

Vorlesung 10

Allgemeine Relativitätstheorie mit dem Computer

PC-Pool Raum 01.120 Johann Wolfgang Goethe Universität 20. Juni, 2016

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Allgemeines

Ort und Zeit:

PC-Pool Raum 01.120, immer Montags von 16.15 bis 17.45 Uhr

Zusätzlicher, freiwilliger Übungstermin 15.00 bis 16.15 Uhr

Vorlesungs-Materialien und *Lon Capa* Online-Lernplattform:

<http://th.physik.uni-frankfurt.de/~hанаuske/VARTC/>

<http://lon-capa.server.uni-frankfurt.de/>

Vortrag am Freitag (24. Juni 2016 um 20:00 Uhr) im Physikalischen Verein (Hörsaal BiK-F, Georg-Voigt-Str. 14-16): “[Neutronensterne auf Kollisionskurs](#)”

Plan für die heutige Vorlesung:

Die Kerr Metrik mit Maple

- Wiederholung:

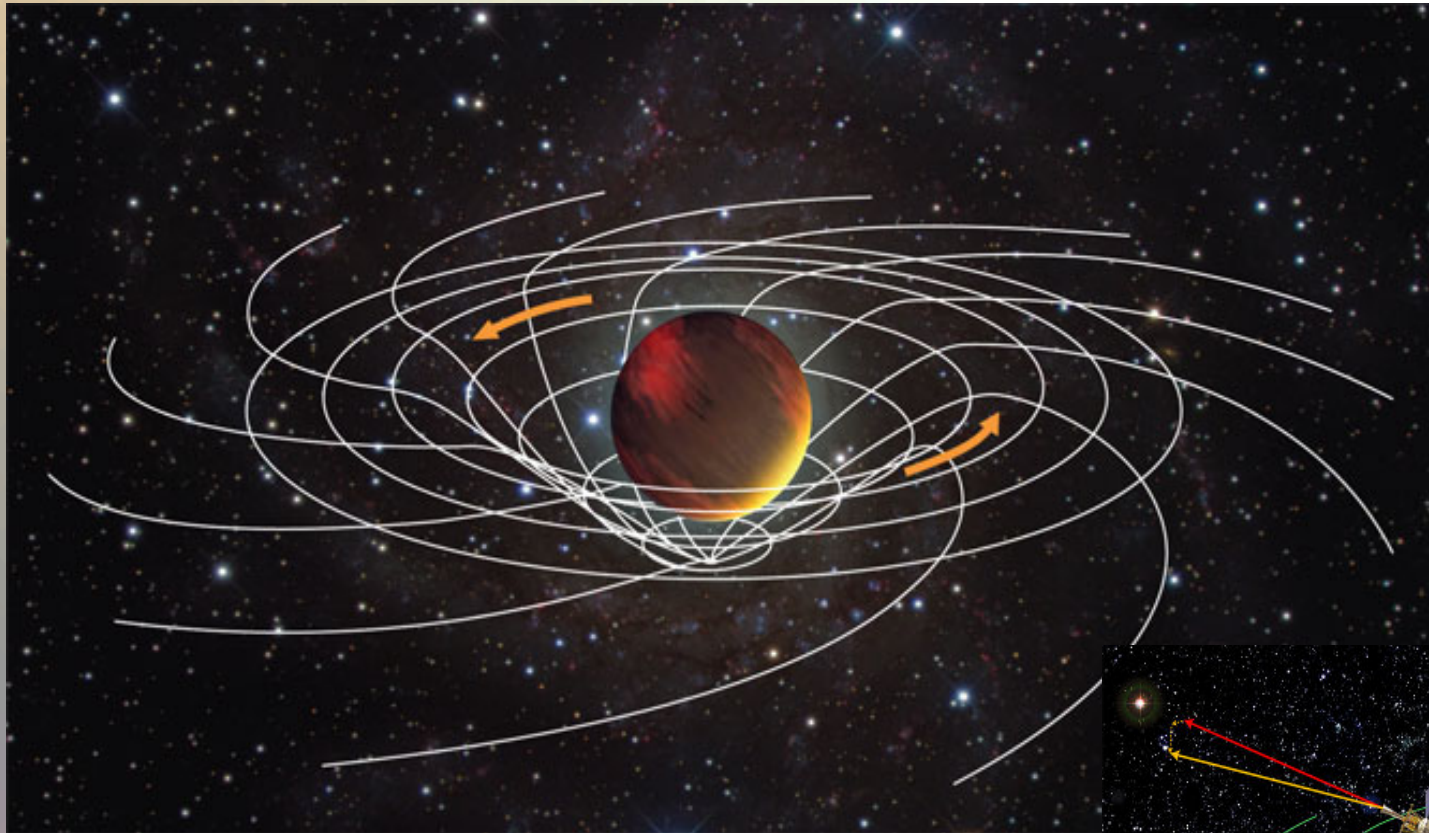
Eigenschaften der Kerr Metrik, “Frame-Dragging”, gravitomagnetischer Effekt

- Klassifizierung der möglichen Bahnbewegungen mittels eines effektiven Potentials und die letzte quasistabile Kreisbahn (ISCO) eines Probekörpers

Teil III: Computersimulationen mit dem Einstein-Toolkit (ET)

- Die Einsteingleichung im (3+1)-Split, Download und Kompilierung des ET

Der Frame-Dragging Effekt

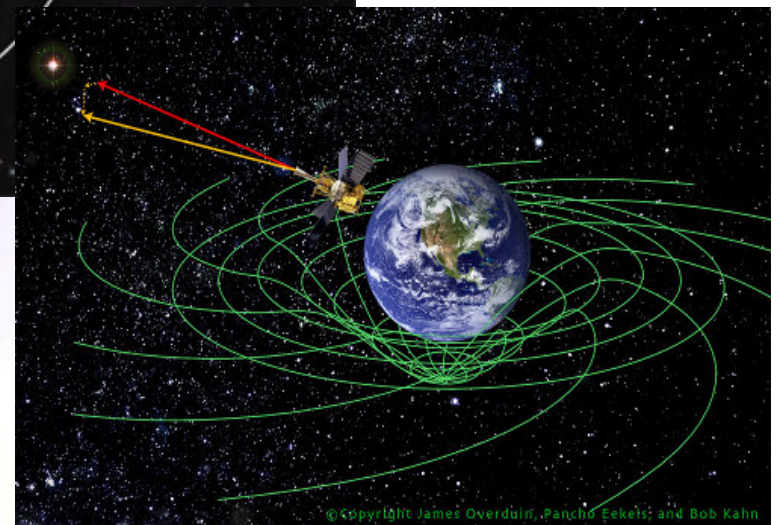


Experimente zur
Überprüfung des
Lense-Thirring
Effektes:

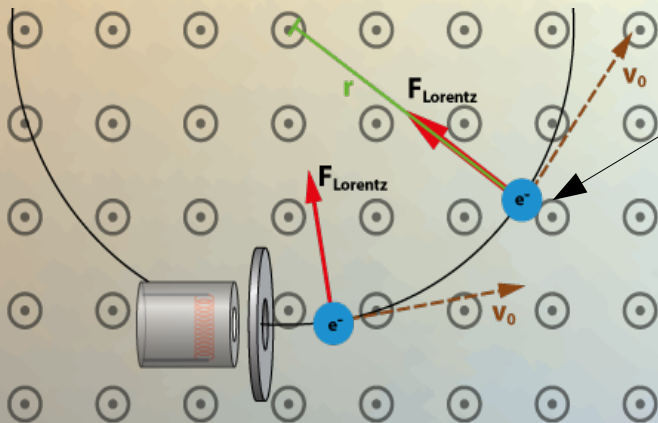
LARES

Gravity Probe B

$$\omega(r, \theta) = \frac{d\phi}{dt} = \frac{\frac{d\phi}{d\tau}}{\frac{dt}{d\tau}} = \frac{u^\phi}{u^t} = \frac{g^{t\phi}}{g^{tt}}$$



Der gravitomagnetische Effekt



Elektromagnetischer Effekt der Lorentzkraft

Gravito-
magnetischer
Effekt

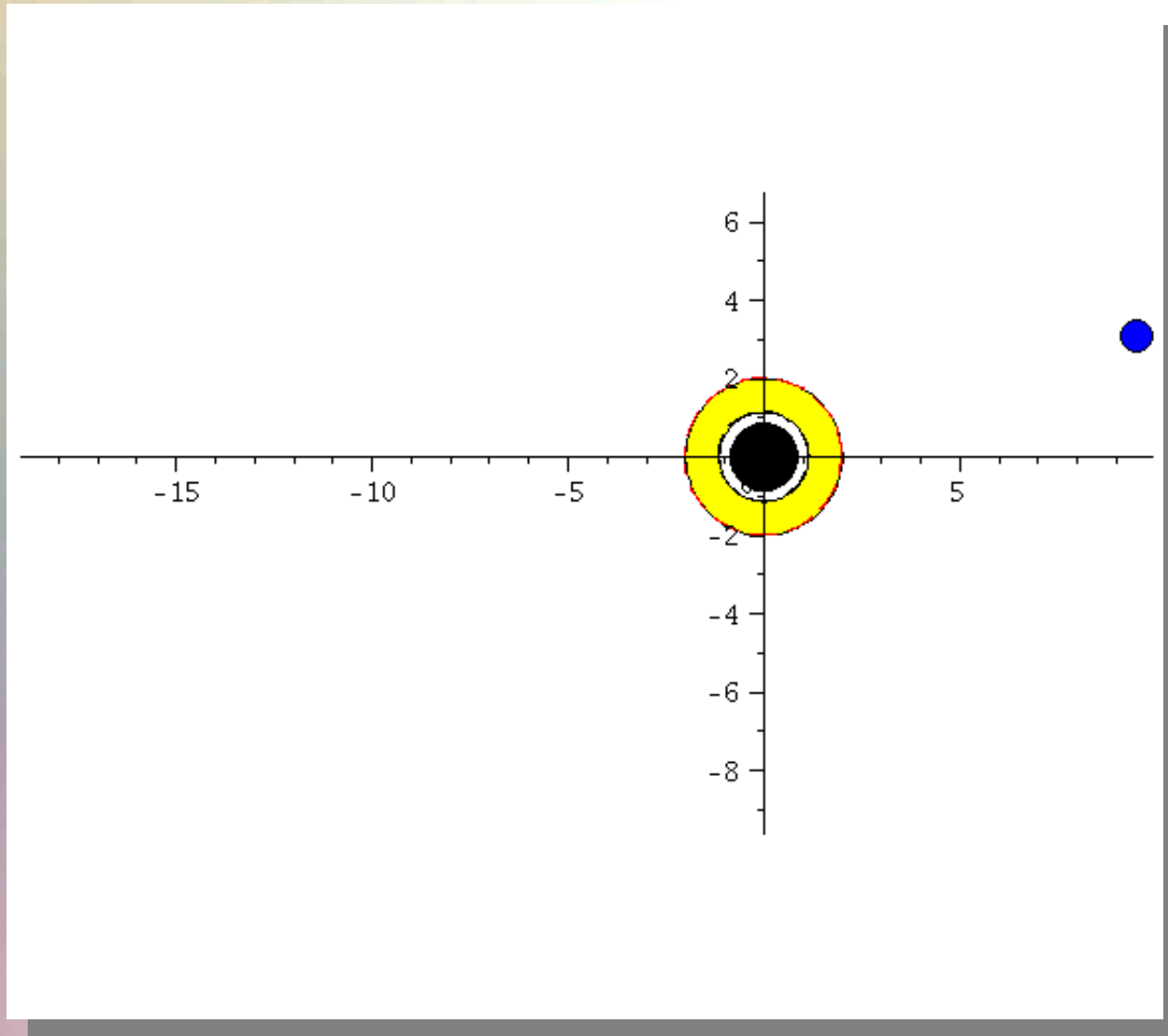
$$\omega(r, \theta) = \frac{d\phi}{dt} = \frac{\frac{d\phi}{d\tau}}{\frac{dt}{d\tau}} = \frac{u^\phi}{u^t} = \frac{g^{t\phi}}{g^{tt}}$$

Diese "Frame dragging" Frequenz wirkt in ähnlicher Weise auf die Geschwindigkeit von Probekörpern, wie das Magnetfeld in der Elektrodynamik die Lorentzkraft verursacht. Im Fall 1 (grün) ist das gravitomagnetische Feld Null, im Fall 2 (blau) ist es aus der äquatorialen Ebene nach oben gerichtet und im Fall 3 zeigt es nach unten. In erster Näherung (siehe Fließbach Buch, S:172) ist die gravitomagnetische Lorentzkraft gleich $\sim 2(\boldsymbol{\omega} \times \mathbf{v})$, wobei \times das Kreuzprodukt, $\boldsymbol{\omega}$ der axiale Vektor der "Frame dragging" Frequenz und \mathbf{v} der Geschwindigkeitsvektor des Probekörpers ist. Die Änderung des Geschwindigkeitsvektors nimmt in Schwachfeldnäherung dann die folgende Gestalt an:

$$\frac{d\mathbf{v}}{d\tau} = \underbrace{-\text{grad } \Phi(\mathbf{r})}_{\text{gewöhnlicher Teil der gravitativen Kraft}} + \underbrace{2\boldsymbol{\omega}(\mathbf{r}) \times \mathbf{v}}_{\text{gravitomagnetische Lorentzkraft}} + \mathcal{O}(v^2/c^2),$$

wobei $\Phi(\mathbf{r})$ das Newtonsche Gravitationspotential und $\mathbf{v} = (v^r, v^\theta, v^\phi)$ der Geschwindigkeitsvektor des Probekörpers ist. Die unten abgebildete Grafik zeigt die "Frame dragging" Frequenz $\boldsymbol{\omega} = \omega_z(r)$ für die Kerr Metrik, wobei bei der schwarzen Kurve $a=0$, bei der blauen Kurve $a=0.99$ und bei der roten Kurve $a=-0.99$ ist.

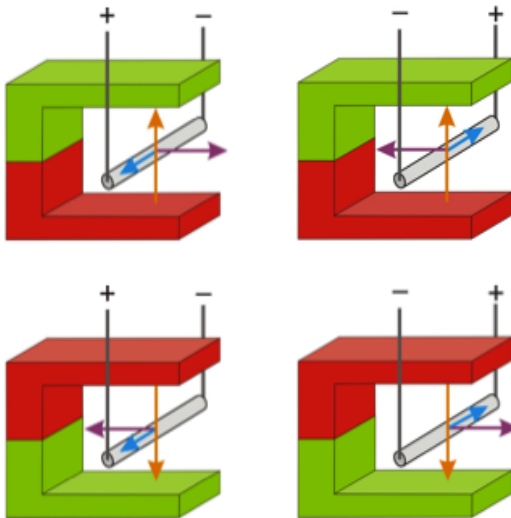
Der gravitomagnetische Effekt



Der gravitomagnetische Effekt

Beobachtung

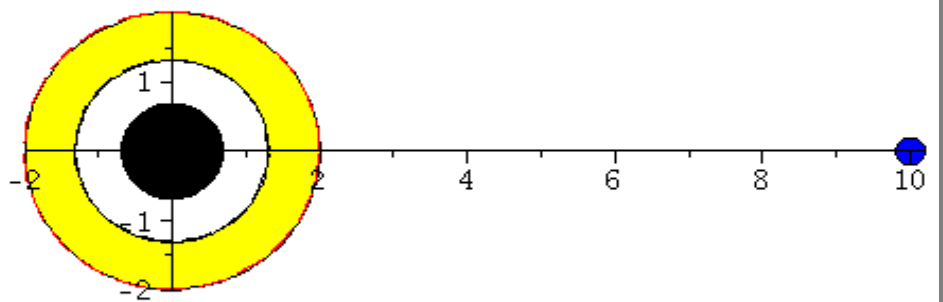
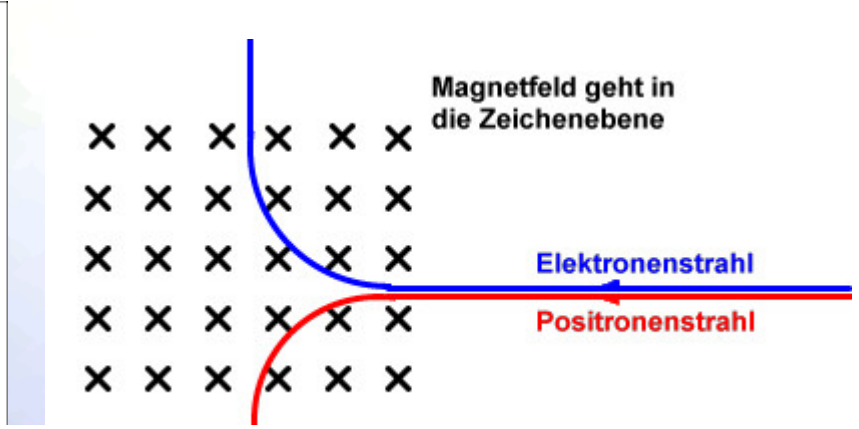
a) + b)



Versuche eine Regel mit Daumen, Zeigefinger und Mittelfinger deiner linken Hand zu formulieren.

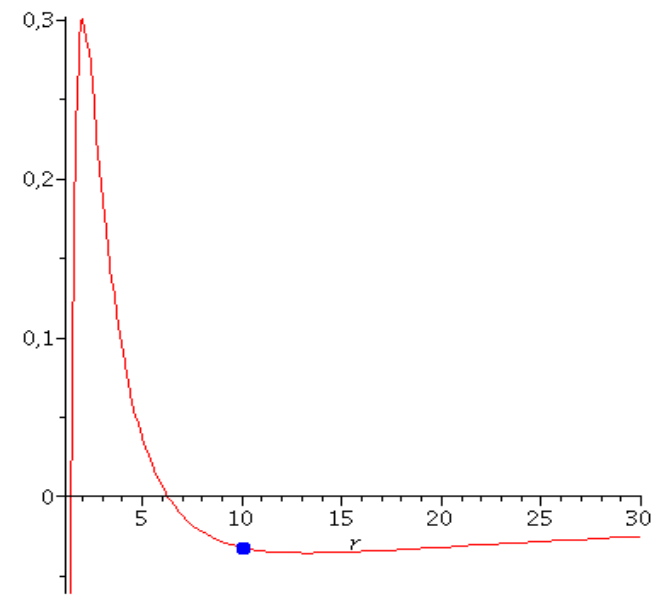
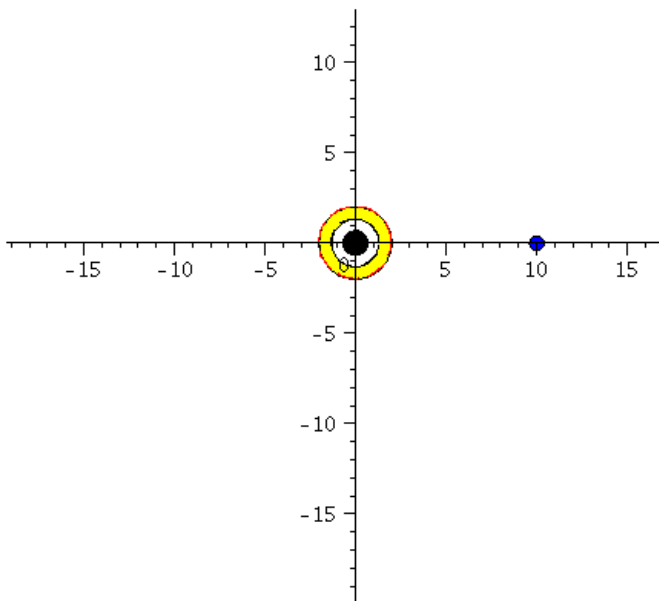


(C) Lorenz K



Kerr Metrik: Effektives Potential

$$V_{eff}(r, M, l, a, E) = -\frac{M}{r} + \frac{l^2 - a^2(E^2 - 1)}{2r^2} - \frac{M(l - aE)^2}{r^3}$$

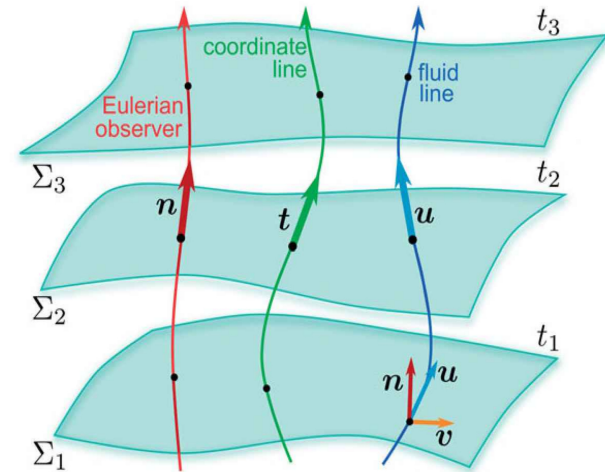


Relativistic Hydrodynamics and Numerical General Relativity

A realistic numerical simulation of a twin star collapse, a merger of two compact stars or a collapse to a black hole, needs to go beyond a static, spherically symmetric TOV-solution of the Einstein- and Hydrodynamical equations.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$

$$\begin{aligned} \nabla_{\mu}(\rho u^{\mu}) &= 0, \\ \nabla_{\nu}T^{\mu\nu} &= 0. \end{aligned}$$

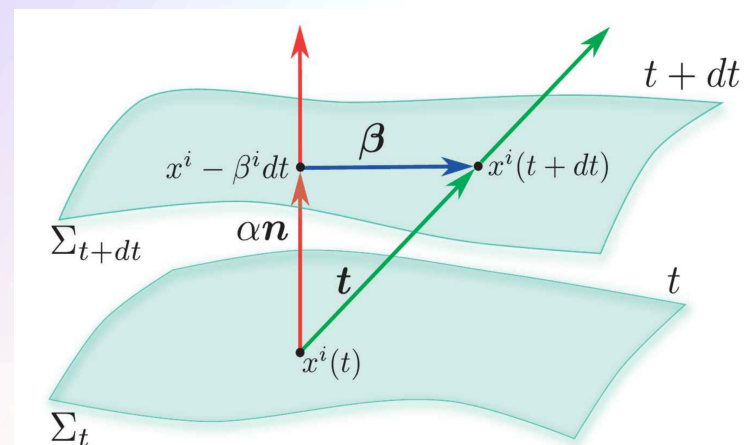


$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

(3+1)
decomposition
of spacetime

$$d\tau^2 = \alpha^2(t, x^j) dt^2$$

$$x^i_{t+dt} = x^i_t - \beta^i(t, x^j) dt$$



The ADM equations

The ADM (Arnowitt, Deser, Misner) equations come from a reformulation of the Einstein equation using the (3+1) decomposition of spacetime.

$$\begin{aligned}\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij} \\ &= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i\end{aligned}$$

$$\begin{aligned}\partial_t K_{ij} &= -D_i D_j \alpha + \beta^k \partial_k K_{ij} + K_{ik} \partial_j \beta^k + K_{kj} \partial_i \beta^k \\ &\quad + \alpha \left({}^{(3)}R_{ij} + K K_{ij} - 2K_{ik} K^k_j \right) + 4\pi \alpha [\gamma_{ij} (S - E) - 2S_{ij}]\end{aligned}$$

← Time evolving part of ADM

$$D_j (K^{ij} - \gamma^{ij} K) = 8\pi S^i$$

$${}^{(3)}R + K^2 - K_{ij} K^{ij} = 16\pi E$$

← Constraints on each hypersurface

Three dimensional covariant derivative

$$D_\nu := \gamma^\mu_\nu \nabla_\mu = (\delta^\mu_\nu + n_\nu n^\mu) \nabla_\mu$$

Three dimensional Riemann tensor

$${}^{(3)}R^\mu_{\nu\kappa\sigma} = \partial_\kappa {}^{(3)}\Gamma^\mu_{\nu\sigma} - \partial_\sigma {}^{(3)}\Gamma^\mu_{\nu\kappa} + {}^{(3)}\Gamma^\mu_{\lambda\kappa} {}^{(3)}\Gamma^\lambda_{\nu\sigma} - {}^{(3)}\Gamma^\mu_{\lambda\sigma} {}^{(3)}\Gamma^\lambda_{\nu\kappa}$$

$${}^{(3)}\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \gamma^{\alpha\delta} (\partial_\beta \gamma_{\gamma\delta} + \partial_\gamma \gamma_{\delta\beta} - \partial_\delta \gamma_{\beta\gamma})$$

Spatial and normal projections of the energy-momentum tensor:

$$S_{\mu\nu} := \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta},$$

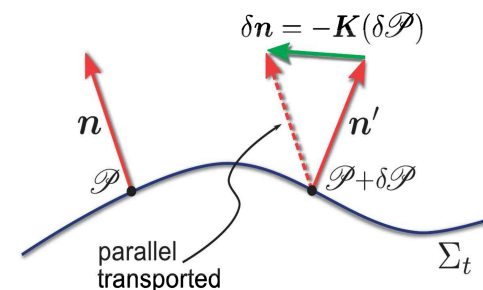
$$S_\mu := -\gamma^\alpha_\mu n^\beta T_{\alpha\beta},$$

$$S := S^\mu_\mu,$$

$$E := n^\alpha n^\beta T_{\alpha\beta},$$

Extrinsic Curvature:

$$K_{\mu\nu} := -\gamma^\lambda_\mu \nabla_\lambda n_\nu$$



From ADM to BSSNOK

Unfortunately the ADM equations are only weakly hyperbolic (mixed derivatives in the three dimensional Ricci tensor) and therefore not "well posed". It can be shown that by using a conformal traceless transformation, the ADM equations can be written in a hyperbolic form. This reformulation of the ADM equations is known as the BSSNOK (Baumgarte, Shapiro, Shibata, Nakamuro, Oohara, Kojima) formulation of the Einstein equation. Most of the numerical codes use this (or the CCZ4) formulation.

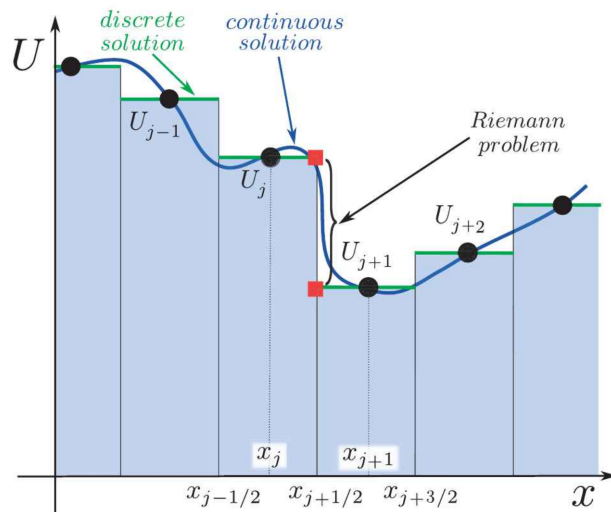
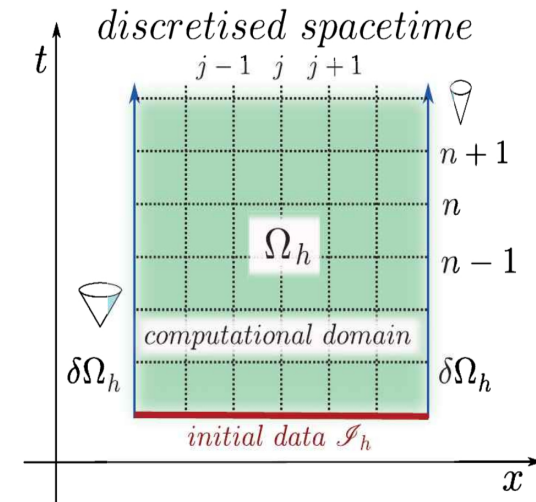
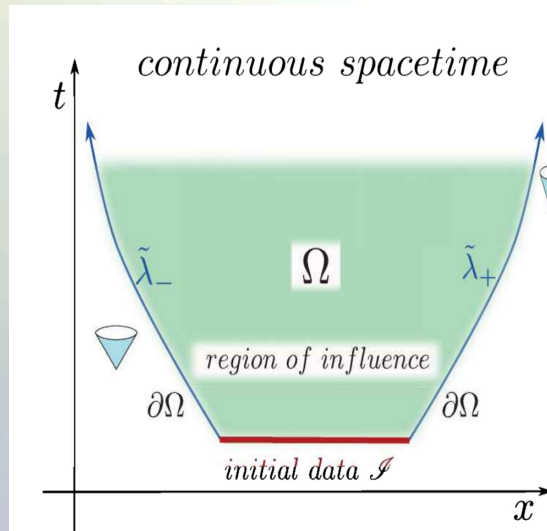
The 3+1 Valencia Formulation of the Relativistic Hydrodynamic Equations

$$\begin{aligned}\nabla_{\mu}(\rho u^{\mu}) &= 0, \\ \nabla_{\nu}T^{\mu\nu} &= 0.\end{aligned}$$

To guarantee that the numerical solution of the hydrodynamical equations (the conservation of rest mass and energy-momentum) converge to the right solution, they need to be reformulated into a conservative formulation. Most of the numerical "hydro codes" use here the 3+1 Valencia formulation.

Finite difference methods

Discretisation of a hyperbolic initial value boundary problem.



High resolution shock capturing methods (HRSC methods) are needed, when Riemann problems of discontinuous properties and shocks needs to be evolved accurately

The Einstein Toolkit



einstein
toolkit



WELCOME

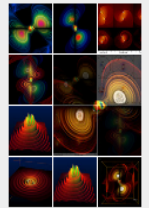
The Einstein Toolkit Consortium is developing and supporting open software for relativistic astrophysics. Our aim is to provide the core computational tools that can enable new science, broaden our community, facilitate interdisciplinary research and take advantage of emerging petascale computers and advanced cyberinfrastructure.

Please read our pages [about](#) the Einstein Toolkit, its [governance](#), and how to [get started](#) with the toolkit for more information.

Download

November 2014: We are pleased to [announce the tenth release](#) (code name "[Herschel](#)") of the Einstein Toolkit, an open, community developed software infrastructure for relativistic astrophysics.

<https://www.youtube.com/watch?v=EO4d32ch6OI>
<https://www.youtube.com/watch?v=p5bq2iUO3DE>
https://www.youtube.com/watch?v=MNpyd_o0MT4
<https://www.youtube.com/watch?v=Qg6PwRI2uS8>
<https://www.youtube.com/watch?v=ZW3aV7U-aik>



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