DECAYS IN QFT - WS 2012/2013

Sheet 3 **9/11/2012**

Exercise 1: Decay width (15 points = 4 + 3 + 4 + 4)

A system is described by the Hamiltonian

$$H = H_0 + H_1 \tag{1}$$

with

$$H_0 = M_0 \left|S\right\rangle \left\langle S\right| + \sum_{k=2\pi n/L} \omega(k) \left|k\right\rangle \left\langle k\right| \ , \tag{2}$$

$$H_1 = g \sum_{k=2\pi n/L} \frac{f(k)}{\sqrt{L}} \left(\left|S\right\rangle \left\langle k\right| + \left|k\right\rangle \left\langle S\right|\right) \ . \tag{3}$$

1. Make the following Ansatz:

$$\left|s(t)\right\rangle = a(t) e^{-iM_0 t} \left|S\right\rangle + \sum_{k=2\pi n/L} r_k(t) e^{-i\omega(k)t} \left|k\right\rangle \ . \tag{4}$$

   (Again, we have factorized out the phases. This is the so-called interaction picture).

   Determine the first-order differential equations for $a(t)$ and $r_q(t)$ which must be fulfilled, in order that the state $\left|s(t)\right\rangle$ is a solution of the Schrödinger equation

$$i\frac{d}{dt}\left|s(t)\right\rangle = H\left|s(t)\right\rangle \ . \tag{5}$$

   Hint: The first-order differential equation for $r_q(t)$ should be of the form

$$\dot{r}_q \propto a(t) e^{i(\omega(q)-M_0)} \left\langle q\left|H_1\right|S\right\rangle \ . \tag{6}$$

2. Suppose that for $t = -T/2$ the system is described by $\left|S\right\rangle$, that is $a(-T/2) = 1$. The aim is to evaluate the probability that the system is in the state $\left|q\right\rangle$ at $t = T/2$. Assume that T is not too large, so that in Eq. (6) we can set $a(t) \simeq 1$. Then, Eq. (6) can be easily solved. To this end, one can recognize the emergence of the δ-function $\delta(\omega(q) - M_0) = \int_{-T/2}^{T/2} \frac{dt}{2\pi} e^{i(\omega(q)-M_0)}$. (The latter is now valid for T large enough.)

3. The probability that the state has decayed at $t = T/2$ (that is, it is not in the state $\left|S\right\rangle$) is the sum of all the channels:

$$1 - p(t = T/2) = \sum_{q=2\pi n/L} \left|r_q(t = T/2)\right|^2 \ . \tag{7}$$

   One sees that the 'awful quantity' $\left[\delta(\omega(q) - M_0)\right]^2$ appears. Usual Fermi-trick:

$$\left[\delta(\omega(q) - M_0)\right]^2 = \left[\delta(\omega(q) - M_0)\right]\delta(0) \equiv \left[\delta(\omega(q) - M_0)\right] T/2\pi \ . \tag{8}$$

   Thus, the expression (7) takes the form

$$1 - p(t) = \Gamma T \tag{9}$$

   where Γ is the decay width. Determine Γ. As the very last step, perform the continuos limit $L \to \infty$.

4. Determine Γ for the specific choice $\omega(q) = 2\sqrt{q^2 + m^2}$.

Exercise 2: Recall of complex analysis (10 points = 4+ 3 + 3)

1. Consider the complex function
$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} \tag{10}$$
Determine the poles and the residues of $f(z)$. Calculate $\int_{-\infty}^{+\infty} f(x)dx$ using the residue calculus.
2. Evaluate the principal part of the integral
$$\int_a^b \frac{dx}{x - x_0} \tag{11}$$
whereas $a < x_0 < b$.
3. Evaluate the principal part of the integral
$$\int_a^b \frac{f(x)dx}{x - x_0} \tag{12}$$
with $f(x) = (1 + x^2)^{-1}$.

Exercise 3: A bomb-like setup (5 points)

A photon can have two polarizations, $|H\rangle$ and $|V\rangle$. Consider the filter F which transforms the photon polarization as it follows:
$$|H\rangle \to \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \ , \ |V\rangle \to \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \ . \tag{13}$$
The setup of the system is like this: a photon $|H\rangle$ comes from the left along the x-axis, it goes through a filter of the type F placed at $x = 0$ and, later on at $x = L > 0$, to a second filter of the same type F. Then, the photon goes into a detector which measures its polarization. What will it measure?
Now suppose to put at $x = L/2$ a bomb, which is activated by a photon detector which only reveals a photon when the polarization is $|H\rangle$ (and, then the photon gets absorbed). Conversely, if the photon is in the state $|V\rangle$, it simply goes through the detector of the bomb and nothing happens. Describe this system.
Suppose in the end to have many equal bombs of the type described above, which are mixed with fake bombs. The fake bomb has also a fake detector, which leaves both polarizations $|H\rangle$ and $|V\rangle$ go through undisturbed. How can I reveal if the bomb is true or not?