Experiments in Computer Simulations :
Collapse of an unstable Neutron Star to a Black Hole

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0   Class Information

- application form :

  - if you want to do this experiment, please register via e-mail to me no later than on the last Wednesday before the week in which you want to do the experiment; your e-mail should include the following information: (1) student number, (2) full name, (3) e-mail address.

- intensive course :

  - 12 [hours] = 6 [hours/week] × 2 [weeks].

- when :

  - Monday 9-16, two subsequent weeks upon individual arrangement with me; other time slots may be arranged with me individually.

- where :

  - Pool Room 01.120.

- preparation :

  - an account for you on the "FUCHS" cluster of the CSC (http://csc.uni-frankfurt.de/) will be provided; please read the quick starting guide on the CSC web pages before starting the simulation.

- required skill :

  - basic Linux knowledge.

- using software :

  - `Einstein Toolkit` [5].
  - `gnuplot` (http://www.gnuplot.info/)
  - `pygraph` (https://bitbucket.org/dradice/pygraph).
  - `python` (https://www.python.org/) and `matplotlib` (http://matplotlib.org/).
  - `Mathematica` (http://www.wolfram.com/mathematica/).

1 Introduction

Neutron stars are beside white dwarfs and black holes the potential final states of the evolution of a normal star. These extremely dense astrophysical objects, which are formed in the center of a supernova explosion, represent the last stable state before the matter collapses to a black hole. Due to their large magnetic fields (up to 10^{11} Tesla) and fast rotation (up to 640 rotations in one second) neutron stars emit a certain electromagnetic spectrum. In our galaxy we currently know about 2500 of these exotic objects, which are mainly observed as pulsars by radio telescopes. Among the known neutron stars, there are some which are in binary systems where the companion of the neutron star is either a normal star, a planet, a white dwarf or again a neutron star; unfortunately a neutron star - black hole binary system has not been found until now.

The most impressive binary system right now is the so called *Double Pulsar: PSR J0737-3039A/B*, which has been discovered in 2003 (see [18, 16, 11] for details). The two neutron stars in this binary system, which are only separated by 800,000 km, orbit around each other with an orbital period of 147 minutes and a mean velocities of one million km/h. This system is excellent for testing Einstein's theory of general relativity and alternative theories of gravity in the strong-field regime [17]. Another interesting feature is the observation of the periodic eclipse of one pulsar by the other [10], which is a direct consequence of the almost edge-on geometry of the orbit (see [1] for an animation of the orbit).

Due to the emission of gravitational waves, the distance between the two neutron stars decreases with time and finally the two objects need to merge. The emitted gravitational waves travel, almost unaffected, with the speed of light trough our galaxy. These little space-time oszillations (the gravitational waves) could be detected on earth if the emitting source (the merging neutron star binary system) is not to far away. Several gravitational wave antennas are trying to detect at the moment these little space-time ripples [7, 6]. Right after the neutron star merger process a new supramassive or hypermassive neutron star is formed, which could be stable for longer times or collapse almost immediately to a black hole. During this process a short gamma ray burst is emitted, releasing in less than one second the energy emitted by our Galaxy over one year [23]. Space-based gamma-ray telescopes (e.g., the Swift gamma-ray burst mission [12]) detect on average approximately one gamma-ray burst per day - however, fortunately all of them appeared in distant galaxies. Last but not least it should be mentioned that a triple system has also been observed, where two white dwarfs are orbiting around a neutron star [20, 2].

In this computer experiment we try to simulate the collapse of a hypermassive neutron star to a black hole. An analytic treatment of such a collopse is only possible if one simplifies the real scenario in a rather crude way. The Oppenheimer-Snyder collapse (for details see e.g., [21] page 612f) is the most simple description of a collapse of a star composed of uniform-density pressurless dust - however, we cannot use this approximation in this experiment, as the neutron star fluid is far from beeing pressureless. In our numerical approach, we will simulate the collapse of a non-rotating neutron star and assume a spherical symmetry of the star and do not include magnetic fields.

In the following the main theoretical concepts and basic equations are summarized, which are needed to understand the background of the computer experiment (for details see [22],[19] page 842f and [21] page 612f). Unless explicitly stated, we use in the following units in which $c = G = M_\odot = 1$.

2 Theoretical Background

In the following the main theoretical concepts and basic equations are summarized, which are needed to understand the background of the computer experiment. Einstein's theory of general relativity in connection with the conservation laws for energy-momentum and rest mass are the groundings of the differential equations which we need to solve. The Einstein equation and the conservation laws are summarized in the following set of highly non-linear differential equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;=\; 8\pi\, T_{\mu\nu}\;,\;\;\; \nabla_\mu T^{\mu\nu} \;=\; 0\;,\;\;\; \nabla_\mu\left(\rho\, u^\mu\right) \;=\; 0\;\;\;, \tag{1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $R_{\mu\nu}$ is the Ricci tensor, which contains first and second derivatives of the space-time metric $g_{\mu\nu}$, ∇_μ is the covariant derivative and u^μ is the four velocity of the star's fluid. The Einstein equation (first equation in (1)) describes in which way the space-time structure need to bend (left hand side of the equation) if energy-momentum is present (right hand side of the equation). In this experiment the energy-momentum, which curves space-time, arises from the large energy amount of the neutron star matter.

In the following subsection we will describe the most simple model of a static, spherically symmetric neutron star having an energy-momentum tensor of an ideal fluid.

2.1 The Tollman-Openheimer-Volkov equations

In a static, spherical symmetrical condition, the metric $g_{\mu\nu}$ and the infinitesimal line element ds can be written as

$$ds^2 \;=\; g_{\mu\nu}\, dx^\mu dx^\nu \;=\; -e^{2\nu(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\; d\phi^2 \quad. \tag{2}$$

In the following we will use the expression of an ideal fluid energy-momentum tensor to describe the neutron star matter

$$T_{\mu\nu} \;=\; (e+p)\,u_\mu u_\nu + p\, g_{\mu\nu}\;,\;\;\; e \;=\; \rho\,(1+\epsilon) \quad, \tag{3}$$

where e is the energy density, p the pressure, ρ the rest mass density and ϵ the internal energy density of the neutron star fluid.

By inserting the Ansatz of the metric (2) together with the expression for the energy momentum tensor (3) into the equations (1) one derives the Tollman-Openheimer-Volkov (TOV) equations (see e.g., [21]):

$$\frac{dm}{dr} \;=\; 4\pi r^2\, e\;,\;\;\; \frac{d\nu}{dr} \;=\; \frac{m + 4\pi r^3 p}{r\,(2m-r)}\;,\;\;\; \frac{dp}{dr} \;=\; -(e+p)\frac{d\nu}{dr} \quad. \tag{4}$$

For a given equation of state (EoS: a function $p(e)$) the TOV equation can be solved numerically by fixing the central pressure p_c and integrating outwards to the star's surface where the pressure is zero. It can be shown (Birkov Theorem, see e.g., [19], page 843f) that the metrical structure outside the neutron star is identical to the Schwarzschild metric of a black hole (see subsection 2.2). The left picture of figure 1 shows the energy density profiles calculated within a realistic EoS for three different central values of the energy density. Each star has a different mass and radius, which is marked on the mass-radius curve (see right picture on figure 1). From this mass-radius relation we see that there exists a maximum value of the total mass of the star ($M_{\max}$), which cannot be exceeded. One can show (for details see [8]) that if the energy density is higher than the energy density of the maximum mass star $e_{\max}$, the neutron stars become unstable and either need to collapse to a black hole or migrate back to a stable neutron star (see figure 5).
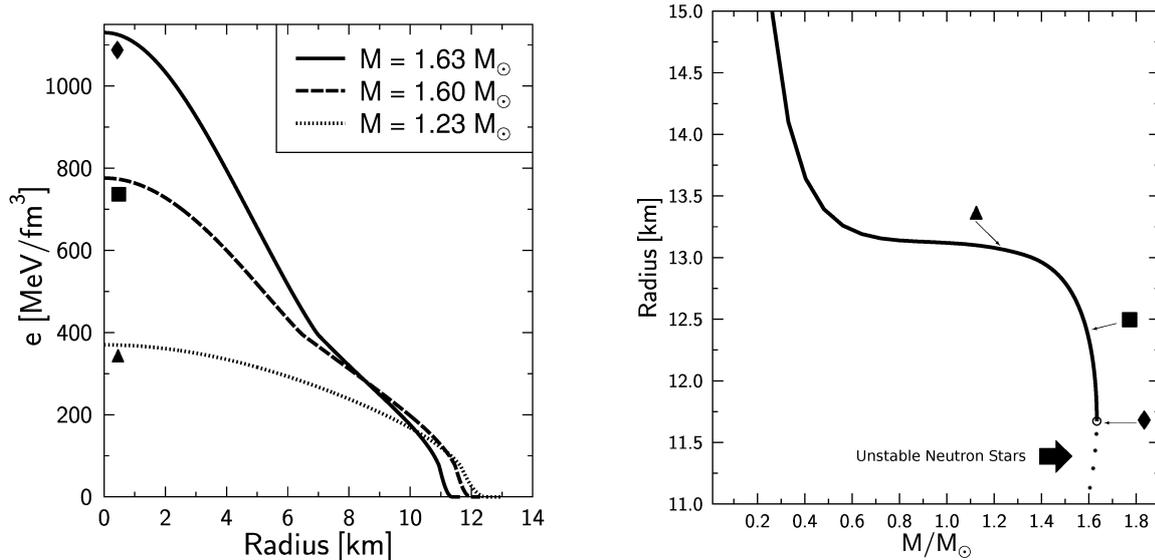
Figure 1: Left: The energy density profiles of three neutron stars. Right: The neutron star mass-radius relation (see [15, 14] for details).

In Task 2 of this computer experiment (see subsection 3.2) we will calculate the maximum mass value for a given polytropic equation of state, ($p = K\rho^\Gamma$, $\Gamma = 2$ and $K = 100$) and identify the mass-radius region of stability.

2.2 Black holes

The metric of a non-rotating, chargeless black hole is given by the Schwarzschild metric (see e.g., [19] page 820f)

$$g_{\mu\nu} \;\; = \;\; \text{diag}\left(-\left(1 - \frac{2M}{r}\right) \;,\; \left(1 - \frac{2M}{r}\right)^{-1} \;,\; r^2 \;,\; r^2\sin^2\theta\right) \;\;,\; \tag{5}$$

where M defines the total mass of the black hole. A real singularity occurs at the center of the black hole ($r = 0$), where all of the mass is located and the energy density becomes infinite. The Schwarzschild metric describes the properties of a black hole in a gauge which can be regarded as the viewpoint of an observer at rest which is located at spatial infinity. Apart from the real singularity at $r = 0$, a coordinate singularity at $r = 2M$ excists where space and time changes their properties. The left diagram in figure 2 illustrates this strange property of the Schwarzschild black hole by visualizing the light cones of in- and outgoing photons in a space-time diagram. A particle which is falling in such a Schwarzschild black hole will never reach the inner area ($r < 2M$) of the black hole and for the outside observer the object seems to become frozen at the event horizon at $r = 2M$ (see [13] for an illustration of these effects). However, the coordinate singularity of the Schwarzschild metric is not a real singularity and a particle falling into a black hole will indeed travel trough the event horizon and reach the center of the black hole in a finite time. This fact can be easily understood if one transforms the coordinate system into a coordinate frame moving with the particle, e.g., the Eddington-Finkelstein coordinates describe a photon falling with the speed of light radially into a black hole from the photon's viewpoint (see right diagram of figure 2). The space-time diagram clearly shows, that the particle travels trough the event horizon, but has no way to communicate with the outer world if it is inside the black hole at $r < 2M$.
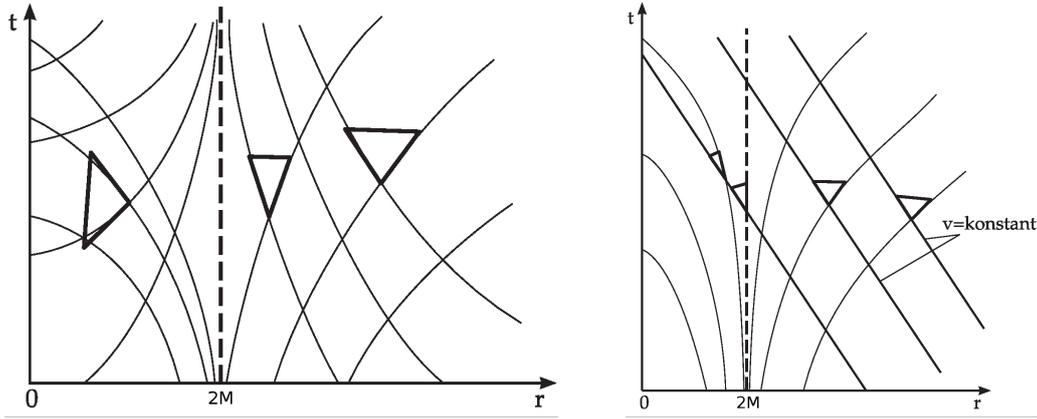
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Figure 2: Space-time diagram of a black hole in Schwarzschild (left picture) and Eddington-Finkelstein (right picture) coordinates.

2.3 The gravitational collapse of a neutron star to a black hole

The last two subsections both were dealing with static solutions of the Einstein equation. However, to simulate the evolution of a collapse of a neutron star, we need to reformulate the equations 1 and solve the time dependent problem numerically. This reformulation, the so called $(3+1)$-split, starts by slicing the 4-dimensional manifold $\mathcal{M}$ into 3-dimensional space-like hypersurfaces Σ_t. The space-time metric $g_{\mu\nu}$ is then also divided into a purely spatial metric γ_{ij} and a lapse function α and a shift vector β_i:

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} \qquad (6)$$

The lapse function α describes the difference between the coordinate time t and the proper time of a fluid particle τ ($d\tau = \alpha\, dt$). The shift vector β_i measures how the coordinates are shifted on the spatial slice if the fluid particle moves an infinitesimal time step further (see figure 3)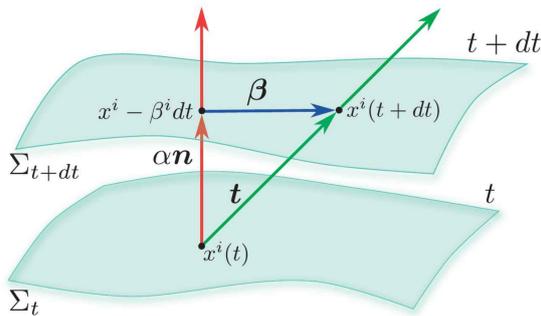. By inserting the metric (6) into the Einstein equation one can reformulate the equations into a system of first order differential

Figure 3: Visualization of the (3+1)-formulation of general relativity (picture reference: [21]).

equations, the so called ADM equations. As the ADM equations are not 'well posed' (for details see [21]), the equations need to be further transformed into the Baumgarte-Shapiro-Shibata-Nakamura-Oohara-Kojima (BSSNOK) formulation [24, 9]. This BSSNOK formulation of general relativity together with the relativistic hydro equations are finally used as the grounding equations in our computer program.

If the central density of a neutron star exceeds the maximally allowed value ρ_c and the star collapses to a black hole, its radius shrinks until it reaches the value of the location of the event horizon of the corresponding black hole at $r = 2M$. Similar as already described in subsection 2.2, the final outcome depends highly on the chosen gauge (which means on the observer's viewpoint). In our simulations, a specific gauge is used, which is neither similar to the Schwarzschild nor to the Eddington-Finkelstein case. To slow down the evolution near the real singularity a singularity avoiding gauge, the so called '1-log'-slicing condition is used. In addition, the 'gamma driver'-shift condition has been used to push the spatial coordinates near

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the singularity outwards (for details see [21], page 343f). In this gauge no real or coordinate singularity could appear during the numerical evolution.
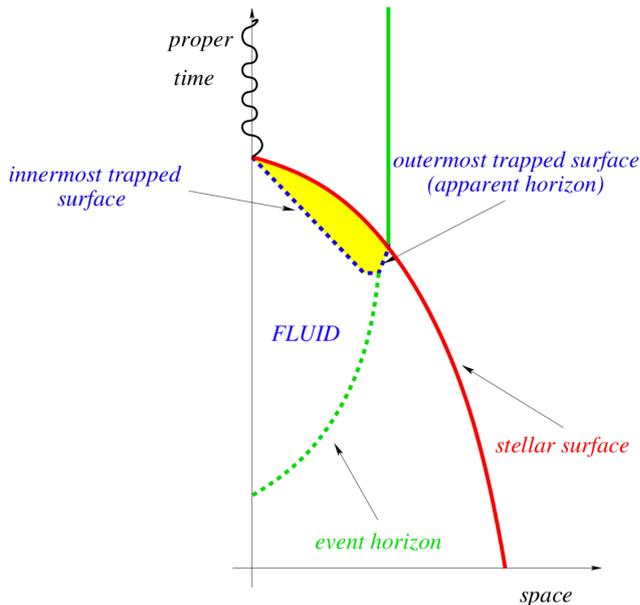

Figure 4: Schematic evolution of the relevant surfaces in the collapse of an ideal fluid (picture reference [22]).

Fig. 4 depicts the global szenario of a collapsing neutron star in a space-time diagram (for details see [21], page 612f). The radius R of the star is decreasing with time and at the point where the star's surface cross the event horizon (EH) the black hole arises. The event horizon actually comes into existence somewhat before the black hole is formed. The EH grows from zero size until it reaches $r = 2M$, which is exactly the point where the star's surface crosses the EH. After the formation of the EH in the stars center, two trapped surfaces form at a certain distance. The outermost trapped surface, the apparent horizon (AH) is always inside the event horizon (for details see [21], page 615f). In contrast to the EH, which is a global property of the whole manifold $\mathcal{M}$, the AH can be defined on each space-like hypersurface Σ_t.

3   Tasks

3.1   Task 1: Set Up Laboratory

In practice we will use the publicly available `Einstein Toolkit` [5] (see also [3] for an animation of the history of the code structure and [4] for recent developements) and compile and run the simulations on the CSC-FUCHS cluster (http://csc.uni-frankfurt.de/).

- Download the Einstein-Toolkit on your Fuchs account.

- Compile the code and perform a test run.

- Check the results and visualize some properties using gnuplot.

3.2   Task 2: Maximum mass of a TOV star

In this task the value for the maximum mass $M_{\max}$ of a non-rotating neutron star using a polytropic equation of state ($K = 100$, $\Gamma = 2$) is calculated. By knowing the maximum mass value, the maximal possible central density $e_{\max}$ and the minimal radius of the star $R_{\min}$ is also defined. These values separate the star's mass-radius relation into a stable and unstable branch. The following subtasks should be carried out:

- Please calculate the following values: $M_{\max}$ in units of $[M_\odot]$ and [km], $\rho_{\max}$ in [g/cm^3] and $R_{\min}$ in [km].

- Compare the maximum mass value with the known observational masses of neutron stars and discuss your results.

- Please plot the following diagrams: Mass-radius relation $M(R)$ in units of $[M_\odot]$-[km], mass-ρ_c diagram in $[M_\odot]$-$[g/cm^3]$ and mark the stable and unstable regions in both plots.

- Please show three different neutron star density profiles ($\rho(r)$, in units of [km]-$[g/cm^3]$) in one picture. Choose one profile within the stable region ($\rho_c < \rho_{max}$), one within the unstable branch ($\rho_c > \rho_{max}$) and one profile should be the profile of the maximum mass star ($\rho_c \equiv \rho_{max}$).

3.3 Task 3: Migration of an unstable neutron star to a stable configuration

Please use in this task an initial value of the central density, which is far beyond the stability threshold ($\rho_c >> \rho_{max}$) and evolve the neutron star with time. Show that by using the Einstein Toolkit, the natural evolution of this unstable neutron star follows a migration to a corresponding stable star which has a much smaller central density and a larger radius (see figure 5). The following subtasks should be carried out:

- Please show the time dependence of the central density ($\rho_c(t)$, in units of [ms]-$[g/cm^3]$) in order to visualize the migration of the unstable neutron star to the stable star.

- Please show the density profile of the initial unstable neutron star ($\rho(r, t = 0)$) and the final star at the endpoint of your simulation ($\rho(r, t = t_{end})$, in units of [km]-$[g/cm^3]$) in one picture.

Figure 5: The mass of the neutron star versus the central density (in units of $\rho_0 := 2.705\,10^{14}$ g/cm^3). An unstable neutron star could migrate to stable star or collapse to a black hole.

- In order to visualize the evolution of the star and the underlying space-time structure, please generate a contour-plot movie of the following quantities on the equatorial plane: $\rho(x, y, t)$ and $\alpha(x, y, t)$, with $t \in [0, t_{end}]$.

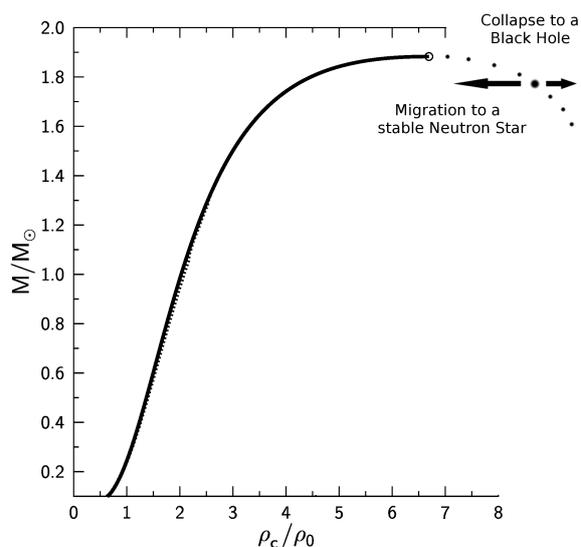## 3.4 Task 4: Collapse of a neutron star to a black hole

Please use in this task again an initial value of the central density which is beyond the stability threshold ($\rho_c > \rho_{max}$) and in addition, perturb the star's density profile to have an overall higher value than the profile of the TOV star. Show that by using the Einstein Toolkit, the evolution of this perturbed neutron star follows a gravitational collapse to a black hole. Please perform in addition a second simulation, where the parameter 'excision' is turned on.[1] The following subtasks should be carried out:

- To visualize the collapse, please show the time dependence of the central density ($\rho_c(t)$, in units of [ms]-$[g/cm^3]$) and the time evolution of the central value of the lapse function $\alpha_c(t)$.

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[1]If the parameter 'excision' is turned on, the program sets (after it has detected the apparent horizon for the first time) the density inside the apparent horizon to atmospheric values.

- Follow the evolution of the outermost and innermost apparent horizon and plot the radius of these two surfaces as a function of time.

- Plot a combined space-time diagram, where you display the outermost and innermost apparent horizon and the trajectories of three fixed density values. Choose for the lowest density a value which is near to the atmospheric density to show the time evolution of the surface of the neutron star.

- When will the event horizon be formed? Compare the radial position of the event horizon (R_{EH}) with the initial neutron star mass and check whether $R_{\text{EH}} = 2M$ - discuss your results.

4 Final Report

You have to submit the report within four weeks after performing the simulation. Please summarize what you have learned and include the results of all the tasks and a discussion of them in your final report.

References

[1] https://www.youtube.com/watch?v=__9Gq3U3mq4. Animation of the double pulsar.

[2] https://www.youtube.com/watch?v=oDgfqq_W_uM. Animation of the pulsar triple system.

[3] https://www.youtube.com/watch?v=EO4d32ch6OI. Animation of the history of the Einstein Toolkit.

[4] http://agenda.albanova.se/conferenceDisplay.py?confId=4936. Einstein Toolkit Workshop 2015.

[5] http://einsteintoolkit.org.

[6] J Aasi, BP Abbott, R Abbott, T Abbott, MR Abernathy, K Ackley, C Adams, T Adams, P Addesso, RX Adhikari, et al. Advanced ligo. *Classical and quantum gravity*, 32(7):074001, 2015.

[7] Peter Aufmuth and Karsten Danzmann. Gravitational wave detectors. *New Journal of Physics*, 7(1):202, 2005.

[8] J. M. Bardeen, K. S. Thorne, and D. W. Meltzer. A Catalogue of Methods for Studying the Normal Modes of Radial Pulsation of General-Relativistic Stellar Models. *Astrophys. J.*, 145:505, August 1966.

[9] T. W. Baumgarte and S. L. Shapiro. Numerical integration of Einstein's field equations. *Phys. Rev. D*, 59(2):024007, January 1999.

[10] RP Breton, VM Kaspi, MA McLaughlin, M Lyutikov, M Kramer, IH Stairs, SM Ransom, RD Ferdman, F Camilo, and A Possenti. The double pulsar eclipses. i. phenomenology and multi-frequency analysis. *The Astrophysical Journal*, 747(2):89, 2012.

[11] M Burgay, M Kramer, and MA McLaughlin. The double pulsar j0737–3039a/b: a decade of surprises. *Bulletin of the Astronomical Society of India*, 42:101–119, 2014.

[12] Neil Gehrels, G Chincarini, P Giommi, KO Mason, JA Nousek, AA Wells, NE White, SD Barthelmy, DN Burrows, LR Cominsky, et al. The swift gamma-ray burst mission. *The Astrophysical Journal*, 611(2):1005, 2004.

[13] Matthias Hanauske. Black holes and the german reichstag. *Physics World*, 18(10):64, 2005. `http://th.physik.uni-frankfurt.de/~hanauske/LateralThoughts.html`.

[14] Matthias Hanauske. *Eigenschaften von kompakten Sternen in QCD-motivierten Modellen*. PhD thesis, Johann Wolfgang Goethe-University Frankfurt am Main, 2005. `http://publikationen.ub.uni-frankfurt.de/frontdoor/index/index/docId/4937`.

[15] Matthias Hanauske and Walter Greiner. Neutron star properties in a qcd-motivated model. *General Relativity and Gravitation*, 33(5):739–755, 2001.

[16] M Kramer and IH Stairs. The double pulsar. *Annu. Rev. Astron. Astrophys.*, 46:541–572, 2008.

[17] Michael Kramer, Ingrid H Stairs, RN Manchester, MA McLaughlin, AG Lyne, RD Ferdman, M Burgay, DR Lorimer, A Possenti, N D'Amico, et al. Tests of general relativity from timing the double pulsar. *Science*, 314(5796):97–102, 2006.

[18] AG Lyne, Marta Burgay, M Kramer, A Possenti, RN Manchester, F Camilo, MA McLaughlin, DR Lorimer, N D'Amico, BC Joshi, et al. A double-pulsar system: A rare laboratory for relativistic gravity and plasma physics. *Science*, 303(5661):1153–1157, 2004.

[19] Charles W. Misner, Kip S. Thorne, and John A. Wheeler. *Gravitation*. W. H. Freeman, San Francisco, 1973.

[20] Scott M Ransom, IH Stairs, AM Archibald, JWT Hessels, DL Kaplan, MH van Kerkwijk, J Boyles, AT Deller, S Chatterjee, A Schechtman-Rook, et al. A millisecond pulsar in a stellar triple system. *Nature*, 505(7484):520–524, 2014.

[21] L. Rezzolla and O. Zanotti. *Relativistic Hydrodynamics*. Oxford University Press, Oxford, UK, 2013.

[22] Luciano Rezzolla. An introduction to gravitational collapse to black holes. `http://www.aei.mpg.de/~rezzolla/lnotes/mondragone/collapse.pdf`, 2004. Lectures given at the Villa Mondragone International School of Gravitation and Cosmology.

[23] Luciano Rezzolla, Bruno Giacomazzo, Luca Baiotti, Jonathan Granot, Chryssa Kouveliotou, and Miguel A Aloy. The missing link: Merging neutron stars naturally produce jet-like structures and can power short gamma-ray bursts. *The Astrophysical Journal Letters*, 732(1):L6, 2011.

[24] M. Shibata and T. Nakamura. Evolution of three-dimensional gravitational waves: Harmonic slicing case. *Phys. Rev. D*, 52:5428–5444, November 1995.