1 |S\rangle + \sum_k r_k e^{-i\omega_k t} H_1 |k\rangle$$

$\langle S|$

$$i\dot{a}\, e^{-iM_0 t} = a(t)\, e^{-iM_0 t} \langle S|H_1|S\rangle + \sum_k r_k e^{-i\omega_k t} \langle S|H_1|k\rangle$$

$\langle r_q|$

$$i\dot{r}_q e^{-i\omega_q t} = a(t)\, e^{-iM_0 t} \langle q|H_1|S\rangle + \sum_k r_k e^{-i\omega_k t} \langle q|H_1|k\rangle$$

Let us suppose that $\langle S|H_1|S\rangle = \langle q|H_1|q\rangle = 0$

$$\begin{cases} i\dot{a}\,e^{-iM_0 t} = \sum_k r_k e^{-i\omega_k t}\langle S|H_1|k\rangle \\ i\dot{r}_q e^{-i\omega_q t} = a(t)\, e^{-iM_0 t}\langle q|H_1|S\rangle \end{cases}$$

$t=0$

$a(0) = 1,\ r_q(0) = 0$

Let us make the "extreme" simplification

$$i\dot{r}_q e^{-i\omega_q t} = e^{-iM_0 t}\langle q|H_1|S\rangle$$

$$\dot{r}_q = -i\, e^{i(\omega_q - M_0)t}\langle q|H_1|S\rangle$$

$r_q(T)$ is the probability amplitude to find "$|q\rangle$" after the time-interval T.

In this approximation we get:

$$r_q(T) = -i\int_0^T dt'\, e^{i(\omega_q - M_0)t}\langle q|H_1|S\rangle =$$

$$\boxed{r_q(T) = -i(2\pi)\,\delta(\omega_q - M_0)\,\langle q|H_1|S\rangle\ !}$$

Or better:

$$r_{q}(T/2) = -i \int_{-T/2}^{T/2} dt' \, e^{i(\omega_q - M_0)t} \langle q|H_1|S\rangle =$$

$$r_q(-T/2) = -i\,(2\pi)\,\delta(\omega_q - M_0)\langle q|H_1|S\rangle$$

$$\delta(\omega_q - M_0)^2 = \delta(\omega_q - M_0)\cdot\delta(0) = T\cdot\delta(\omega_1 - M_0)$$

δ rγo:

$$|r_q(T/2)|^2 = (2\pi)^2\,|\langle q|H_1|S\rangle|^2\,\delta(\omega_q - M_0)\,T$$

At this point: integrate...

$$\Sigma(E) = -\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{f^2(k)}{E-\omega(k)+i\varepsilon}$$

$$\mathrm{Im}\,\Sigma(E) =$$

$$L\int\frac{dq}{2\pi}\,|r_q(T/2)|^2 = (2\pi)^2\,\delta(E-\omega_q)$$

$\omega_q = 2\sqrt{q^2+m^2}$

What do we get for the decay?

$$\mathrm{Im}\,\Sigma = (2\pi)^2\int\frac{dq}{2\pi}\,f^2(q)\,\delta(E-\omega_q)$$

$$H_1 = \sum_k \frac{g\, f(k)}{\sqrt{L}} \left(|k\rangle\langle S| + |S\rangle\langle k| \right)$$

$$\langle q|H_1|S\rangle = \sum_k \frac{g\, f(q)}{\sqrt{L}} = \frac{1}{\sqrt{L}} \sum_k g$$

$$|\langle q|H_1|S\rangle|^2 = \frac{g^2 f^2(q)}{L}$$

$$r_q = -i(2\pi)\,\delta(\omega_q - M_0)\,\frac{g\, f(q)}{L}$$

$$\boxed{|r_q|^2 = (2\pi)^2\,\delta(\omega_q - M_0)\,T\,\frac{g^2 f^2(q)}{L}}$$

$$\sum_q |r_q|^2 = \cancel{L} \int_{-\infty}^{\infty} \frac{dq}{\cancel{2\pi}}\,(2\pi)^{\cancel{2}}\,\delta(\omega_q - M_0)\,T\,\frac{g^2 f^2(q)}{\cancel{L}}$$

$$= (2\pi)\,T \int_{-\infty}^{\infty} dq\,\delta(\omega_q - M_0)\,g^2 f^2(q)$$

$$\boxed{= 2\pi\, g^2 \int_{-\infty}^{\infty} dq\, f^2(q)\,\delta(\omega_q - M_0)}$$

$$\omega_q = 2|q|$$

$$2\pi g^2 \int_{-\infty}^{\infty} dq\, f^2(q)\,\delta(2|q| - M_0)$$

$$\Sigma(E) = -\int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{f^2(k)}{E - \omega(k) + i\varepsilon} \qquad 2\pi g^2\, \mathrm{Im}\,\Sigma(E)$$

$$r_q(t) = -i\langle q|K_1|S\rangle \left[\frac{1}{i(\omega_1-\mu_0)}\left(e^{-i(\omega_1-\mu_0)T}-1\right)\right]$$

$$= -i\langle q|K_1|S\rangle \left(\frac{\cancel{1}-i(\omega_1-\mu_0)T-\cancel{1}}{i(\omega_1-\mu_0)}\right) = +i\langle q|K_1|S\rangle T$$

$$\Gamma = \frac{|r_q|^2}{T} = 2\pi|\langle q|H_1|S\rangle|^2\,\delta(\omega-\mu_0)\ ?$$

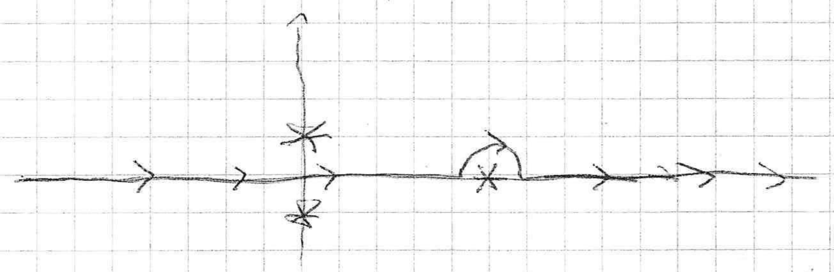
Wulle ... but how much shall we wait?

example:

$$\rho(x) = \frac{1}{1+x^2} \qquad I = \int_{-\infty}^{\infty} \frac{\rho(x)}{(x-x_0)}\, dx$$

→ calculate $P(I)$

$\rho(x)$ has two pole, for $x = \pm i$;



$$I_1 = 2\pi i \lim_{z \to i} \frac{\cancel{(z-i)}}{(z-x_0)\cancel{(z-i)}(z+i)} = \frac{2\pi \cancel{i}}{(i-x_0)2\cancel{i}}$$

$$= \frac{\pi}{(i-x_0)}$$

Therefore:

$$P(I) = \frac{\pi}{i-x_0} + i\pi \frac{1}{1+x_0^2} =$$

$$= \frac{\pi}{i-x_0}\,\frac{i+x_0}{i+x_0} + \frac{i\pi}{1+x_0^2} = -\frac{\pi}{(1+x_0^2)}(i+x_0) + \frac{i\pi}{1+x_0^2}$$

$$P(I(x_0)) = \frac{-\pi x_0}{1+x_0^2} = -\pi x_0\, \rho(x_0^2)$$ (general result!)

$P(I)$ is now real, as expected, beeing an area...

but it depends on $\rho(x)$, of course...

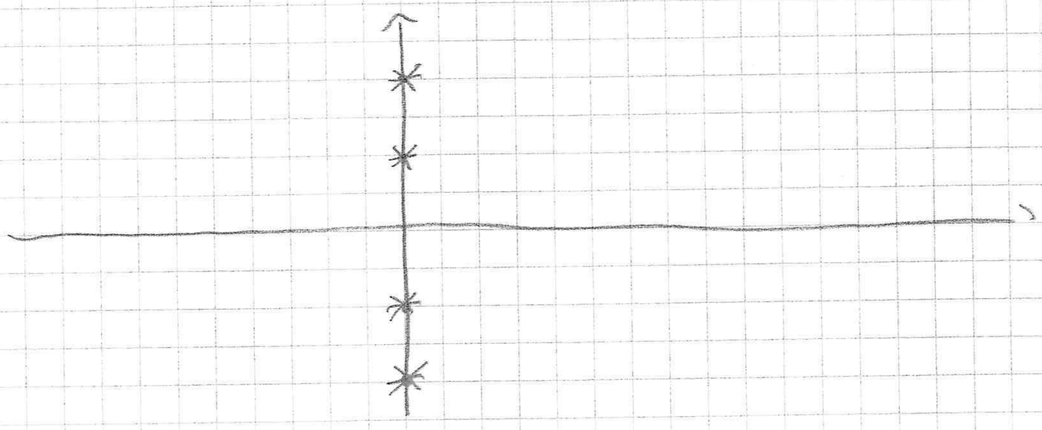
• Calculate the residues of

$$F(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

Poles for:

$z^2+1 = (z-i)(z+i) \implies z = \pm i$

$z^2+4 = (z-2i)(z+2i) \implies z = \pm 2i$



$$F(z) = \frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)}$$

$$\operatorname{Res} F(z)\Big|_{z=i} = \lim_{z\to i} (z-i)F(z) = \frac{(i)^2}{2i\cdot(-i)(3i)} =$$

$$= \frac{-1}{6i} = \frac{i}{6}$$

$$\operatorname{Res} F(z)\Big|_{z=-i} = \frac{(-i)^2}{(-2i)(-3i)(i)} = -\frac{i}{6}$$

$$\operatorname{Res} F(z)\Big|_{z=2i} = \frac{4(i)^2}{i(3i)4i} = \frac{-1}{i\cdot(-3)} = -\frac{i}{3} = -\operatorname{Res} F(z)\Big|_{z=-2i}$$

$$\sum_{\substack{\text{res}\\ \text{in the finite}}} F(z) = 0 \quad \mapsto \quad \text{Res}\, F(z)\Big|_{z=\infty} = 0$$

$$z = 1/t$$

$$F(z) \xrightarrow[z \mapsto \infty]{} \frac{1}{z^2}$$

$$z = 1/t$$

$$t^2 \times \frac{1}{t^2} = 1 \mapsto \underline{\underline{\text{no pole}}}$$

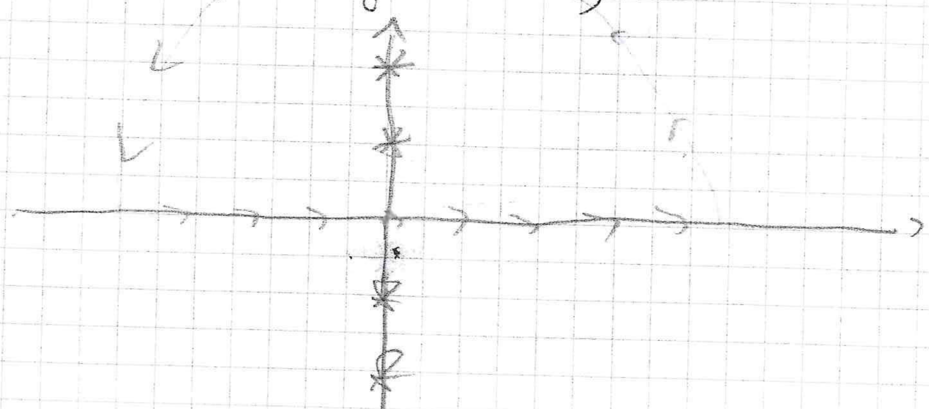
We find, as expected:

$$\text{Res}\, F(z)\Big|_{z=\infty} = 0$$

$$\int_{-\infty}^{\infty} dz\, F(z) = 2\pi i \left(\text{Res}\, F(z)\Big|_{i} + \text{Res}\, F(z)\Big|_{2i} \right) =$$

$$= 2\pi i \left(\frac{i}{6} - \frac{i}{3} \right) = +2\pi \left(\frac{1}{3} - \frac{1}{6} \right) =$$

$$= 2\pi \, \frac{2-1}{6} = \frac{\pi}{3}$$



(note that the Jordan condition is satisfied, being $F(z) \propto 1/z^2$!!) $\left(\frac{1}{z^\alpha} \; \alpha > 1 \text{ is enough...} \right.$

$\alpha = 1$ need oscillating

Sheet 4, ex. 1:

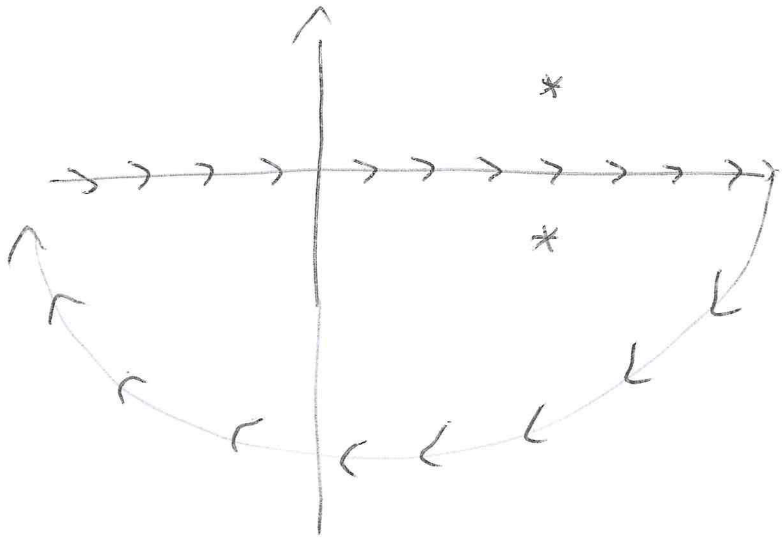
$$N = \frac{\Gamma}{2\pi}$$

Namely:

$$(x-M)^2 + \Gamma^2/4 = 0 \;\rightarrow\; (x-M) = \pm i\Gamma/2 \qquad \rightarrow\; x = M - i\Gamma/2 \text{ is a pole.}$$

close below:

$$1 = \int_{-\infty}^{\infty} d_s(x)\,dx = N \cdot \int_{-\infty}^{\infty} dx \,\frac{1}{(x-(M-i\Gamma/2))(x-(M+i\Gamma/2))}$$



$$= N \cdot (-2\pi i)\, \frac{1}{(M - i\Gamma/2 - M - i\Gamma/2)} = N \cdot \frac{2\pi}{\Gamma} = 1 \longmapsto \boxed{N = \frac{\Gamma}{2\pi}}$$

1.2

$$\Gamma_Z = \frac{1}{\sqrt{\langle H^2\rangle - \langle H\rangle^2}}$$

$$\langle H\rangle = \int_{-\infty}^{\infty} x\, d_S(x)\, dx$$

Achtung... in principle, this is not determined. But, if we do the change of coordinate

$y = x - M$ we get:

$$\langle H\rangle = \int_{-\infty}^{\infty} (y+M)\, \frac{N}{y^2+(\Gamma/2)^2}\, dy = \underbrace{\int_{-\infty}^{\infty} y\, \frac{N}{y^2+\Gamma^2/4}\, dy}_{=0} + M \int_{-\infty}^{\infty} \frac{N}{y^2+\Gamma^2/4}\, dy = M!$$

on the contrary:

$$\langle H^2\rangle = \int_{-\infty}^{\infty} x^2\, d_S(x)\, dx = +\infty$$! No way to cure it.

$$\Gamma_Z = 0.$$

There is no zero - time, because the decay is an exact exponentia

1.3

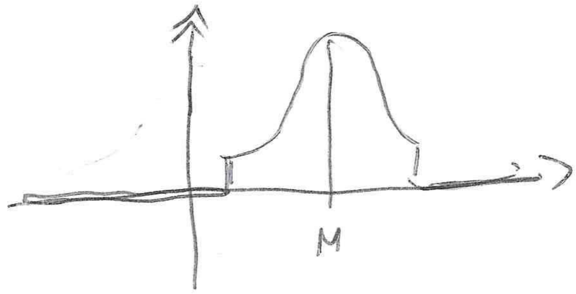
$$a(t) = \int_{-\infty}^{\infty} d_s(x)\, e^{-ixt}\, dx$$

Follow the same steps as before:

$$a(t) = e^{-i(M - i\Gamma/2)t} = e^{-iMt}\, e^{-\Gamma/2\, t}$$

$$P(t) = e^{-\Gamma t}$$

2-1



$$\int_{-\infty}^{\infty} d_s(x)\,dx = \int_{M-\Lambda}^{M+\Lambda} \frac{N}{(x-M)^2+\Gamma^2/4}\,dx = \int_{-\Lambda}^{\Lambda} \frac{N}{y^2+\Gamma^2/4}\,dy = N\frac{4}{\Gamma}\arctan\left(\frac{2\Lambda}{\Gamma}\right) = 1$$

$$\boxed{N = \frac{\Gamma}{4\arctan\left(\frac{2\Lambda}{\Gamma}\right)}}$$

For $\Lambda \mapsto \infty$ we get: $N = \frac{\Gamma}{4\frac{\pi}{2}} = \frac{\Gamma}{2\pi}$ ✓

(Recall that:

$$\int dy\,\frac{1}{y^2+\Gamma^2/4} = \frac{2}{\Gamma}\arctan\left(\frac{2y}{\Gamma}\right)$$
)

)

$$\langle x\rangle = \int_{-\infty}^{\infty} x\, d_S(x)\, dx = M \qquad (y = x - M)$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 d_S(x)\, dx = \int_{-\Lambda}^{\Lambda} (y^2 + M^2 + 2yM)\, \frac{N}{y^2 + \Gamma^2/4}\, dy =$$

$$= \int_{-\Lambda}^{\Lambda} y^2 \frac{N}{y^2+\Gamma^2/4} + M^2 = N\left(2\Lambda - \Gamma \arctan\left(\frac{2\Lambda}{\Gamma}\right)\right) + M^2$$

Ergo:

$$Y_7 = \frac{1}{\sqrt{N\left[2\Lambda - \Gamma \arctan\left(\frac{2\Lambda}{\Gamma}\right)\right]}} =$$

For $\Lambda \mapsto \infty$ we get zero.

(Recall that:

$$\int^{\infty}_{-\infty} \frac{y^2}{y^2+\Gamma^2/4}\, dy = 4\left[\frac{y}{4} - \frac{1}{8}\Gamma \arctan\left(\frac{2y}{\Gamma}\right)\right].$$

$$L = \frac{1}{2} m \dot{x}^2 - V(x,t)$$

$$y = x - \frac{1}{2} a t^2$$

$$\dot{y} = \dot{x} - at$$

$$\Rightarrow \dot{x} = \dot{y} + at$$

$$\boxed{L = \frac{1}{2} m (\dot{y} + at)^2 - \tilde{U}(y,t)}$$ where $\tilde{U}(y,t) = V(y + \frac{1}{2} a t^2, t)$

$$= \frac{1}{2} m \dot{y}^2 + \underbrace{\frac{1}{2} m a^2 t^2 + m \dot{y} a t}_{} - \tilde{U}(y,t)$$

We can rewrite the term as:

$$\frac{1}{2} m a^2 t^2 + m \dot{y} a t = \frac{d}{dt}\left(\frac{1}{6} m a^2 t^3 + m y a t\right) - may$$

Ergo:

$$L = \frac{1}{2} m \dot{y}^2 - \tilde{U}(y,t) + \underbrace{\frac{d}{dt}\left(\frac{1}{6} m a^2 t^3 + m y a t\right)}_{\text{no influence}} - may$$

$$= \frac{1}{2} m \dot{y}^2 - \tilde{U}(y,t) - may = \frac{1}{2} m \dot{y}^2 - U(y,t)$$

$$U(y,t) = \tilde{U}(y,t) + may$$

Sheet 7, ex. 1

$$f(x) = \sqrt{x^2-4m^2} \ \ln\left[\frac{\sqrt{x^2-4m^2}+x}{\sqrt{x^2-4m^2}-x}\right]$$

For $x \in (0, 2m)$ we have:

$$\sqrt{x^2-4m^2} = i\alpha \quad , \quad \alpha = \sqrt{4m^2-x^2} \in \mathbb{R}^+$$

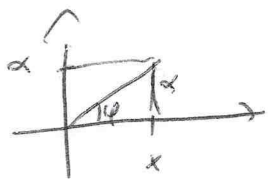
Ergo:

$$i\alpha \ln\left(\frac{i\alpha + x}{i\alpha - x}\right)$$

$$\ln z = \ln |z| e^{i\varphi} = \ln|z| + i\varphi$$

$x = x$
$y = \alpha$

$$z = \frac{x+i\alpha}{x-i\alpha} \qquad |z| = 1$$



$$z = \frac{\sqrt{x^2+y^2}\, e^{i\varphi}}{\sqrt{x^2+y^2}\, e^{-i\varphi}} = e^{i2\varphi}$$

$$\varphi = \arctan\left(\frac{\sqrt{x^2+x^2}}{x}\right)$$

$$\tan\varphi = \frac{\alpha}{x}$$

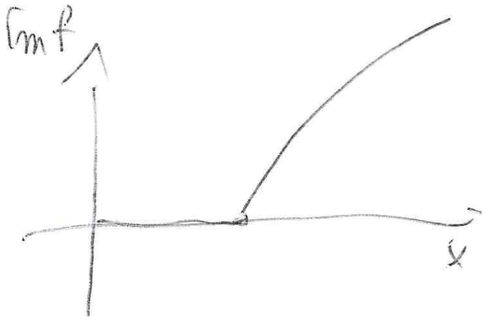
$$\varphi = \arctan\frac{\alpha}{x}$$

Ergo, for $x \in (0, 2m)$:

$$f(x) = i\alpha \cdot \ln z = i\alpha\left(\ln |z| e^{i2\varphi}\right) =$$

$= i\alpha \cdot i2\varphi$ is real....

For $x > 2m$ we get both a real and an imaginary part...

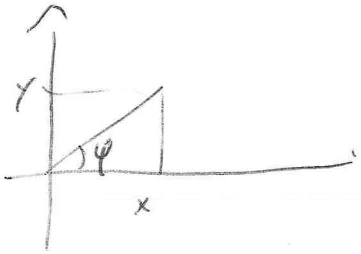


Namely, for $x > 2m$ ... we get

$$\sqrt{x^2-4m^2}\,\ln\left(\frac{\sqrt{x^2-4m^2}+x}{\sqrt{x^2-4m^2}-x}\right) =$$

$$= \sqrt{\quad}\;\ln\left|\quad\right| + i\pi\sqrt{x^2-4m^2}$$

The imaginary part is $\pi\sqrt{x^2-4m^2}$.



$$y = x \tan\varphi$$

$$\frac{y}{x} = \tan\varphi$$

$$z = x + iy = \sqrt{x^2+y^2}\, e^{i\varphi} = \sqrt{x^2+y^2}\, e^{i \arctan\left(\frac{y}{x}\right)}$$

$$\sqrt{x^2-4m^2} + x = 2m\, e^{i \arctan\left(\sqrt{\ }/x\right)}$$

$$\sqrt{x^2-4m^2} - x = -2m\, e^{-i \arctan\left(\sqrt{\ }/x\right.}$$

$$\ln\left(\frac{\sqrt{\ } + x}{\sqrt{\ } - x}\right) = \ln\left(\frac{2m\, e^{i\varphi}}{-2m\, e^{-i\varphi}}\right) = \ln\left(e^{+i\pi} e^{i\varphi}\right)$$

$$= \varphi + \pi$$

1.1

$$\Gamma = \frac{|\vec{K}_1|}{8\pi M_0^2}\, g^2 \qquad \text{where: } [g] = [\mathrm{EV}].$$

$$(M_0, \vec{0}) = \left(\sqrt{\vec{K}_1^2 + m_1^2},\ \vec{K}_1\right) + \left(\sqrt{\vec{K}_2^2 + m_2^2},\ \vec{K}_2\right).$$

$$M_0 = \sqrt{\vec{K}_1^2 + m_1^2} + \sqrt{\vec{K}_2^2 + m_2^2} \qquad \text{και } \vec{K}_1 + \vec{K}_2 = 0 \rightarrow$$

$$M_0^2 = \vec{K}_1^2 + m_1^2 + \vec{K}_1^2 + m_2^2 + 2\sqrt{}\sqrt{}$$

$$M_0^2 - 2\vec{K}_1^2 - (m_1^2 + m_2^2) = 2\sqrt{}\sqrt{}$$

$$M_0^4 + \cancel{4\vec{K}_1^4} + (m_1^2+m_2^2)^2 - 4\vec{K}_1^2 M_0^2 + \cancel{4\vec{K}_1^2 (m_1^2+m_2^2)} - 2M_0^2(m_1^2+m_2^2) =$$

$$4(m_1^2 + \vec{K}_1^2)(m_2^2 + \vec{K}_1^2) = 4m_1^2 m_2^2 + \cancel{4\vec{K}_1^2(m_1^2+m_2^2)} + \cancel{4\vec{K}_1^4}$$

$$M_0^4 + (m_1^2 - m_2^2)^2 - 4\vec{K}_1^2 M_0^2 - 2M_0^2(m_1^2+m_2^2) = 0$$

$$\boxed{|\vec{K}_1| = \frac{1}{2M_0}\sqrt{M_0^4 + (m_1^2 - m_2^2)^2 - 2M_0^2(m_1^2+m_2^2)}} = f_K(M_0, m_1, m_2)$$

check: $m_1 = m_2 \rightarrow |\vec{K}_1| = \frac{1}{2M_0}\sqrt{M_0^4 - 4M_0^2 m^2} = \sqrt{\frac{M^2}{4} - m^2}$ qed.

1.2

There are 3 decays:

$[g] \{E^{-1}\}$

$$\Gamma_{S\mapsto\varphi_1\varphi_1} = \frac{\sqrt{\frac{M_0^2}{4}-m_1^2}}{8\pi M_0^2}\left[\sqrt{2}\, g_1\right]^2$$

$$\Gamma_{S\mapsto\varphi_2\varphi_2} = \sqrt{\frac{M_0^2}{4}-m_2^2}\left[\sqrt{2}\, g_2\right]^2$$

$$\Gamma_{S\mapsto\varphi_1\varphi_2} = \frac{f_{kin}(M_0, m_1, m_2)}{8\pi M_0^2}\left[g\right]^2$$

$$\Gamma_{S,tot} = \Gamma_{S\mapsto\varphi_1\varphi_1} + \Gamma_{S\mapsto\varphi_2\varphi_2} + \Gamma_{S\mapsto\varphi_1\varphi_2}$$

1.3

$$-i\mathcal{M} = ig \cdot iK_{1,\mu}\, iK_2^{\mu} = -ig\,(K_1\cdot K_2) \qquad [g] = [E^{-1}]$$

$$\Gamma_{S\mapsto\varphi_1\varphi_2} = \frac{f_{kin}(M_0, m_1, m_2)}{8\pi M_0^2}\left[g\,(K_1\cdot K_2)\right]^2 = \frac{f_{kin}(M_0, m_1, m_2)}{8\pi M_0^2}\left[g\,\frac{M_0^2-m_1^2-m_2^2}{2}\right.$$

$$P = K_1+K_2 \rightarrow P^2 = K_1^2+K_2^2+2K_1K_2 \rightarrow \boxed{K_1K_2 = \frac{M_0^2-m_1^2-m_2^2}{2}}$$

1.4

$[g] = [E^{-1}]$ 3

$$\mathcal{L} = g(\partial_\mu S)\varphi_1(\partial^\mu \varphi_2)$$

$$-i\mathcal{M} = ig(-iP_\mu)(iK_2^\mu) = ig(P \cdot K_2)$$

$$P = K_1 + K_2$$

$$P - K_2 = K_1$$

$$P^2 + K_2^2 - 2PK_2 = K_1^2$$

$$M_0^2 + m_2^2 - 2P \cdot K_2 = m_1^2$$

$$\boxed{P \cdot K_2 = \frac{M_0^2 + m_2^2 - m_1^2}{2}}$$

$$\Gamma = \frac{f_{km}(M_0, m_1, m_2)}{8\pi M_0^2}\left[g\,\frac{M_0^2 + m_2^2 - m_1^2}{2}\right]^2.$$

1.5)

$$\Gamma = \frac{f_{kin}(M_0, m_1 m_2)}{8\pi M_0^2} \left[g \, (-) - \frac{h\,(M_0^2 - m_1^2 - m_2^2)}{2} \right]^2$$

$$\begin{cases} [g] = [E] \\ [h] = [E^{-1}] \end{cases}$$

$$-i\mathcal{M} = ig + ih\,(ik_{1,\mu})(ik_2^{\mu})$$

$$= ig - ih\,(k_1 \cdot k_2) =$$

$$= i\left[g - h\,\frac{M_0^2 - m_1^2 - m_2^2}{2} \right]$$

$$\rightarrow -i\mathcal{M} = 0 \Rightarrow \boxed{g = \frac{h\,(M_0^2 - m_1^2 - m_2^2)}{2}}$$

$$\tilde{M}_1^2 = M_1^2 c^2 + M_2^2 s^2 + 2g\, s c$$

$$\tilde{M}_2^2 = M_1^2 s^2 + M_2^2 c^2 - 2g\, s c$$

$$\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix}$$

$$\alpha = \frac{1}{2} \arctan\left(\frac{-2g}{M_2^2 - M_1^2} \right)$$

Now, for $M_2 \mapsto M_1 = M_0^2 \Rightarrow \alpha = \frac{1}{2} \arctan(-\infty) = -\frac{\pi}{4}$

Ergo:

$$\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

$$\begin{cases} \tilde{M}_1^2 = \dfrac{M_1^2 + M_2^2}{2} + 2g \dfrac{1}{\sqrt{2}} \left(-\dfrac{1}{\sqrt{2}} \right) = M_0^2 - g \\ \tilde{M}_2 = M_0^2 + g \end{cases}$$

$$\mathcal{L}_1 = g S_1 \varphi_1^2 = g \left(\frac{1}{\sqrt{2}} \tilde{S}_1 + \frac{1}{\sqrt{2}} \tilde{S}_2 \right) \varphi_1^2$$

Ergo:

$$\boxed{\Gamma_{\tilde{S}_1 \to \varphi_1 \varphi_1} = \frac{\sqrt{\frac{\tilde{M}_1^2}{4} - m^2}}{8\pi \tilde{M}_1^2} g^2 , \quad \Gamma_{\tilde{S}_2 \to \varphi\varphi} = \frac{\sqrt{\frac{\tilde{M}_2^2}{4} - m^2}}{8\pi \tilde{M}_2^2} g^2 .}$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu S_1)^2 + \frac{1}{2}(\partial_\mu S_2)^2 + \alpha(\partial_\mu S_1)(\partial^\mu S_2)$$

$$= \frac{1}{2}(\partial_\mu S_1, \partial_\mu S_2)\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} \partial_\mu S_1 \\ \partial_\mu S_2 \end{pmatrix} =$$

$$\theta = -\frac{\pi}{4}$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \mapsto \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2}(1-\alpha)(\partial_\mu \varphi_1)^2 + \frac{1}{2}(1+\alpha)(\partial_\mu \varphi_2)^2$$

$$\begin{cases} \varphi_1 \mapsto \frac{1}{\sqrt{1-\alpha}}\varphi_1 \\ \varphi_2 \mapsto \frac{1}{\sqrt{1+\alpha}}\varphi_2 \end{cases} \Rightarrow |\alpha| < 1$$

διοτι:

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1-\alpha}}\varphi_1 \\ \frac{1}{\sqrt{1+\alpha}}\varphi_2 \end{pmatrix}$$

<u>check</u>: plug it in

$$\frac{1}{2}\left(\frac{1}{\sqrt{2}\sqrt{1-\alpha}}\partial_\mu\varphi_1+\frac{1}{\sqrt{2}\sqrt{1+\alpha}}\partial_\mu\varphi_2\right)^2+\frac{1}{2}\left(-\frac{1}{\sqrt{2}\sqrt{1-\alpha}}\partial_\mu\varphi_1+\frac{1}{\sqrt{2}\sqrt{1+\alpha}}\partial_\mu\varphi_2\right)^2$$

$$+\alpha\left(\frac{1}{\sqrt{2}\sqrt{1-\alpha}}\partial_\mu\varphi_1+\frac{1}{\sqrt{2}\sqrt{1+\alpha}}\partial_\mu\varphi_2\right)\left(-\frac{1}{\sqrt{2}\sqrt{1-\alpha}}\partial_\mu\varphi_1+\frac{1}{\sqrt{2}\sqrt{1+\alpha}}\partial_\mu\varphi_2\right)$$

$$=\frac{1}{2}\left(\frac{1}{2(1-\alpha)}(\partial_\mu\varphi_1)^2+\frac{1}{2(1+\alpha)}(\partial_\mu\varphi_2)^2\right)+\cancel{\frac{1}{2\sqrt{1-\alpha^2}}\partial_\mu\varphi_1\partial^\mu\varphi_2}$$

$$+\frac{1}{2}\,\frac{1}{2(1-\alpha)}(\partial_\mu\varphi_1)^2+\frac{1}{2(1+\alpha)}(\partial_\mu\varphi_2)^2-\cancel{\frac{1}{2\sqrt{1-\alpha^2}}\partial_\mu\varphi_1\partial_\mu\varphi_2}$$

$$+\alpha\left(-\frac{1}{2(1-\alpha)}(\partial_\mu\varphi_1)^2+\frac{1}{2(1+\alpha)}(\partial_\mu\varphi_2)^2\right)=$$

$$\frac{1}{2}(\partial_\mu\varphi_1)^2\left(\underbrace{\frac{1}{2(1-\alpha)}+\frac{1}{2(1-\alpha)}-\frac{2\alpha}{2(1-\alpha)}}_{=\frac{2-2\alpha}{2(1-\alpha)}}\right)+\frac{1}{2}(\partial_\mu\varphi_2)^2$$

qed...

Ergo:

$$\boxed{S_1 = \frac{1}{\sqrt{2}\sqrt{1-\alpha}}\varphi_1 + \frac{1}{\sqrt{2}\sqrt{1+\alpha}}\varphi_2}$$

it follows that:

$$\mathcal{L}_1 = g\varphi S_1^2 = g\varphi\left(\frac{1}{2(1-\alpha)}\varphi_1^2 + \frac{1}{2(1+\alpha)}\varphi_2^2 + \frac{1}{1-\alpha^2}\varphi_1\varphi_2\right)$$

Ergo:

$$\begin{cases} \Gamma_{\varphi \mapsto \varphi_1\varphi_1} = \dfrac{\frac{m}{2}}{8\pi m^2}\left[\dfrac{g\sqrt{2}}{2(1-\alpha)}\right]^2 \\ \Gamma_{\varphi \mapsto \varphi_2\varphi_2} = \dfrac{\frac{m}{2}}{8\pi m^2}\left[\dfrac{g\sqrt{2}}{2(1+\alpha)}\right]^2 \\ \Gamma_{\varphi \mapsto \varphi_1\varphi_2} = \dfrac{\frac{m}{2}}{8\pi m^2}\left[\dfrac{g}{1-\alpha^2}\right]^2 \end{cases}$$

$$\mathcal{L}_1 = g\varphi e_\mu e^\mu$$

$$-i\mathcal{M}^{ab} = i g\, \varepsilon^{a}_{\mu}(k_1)\, \varepsilon^{b,\mu}(k_2)$$

Ergo:

$$\sum_{ab} |-i\mathcal{M}^{ab}|^2 = g^2 \left(-g_{\mu\nu} + \frac{k_{1\mu}k_{1\nu}}{m_e^2}\right)\left(-g^{\mu\nu} + \frac{k_{2\mu}k_{2\nu}}{m_e^2}\right)$$

$$= g^2\left(4 - 1 - 1 + \frac{(k_1 k_2)^2}{m_e^4}\right)$$

$$= g^2\left(2 + \frac{(k_1\cdot k_2)^2}{m_e^2}\right) = g^2\cdot\left(2 + \frac{(m_\varphi^2 - 2m_e^2)^2}{4m_e^2}\right)$$

$$\boxed{\Gamma_{\varphi\mapsto ee} = \frac{\sqrt{\frac{m_\varphi^2}{4} - m_e^2}}{8\pi m_\varphi^2}\,\frac{1}{2}\, g^2 \left(2 + \underbrace{\frac{(m_\varphi^2 - 2m_e^2)^2}{4m_e^2}}\right)} \quad \sim (m_\varphi^2 \ldots$$

because the particles are identical

note: for m_φ very large then is dominant ...

$m_e \mapsto 0$ is not performed!

Γ grows very fast with m_φ! (due to the last term).

$$\mathcal{L}_1 = g\,\varphi\,\partial_\mu \rho_\nu\,\partial^\mu \rho_\nu$$

The same ... there is just an extra $(k_1 k_2)$ from the derivatives:

$$\Gamma = \frac{\sqrt{\frac{m_\varphi^2}{4} - m_\rho^2}}{8\pi m_\varphi^2}\,2\cdot g^2 \left(\frac{m_\varphi^2 - 2m_\rho^2}{2}\right)^2 \left(2 + \frac{(m_\varphi^2 - 2m_\rho^2)^2}{4 m_\rho^2}\right) \theta(m_\varphi - 2m_\rho)$$

$\downarrow$ identical particles

$$\mathcal{L}_1 = g H F_{\mu\nu} F^{\mu\nu} = g H (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu).$$

$$= 2g H \partial_\mu A_\nu \partial^\mu A^\nu - 2g H \partial_\mu A_\nu \partial^\nu A^\mu.$$

$$-i\mathcal{M}^{ab} = (2g)(k_1 k_2)\, \varepsilon^a_\nu(\vec{k}_1) \varepsilon^{b,\nu}(\vec{k}_2) - (2g)\underbrace{k_{1,\mu} \varepsilon^\mu(\vec{k}_2)}_{=0} \underbrace{k_{2,\nu} \varepsilon^\nu(\vec{k}_1)}_{=0}.$$

Ergo:

$$|-i\mathcal{M}^{ab}|^2 = (2g)^2 (k_1 k_2)^2 (-g_{\mu\nu})(-g^{\mu\nu}) \qquad k_1 k_2 = \frac{M_H^2}{2}$$

$$\Gamma = \frac{|\vec{k}|}{8\pi M_H^2} 2 \cdot (2g)^2 \cdot \frac{M_H^4}{\cancel{4}} \cdot \cancel{4} = \frac{M_H^3 g^2}{2\pi}$$

$\Rightarrow$ However, this is wrong...

Namely:

$$\varepsilon^a_\nu(\vec{k}_1) \varepsilon^{b,\nu}(\vec{k}_2) = \delta^{ab}$$

$$\sum_{a,b} |-i\mathcal{M}^{ab}|^2 = (2g)^2 (k_1 k_2)^2 \cdot 2 \; !! \quad \Rightarrow \Gamma = \frac{M_H^3 g^2}{4\pi M}$$

(compare also with the previous case... only 2 polarisations come in).

Namely, the sum is more complicated:

$$\sum_{a=1}^{2} \varepsilon_\mu^a(k)\varepsilon_\nu^a(k) = -g_{\mu\nu} + \eta_\mu\eta_\nu - \bar{k}_\mu\bar{k}_\nu$$

$$\bar{k}^\mu = \frac{k^\mu - (k\cdot\eta)\eta^\mu}{\left[(k\cdot\eta)^2 - k^2\right]^{1/2}} \quad ; \quad \eta_\mu = (1,0,0,0)$$

In the present case: $k = k_1 = (|\vec{k}_1|, \vec{k}_1)$

$$k\cdot\eta = \omega$$

$$\bar{k}^\mu = \frac{k^\mu - \omega\eta^\mu}{\omega^2 - \omega^2 + \vec{k}^2} = \frac{k^\mu - \omega\eta^\mu}{\omega^2} = \frac{1}{\omega^2}(0, \vec{k})$$

$$\left(-g_{\mu\nu} + \underline{\eta^1_\mu\eta^2_\nu} - \bar{k}_{1\mu}\bar{k}_{1\nu}\right)\left(-g^{\mu\nu} + \underline{\eta^{2\mu}\eta^{2\nu}} - \bar{k}_2^\mu\bar{k}_2^\nu\right)$$

$$= 4 - \eta_2^2 + \bar{k}_2^2 - \eta_1^2 + \underline{(\eta^1\cdot\eta^2)}^2 - \underbrace{(\bar{k}_2\cdot\eta)^2}_{\Downarrow\ 0} + \bar{k}_1^2 - (\bar{k}_{1\mu}\bar{k}_2)^2$$

$$= 4 - 1 + 1 - 1 + 1 - 1$$

$$= 4 - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{1} + 0 + 1 + \ldots$$

therefore spacelike. We introduce a timelike vector $\eta_\mu = (1, 0, 0, 0)$, which is orthogonal to ε_μ in the radiation gauge. We then form a tetrad from $\varepsilon_\mu^{(1,2)}$, η_μ and one other spacelike vector, denoted $\bar{k}$:

$$\bar{k}^\mu = \frac{k^\mu - (k \cdot \eta)\eta^\mu}{[(k \cdot \eta)^2 - k^2]^{1/2}}.$$

It is straightforward to verify that $\bar{k}$ is spacelike:

$$\bar{k}^2 = \frac{k^2 - 2(k \cdot \eta)^2 + (k \cdot \eta)^2 \eta^2}{(k \cdot \eta)^2 - k^2} = -1,$$

where we have used $\eta^2 = 1$; and that $\bar{k}$ is orthogonal to ε:

$$\bar{k} \cdot \varepsilon = \frac{k \cdot \varepsilon - (k \cdot \eta)(\eta \cdot \varepsilon)}{[(k \cdot \eta)^2 - k^2]^{1/2}} = 0,$$

since $k \cdot \varepsilon = 0$ and $\eta \cdot \varepsilon = 0$. Having constructed a tetrad, we now have

$$g_{\mu\nu} = \eta_\mu \eta_\nu - \sum_{\lambda=1}^{2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)}(k) - \bar{k}_\mu \bar{k}_\nu,$$

hence

$$\begin{aligned}\sum_{\lambda=1}^{2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{\lambda}(k) &= -g_{\mu\nu} + \eta_\mu \eta_\nu - \bar{k}_\mu \bar{k}_\nu \\ &= -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k \cdot \eta)^2 - k^2} + \frac{(k \cdot \eta)(k_\mu \eta_\nu + \eta_\mu k_\nu)}{(k \cdot \eta)^2 - k^2} \\ &\quad - \frac{k^2 \eta_\mu \eta_\nu}{(k \cdot \eta)^2 - k^2}. \end{aligned} \tag{7.17}$$

This is the desired expression for $\sum_{\lambda=1}^{2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)}(k)$, and when substituted in (7.15) it gives an explicit expression for the propagator for transverse photons.

7.2 Non-Abelian gauge fields and the Faddeev–Popov method

We now want to extend what we have done to non-Abelian gauge fields (Yang–Mills fields). Our aim is to discover the general rules for finding the gauge field propagator. We proceed by developing the formal path-integral method referred to in the last section, based on making Z finite. This method was first devised by Faddeev and Popov.

We saw above that Z is infinite because the functional integration extends over all A_μ, even over those related by a gauge transformation, under which the integrand is invariant. In a schematic way we may write each A_μ as

$$A_\mu \sim \bar{A}_\mu, \Lambda(x) \tag{7.18}$$

We calculate the left-hand side by substituting for $A_\mu(x)$ from (4.64) (or (4.80), but with the sum over $\lambda = 1, 2$ only). Hence

$$\begin{aligned}\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle &= \left\langle 0\left|\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\frac{\mathrm{d}^3k'}{(2\pi)^3 2k'_0}\sum_{\lambda,\lambda'=1}^{2}\varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda')}(k')\right.\right.\\ &\quad\times\{[a^{(\lambda)}(k)\,\mathrm{e}^{-\mathrm{i}kx} + a^{(\lambda)\dagger}(k)\,\mathrm{e}^{\mathrm{i}kx}]\\ &\quad\times[a^{(\lambda')}(k')\,\mathrm{e}^{-\mathrm{i}k'y} + a^{(\lambda')\dagger}(k')\,\mathrm{e}^{\mathrm{i}k'y}]\theta(x_0 - y_0)\\ &\quad+[a^{(\lambda')}(k')\,\mathrm{e}^{-\mathrm{i}k'y} + a^{(\lambda')\dagger}(k')\,\mathrm{e}^{\mathrm{i}k'y}]\\ &\quad\left.\left.\times[a^{(\lambda)}(k)\,\mathrm{e}^{-\mathrm{i}kx} + a^{(\lambda)\dagger}(k)\,\mathrm{e}^{\mathrm{i}kx}]\theta(y_0 - x_0)\}\right|0\right\rangle\\ &= \left\langle 0\left|\int \frac{\mathrm{d}^3k\,\mathrm{d}^3k'}{(2\pi)^6 2k_0 2k'_0}\sum_{\lambda,\lambda'=1}^{2}\varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda')}(k')\right.\right.\\ &\quad\times[a^{(\lambda)}(k)a^{(\lambda')\dagger}(k')\,\mathrm{e}^{\mathrm{i}(k'y-kx)}\theta(x_0 - y_0)\\ &\quad\left.\left.+a^{(\lambda')}(k')a^{(\lambda)\dagger}(k)\,\mathrm{e}^{\mathrm{i}(kx-k'y)}\theta(y_0 - x_0)]\right|0\right\rangle.\end{aligned}$$

The two terms in $aa^\dagger$ may be replaced by their commutators, given by (4.68). The delta functions then enable one to integrate over k' and sum over λ', giving

$$\begin{aligned}\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\sum_{\lambda=1}^{2}\varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)}(k)\\ &\quad\times[\mathrm{e}^{\mathrm{i}k(y-x)}\theta(x_0 - y_0) + \mathrm{e}^{\mathrm{i}k(x-y)}\theta(y_0 - x_0)].\end{aligned} \tag{7.15}$$

Now, from equations (6.14) and (6.56), the Feynman propagator for massless particles is

$$\begin{aligned}\Delta_\mathrm{F}(x, m = 0) &= -\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}[\theta(x_0)\,\mathrm{e}^{-\mathrm{i}kx} + \theta(-x_0)\,\mathrm{e}^{\mathrm{i}kx}]\\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}kx}}{k^2 + \mathrm{i}\varepsilon}.\end{aligned}$$

Hence we may write

$$\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2 + \mathrm{i}\varepsilon}\sum_{\lambda=1}^{2}\varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)}(k)$$

and the propagator for transverse photons is then

$$D_{\mu\nu}^{\mathrm{tr}}(x - y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2 + \mathrm{i}\varepsilon}\sum_{\lambda=1}^{2}\varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)}(k). \tag{7.16}$$

What is $\sum_{\lambda=1}^{2}\varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)}(k)$? Now ε_μ is orthogonal to k_μ, which is lightlike; ε_μ is

Recall of the matrix C

The matrix C is introduced when studying the operation of 'charge conjugation' (particle $\leftrightarrow$ antiparticle)

$$\psi \mapsto \psi_c = C\bar{\psi}^t$$

where C is such that

$$\boxed{C\gamma^{\mu\,t}C^{-1} = -\gamma^\mu}$$

In the Dirac representation one has:

$$C = i\gamma^2\gamma^0$$

it follows that in this representation $\boxed{\psi_c = i\gamma^2\psi^*}$

$$\begin{cases} C^\dagger = C^{-1} = -C \\ C\gamma^5 = \gamma^5 C \end{cases}$$

Namely.

- $C^\dagger = (+i\gamma^2\gamma^0)^\dagger = -i\gamma^0(-\gamma^2) = +i\gamma^0\gamma^2$

  $C^\dagger C = +i\gamma^0\gamma^2 \cdot i\gamma^2\gamma^0 = 1$ q.e.d.

- $C\gamma^5 = i\gamma^2\gamma^0\gamma^5 = \gamma^5 C \Rightarrow [C,\gamma^5] = 0$.

THE LAGRANGIAN

$$\mathcal{L} = g D \psi^t C \gamma^5 \psi + h.c.$$

$$(\psi^t C \gamma^5 \psi)^\dagger = \psi^\dagger \gamma^{5\dagger} C^\dagger \psi^* = -\psi^\dagger \gamma^5 C \psi^* = -\psi^\dagger \gamma^0 \gamma^0 \gamma^5 C \psi^* =$$

$$= +\bar{\psi} \gamma^5 \gamma^0 (i\gamma^2 \gamma^0) \psi^* = -\bar{\psi} \gamma^5 C \gamma^0 \psi^* = -\bar{\psi} \gamma^5 C \bar{\psi}^t = \bar{\psi} C \gamma^5 \bar{\psi}^t$$

(in fact: $\bar{\psi}^t = (\psi^\dagger \gamma^0)^t = \gamma^{0t} \psi^* = \gamma^0 \psi^*$)

Ergo:

$$\boxed{\mathcal{L} = g D \left(\psi^t C \gamma^5 \psi + \bar{\psi} C \gamma^5 \bar{\psi}^t \right)}$$

Achtung: even more generally we can write:

$$\mathcal{L} = g D \psi^t C \gamma^5 \psi + g^* D \bar{\psi} C \gamma^5 \bar{\psi}^t$$

However, this has no difference on the rest of the calculation

$$\mathcal{L}^\dagger = \mathcal{L}$$ by construction (as it must).

FURTHER CHECKS of $\mathcal{L}$:

Parity:

$$\mathcal{L} = q D \psi^t C \gamma^5 \psi \mapsto q D (\gamma^0 \psi)^t C \gamma^5 \gamma^0 \psi =$$

$$= q D \psi^t \gamma^0 C \gamma^5 \gamma^0 \psi = q D \psi^t C \gamma^5 \psi$$

φ ok

charge conjugation:

$$\psi \mapsto i \gamma^2 \psi^* \qquad\qquad \psi \mapsto C \bar{\psi}^t$$

$$q D \psi^t C \gamma^5 \psi \mapsto q D (i \gamma^2 \psi^*)^t C \gamma^5 (i \gamma^2 \psi^*)$$

$$q D (C \bar{\psi}^t)^t C \gamma^5$$

$$= q D \psi^\dagger i \gamma^{2t} C \gamma^5 i \gamma^2 \psi^*$$

$\gamma^{2t} = -\gamma^2$

$$= + q D \psi^\dagger \gamma^2 C \gamma^5 \gamma^2 \psi^*$$

$$= q D \psi^\dagger \gamma^2 (i \gamma^2 \gamma^0) \gamma^5 \gamma^2 \psi^*$$

$$= -q D \psi^t C \gamma^5 \psi^* = -q \bar{\psi} C \gamma^5 \bar{\psi}^t$$

Ergo, the only possible way to fullfill C-invariance and hermiticity is:

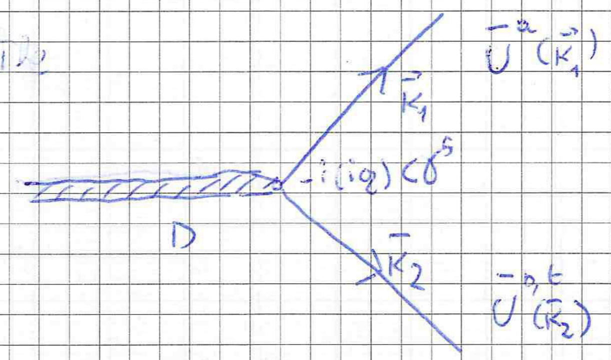
$$\boxed{\mathcal{L} = i q \left(\psi^t C \gamma^5 \psi - \bar{\psi} C \gamma^5 \bar{\psi}^t \right) D}$$

$\mathcal{L}^\dagger = \mathcal{L} \mapsto$ ok... $\qquad \mathcal{L}_C = \mathcal{L} \to$ ok...

The Lagrangian fullfill.. C, P and $\mathcal{L}^\dagger = \mathcal{L}$ is given by

$$\boxed{\mathcal{L} = i\underset{\substack{\downarrow \\ \text{real}}}{g}\, D \left(\psi^t C \gamma^5 \psi - \bar{\psi} C \gamma^5 \bar{\psi}^t \right)}$$

The



$$\boxed{-i\mathcal{M}^{ab} = -i(ig)\,\bar{u}^{(a)}_{(\vec{k}_1)}\, C\gamma^5\, \bar{u}^{(b),t}_{(\vec{k}_2)}}$$

The decay width is given by:

$$\Gamma_{D\to\psi\psi} = \frac{|\vec{k}_1|}{8\pi M_D^2}(4m^2)\sum_{a,b}\overline{\left|-i\mathcal{M}^{ab}\right|}^2 = \frac{|\vec{k}_1|}{8\pi M_D^2}(4m^2)\sum_{a,b} g^2 \left(\bar{u}^a_{(\vec{k}_1)} C\gamma^5 \bar{u}^{(b)t}_{(\vec{k}_2)}\right)^\dagger \left(\bar{u}^a_{(\vec{k}_1)} C\gamma^5 \bar{u}^{b,t}_{(\vec{k}_2)}\right)$$

$$\left(\bar{u}_{(\vec{k}_1)} C\gamma^5 \bar{u}^t_{(\vec{k}_2)}\right)^\dagger = \left[(u^\dagger_{(\vec{k}_2)}\gamma^0)^t\right]^\dagger (C\gamma^5)^\dagger (u^\dagger_{(\vec{k}_1)}\gamma^0)^\dagger = \left[\gamma^0 u^*_{(\vec{k}_2)}\right]^\dagger \gamma^5 C^\dagger \gamma^0 u(\vec{k}_1) = u^t_{(\vec{k}_2)}\gamma^0\gamma^5 C^\dagger \gamma^0 u_{(\vec{k}_1)}$$

$$= u^{(b)t}_{(\vec{k}_2)}(-\gamma^5 C)\, u^{(a)}_{(\vec{k}_1)} = u^{(b)t}_{(\vec{k}_2)}\, C\gamma^5\, u^a_{(\vec{k}_1)}$$

Doing the polarisation sums:

By performing the polarization sums:

$$\boxed{\Gamma = \frac{|\vec{k}_1|}{8\pi M_D^2}\, \mathrm{Tr}\left[(\not{k}_2+m)^t\, C\gamma^5 (\not{k}_1+m)\, C\gamma^5 \right]}$$

$$= \frac{|\vec{k}_1| g^2}{8\pi M_D^2}\left\{ \mathrm{Tr}\left[C^{-1}\not{k}_2^t C\, \gamma^5 \not{k}_1 \gamma^5 \right] + \mathrm{Tr}\left[m^2 C\gamma^5 C\gamma^5 \right] \right\}$$

$$= \frac{|\vec{k}_1| g^2}{8\pi M_D^2}\left\{ -\mathrm{Tr}\left[\not{k}_2\gamma^5 \not{k}_1 \gamma^5 \right] + m^2\, \mathrm{Tr}\left[C\gamma^5 C\gamma^5 \right] \right\}$$

$$= \frac{|\vec{k}_1| g^2}{8\pi M_D^2}\left\{ 4(k_1\cdot k_2) + m^2\, \mathrm{Tr}\left[-CC \right] \right\} = \frac{|\vec{k}_1|}{8\pi M_D^2}\left\{ 4(k_1 k_2) + 4m^2 \right\}$$

$$= \frac{|\vec{k}_1|}{8\pi M_D^2} g^2 \left\{ \cancel{4}^{\,2}\, \frac{M_D^2 - 2m^2}{\cancel{2}_{\,1}} + 4m^2 \right\} = \frac{|\vec{k}_1|}{4\pi M_D^2} M_D^2 g^2 = \frac{|\vec{k}_1|}{4\pi} g^2 .$$

1

$$Z_\mu \bar{\psi} \gamma^\mu \psi$$

$$-iM^{abc} = +ig\, \varepsilon^a_\mu(\vec{p})\, \bar{U}^b(\vec{k}_1)\gamma^\mu V^c(\vec{k}_2) \qquad (\bar{U}\gamma^\mu V)^\dagger = V^\dagger\gamma^0\gamma^\mu\gamma^0(U^\dagger\gamma^0)^\dagger = \bar{V}\gamma^\mu U$$

$$\sum_{a,b,c} g^2\, \bar{V}^c_\alpha(\vec{k}_2)\gamma^\mu_{\alpha\beta} U^b_\beta(\vec{k}_1)\, \varepsilon^a_\mu(\vec{p}) \cdot \bar{U}^b_\delta(\vec{k}_1)\gamma^\nu_{\delta\varepsilon} V^c_\varepsilon(\vec{k}_2)\, \varepsilon^a_\nu(\vec{p})$$

$$\boxed{= \frac{g^2}{4m^2} \mathrm{Tr}\left[(\not{k}_2 - m)\gamma^\mu(\not{k}_1 + m)\gamma^\nu\right]\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M_Z^2}\right).}$$

$$= \frac{g^2}{4m^2}\left(\mathrm{Tr}\left[\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu\right] - m^2 4g^{\mu\nu}\right)\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M_Z^2}\right).$$

$$= \frac{g^2}{4m^2}\left(k_{2,\rho}k_{1,\sigma}\mathrm{Tr}\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right] - 4m^2 g^{\mu\nu}\right)\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M_Z^2}\right) =$$

2

$$= \frac{g^2}{4m^2}\left(K_{2\rho}K_{1,\sigma}\left(4g^{\rho\mu}g^{\sigma\nu} - 4g^{\rho\sigma}g^{\mu\nu} + 4g^{\rho\nu}g^{\mu\sigma}\right) - 4m^2 g^{\mu\nu}\right)\left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{M_Z^2}\right) =$$

$$= \frac{g^2}{4m^2}\left(4K_2^\mu K_1^\nu - 4(K_1K_2)g^{\mu\nu} + 4K_2^\nu K_1^\mu - 4m^2 g^{\mu\nu}\right)\left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{M_Z^2}\right) =$$

$$= \frac{g^2}{4m^2}\left(-4(K_1K_2) + 4\frac{(K_1\cdot P)(K_2\cdot P)}{M_Z^2} - 4(K_1K_2)\cdot\left(-4 + \frac{P^2}{M_Z^2}\right) - 4(K_1K_2) + 4\frac{(K_1\cdot P)(K_2\cdot P)}{M_Z^2} - 4m^2\left(-4 + \frac{P^2}{M_Z^2}\right)\right)$$

$$= \frac{g^2}{4m^2}\left[\underbrace{-4(K_1\cdot K_2) + 16(K_1K_2) - 4(K_1K_2) + 16m^2}_{\text{for } -g^{\mu\nu}} + \frac{4}{M_Z^2}(K_1\cdot P)(K_2P) - \frac{4P^2}{M_Z^2} + \frac{4}{M_Z^2}(K_1P)(K_2P) - 4m^2\frac{P^2}{M_Z^2}\right]$$

$K_1 \cdot P = ?$ $\quad P = K_1 + K_2 \quad P - K_1 = K_2$

$P^2 + K_1^2 - 2PK_1 = K_2^2$

$P^2 + \cancel{m^2} - 2PK_1 = \cancel{m^2} \rightarrow P^2 = 2PK_1 \rightarrow PK_1 = \frac{P^2}{2}$

$$\frac{g^2}{4m^2}\left[8(K_1K_2)+16m^2+\left(\frac{4}{M_Z^2}\frac{P^2}{2}\frac{P^2}{2}-\frac{4P^2(K_1K_2)}{M_Z^2}+\frac{4}{M_Z^2}\frac{P^2}{2}\frac{P^2}{2}-4m^2\frac{P^2}{M_Z^2}\right)\right]=$$

$$=\frac{g^2}{4m^2}\left[8(K_1K_2)+16m^2+\left[M_Z^2-4(K_1K_2)+M_Z^2-4m^2\right]\right]$$

$$=\frac{g^2}{4m^2}\left[8(K_1K_2)+16m^2+\underbrace{\left(M_Z^2-4\left(\frac{M_Z^2-2m^2}{2}\right)+M_Z^2-4m^2\right)}_{}\right]$$

$$M_Z^2-2M_Z^2+4m^2+M_Z^2-4m^2=0 \; !!!$$

On shell it is transversal...

The longitudinal mode does not contribute !!!!!

$$=\frac{g^2}{4m^2}\left[8(K_1K_2)+16m^2\right]=\frac{g^2}{4m^2}\left[8\,\frac{M_Z^2-2m^2}{2}+16m^2\right]=\frac{g^2}{4m^2}\cdot\left[4M_Z^2-8m^2+16m^2\right]=\frac{g^2}{4m^2}\left[4M_Z^2+8m^2\right]=\frac{g^2}{m^2}\left[M_Z^2+2m^2\right]$$

Check:

$$\frac{g^2}{4m^2}\, \mathrm{Tr}\left[(K_2 - m)\gamma^\mu (K_1 + m)\gamma^\nu \right] \left(-g_{\mu\nu} \right) =$$

$$= \frac{g^2}{4m^2}\left[1\, K_2^\mu K_1^\nu - 1\,(K_1 K_2)\, g^{\mu\nu} + 1\, K_2^\nu K_1^\mu - 4m^2 g^{\mu\nu} \right] \left[-g_{\mu\nu} \right] =$$

$$= \frac{g^2}{4m^2}\left[-1\,(K_1 K_2) + 16\,(K_1 K_2) - 4\,(K_1 K_2) + 16m^2 \right] \;!!$$

5

NB: The longit. term is (outofshell):

$$L = 2\frac{P^2}{M_Z^2}P^2 - \frac{4P^2}{M_Z^2}\frac{(P^2-2m^2)}{2} - 4m^2\frac{P^2}{M_Z^2} = 2\frac{P^4}{M_Z^2} - 2\frac{P^4}{M_Z^2} + \frac{4m^2P^2}{M_Z^2} - \frac{4m^2P^2}{M_Z^2} = 0$$ is still zero ... also outofshell.

$$\Gamma = \frac{\sqrt{\frac{M_Z^2}{4}-m^2}}{2 \cdot 8\pi M_Z^2} 4 g^2 \left(M_Z^2 + 2m^2\right) = \frac{\sqrt{\frac{M_Z^2}{4}-m^2}}{2\pi M_Z^2} g^2 \left(M_Z^2 + 2m^2\right) \qquad \Gamma(x) = \frac{\sqrt{\frac{x^2}{4}-m^2}}{2\pi x^2} g^2 (x^2 + 2m^2)$$

For large x: $\Gamma \sim \frac{x}{x^2} x^2 \sim x$!!! Ergo, $\rho(x) \simeq \frac{1}{x}$.

ex 1

$$E = \sqrt{p^2+m^2}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$, that is $\rightarrow$ we need v!

$$E = \sqrt{p^2+m^2} = \frac{m}{\sqrt{1-v^2}} \rightarrow E\sqrt{1-v^2} = m$$

$$\rightarrow \sqrt{1-v^2} = \frac{m}{E} \rightarrow 1-v^2 = \frac{m^2}{E^2}$$

$$\boxed{v = \sqrt{1-\frac{m^2}{E^2}}}$$

Or, even easier:

$$E = \gamma m \longmapsto \gamma = \frac{E}{m}$$

$$e^{-t_{1/2}/\tau} = \frac{1}{2}$$

$$-t_{1/2}/\tau = -\ln 2$$

$$t_{1/2} = \ln 2 \cdot \tau$$

Then, adding:

$$t_{1/2} = (\ln 2)\cdot \tau$$

$$t_{1/2}^{\text{our frame}} = \gamma \cdot \ln 2 \cdot \tau_{\text{Neutron}};$$

$$\boxed{\ln 2 = 0.693}$$

$$t_{1/2}^{\text{in our frame}} = \gamma\, t_{1/2}$$

$$\begin{cases} t_{1/2} = 611.0 \pm 1.0 \text{ sec} \\ \tau = 881.5 \pm 1.5 \text{ s} \end{cases}$$

$$\boxed{t_{1/2}^{\text{in our frame}} = \frac{E}{m}\, t_{1/2}}$$

Then:

1) $\frac{E}{m} = \frac{10^3}{10^3} \longrightarrow \gamma = 1 \Rightarrow t_{1/2} \simeq 1000 \text{ min}$

2) $\frac{E}{m} = 100 \cdot 10^3 = 10^5 \qquad t_{1/2} = 10^6 \text{ min}$

3) $\frac{E}{m} = \frac{10^{18} \text{ eV}}{1 \text{ GeV}} = \frac{10^{18} \cdot 10^{-9} \text{ GeV}}{1 \text{ GeV}} \simeq 10^9 \text{ min}$

Note, how many km can the n do before it decays?

$S = c \cdot t = 30$

$$S_{anni\ luce} = \frac{c \cdot t}{c\, t_{anno}} = \frac{t}{t_{anno}}$$

1) 0.001 a.l.

2) ~ 2 anni

3) ~ 1900 anni

ex 2

$$P = K_1 + K_2$$

$$P - K_1 = K_2 \quad ; \quad P = (M, \vec{0})$$

Square:

$$M^2 + m_1^2 - 2E_1 M = m_2^2 \qquad E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

similarly from

$$P - K_2 = K_1 \rightarrow \qquad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}$$

Kinetic energies:

$$\begin{cases} T_1 = E_1 - m_1 = \dfrac{M^2 + m_1^2 - m_2^2 - 2Mm_1}{2M} = \dfrac{(M - m_1)^2 - m_2^2}{2M} \\[2ex] T_2 = E_2 - m_2 = \dfrac{(M - m_2)^2 - m_1^2}{2M} \end{cases}$$

Then, if $m_1 \ll m_2 \simeq M$ we get:

$$T_1 = \frac{(M-m_1)^2 - m_2^2}{2M} \simeq \frac{M^2 - m_2^2}{2M} \simeq \frac{(M-m_2)(M+m_2)}{2M}$$

$$\simeq M - m_2$$

$T_2 \simeq 0$!!!

$$\frac{T_2}{T_1} = \frac{(M-m_2)^2 - m_1^2}{(M-m_1)^2 - m_2^2}$$

Suppose that $m_1 \simeq 0$

$$\frac{T_2}{T_1} = \frac{(M-m_2)^2}{M^2 - m_2^2} = \frac{M-m_2}{M+m_2} \ll 1 \quad \text{if } m_2 \simeq M.$$

β decay:

electron

$$P = K_1 + K_2 + K_3$$

$$P - K_3 = K_1 + K_2$$

$$M_n^2 + m_e^2 - 2E_3 M_n = M_p^2 + m_\nu^2 + 2K_1 \cdot K_2$$

$$M_n^2 + m_e^2 - 2E_3 M_n = M_p^2 + m_\nu^2 + 2E_p m_\nu$$

m_ν is very small...

$E_p \simeq M_p$

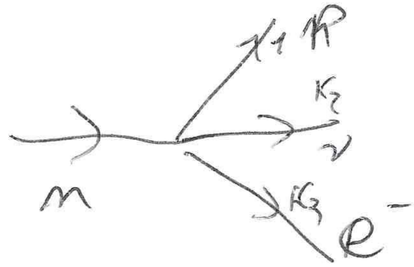
"almost good..."

Note:

$$P - K_2 = K_1 + K_3$$

$$(M - m_\nu)^2 = M_p^2 + m_e^2 + 2(E_p E_e - \vec{k}^2)$$

But the best trick is to form:



$$q = p - K_2 = (M_n - m_\nu, \vec{0})$$

$$q - K_3 = K_1$$

Then square:

$$\boxed{E_{el} = \frac{(M_n - m_\nu)^2 + m_e^2 - M_p^2}{2(M_n - m_\nu)} \quad ; \quad T_{el} = E_{el} - m_e}$$

$$\left(\simeq \frac{M_n^2 + m_\nu^2 - 2M_n m_\nu + m_e^2 - M_p^2}{2M_n} \right)$$

$$T_{el} = E_{el} - m_e = \frac{M_n^2 + m_\nu^2 - 2M_n m_\nu + m_e^2 - M_p^2 - 2M_n m_e}{2(M_n - m_\nu)}$$

$$\simeq (M_n \, m_\nu^2 - m_e - m_p$$

Ex 3

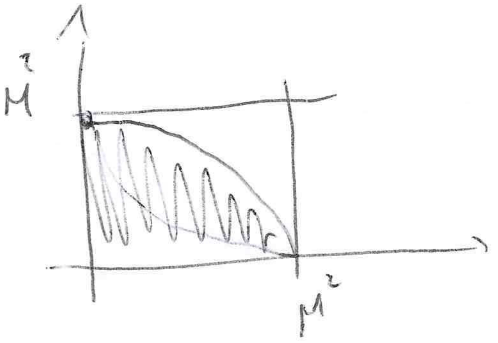
1) $m_{12}^2 + m_{23}^2 + m_{13}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$

in fact:

$(P-K_3)^2 + (P-K_1)^2 + (P-K_2)^2 =$

$$= 3M^2 + m_1^2 + m_2^2 + m_3^2 - 2M(\underbrace{E_1 + E_2 + E_3}_{M}) = M^2 + m_1^2 + m_2^2 + m_3^2$$

2) $m_1 = m_2 = m_3 = 0$!



$$(m_{23}^2)_{Max} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2}} - \sqrt{E_3^*}\right)^2$$

$E_2^* = \frac{m_{12}}{2}$

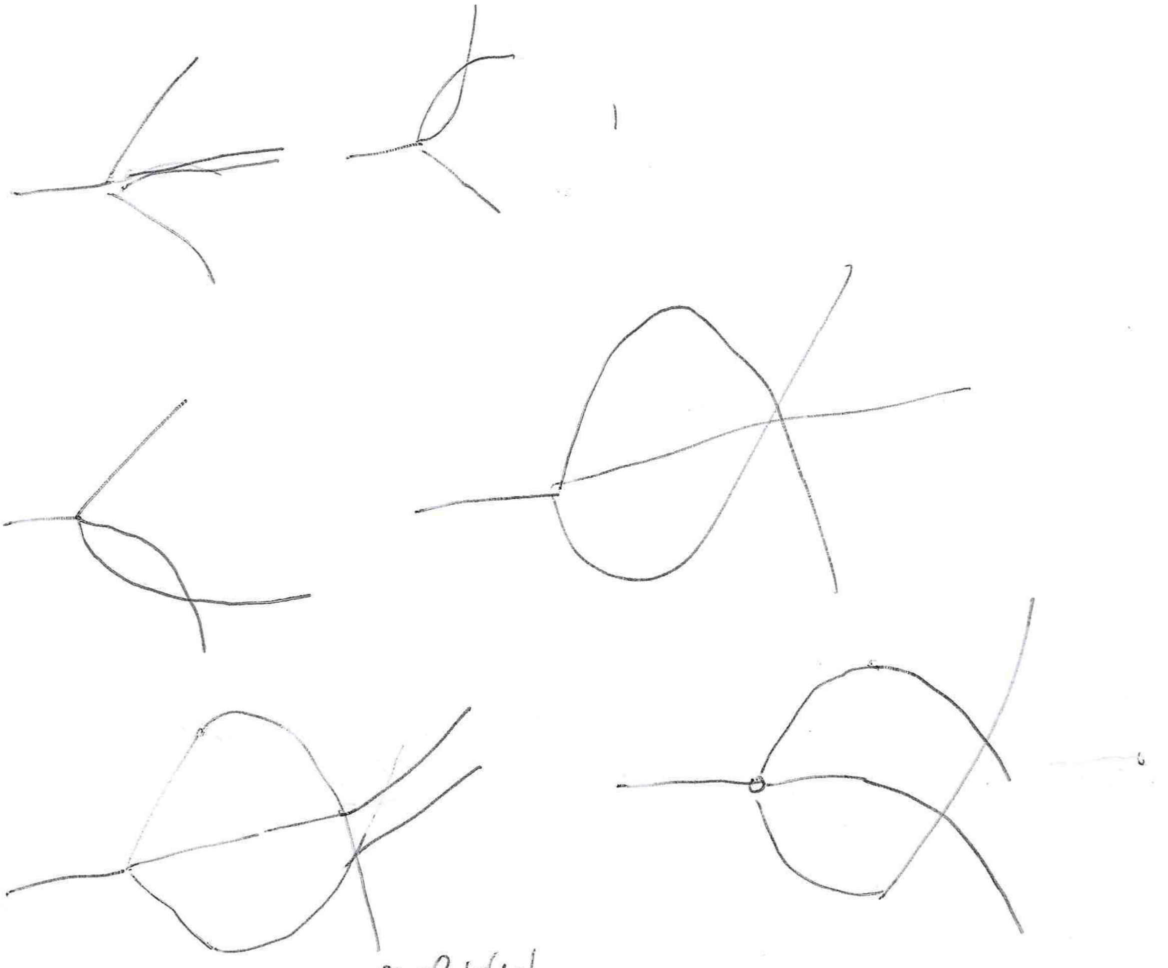
$E_3^* = \frac{M^2 - m_{12}^2}{2m_{12}}$

$$= \cancel{\left(\frac{m_{12}}{2} + \frac{M^2 - m_{12}^2}{2m_{12}}\right)^2 - \left(\sqrt{} - \right.}$$

$$= (E_2^* + E_3^*)^2 - (E_2^* - E_3^*)^2 = 4E_1^* E_3^* =$$

$$= 4\,\frac{m_{12}}{2}\,\frac{M^2 - m_{12}^2}{2m_{12}} = M^2 - m_{12}^2$$

3) 3 identical particles:



equal ident.
6 amplitudes !!!

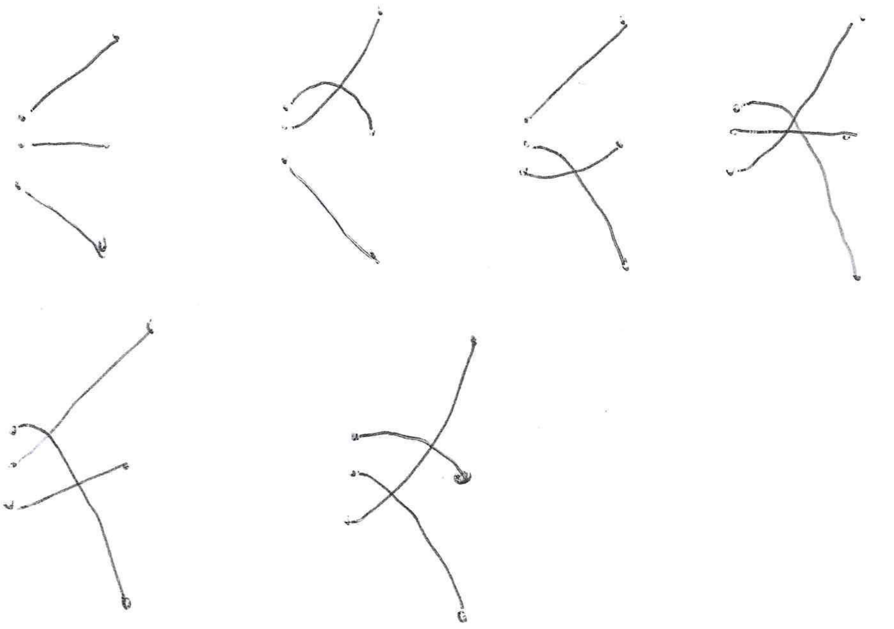
$|k_1 k_2 k_3\rangle \to$ 6 permutations...

$\frac{1}{\sqrt{6}}$ needed to normalize.

$\left(\frac{1}{\sqrt{6}}\right)^2 \cdot 6^2 = \frac{36}{6} = 6$!!! overall needed factor

$\Gamma_{S\to\varphi\varphi\varphi} = 6\,\Gamma_{S\to\varphi_1\varphi_2\varphi_3}$ where $m = m_1 = m_2 = m_3$!

Better shown:



1) $$\Gamma_{P_1 \to S\varphi_3} = \frac{|\vec{K}_3|}{8\pi M_P^2}\,\alpha^2$$

$|\vec{K}_3|$ can be explicitly determined as:

$$P = K_S + K_3$$

$$P - K_3 = K_S$$

$$M_P^2 + 0 - 2E_3 \cdot M_P = M_S^2$$

But $E_3 = |\vec{K}_3|$

so:

$$\boxed{|\vec{K}_3| = \frac{M_P^2 - M_S^2}{2M_P}}$$

In Book:

$$\frac{1}{2M_P}\sqrt{M_P^4 + (M_S^2 - m_3^2)^2 - 2M_P^2(M_S^2 + m_3^2)} = \frac{1}{2M_P}\sqrt{M_P^4 + M_S^4 - 2M_P^2 M_S^2}$$

$$= \frac{M_P^2 - M_S^2}{2M_P} \qquad \text{qed}$$

2)

$$\Gamma_{S \to \varphi_1 \varphi_2} = \frac{|\vec{K}_1|}{8\pi M_S^2} g^2$$

$$|\vec{K}_1| = \frac{1}{2M_S} \sqrt{M_S^4 + (m_1^2 - m_2^2)^2 - 2M_S^2(m_1^2 + m_2^2)}$$

N is determined by imposing the normalization

$$N \int_0^\infty dx\, d_S(x) = 1$$

3)

$$\Gamma_{P \to \varphi_1 \varphi_2 \varphi_3} = \int_0^\infty \Gamma_{P \to S\varphi_3}(x)\, d_S(x)\, dx$$

where:

$$\Gamma_{P \to S\varphi_3}(x) = \frac{M_P^2 - x^2}{2M_P} \cdot \frac{1}{8\pi M_P^2} \alpha^2\, \theta(M_P - x)$$

Ergo:

$$\Gamma_{P\mapsto\varphi_1\varphi_2\varphi_3} = \int_{m_1+m_2}^{M_P} \left(\frac{M_P^2 - x^2}{2M_P} \frac{1}{8\pi M_P^2} \right) \frac{N}{(x^2-M_S^2)^2 + \underbrace{(M_S\Gamma_S(x=M_S))^2}_{\text{this is an approx.}}}$$

4) when g is very small

$$d_S(x) \simeq \delta(x - M_S)$$

Ergo:

$$\boxed{\Gamma_{P\mapsto\varphi_1\varphi_2\varphi_3} = \Gamma_{P\mapsto S\varphi_3}(x = M_S)}.$$

2) When a particle is emitted in the 1-3 channel it was not the best choice of variable..

one has in the amplitude

$$\frac{1}{m_{13}^2 - M_S^2 + i(\Gamma_S M_S)}$$

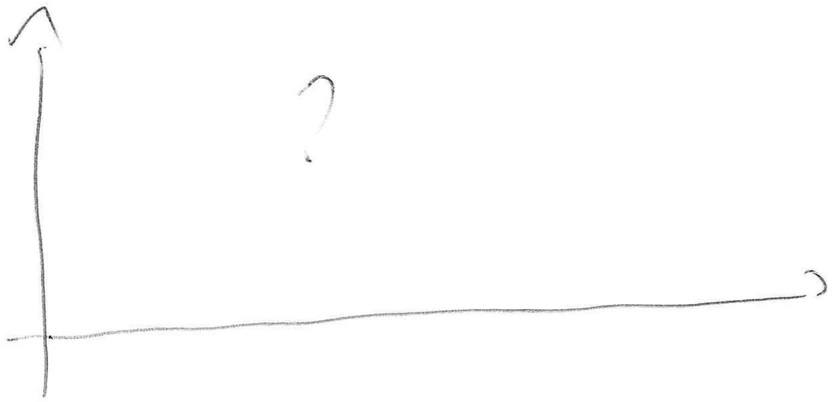
Then squared

$$\frac{1}{(m_{13}^2 - M_S^2) + (\Gamma_S M_S)^2}$$

$$m_{12}^2 + m_{23}^2 + m_{13}^2 = m_1^2 + m_2^2 + m_3^2 + M_P^2$$

Ergo:

$$\boxed{m_{13}^2 = m_1^2 + m_2^2 + m_3^2 + M_P^2 - m_{12}^2 - m_{23}^2}$$



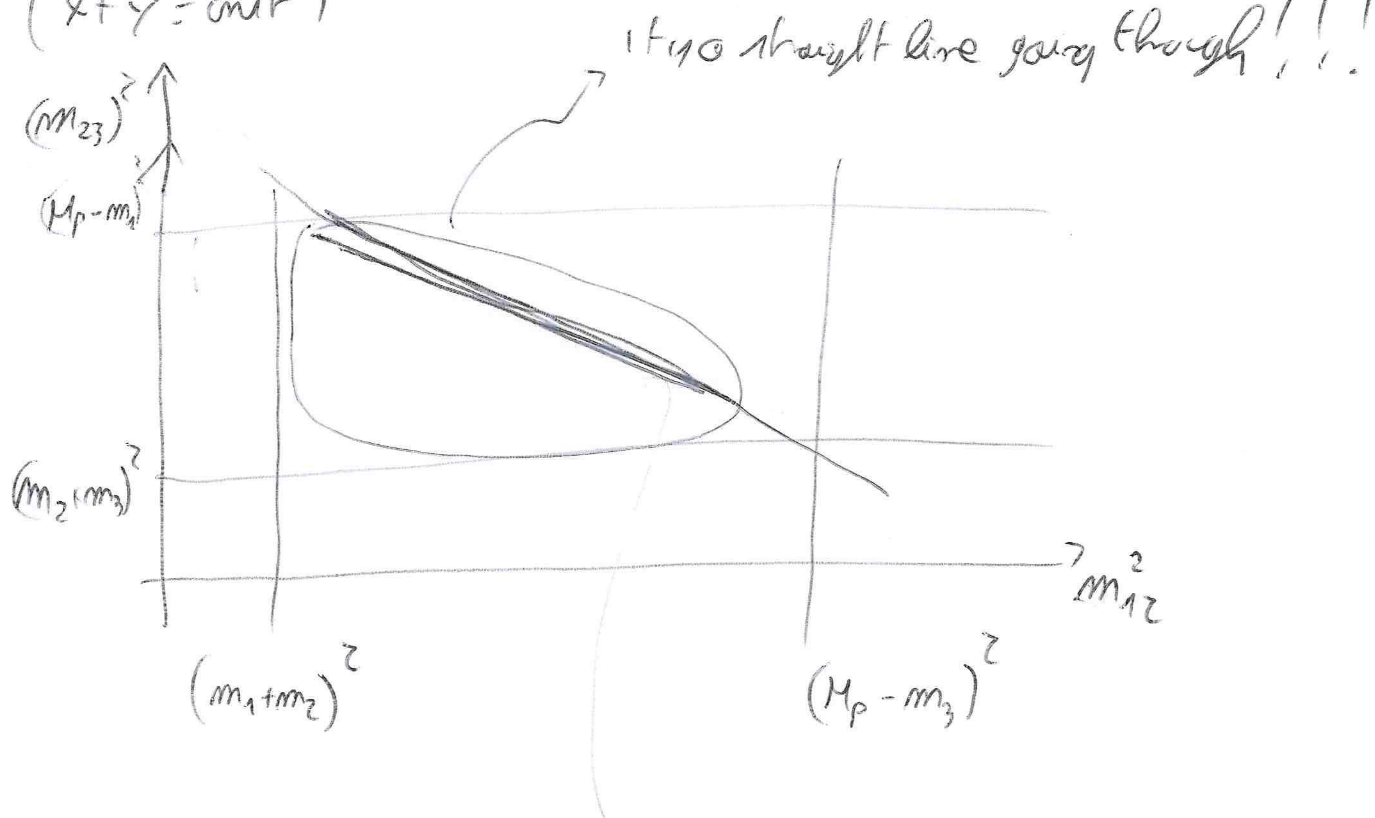
This means

$$\frac{1}{(\text{const} - m_{12}^2 - m_{23}^2 - M_S^2)^2 + (\Gamma_S M_S)^2}$$

The enhancement is for

$$m_1^2 + m_2^2 + m_3^2 + M_P^2 - M_S^2 = m_{12}^2 + m_{23}^2$$

$(x + y = \text{const})$



$m_{12} = (m_1 + m_2)^2$

Sol 11

1) $$V = \frac{\lambda}{4}\left(\sigma^2+\pi^2-F^2\right)^2 - \varepsilon\sigma$$

$$\partial_\sigma V = \frac{\lambda}{4}\cdot 2\sigma\cdot 2\left(\sigma^2+\pi^2-F^2\right) - \varepsilon = 0$$

$$\pi = 0, \quad \sigma = \phi$$

$$\boxed{\lambda\phi\left(\phi^2-F^2\right) = \varepsilon}$$

Indeed 3 solutions

2) $$\sigma \mapsto \phi + \sigma$$

$$V = \frac{\lambda}{4}\left(\phi^2+\sigma^2+2\sigma\phi+\pi^2-F^2\right)^2 - \varepsilon(\phi+\sigma)$$

$$\frac{M_\pi^2}{2}\pi^2 = \frac{\lambda}{4}\cdot 2\left(\phi^2-F^2\right)\pi^2$$

$$M_\pi^2 = \lambda\left(\phi^2-F^2\right) = \varepsilon/\phi$$

$$\varepsilon \mapsto 0 \;\dots\; M_\pi^2 \mapsto 0$$

$$\varepsilon \propto m_q \qquad M_\pi^2 \sim m_q \;!!!$$

$$\frac{M_\sigma^2}{2}\sigma^2 = \frac{\lambda}{4}\cdot 4\sigma^2\phi^2 + \frac{\lambda}{4}\left(\phi^2-F^2\right)2\sigma^2$$

$$\frac{M_\sigma^2}{2} = \lambda\phi^2 + \frac{M_\pi^2}{2}$$

$$\boxed{M_\sigma^2 = 2\lambda\phi^2 + M_\pi^2}$$

$\phi = 92.4$, $M_\sigma = 1\,\text{GeV}$, $M_\pi = 139\,\text{MeV}$

$$\begin{cases} M_\pi^2 = \varepsilon/\phi \\ \varepsilon = M_\pi^2 \phi = 0.00178\ \text{GeV}^3 \end{cases}$$

$$\lambda = \frac{M_\sigma^2 - M_\pi^2}{2\phi^2} = 57.43$$

$$\lambda(\phi^2 - F^2) = M_\pi^2$$

$$\lambda\phi^2 - \lambda F^2 = M_\pi^2 \rightarrow \lambda F^2 = \lambda\phi^2 - M_\pi^2$$

$$F^2 = \frac{\lambda\phi^2 - M_\pi^2}{\lambda}$$

$$F = \sqrt{\frac{\lambda\phi^2 - M_\pi^2}{\lambda}} \simeq 0.090$$

3) The decay term is given by

$$\frac{\lambda}{4}\cdot 2\,(2\sigma\phi)\cdot\pi^2 = \frac{\lambda}{2}\phi\,(\sigma\pi^2)$$

$$\Gamma = \frac{\sqrt{\frac{M_\sigma^2}{4} - M_\pi^2}}{8\pi M_\sigma^2}\cdot 2\,(\lambda\phi)^2$$

$\Gamma \simeq 1$ GeV

(with 3 pions it would be 3 GeV)

(remember the discussion done 2 weeks ago).

Nb: for $\varepsilon = 0 \rightarrow M_\pi = 0$

$$M_\sigma^2 = 2\lambda f_\pi^2$$

$$\Gamma = \frac{\frac{M_\sigma}{8}}{8\pi M_\sigma^2}\cdot 2\cdot\left(\frac{M_\sigma^2}{2 f_\pi^2} f_\pi\right)^2 = \frac{M_\sigma}{8\pi M_\sigma^2}\cdot\frac{M_\sigma^{2}}{4 f_\pi^2} = \frac{M_\sigma^3}{32\pi f_\pi^2}\ !!!$$

ex2

$$\begin{cases} \sigma = \rho\cos\varphi \\ \pi = \rho\sin\varphi \end{cases} \qquad \sigma^2+\pi^2 = \rho^2$$

φ must be dimensionless…

$$\begin{cases} \partial_\mu\sigma = \partial_\mu\rho\cos\varphi + \rho\cdot\partial_\mu\varphi(-\sin\varphi) \\ \partial_\mu\pi = \partial_\mu\rho\cdot\sin\varphi + \rho\,\partial_\mu\varphi(\cos\varphi) \end{cases} \qquad \varphi = \eta/F$$

$$\frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2$$

$$= \frac{1}{2}(\partial_\mu\rho)^2\cos^2\varphi + \frac{1}{2}\rho^2(\partial_\mu\varphi)^2\sin^2\varphi + \cancel{2(\partial_\mu\rho)\cos\varphi\,\rho(\partial^\mu\varphi)(-\sin\varphi)}$$

$$+\frac{1}{2}(\partial_\mu\rho)^2\sin^2\varphi + \frac{1}{2}\rho^2(\partial_\mu\varphi)^2\sin^2\varphi + \cancel{2(\partial_\mu\rho)\sin\varphi\,\rho(\partial^\mu\varphi)\cos\varphi}$$

$$= \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\rho^2(\partial_\mu\varphi)^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\rho^2(\partial_\mu\varphi)^2 - \frac{\lambda}{4}(\rho^2-F^2)^2. \qquad (\varepsilon=0).$$

$$\rho \mapsto F+\rho \qquad\qquad M_\rho = M_\sigma$$

$$\mathcal{L}_{int} = \frac{1}{8}\cdot 8F\rho(\partial_\mu\varphi)^2$$

Ergo:

$$\cancel{\Gamma(\rho\to\varphi\varphi) = \frac{M_\rho^3}{8\pi v_\sigma^2}\,2\cdot\Big[}$$

N.B.

ρ has dim. energy $\qquad \rho \mapsto F+\rho$

φ is dimensionless!!!

$\partial_\mu \rho \;, \partial_\mu \varphi$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \underbrace{\frac{1}{2} F^2 (\partial_\mu \varphi)^2}_{\text{no can. norm.}} + \frac{F}{2}\rho(\partial_\mu \varphi)^2 - \frac{\lambda}{4}(\cancel{F^2} + \rho^2 - 2F\rho - \cancel{F^2})^2$$

$\rho \mapsto \varphi\varphi$

$\varphi \mapsto F\varphi/F \qquad (\varphi = \eta/F)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}(\partial_\mu \eta)^2 + \frac{\rho}{F}(\partial_\mu \eta)^2 - \frac{\lambda}{4}(\rho^4 + 4F^2\rho^2 - 4F\rho^3)$$

$M_\rho^2 = 2\lambda F^2 = M_\sigma^2$

$M_\eta = 0$

$$\Gamma_{\rho \mapsto \eta\eta} = \frac{M_\rho/2}{8\pi M_\rho^2}\, 2 \cdot \left[\frac{1}{F} \cdot (k_1 k_2)\right]^2$$

$$= \frac{M_\rho}{8\pi M_\rho^2}\left[\frac{1}{f_\pi}\cdot\frac{M_\rho^2}{2}\right]^2 = \frac{M_\rho}{8\pi M_\rho^2}\frac{1}{f_\pi^2}\frac{M_\rho^4}{4} = \frac{M_\rho^3}{32\pi f_\pi^2}\ !$$