30. 11. 2012

Lecture: Decays in QFT

- main results needed from complex analysis:
  - complex differentiation: $f: \Omega \to \mathbb{C}$, has a derivative in $a \in \Omega$ if the limit
    $$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \text{ exists}$$
  - holomorphic function: $f: \Omega \to \mathbb{C}$ is called holomorphic if it is differentiable in every $a \in \Omega$ with $z \mapsto f'(z)$, a continuous function;

    in fact it is holomorphic if the Cauchy-Riemann differential equations are fullfilled for $f: z = x+iy \mapsto u(x,y) + i v(x,y)$,
    $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

    examples: $f(z) = e^z$, $f(z) = \sin z$ $\quad (\Omega \equiv \mathbb{C})$

    not holomorphic in any point is $f(z) = |z|$

- infinite power series: if a function f can be defined in an open subset $\Omega \subset \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, then f is infinitely differentiable

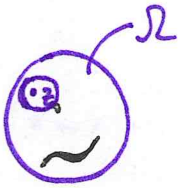
- analytic function: $f: \Omega \to \mathbb{C}$ is called <u>analytic</u> if for all $z_0 \in \Omega$ the power series $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges in a neighborhood of z_0; in this case it is infinitely differentiable

$\Rightarrow$ one main point to show in complex analysis is the fact, that a holomorphic function is analytic and vice versa...

$\Rightarrow$ one should not make the mistake to think that holomorphic functions aren't special; e.g. a holomorphic function is automatically infinitely differentiable, which is not true for real numbers...

- identity theorem (for holomorphic functions):

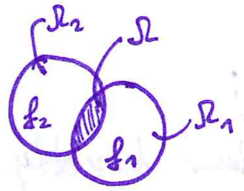
given two functions f_1 and f_2 both holomorphic on Ω, if $f_1(z) = f_2(z)$ on some line segment $U \subset \Omega$, then $f_1(z) = f_2(z)$ on the whole domain Ω



$\Rightarrow$ a holomorphic function on Ω is completely determined by its values on a neighborhood / line segment in Ω

$\Rightarrow$ real functions, if the holomorphic extension to complex numbers is possible, can be <u>uniquely</u> continued to the complex numbers

- analytic continuation: two holomorphic functions f_1 and f_2 each defined on domains $\Omega_1, \Omega_2 \subset \mathbb{C}$ such that $\Omega = \Omega_1 \cap \Omega_2$ with $f_1(z) = f_2(z)\ \forall z \in \Omega$; then f_1 and f_2 are each analytic in their domains:

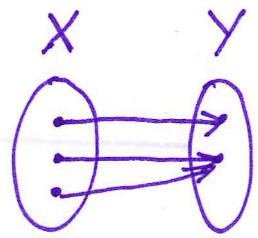


$$f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n \quad , \quad f_2(z) = \sum_{n=0}^{\infty} b_n (z - z_2)^n$$

$\Rightarrow$ either of the two expressions is valid in Ω and so is the series representation of <u>one and the same analytic function</u> f around z_1 and z_2, respectively

- multi-valued complex functions:

  - for real numbers one can define a function f as a mapping $X \rightarrow Y$ such that each input value $x \in X$ needs to be associated with exactly one output value $y \in Y$



  - a classical <u>multi-valued</u> function is for example $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, $f(x) =$

$\rightarrow$ modify: $f_+ : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $f_+(x) = +\sqrt{x}$
$f_- : \mathbb{R}^+ \rightarrow \mathbb{R}^-$, $f_-(x) = -\sqrt{x}$

- for complex numbers we are confronted with a new kind of multi-valued character:

$$f: \mathbb{C} \to \mathbb{C}, \quad f(z) = \sqrt{z} = \omega$$
$$= \sqrt{\rho}\, e^{i\frac{\varphi}{2}}, \quad \varphi \in (-\pi, \pi]$$

this is not single-valued because (→ Fig. 2.1); on the other hand we can investigate its behaviour directly on the negative real axis:

$$\lim_{\varepsilon \to 0^+} f(-\rho + i\varepsilon) = \sqrt{\rho}\, e^{i\frac{\pi}{2}} = i\sqrt{\rho}$$

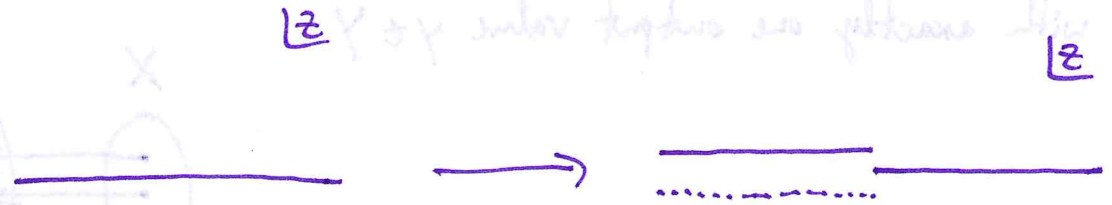
$$\lim_{\varepsilon \to 0^+} f(-\rho - i\varepsilon) = \sqrt{\rho}\, e^{-i\frac{\pi}{2}} = -i\sqrt{\rho}$$

→ can we manipulate f such that it becomes single-valued?

- we mathematical scissor cut along neg. real axis:

$$f: \mathbb{C} \to \mathbb{C} \setminus \{ z \in \mathbb{R} : z < 0 \}, \quad f(z) = \sqrt{z} = \omega$$

(→ principal branch)



it is now especially not possible to cross this horizone, called branch cut

- another example is the complex logarithm,

$$f : \mathbb{C} \to \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$$

which has also a branch cut on neg. real axis (including origin); note any arbitrary line can be chosen, only the branch points are the independent objects (the circling around said points reveals a multi-valued character (→ tipp for the exercise); on the other hand:

$$\lim_{\varepsilon \to 0^+} f(-\rho + i\varepsilon) = \ln \rho e^{i\pi} = \ln \rho + i\pi$$

$$\lim_{\varepsilon \to 0^+} f(-\rho - i\varepsilon) = \ln \rho e^{-i\pi} = \ln \rho - i\pi$$

⟹ simple-valued function can be defined by

$$\ln(z; k) = \ln|z| + i(\arg z + 2\pi k) \quad , \quad k \in \mathbb{Z}$$

- Riemann sheets:

  - regarding Fig. 2.1, the whole problem is only a question of choosing one of the z-planes, already suggested with the new "coordinate" k

→ redefinition: f is a mapping from those two complex z-planes onto a single w-plane

what then happens to the branch cut?

- the branch cut can now be understood as the connection between the different z-planes ($\rightarrow$ spiral stairway for complex root)

  $\rightarrow$ Fig. 2.4

  the new structure ensures that there is a continuous transition between the planes

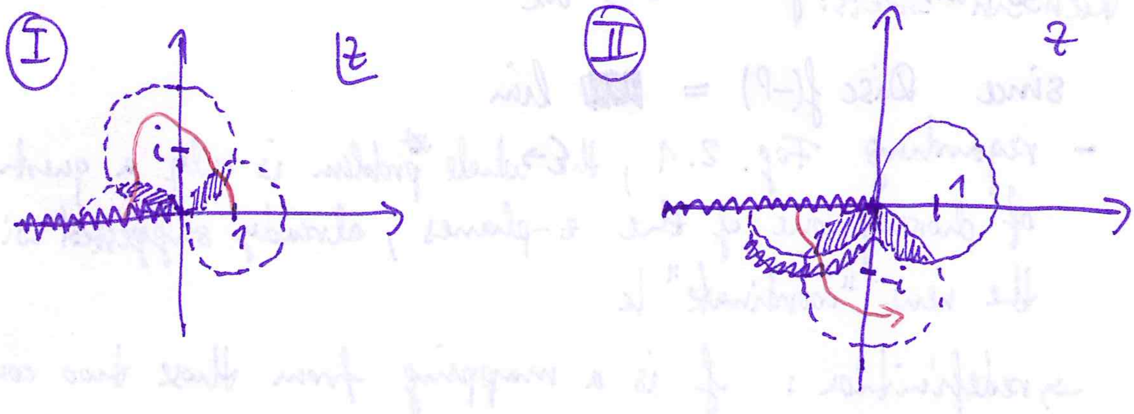
  $\rightarrow$ individual plane is called Riemann sheet

  $\rightarrow$ structure is called Riemann surface, an one-dimensional complex manifold:

$$f : X \rightarrow \mathbb{C}, \quad f(z) = w$$

  $\rightarrow$ it can be proved that the residue theorem is still valid on a Riemann surface X; the identity theorem also holds true!

- analytic continuation through the branch cut: (zeige explizit für Wurzel!)
  - now this is pretty clear:



by expanding into power serieses we perform the continuation into diffent Riemann sheets

- explicitely this is done by using the discontinuity:

$$f(z) = \ln z \quad \rightsquigarrow \quad \mathrm{Disc} f(-\rho) = \lim_{\varepsilon\to 0^+} \left[f(-\rho+i\varepsilon) - f(-\rho-i\varepsilon) \right]$$

$$= \ln\rho + i\pi - (\ln\rho - i\pi)$$

$$= 2\pi i$$

we require: $$\lim_{\varepsilon\to 0^+} f_{II}(-\rho-i\varepsilon) = \lim_{\varepsilon\to 0^+} f(-\rho+i\varepsilon)$$

$$= \lim_{\varepsilon\to 0^+} f(-\rho-i\varepsilon) + \underbrace{\mathrm{Disc}\, f(-\rho)}_{=2\pi i}$$

$$\Rightarrow \quad f_{II}(z) = \ln z + 2\pi i$$

- in general for $f(z) = f^*(z^*)$ it is (zeigen!)

$$\mathrm{Disc}\, f(x) = 2i \lim_{\varepsilon\to 0^+} \mathrm{Im}\, f(x+i\varepsilon)$$

for $a < x < b$, where a and b mark the branch points

→ from this we immediately obtain dispersion relations like

$$G_S(E) = \int_{-\infty}^{\infty} d\omega \, \frac{\rho(\omega)}{E-\omega+i\varepsilon}, \quad \rho(\omega) = -\frac{1}{\pi} \mathrm{Im}\, G_S(\omega+i\varepsilon)$$

- application to decays in QM:
  - we have found for the non-relativistic Lee model,

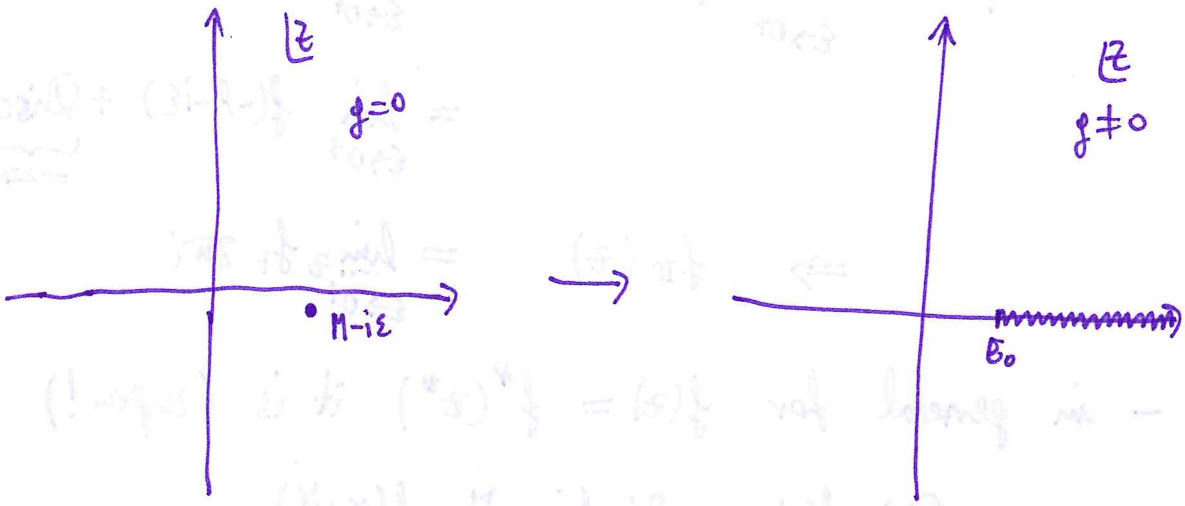
$$G_S(E) = \langle S | \frac{1}{E-\hat{H}+i\varepsilon} | S \rangle = \frac{1}{E - M_0 + g^2 \Sigma(E) + i\varepsilon}$$

with $$\Sigma(E) = -\int \frac{dk}{2\pi} \frac{f^2(k)}{E-\omega(k)+i\varepsilon}$$

→ obviously, the mass of the stable particle ($g=0$) is represented as a simple pole of the propagator

→ it is consensus to identify the pole(s) of the full propagator as $z_{pole} = M - i\frac{\Gamma}{2}$

→ nevertheless, usually there are no poles to find on the complex plane where $G_s(z)$ is defined ($E \to z = x+iy$)



- the problem is that poles are situated in the second Riemann sheet, because $G_s(z)$ is multi-valued due to $\Sigma(z)$:

simplest model with $f(k) = \Theta(2k-E_0)\,\Theta(\Lambda-2k)$

$$\rightsquigarrow \Sigma(E) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, \frac{\Theta(2k-E_0)\,\Theta(\Lambda-2k)}{E-2k+i\varepsilon}$$

$$\overset{k\to 2k}{=} \frac{1}{4\pi}\int_{E_0}^{\Lambda} dk \, \frac{1}{k-E-i\varepsilon}$$

$$= \begin{cases} \frac{1}{4\pi}\left(\ln\frac{\Lambda-E}{E-E_0} + i\pi\right), & E_0 < E < \Lambda \\ \frac{1}{4\pi}\ln\frac{\Lambda-E}{E_0-E}, & \text{else} \end{cases}$$

$$\text{or} \quad \Sigma(z) = \frac{1}{4\pi}\left(\ln(\Lambda-z) - \ln(E_0-z)\right)$$

- we take the simple way in order to calculate the discontinuity:

$$\mathrm{Disc}\,\Sigma(E) = \Sigma(E+i\varepsilon) - \Sigma(E-i\varepsilon)$$

$$= \frac{1}{4\pi}\int_{E_0}^{\Lambda} dk \left(\frac{1}{k-E-i\varepsilon} - \frac{1}{k-E+i\varepsilon} \right)$$

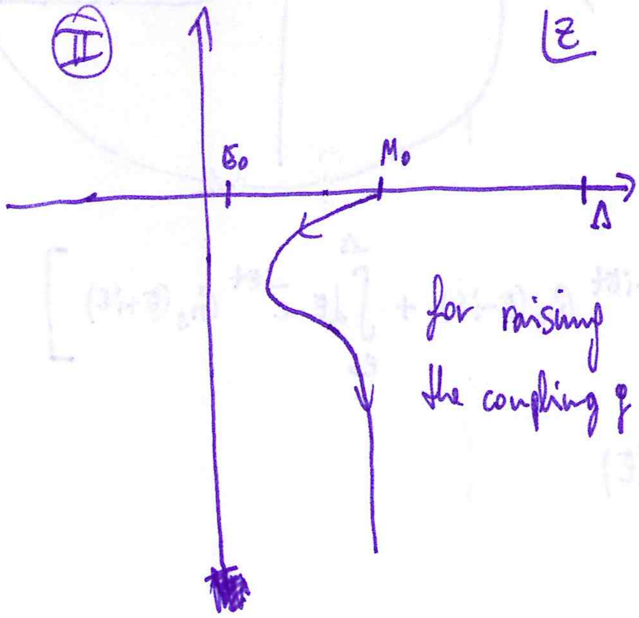
$$= \frac{1}{4\pi}\int_{E_0}^{\Lambda} dk \, \frac{2i\varepsilon}{(k-E)^2+\varepsilon^2} = \frac{1}{2\pi}\int_{E_0}^{\Lambda} dk \; i\pi\,\delta(k-E)$$

$$= \frac{i}{2}\,, \quad E_0 < E < \Lambda$$

it has to be: $$G_{\mathrm{II}}(z) = \frac{1}{z - M_0 + g^2 \Sigma_{\mathrm{II}}(z)}$$

$$\rightsquigarrow \Sigma_{\mathrm{II}}(z) = \Sigma(z) + \frac{i}{2}$$

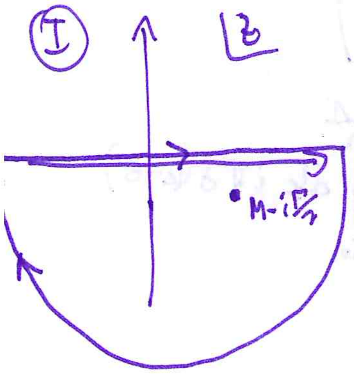
- for $M_0 = 2E_0$, $\Lambda = 10E_0$, $E_0 = 1$ we find:



for high couplings the spectral function loses its Breit-Wigner shape

- can we make connection to non-exponential decay behaviour(s)?

we know:

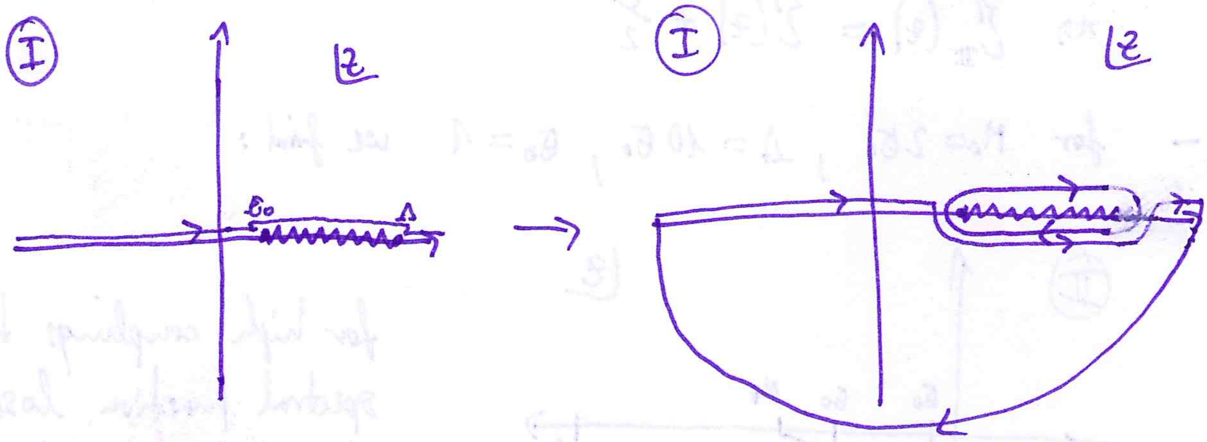


$$a(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dE \, \frac{e^{-iEt}}{E - z_{pole}} \, , \qquad z_{pole} = M - i\frac{\Gamma}{2}$$

$$= \frac{i}{2\pi}\oint_C dz \, \frac{e^{-izt}}{z - z_{pole}} = e^{-iMt - \frac{\Gamma}{2}t}$$

$$\Rightarrow \quad p(t) = e^{-\Gamma t}$$

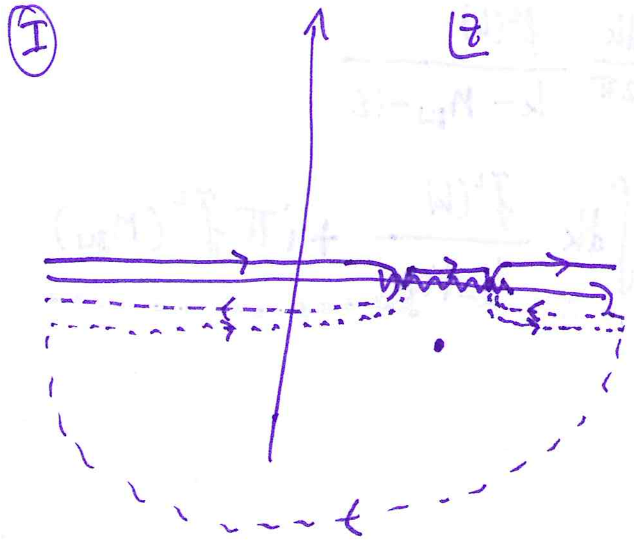
this is <u>not the case here</u> since the above integration is performed without including the branch cut; we need to modify this to get the <u>full</u> amplitude:



$$\Rightarrow \quad a(t) = \frac{i}{2\pi}\left[-\int_{E_0}^{\Lambda} dE \, e^{-iEt} \, G_s(E - i\varepsilon) + \int_{E_0}^{\Lambda} dE \, e^{-iEt} \, G_s(E + i\varepsilon) \right]$$

$$= \int_{E_0}^{\Lambda} dE \, e^{-iEt} \rho(E)$$

- one often studies the amplitude for intermediate times in the weak coupling limit by performing:



→ by this one can compare the contributions to amplitude and probability (to obtain exponential decay for intermediate times the pole contribution must dominate ...)

- we can reproduce the exponential decay law quicker if the pole lies just below the cut (weak couplings):

$$z_{pole} = M - i\frac{\Gamma}{2} \approx M_{BW} + i\varepsilon \quad , \quad M_{BW} = M_0 - \delta M \quad , \quad \Gamma \ll M$$

$$\Rightarrow \Sigma_{II}(M - i\tfrac{\Gamma}{2}) \approx \Sigma(M_{BW} + i\varepsilon)$$

$$\Rightarrow G_S(z)\Big|_{U(z_{pole})} \approx \frac{1}{z - M_0 + g^2 \Sigma(M_{BW} + i\varepsilon)}$$

$$= \frac{1}{x - M_0 + g^2 \operatorname{Re}\Sigma(M_{BW}+i\varepsilon) + i\left(y + g^2 \operatorname{Im}\Sigma(M_{BW}+i\varepsilon)\right)}$$

$$\boxed{\begin{aligned} \Rightarrow \delta M &= g^2 \operatorname{Re}\Sigma(M_{BW}+i\varepsilon) && (\to \text{renormalized mass shift}) \\ \Gamma_{BW} &= 2g^2 \operatorname{Im}\Sigma(M_{BW}+i\varepsilon) \\ &= 2\pi g^2 \tilde{f}^2(M_{BW}) && (\to \text{Fermi's golden rule}) \end{aligned}}$$

where the latter comes from:

$$\Sigma_{II}(M - i\tfrac{\Gamma}{2}) \approx \Sigma(M_{BW} + i\varepsilon)$$

$$= \int \frac{dk}{2\pi} \frac{f^2(k)}{k - M_{BW} - i\varepsilon}$$

$$\overset{\tilde{f}(k) = \frac{f(k)}{2\pi}}{=} \mathcal{P}\int dk \, \frac{\tilde{f}^2(k)}{k - M_{BW}} + i\pi \tilde{f}^2(M_{BW})$$

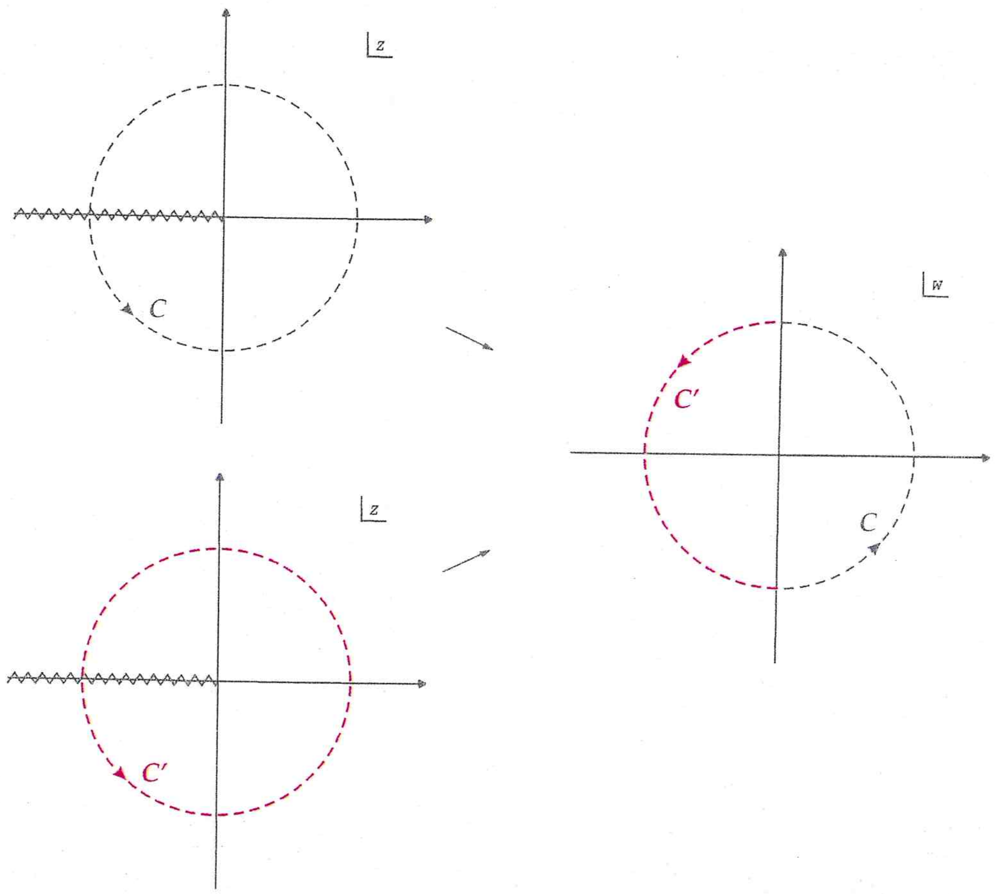


Figure 2.1: Multi-valued character of $f(z) = \sqrt{z}$ with paths C (black, dashed) and C' (red, dashed) in the complex z- and w-planes.

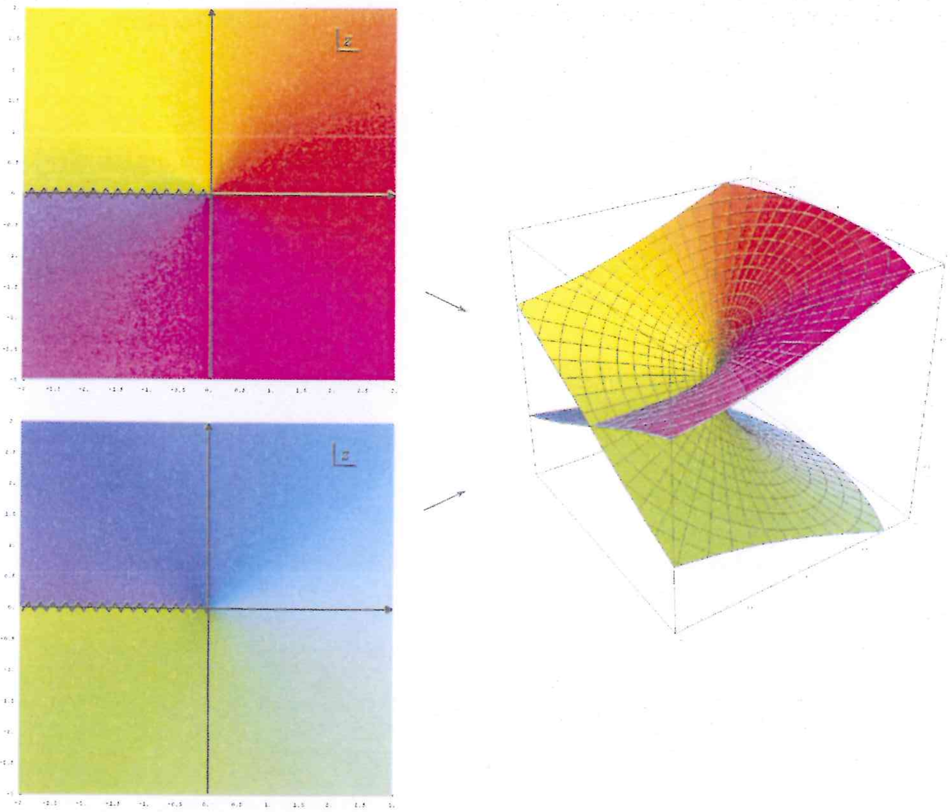


Figure 2.4: Riemann surface of the complex root function. Each complex value w is represented as a particular color: the arg of the complex number is encoded as the hue of the color, the modulus as its saturation (the colored background graphics on the left as well as the figure on the right were created by Jan Homann from the University of Pennsylvania).

of the complex root $\sqrt{z} = f(z)$,

analytic continuation along a path $e^{i\varphi}$ from $\varphi = 0$ to $\varphi = 2\pi$ ~~across~~ through the branch cut:

$$\sqrt{z} = z^{\frac{1}{2}} = \sqrt{z_0} + \frac{1}{2\sqrt{z_0}}(z - z_0) - \frac{1}{8}\frac{1}{z_0^{\frac{3}{2}}}(z - z_0)^2 + \dots$$

1) $z_0 = 1 \ (\varphi = 0)$: $\sqrt{z} = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \dots$

$z_0 = -1 \ (\varphi = \pi)$: $\sqrt{z} = i + \frac{1}{2i}(z+1) + \frac{1}{8i}(z+1)^2 + \dots$

2) $z_0 = 1 \ (\varphi = 2\pi)$: $\sqrt{z} = -1 - \frac{1}{2}(z-1) + \frac{1}{8}(z-1)^2 + \dots$

$\Rightarrow$ obviously, $f_{II}(z)$ in a disc with radius one around the point $z_0 = 1$ in the second Riemann sheet has values $f_{II}(z) = -\sqrt{z} = -f(z)$