Deriving the Kerr metric

(i.e. paint-by-numbers feat. Chandrasekhar)

David Wagner

dwagner@th.physik.uni-frankfurt.de

September 23, 2021

In this document we will attempt to derive the Kerr metric (in natural units $G = c = 1$), describing spacetime outside a uniformly rotating massive body, e.g. a black hole. This calculation is by no means simple and will take a while. We will follow Chandrasekhar tightly, who presented a derivation in his paper from 1978 [1].

Contents

1 Preliminaries and goal

In general the line element ds^2 is given as

$$ds^2 = dx^\mu dx_\mu = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{1}$$

Since we want a vacuum solution for an axisymmetric mass distribution (this differs from spherical symmetry in such a way that it incorporates flattening due to centrifugal forces, such that azimuthal symmetry is broken), we can simplify this generic form considerably. Firstly, we will for now work in spherical coordinates t, r, θ, ϕ. Now due to the rotational symmetry and the fact that spacetime should be static, we can infer that ds^2 should be invariant under changes $t \to -t, \phi \to -\phi$ (this corresponds to reversing the time direction and thus the rotation). This excludes the $(t,\theta), (t,r), (\theta,\phi), (r,\phi)$ components of the metric, leaving only the $(t,t), (r,r), (\theta,\theta), (\phi,\phi)$ and (t,ϕ) components. Notice that in contrast to the Schwarzschild case, there is one off-diagonal term.
Furthermore, due to axisymmetry the metric should only depend on r and θ (just as in electrodynamics). This dependence in combination with the off-diagonal term is what complicates the field equations immensely, but will not deter us from obtaining a solution.
So far we thus have reduced the line element to

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2 + g_{t\phi} dt d\phi \ . \tag{2}$$

After rewriting $g_{tt} := -e^{2\nu} + e^{2\psi}\omega^2$, $g_{t\phi} := -2\omega e^{2\psi}$, $g_{\phi\phi} := e^{2\psi}$, $g_{rr} := e^{2\mu_2}$ and $g_{\theta\theta} := e^{2\mu_3}$ (where the terms μ_2 and μ_3 originate from Chandrasekhar numbering of coordinates), we can write

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_2} dr^2 + e^{2\mu_3} d\theta^2 \ , \tag{3}$$

where ν, ψ, ω, μ_2 and μ_3 are functions of r and θ. Notice that we are obviously working to get a result in the $(-+++)$ convention; this however, can in the end easily reverted by multiplying with -1.
In the end, our goal will be to obtain the following values for the interesting functions:

$$e^{2\nu} = \frac{\rho^2 \Delta}{\Sigma^2}$$

$$
\begin{aligned}
e^{2\psi} =& \frac{\Sigma^2 \sin^2(\theta)}{\rho^2} \\
e^{2\mu_2} =& \frac{\rho^2}{\Delta} \\
e^{2\mu_3} =& \rho^2 \\
\omega =& \frac{2aMr}{\Sigma^2}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta =& r^2 + a^2 - 2Mr \\
\rho^2 =& r^2 + a^2 \cos^2(\theta) \\
\Sigma^2 =& (r^2 + a^2)^2 - a^2 \Delta \sin^2(\theta) \; .
\end{aligned}
$$

2 Basic quantities characterising spacetime

The Einstein field equations in vacuum imply that the Ricci-Tensor vanishes:

$$R_{\mu\nu} = 0 \; . \tag{4}$$

Now one could go on to explicitly compute the different components of the Ricci tensor using the general metric given by Eq. 3, however this is not practical, as the equations become unwieldy (which is one of the reasons why it has taken nearly 50 years to arrive at an axisymmetric solution).
Instead we use Cartan's calculus of exterior forms and apply its structure equations. Firstly, we decompose the metric into tetrads as

$$g_{\mu\nu} := \mathbf{e}^a_\mu \mathbf{e}^b_\nu \eta_{ab} \; .$$

Here latin indices are raised and lowered by the Minkowski metric η_{ab} (here defined in the $(-+++)$ convention for reasons mentioned above). These tetrads are matrices transforming the coordinate basis dx^μ to an orthonormal basis of the cotangent space $T^*_x M$ of the spacetime manifold M at point x [3]

$$\mathbf{e}^a := \mathbf{e}^a_\mu dx^\mu \; .$$

Furthermore, their inverse $\mathbf{E}^\mu_a$ transforms to an orthonormal basis of the respective tangent space $T_x M$

$$\mathbf{E}_a := \mathbf{E}^\mu_a \partial_\mu \; .$$

The tetrads thus give a coordinate system at each spacetime point that is locally flat and thus easier to work with. The price to pay is of course that the basis vectors are now dependent on the spacetime point x we evaluate them at.
We now need to find the connection ω_{ab} on our manifold, which is related to the Christoffel symbols and determines the curvature.
Since we assume a torsion-free connection (i.e. the absence of spin), the connection has to fulfil the Cartan structure equations:

$$0 = d\mathbf{e}^a + \omega^a{}_b \wedge \mathbf{e}^b \tag{5}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \; . \tag{6}$$

From Eq. 3 we can read off the tetrad one-forms as

$$
\begin{aligned}
\mathbf{e}^0 =& e^\nu dt & dt =& e^{-\nu} \mathbf{e}^0 \\
\mathbf{e}^1 =& e^\psi (d\phi - \omega dt) & d\phi =& \omega e^{-\nu} \mathbf{e}^0 + e^{-\psi} \mathbf{e}^1
\end{aligned}
$$

$$\mathbf{e}^2 = e^{\mu_2} dr \qquad dr = e^{-\mu_2}\mathbf{e}^2$$
$$\mathbf{e}^3 = e^{\mu_3} d\theta \qquad d\theta = e^{-\mu_3}\mathbf{e}^3 \; .$$

The exterior derivative of a one-form reads

$$d \circ (f_i dx^i) = \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^i \; . \tag{7}$$

For the zeroth differential form, we get

$$\begin{aligned} d \circ \mathbf{e}^0 =& d \circ (e^\nu dt) \\ =& e^\nu \left[\frac{\partial \nu}{\partial r} dr \wedge dt + \frac{\partial \nu}{\partial \theta} d\theta \wedge dt \right] \\ =& (\partial_r \nu) e^{-\mu_2} \mathbf{e}^2 \wedge \mathbf{e}^0 + (\partial_\theta \nu) e^{-\mu_3} \mathbf{e}^3 \wedge \mathbf{e}^0 \; . \end{aligned} \tag{8}$$

Likewise, we obtain for the first form

$$\begin{aligned} d \circ \mathbf{e}^1 =& d \circ \left[e^\psi d\phi - e^\psi \omega dt \right] \\ =& e^\psi \left[\partial_r \psi dr \wedge d\phi + \partial_\theta \psi d\theta \wedge d\phi - (\omega \partial_r \psi + \partial_r \omega)\, dr \wedge dt - (\omega \partial_\theta \psi + \partial_\theta \omega)\, d\theta \wedge dt \right] \\ =& e^\psi \Big[\partial_r \psi e^{-\mu_2} \mathbf{e}^2 \wedge \left(\omega e^{-\nu} \mathbf{e}^0 + e^{-\psi} \mathbf{e}^1 \right) + \partial_\theta \psi e^{-\mu_3} \mathbf{e}^3 \wedge \left(\omega e^{-\nu} \mathbf{e}^0 + e^{-\psi} \mathbf{e}^1 \right) \\ & - (\omega \partial_r \psi + \partial_r \omega)\, e^{-\mu_2} \mathbf{e}^2 \wedge e^{-\nu} \mathbf{e}^0 - (\omega \partial_\theta \psi + \partial_\theta \omega)\, e^{-\mu_3} \mathbf{e}^3 \wedge e^{-\nu} \mathbf{e}^0 \Big] \\ =& \partial_r \psi e^{-\mu_2} \mathbf{e}^2 \wedge \mathbf{e}^1 + \partial_\theta \psi e^{-\mu_3} \mathbf{e}^3 \wedge \mathbf{e}^1 - \partial_r \omega e^{\psi - \nu - \mu_2} \mathbf{e}^2 \wedge \mathbf{e}^0 - \partial_\theta \omega e^{\psi - \nu - \mu_3} \mathbf{e}^3 \wedge \mathbf{e}^0 \; . \end{aligned} \tag{9}$$

The differential of the second one-form reads

$$\begin{aligned} d\mathbf{e}^2 =& e^{\mu_2} \left[\partial_r \mu_2 dr \wedge dr + \partial_\theta \mu_2 d\theta \wedge dr \right] \\ =& e^{-\mu_3} \partial_\theta \mu_2 \mathbf{e}^3 \wedge \mathbf{e}^2 \; . \end{aligned} \tag{10}$$

Here we used that the wedge product of a one-form with itself vanishes, since it is antisymmetric. The differential of the last one-form gives

$$\begin{aligned} d\mathbf{e}^3 =& e^{\mu_3} \left[\partial_r \mu_3 dr \wedge d\theta + \partial_\theta \mu_3 d\theta \wedge d\theta \right] \\ =& e^{-\mu_2} \partial_r \mu_3 \mathbf{e}^2 \wedge \mathbf{e}^3 \; . \end{aligned} \tag{11}$$

Combining these differentials with the first structure equation, we obtain

$$-\omega^0{}_b \wedge \mathbf{e}^b = (\partial_r \nu) e^{-\mu_2} \mathbf{e}^2 \wedge \mathbf{e}^0 + (\partial_\theta \nu) e^{-\mu_3} \mathbf{e}^3 \wedge \mathbf{e}^0 \tag{12}$$
$$-\omega^1{}_b \wedge \mathbf{e}^b = \partial_r \psi e^{-\mu_2} \mathbf{e}^2 \wedge \mathbf{e}^1 + \partial_\theta \psi e^{-\mu_3} \mathbf{e}^3 \wedge \mathbf{e}^1 - \partial_r \omega e^{\psi - \nu - \mu_2} \mathbf{e}^2 \wedge \mathbf{e}^0 - \partial_\theta \omega e^{\psi - \nu - \mu_3} \mathbf{e}^3 \wedge \mathbf{e}^0 \tag{13}$$
$$-\omega^2{}_b \wedge \mathbf{e}^b = e^{-\mu_3} \partial_\theta \mu_2 \mathbf{e}^3 \wedge \mathbf{e}^2 \tag{14}$$
$$-\omega^3{}_b \wedge \mathbf{e}^b = e^{-\mu_2} \partial_r \mu_3 \mathbf{e}^2 \wedge \mathbf{e}^3 \; . \tag{15}$$

Furthermore, $\omega_{ab} = -\omega_{ba}$ has to be fulfilled (this is equivalent to the metric compatibility condition), such that ω_{ab} has six independent components. To solve these equations, it is helpful decomposing the connections into their respective tetrad components:

$$\omega^a{}_b = (\omega^a{}_b)_c \mathbf{e}^c \; . \tag{16}$$

This decomposition gives (due to the fact that the tetrads are orthogonal) 24 equations for 24 variables. From the first two equations we get

$$(\omega^0{}_1)_0 = 0 \qquad (\omega^1{}_0)_1 = 0$$

$$
\begin{aligned}
(\omega^0{}_2)_0 &= e^{-\mu_2}\partial_r\nu \\
(\omega^0{}_3)_0 &= e^{-\mu_3}\partial_\theta\nu \\
(\omega^0{}_1)_2 - (\omega^0{}_2)_1 &= 0 \\
(\omega^0{}_2)_3 - (\omega^0{}_3)_2 &= 0 \\
(\omega^0{}_3)_1 - (\omega^0{}_1)_3 &= 0
\end{aligned}
\qquad
\begin{aligned}
(\omega^1{}_0)_2 - (\omega^1{}_2)_0 &= e^{\psi-\nu-\mu_2}\partial_r\omega \\
(\omega^1{}_0)_3 - (\omega^1{}_3)_0 &= e^{\psi-\nu-\mu_3}\partial_\theta\omega \\
(\omega^1{}_2)_1 &= e^{-\mu_2}\partial_r\psi \\
(\omega^1{}_3)_2 - (\omega^1{}_2)_3 &= 0 \\
(\omega^1{}_3)_1 &= e^{-\mu_3}\partial_\theta\psi \ ,
\end{aligned}
$$

while the third and fourth equations give us

$$
\begin{aligned}
(\omega^2{}_0)_1 - (\omega^2{}_1)_0 &= 0 \\
(\omega^2{}_0)_2 &= 0 \\
(\omega^2{}_0)_3 - (\omega^2{}_3)_0 &= 0 \\
(\omega^2{}_1)_2 &= 0 \\
(\omega^2{}_3)_2 &= e^{-\mu_3}\partial_\theta\mu_2 \\
(\omega^2{}_1)_3 - (\omega^2{}_3)_1 &= 0
\end{aligned}
\qquad
\begin{aligned}
(\omega^3{}_0)_1 - (\omega^3{}_1)_0 &= 0 \\
(\omega^3{}_0)_2 - (\omega^3{}_2)_0 &= 0 \\
(\omega^3{}_0)_3 &= 0 \\
(\omega^3{}_1)_2 - (\omega^3{}_2)_1 &= 0 \\
(\omega^3{}_2)_3 &= e^{-\mu_2}\partial_r\mu_3 \\
(\omega^3{}_1)_3 &= 0 \ .
\end{aligned}
$$

The solution of this system of equations reads

$$\omega^0{}_1 = \omega^1{}_0 = \frac{1}{2}e^{\psi-\nu-\mu_2}\partial_r\omega\mathbf{e}^2 + \frac{1}{2}e^{\psi-\nu-\mu_3}\partial_\theta\omega\mathbf{e}^3 \tag{17}$$

$$\omega^0{}_2 = \omega^2{}_0 = e^{-\mu_2}\partial_r\nu\mathbf{e}^0 + \frac{1}{2}e^{\psi-\nu-\mu_2}\partial_r\omega\mathbf{e}^1 \tag{18}$$

$$\omega^0{}_3 = \omega^3{}_0 = e^{-\mu_3}\partial_\theta\nu\mathbf{e}^0 + \frac{1}{2}e^{\psi-\nu-\mu_3}\partial_\theta\omega\mathbf{e}^1 \tag{19}$$

$$\omega^1{}_2 = -\omega^2{}_1 = -\frac{1}{2}e^{\psi-\nu-\mu_2}\partial_r\omega\mathbf{e}^0 + e^{-\mu_2}\partial_r\psi\mathbf{e}^1 \tag{20}$$

$$\omega^1{}_3 = -\omega^3{}_1 = -\frac{1}{2}e^{\psi-\nu-\mu_3}\partial_\theta\omega\mathbf{e}^0 + e^{-\mu_3}\partial_\theta\psi\mathbf{e}^1 \tag{21}$$

$$\omega^2{}_3 = -\omega^3{}_2 = e^{-\mu_3}\partial_\theta\mu_2\mathbf{e}^2 - e^{-\mu_2}\partial_r\mu_3\mathbf{e}^3 \ . \tag{22}$$

Next we employ Eq. 6 to construct the components of the curvature 2-form, which will give us the Ricci tensor at once.
Explicitly we have to evaluate

$$
\begin{aligned}
R^0{}_1 = R^1{}_0 &= d\omega^0{}_1 + \omega^0{}_b \wedge \omega^b{}_1 \\
R^0{}_2 = R^2{}_0 &= d\omega^0{}_2 + \omega^0{}_b \wedge \omega^b{}_2 \\
R^0{}_3 = R^3{}_0 &= d\omega^0{}_3 + \omega^0{}_b \wedge \omega^b{}_3 \\
R^1{}_2 = -R^2{}_1 &= d\omega^1{}_2 + \omega^1{}_b \wedge \omega^b{}_2 \\
R^1{}_3 = -R^3{}_1 &= d\omega^1{}_3 + \omega^1{}_b \wedge \omega^b{}_3 \\
R^2{}_3 = -R^3{}_2 &= d\omega^2{}_3 + \omega^2{}_b \wedge \omega^b{}_3 \ .
\end{aligned}
$$

Since we will spell out everything here in detail to give a thorough calculation, we will start by evaluating the exterior derivatives:

$$d\omega^0{}_1 = \frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\left(\partial_\theta\psi - \partial_\theta\nu\right)\partial_r\omega + \partial_r^2\omega - \left(\partial_r\psi - \partial_r\nu\right)\partial_\theta\omega - \partial_\theta^2\omega\right]\mathbf{e}^3 \wedge \mathbf{e}^2 \ . \tag{23}$$

The second form is more work:

$$d\omega^0{}_2 = e^{-2\mu_2}\left[\partial_r(\nu-\mu_2)\partial_r\nu + \partial_r^2\nu\right]\mathbf{e}^2 \wedge \mathbf{e}^0 + e^{-\mu_2-\mu_3}\left[\partial_\theta(\nu-\mu_2)\partial_r\nu + \partial_r\partial_\theta\nu\right]\mathbf{e}^3 \wedge \mathbf{e}^0$$

$$
\begin{aligned}
&+\frac{1}{2}e^{2\psi-\nu-\mu_2}\Big\{\left[\partial_r(2\psi-\nu-\mu_2)\partial_r\omega+\partial_r^2\omega\right]dr\wedge d\phi\\
&+\left[\partial_\theta(2\psi-\nu-\mu_2)\partial_r\omega+\partial_\theta\partial_r\omega\right]d\theta\wedge d\phi-\left[\partial_r(2\psi-\nu-\mu_2)\omega\partial_r\omega+(\partial_r\omega)^2+\omega\partial_r^2\omega\right]dr\wedge dt\\
&+\left[\partial_\theta(2\psi-\nu-\mu_2)\omega\partial_r\omega+\partial_\theta\omega\partial_r\omega+\omega\partial_\theta\partial_r\omega\right]d\theta\wedge dt\Big\}\\
=&e^{-2\mu_2}\left[\partial_r(\nu-\mu_2)\partial_r\nu+\partial_r^2\nu\right]\mathbf{e}^2\wedge\mathbf{e}^0+e^{-\mu_2-\mu_3}\left[\partial_\theta(\nu-\mu_2)\partial_r\nu+\partial_r\partial_\theta\nu\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+\frac{1}{2}e^{2\psi-\nu-\mu_2}\Big\{\left[\partial_r(2\psi-\nu-\mu_2)\partial_r\omega+\partial_r^2\omega\right]e^{-\mu_2}\left[\omega e^{-\nu}\mathbf{e}^2\wedge\mathbf{e}^0+e^{-\psi}\mathbf{e}^2\wedge\mathbf{e}^1\right]\\
&+\left[\partial_\theta(2\psi-\nu-\mu_2)\partial_r\omega+\partial_\theta\partial_r\omega\right]e^{-\mu_3}\left[\omega e^{-\nu}\mathbf{e}^3\wedge\mathbf{e}^0+e^{-\psi}\mathbf{e}^3\wedge\mathbf{e}^1\right]\\
&-\left[\partial_r(2\psi-\nu-\mu_2)\omega\partial_r\omega+(\partial_r\omega)^2+\omega\partial_r^2\omega\right]e^{-\mu_2-\nu}\mathbf{e}^2\wedge\mathbf{e}^0\\
&+\left[\partial_\theta(2\psi-\nu-\mu_2)\omega\partial_r\omega+\partial_\theta\omega\partial_r\omega+\omega\partial_\theta\partial_r\omega\right]e^{-\mu_3-\nu}\mathbf{e}^3\wedge\mathbf{e}^0\Big\}\\
=&e^{-2\mu_2}\left[\partial_r(\nu-\mu_2)\partial_r\nu+\partial_r^2\nu-(\partial_r\omega)^2\frac{1}{2}e^{2\psi-2\nu}\right]\mathbf{e}^2\wedge\mathbf{e}^0\\
&+e^{-\mu_2-\mu_3}\left[\partial_\theta(\nu-\mu_2)\partial_r\nu+\partial_r\partial_\theta\nu-\partial_\theta\omega\partial_r\omega\frac{1}{2}e^{2\psi-2\nu}\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+\frac{1}{2}e^{\psi-\nu-2\mu_2}\left[\partial_r(2\psi-\nu-\mu_2)\partial_r\omega+\partial_r^2\omega\right]\mathbf{e}^2\wedge\mathbf{e}^1\\
&+\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_\theta(2\psi-\nu-\mu_2)\partial_r\omega+\partial_\theta\partial_r\omega\right]\mathbf{e}^3\wedge\mathbf{e}^1\ .
\end{aligned}
\tag{24}
$$

The third one-form looks similar:

$$
\begin{aligned}
d\omega^0{}_3=&e^{-\mu_2-\mu_3}\left[\partial_r(\nu-\mu_3)\partial_\theta\nu+\partial_r\partial_\theta\nu\right]\mathbf{e}^2\wedge\mathbf{e}^0+e^{-2\mu_3}\left[\partial_\theta(\nu-\mu_3)\partial_\theta\nu+\partial_\theta^2\nu\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+\frac{1}{2}e^{2\psi-\nu-\mu_3}\Big\{\left[\partial_r(2\psi-\nu-\mu_3)\partial_\theta\omega+\partial_r\partial_\theta\omega\right]dr\wedge d\phi\\
&+\left[\partial_\theta(2\psi-\nu-\mu_3)\partial_\theta\omega+\partial_\theta^2\omega\right]d\theta\wedge d\phi-\left[\partial_r(2\psi-\nu-\mu_3)\omega\partial_\theta\omega+\partial_r\omega\partial_\theta\omega+\omega\partial_r\partial_\theta\omega\right]dr\wedge dt\\
&+\left[\partial_\theta(2\psi-\nu-\mu_3)\omega\partial_\theta\omega+(\partial_\theta\omega)^2+\omega\partial_\theta^2\omega\right]d\theta\wedge dt\Big\}\\
=&e^{-\mu_2-\mu_3}\left[\partial_r(\nu-\mu_3)\partial_\theta\nu+\partial_r\partial_\theta\nu\right]\mathbf{e}^2\wedge\mathbf{e}^0+e^{-2\mu_3}\left[\partial_\theta(\nu-\mu_3)\partial_\theta\nu+\partial_\theta^2\nu\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+\frac{1}{2}e^{2\psi-\nu-\mu_3}\Big\{\left[\partial_r(2\psi-\nu-\mu_3)\partial_\theta\omega+\partial_r\partial_\theta\omega\right]e^{-\mu_2}\left[\omega e^{-\nu}\mathbf{e}^2\wedge\mathbf{e}^0+e^{-\psi}\mathbf{e}^2\wedge\mathbf{e}^1\right]\\
&+\left[\partial_\theta(2\psi-\nu-\mu_3)\partial_\theta\omega+\partial_\theta^2\omega\right]e^{-\mu_3}\left[\omega e^{-\nu}\mathbf{e}^3\wedge\mathbf{e}^0+e^{-\psi}\mathbf{e}^3\wedge\mathbf{e}^1\right]\\
&-\left[\partial_r(2\psi-\nu-\mu_3)\omega\partial_\theta\omega+\partial_r\omega\partial_\theta\omega+\omega\partial_r\partial_\theta\omega\right]e^{-\mu_2-\nu}\mathbf{e}^2\wedge\mathbf{e}^0\\
&+\left[\partial_\theta(2\psi-\nu-\mu_3)\omega\partial_\theta\omega+(\partial_\theta\omega)^2+\omega\partial_\theta^2\omega\right]e^{-\mu_3-\nu}\mathbf{e}^3\wedge\mathbf{e}^0\Big\}\\
=&e^{-\mu_2-\mu_3}\left[\partial_r(\nu-\mu_3)\partial_\theta\nu+\partial_r\partial_\theta\nu-\partial_r\omega\partial_\theta\omega\frac{1}{2}e^{2\psi-2\nu}\right]\mathbf{e}^2\wedge\mathbf{e}^0\\
&+e^{-2\mu_3}\left[\partial_\theta(\nu-\mu_3)\partial_\theta\nu+\partial_\theta^2\nu-(\partial_\theta\omega)^2\frac{1}{2}e^{2\psi-2\nu}\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_r(2\psi-\nu-\mu_3)\partial_\theta\omega+\partial_r\partial_\theta\omega\right]\mathbf{e}^2\wedge\mathbf{e}^1\\
&+\frac{1}{2}e^{\psi-\nu-2\mu_3}\left[\partial_\theta(2\psi-\nu-\mu_3)\partial_\theta\omega+\partial_\theta^2\omega\right]\mathbf{e}^3\wedge\mathbf{e}^1\ .
\end{aligned}
\tag{25}
$$

The fourth one-form's differential gives

$$\begin{aligned}
d\omega^1{}_2 =& -\frac{1}{2}e^{\psi-\nu-2\mu_2}\left[\partial_r(\psi-\mu_2)\partial_r\omega+\partial_r^2\omega\right]\mathbf{e}^2\wedge\mathbf{e}^0-\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_\theta(\psi-\mu_2)\partial_r\omega+\partial_r\partial_\theta\omega\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+e^{\psi-\mu_2}\Big\{\left[\partial_r(\psi-\mu_2)\partial_r\psi+\partial_r^2\psi\right]e^{-\mu_2}\left[\omega e^{-\nu}\mathbf{e}^2\wedge\mathbf{e}^0+e^{-\psi}\mathbf{e}^2\wedge\mathbf{e}^1\right]\\
&+\left[\partial_\theta(\psi-\mu_2)\partial_r\psi+\partial_r\partial_\theta\psi\right]e^{-\mu_3}\left[\omega e^{-\nu}\mathbf{e}^3\wedge\mathbf{e}^0+e^{-\psi}\mathbf{e}^3\wedge\mathbf{e}^1\right]\\
&-\left[\partial_r(\psi-\mu_2)\partial_r\psi\omega+\partial_r^2\psi\omega+\partial_r\psi\partial_r\omega\right]e^{-\nu-\mu_2}\mathbf{e}^2\wedge\mathbf{e}^0\\
&-\left[\partial_\theta(\psi-\mu_2)\partial_r\psi\omega+\partial_r\partial_\theta\psi\omega+\partial_r\psi\partial_\theta\omega\right]e^{-\nu-\mu_3}\mathbf{e}^3\wedge\mathbf{e}^0\Big\}\\
=& -e^{\psi-\nu-\mu_2}\left\{\frac{1}{2}e^{-\mu_2}\left[\partial_r(\psi-\mu_2)\partial_r\omega+\partial_r^2\omega\right]+e^{-\mu_2}\partial_r\psi\partial_r\omega\right\}\mathbf{e}^2\wedge\mathbf{e}^0\\
&-e^{\psi-\nu-\mu_3}\left\{\frac{1}{2}e^{-\mu_2}\left[\partial_\theta(\psi-\mu_2)\partial_r\omega+\partial_r\partial_\theta\omega\right]+e^{-\mu_2}\partial_r\psi\partial_\theta\omega\right\}\mathbf{e}^3\wedge\mathbf{e}^0\\
&+e^{-\mu_2-\mu_3}\left[\partial_\theta(\psi-\mu_2)\partial_r\psi+\partial_r\partial_\theta\psi\right]\mathbf{e}^3\wedge\mathbf{e}^1+e^{-2\mu_2}\left[\partial_r(\psi-\mu_2)\partial_r\psi+\partial_r^2\psi\right]\mathbf{e}^2\wedge\mathbf{e}^1\ . \qquad (26)
\end{aligned}$$

The fifth form is very similar:

$$\begin{aligned}
d\omega^1{}_3 =& -\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_r(\psi-\mu_3)\partial_\theta\omega+\partial_r\partial_\theta\omega+\partial_\theta\psi\partial_r\omega\right]\mathbf{e}^2\wedge\mathbf{e}^0\\
&-\frac{1}{2}e^{\psi-\nu-2\mu_3}\left[\partial_\theta(\psi-\mu_3)\partial_\theta\omega+\partial_\theta^2\omega+\partial_\theta\psi\partial_\theta\omega\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+e^{-\mu_2-\mu_3}\left[\partial_r(\psi-\mu_3)\partial_\theta\psi+\partial_r\partial_\theta\psi\right]\mathbf{e}^2\wedge\mathbf{e}^1+e^{-2\mu_3}\left[\partial_\theta(\psi-\mu_3)\partial_\theta\psi+\partial_\theta^2\psi\right]\mathbf{e}^3\wedge\mathbf{e}^1\ . \qquad (27)
\end{aligned}$$

For the last form's exterior derivative we obtain

$$d\omega^2{}_3 = e^{-\mu_2-\mu_3}\left\{e^{\mu_2-\mu_3}\left[\partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2+\partial_\theta^2\mu_2\right]+e^{\mu_3-\mu_2}\left[\partial_r(\mu_2-\mu_3)\partial_r\mu_3+\partial_r^2\mu_3\right]\right\}\mathbf{e}^3\wedge\mathbf{e}^2\ . \qquad (28)$$

Next we evaluate the wedge products appearing in the second structure equations one by one:

$$\begin{aligned}
\omega^0{}_b\wedge\omega^b{}_1 =& \omega^0{}_2\wedge\omega^2{}_1+\omega^0{}_3\wedge\omega^3{}_1\\
=& \left[\left(\frac{1}{2}e^{\psi-\nu-\mu_2}\partial_r\omega\right)^2+\left(\frac{1}{2}e^{\psi-\nu-\mu_3}\partial_\theta\omega\right)^2+e^{-2\mu_2}\partial_r\nu\partial_r\psi+e^{-2\mu_3}\partial_\theta\nu\partial_\theta\psi\right]\mathbf{e}^1\wedge\mathbf{e}^0\ , \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\omega^0{}_b\wedge\omega^b{}_2 =& \omega^0{}_1\wedge\omega^1{}_2+\omega^0{}_3\wedge\omega^3{}_2\\
=& \left[e^{-2\mu_3}\partial_\theta\nu\partial_\theta\mu_2-\left(\frac{1}{2}e^{\psi-\nu-\mu_2}\partial_r\omega\right)^2\right]\mathbf{e}^2\wedge\mathbf{e}^0\\
&-e^{-\mu_2-\mu_3}\left[\partial_r\mu_3\partial_\theta\nu+\left(\frac{1}{2}e^{\psi-\nu}\right)^2\partial_r\omega\partial_\theta\omega\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[e^{\mu_2-\mu_3}\partial_\theta\omega\partial_\theta\mu_2+e^{\mu_3-\mu_2}\partial_r\omega\partial_r\psi\right]\mathbf{e}^2\wedge\mathbf{e}^1\\
&+\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[-\partial_\theta\omega\partial_r\mu_3+\partial_\theta\omega\partial_r\psi\right]\mathbf{e}^3\wedge\mathbf{e}^1\ , \qquad (30)
\end{aligned}$$

$$\begin{aligned}
\omega^0{}_b\wedge\omega^b{}_3 =& \omega^0{}_1\wedge\omega^1{}_3+\omega^0{}_2\wedge\omega^2{}_3\\
=& -e^{-\mu_2-\mu_3}\left[\left(\frac{1}{2}e^{\psi-\nu}\right)^2\partial_r\omega\partial_\theta\omega+\partial_r\nu\partial_\theta\mu_2\right]\mathbf{e}^2\wedge\mathbf{e}^0\\
&+\left[-\left(\frac{1}{2}e^{\psi-\nu-\mu_3}\partial_\theta\omega\right)^2+e^{-2\mu_2}\partial_r\nu\partial_r\mu_3\right]\mathbf{e}^3\wedge\mathbf{e}^0
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_r\omega\partial_\theta\psi-\partial_r\omega\partial_\theta\mu_2\right]\mathbf{e}^2\wedge\mathbf{e}^1\\
&+\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[e^{\mu_3-\mu_2}\partial_r\omega\partial_r\mu_3+e^{\mu_2-\mu_3}\partial_\theta\omega\partial_\theta\psi\right]\mathbf{e}^3\wedge\mathbf{e}^1\ ,
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\omega^1{}_b\wedge\omega^b{}_2 =&\omega^1{}_0\wedge\omega^0{}_2+\omega^1{}_3\wedge\omega^3{}_2\\
=&\frac{1}{2}e^{\psi-\nu}\left[e^{-2\mu_2}\partial_r\omega\partial_r\nu-e^{-2\mu_3}\partial_\theta\omega\partial_\theta\mu_2\right]\mathbf{e}^2\wedge\mathbf{e}^0+\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\partial_\theta\omega\left[\partial_r\nu+\partial_r\mu_3\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+\left[\left(\frac{1}{2}e^{\psi-\nu-\mu_2}\partial_r\omega\right)^2+e^{-2\mu_3}\partial_\theta\mu_2\partial_\theta\psi\right]\mathbf{e}^2\wedge\mathbf{e}^1\\
&+\left[\left(\frac{1}{2}e^{\psi-\nu}\right)^2e^{-\mu_2-\mu_3}\partial_\theta\omega\partial_r\omega-e^{-\mu_2-\mu_3}\partial_r\mu_3\partial_\theta\psi\right]\mathbf{e}^3\wedge\mathbf{e}^1\ ,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\omega^1{}_b\wedge\omega^b{}_3 =&\omega^1{}_0\wedge\omega^0{}_3+\omega^1{}_2\wedge\omega^2{}_3\\
=&\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\partial_r\omega\left[\partial_\theta\nu+\partial_\theta\mu_2\right]\mathbf{e}^2\wedge\mathbf{e}^0+\frac{1}{2}e^{\psi-\nu}\left[e^{-2\mu_3}\partial_\theta\omega\partial_\theta\nu-e^{-2\mu_2}\partial_r\omega\partial_r\mu_3\right]\mathbf{e}^3\wedge\mathbf{e}^0\\
&+\left[\left(\frac{1}{2}e^{\psi-\nu}\right)^2e^{-\mu_2-\mu_3}\partial_r\omega\partial_\theta\omega-e^{-\mu_2-\mu_3}\partial_\theta\mu_2\partial_r\psi\right]\mathbf{e}^2\wedge\mathbf{e}^1\\
&+\left[\left(\frac{1}{2}e^{\psi-\nu-\mu_3}\partial_\theta\omega\right)^2+e^{-2\mu_2}\partial_r\mu_3\partial_r\psi\right]\mathbf{e}^3\wedge\mathbf{e}^1\ ,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\omega^2{}_b\wedge\omega^b{}_3 =&\omega^2{}_0\wedge\omega^0{}_3+\omega^2{}_1\wedge\omega^1{}_3\\
=&\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_r\omega\partial_\theta\nu-\partial_r\nu\partial_\theta\omega+\partial_r\psi\partial_\theta\omega+\partial_\theta\psi\partial_r\omega\right]\mathbf{e}^1\wedge\mathbf{e}^0\ .
\end{aligned}
\tag{34}
$$

Having computed all of this, we are able to read off the Riemann tensor by considering how it is related to the curvature two-form:

$$
R^a{}_b=R^a{}_{bcd}\mathbf{e}^c\wedge\mathbf{e}^d\ . \tag{35}
$$

- **Note:** There might be a factor of 1/2 missing here, which will not bother us anyway, however, since we are looking for vacuum solutions.

Furthermore, we should remember some symmetries of the Riemann tensor:

$$
R_{abcd}=-R_{bacd}=-R_{abdc}=R_{cdab}\ .
$$

We obtain thus

$$
R^0{}_{110}=e^{-\mu_2-\mu_3}\left\{\frac{1}{4}e^{2\psi-2\nu}\left[e^{\mu_3-\mu_2}(\partial_r\omega)^2+e^{\mu_2-\mu_3}(\partial_\theta\omega)^2\right]+e^{\mu_3-\mu_2}\partial_r\nu\partial_r\psi+e^{\mu_2-\mu_3}\partial_\theta\nu\partial_\theta\psi\right\} \tag{36}
$$

$$
R^0{}_{132}=\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\left(\partial_\theta\psi-\partial_\theta\nu\right)\partial_r\omega+\partial_r^2\omega-\left(\partial_r\psi-\partial_r\nu\right)\partial_\theta\omega-\partial_\theta^2\omega\right] \tag{37}
$$

$$
R^0{}_{220}=e^{-\mu_2-\mu_3}\left\{e^{\mu_3-\mu_2}\left[\partial_r(\nu-\mu_2)\partial_r\nu+\partial_r^2\nu\right]-(\partial_r\omega)^2\frac{3}{4}e^{2\psi-2\nu+\mu_3-\mu_2}+e^{\mu_2-\mu_3}\partial_\theta\nu\partial_\theta\mu_2\right\} \tag{38}
$$

$$
R^0{}_{230}=e^{-\mu_2-\mu_3}\left[\partial_\theta(\nu-\mu_2)\partial_r\nu+\partial_r\partial_\theta\nu-\partial_\theta\omega\partial_r\omega\frac{3}{4}e^{2\psi-2\nu}-\partial_r\mu_3\partial_\theta\nu\right] \tag{39}
$$

$$
R^0{}_{221}=\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[e^{\mu_3-\mu_2}\partial_r(3\psi-\nu-\mu_2)\partial_r\omega+e^{\mu_3-\mu_2}\partial_r^2\omega+e^{\mu_2-\mu_3}\partial_\theta\omega\partial_\theta\mu_2\right] \tag{40}
$$

$$
R^0{}_{231}=\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_\theta(2\psi-\nu-\mu_2)\partial_r\omega+\partial_\theta\partial_r\omega-\partial_\theta\omega\partial_r\mu_3+\partial_\theta\omega\partial_r\psi\right] \tag{41}
$$

$$
R^0{}_{320}=e^{-\mu_2-\mu_3}\left[\partial_r(\nu-\mu_3)\partial_\theta\nu+\partial_r\partial_\theta\nu-\partial_r\omega\partial_\theta\omega\frac{3}{4}e^{2\psi-2\nu}-\partial_r\nu\partial_\theta\mu_2\right] \tag{42}
$$

$$R^0{}_{330} = e^{-\mu_2-\mu_3}\left\{e^{\mu_2-\mu_3}\left[\partial_\theta(\nu-\mu_3)\partial_\theta\nu+\partial_\theta^2\nu\right]-(\partial_\theta\omega)^2\frac{3}{4}e^{2\psi-2\nu+\mu_2-\mu_3}+e^{\mu_3-\mu_2}\partial_r\nu\partial_r\mu_3\right\} \tag{43}$$

$$R^0{}_{321} = \frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_r(2\psi-\nu-\mu_3)\partial_\theta\omega+\partial_r\partial_\theta\omega+\partial_r\omega\partial_\theta\psi-\partial_r\omega\partial_\theta\mu_2\right] \tag{44}$$

$$R^0{}_{331} = \frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[e^{\mu_2-\mu_3}\partial_\theta(3\psi-\nu-\mu_3)\partial_\theta\omega+e^{\mu_2-\mu_3}\partial_\theta^2\omega+e^{\mu_3-\mu_2}\partial_r\omega\partial_r\mu_3\right] \tag{45}$$

$$R^1{}_{220} = -e^{\psi-\nu-\mu_2-\mu_3}\left\{\frac{1}{2}e^{\mu_3-\mu_2}\left[\partial_r(3\psi-\nu-\mu_2)\partial_r\omega+\partial_r^2\omega\right]+\partial_\theta\psi\partial_r\omega+\frac{1}{2}e^{\mu_2-\mu_3}\partial_\theta\omega\partial_\theta\mu_2\right\} \tag{46}$$

$$R^1{}_{230} = -e^{\psi-\nu-\mu_2-\mu_3}\left\{\frac{1}{2}\left[\partial_\theta(\psi-\mu_2)\partial_r\omega+\partial_r\partial_\theta\omega\right]+\left[e^{\mu_2-\mu_3}\partial_\theta\psi+\frac{1}{2}\partial_r\left(2\psi-\nu-\mu_3\right)\right]\partial_\theta\omega\right\} \tag{47}$$

$$R^1{}_{231} = e^{-\mu_2-\mu_3}\left\{\partial_\theta(\psi-\mu_2)\partial_r\psi+\partial_r\partial_\theta\psi-\partial_r\mu_3\partial_\theta\psi+\frac{1}{4}e^{2\psi-2\nu}\partial_\theta\omega\partial_r\omega\right\} \tag{48}$$

$$R^1{}_{221} = e^{-\mu_2-\mu_3}\left\{e^{\mu_2-\mu_3}\partial_\theta\mu_2\partial_\theta\psi+e^{\mu_3-\mu_2}\left[\partial_r(\psi-\mu_2)\partial_r\psi+\partial_r^2\psi\right]+\frac{1}{4}e^{2\psi-2\nu+\mu_3-\mu_2}(\partial_r\omega)^2\right\} \tag{49}$$

$$R^1{}_{320} = -\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_r(\psi-\mu_3)\partial_\theta\omega+\partial_r\partial_\theta\omega-\partial_r\omega\partial_\theta\nu-\partial_r\omega\partial_\theta\mu_2\right] \tag{50}$$

$$R^1{}_{330} = -\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left\{e^{\mu_2-\mu_3}\left[\partial_\theta(\psi-\nu-\mu_3)\partial_\theta\omega+\partial_\theta^2\omega\right]+e^{\mu_3-\mu_2}\partial_r\omega\partial_r\mu_3\right\} \tag{51}$$

$$R^1{}_{321} = e^{-\mu_2-\mu_3}\left[\frac{1}{4}e^{2\psi-2\nu}\partial_r\omega\partial_\theta\omega-\partial_\theta\mu_2\partial_r\psi+\partial_r(\psi-\mu_3)\partial_\theta\psi+\partial_r\partial_\theta\psi\right] \tag{52}$$

$$R^1{}_{331} = e^{-\mu_2-\mu_3}\left\{\frac{1}{4}e^{2\psi-2\nu+\mu_2-\mu_3}(\partial_\theta\omega)^2+e^{\mu_3-\mu_2}\partial_r\mu_3\partial_r\psi+e^{\mu_2-\mu_3}\left[\partial_\theta(\psi-\mu_3)\partial_\theta\psi+\partial_\theta^2\psi\right]\right\} \tag{53}$$

$$R^2{}_{332} = e^{-\mu_2-\mu_3}\left\{e^{\mu_2-\mu_3}\left[\partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2+\partial_\theta^2\mu_2\right]+e^{\mu_3-\mu_2}\left[\partial_r(\mu_2-\mu_3)\partial_r\mu_3+\partial_r^2\mu_3\right]\right\} \tag{54}$$

$$R^2{}_{310} = \frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[\partial_r\omega\partial_\theta\nu-\partial_r\nu\partial_\theta\omega+\partial_r\psi\partial_\theta\omega+\partial_\theta\psi\partial_r\omega\right] . \tag{55}$$

From these quantities we can obtain all other non-vanishing components of the Riemann tensor as well since the first pair of indices commutes if one of them is nonzero, while it anticommutes if both are nonzero. The second pair of indices always anticommutes. The reason for these simple symmetries even for the Riemann tensor of the first kind (i.e. with one upper index) lies in the use of tetrads, which lets us pull the first two indices of this tensor with the Minkowski metric. This can be seen by considering Ref. [3], section 3.2.
Now the components of the Ricci tensor are

$$\begin{aligned}R_{00} =& R^1{}_{010}+R^2{}_{020}+R^3{}_{030}\\ =& e^{-\mu_2-\mu_3}\Big\{-\frac{1}{2}e^{2\psi-2\nu}\left[e^{\mu_3-\mu_2}(\partial_r\omega)^2+e^{\mu_2-\mu_3}(\partial_\theta\omega)^2\right]\\ &+e^{\mu_3-\mu_2}\left[\partial_r(\psi+\nu+\mu_3-\mu_2)\partial_r\nu+\partial_r^2\nu\right]+e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\nu+\mu_2-\mu_3)\partial_\theta\nu+\partial_\theta^2\nu\right]\Big\}\end{aligned} \tag{56}$$

$$\begin{aligned}R_{11} =& R^0{}_{101}+R^2{}_{121}+R^3{}_{131}\\ =& -e^{-\mu_2-\mu_3}\Big\{\frac{1}{2}e^{2\psi-2\nu}\left[e^{\mu_3-\mu_2}(\partial_r\omega)^2+e^{\mu_2-\mu_3}(\partial_\theta\omega)^2\right]\\ &+e^{\mu_3-\mu_2}\left[\partial_r(\psi+\nu+\mu_3-\mu_2)\partial_r\psi+\partial_r^2\psi\right]+e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\nu+\mu_2-\mu_3)\partial_\theta\psi+\partial_\theta^2\psi\right]\Big\}\end{aligned} \tag{57}$$

$$\begin{aligned}R_{01} =& R^2{}_{021}+R^3{}_{031}\\ =& \frac{1}{2}e^{-2\psi+\nu-\mu_2-\mu_3}\left[\partial_r\left(e^{3\psi-\nu+\mu_3-\mu_2}\partial_r\omega\right)+\partial_\theta\left(e^{3\psi-\nu+\mu_2-\mu_3}\partial_\theta\omega\right)\right]\end{aligned} \tag{58}$$

$$R_{22} = R^0{}_{202}+R^1{}_{212}+R^3{}_{232}$$

$$\begin{aligned}
=& - e^{-\mu_2-\mu_3}\Big\{ e^{\mu_3-\mu_2}\left[\partial_r(\nu-\mu_2)\partial_r\nu+\partial_r^2\nu\right]-(\partial_r\omega)^2\frac{1}{2}e^{2\psi-2\nu+\mu_3-\mu_2} \\
&+e^{\mu_2-\mu_3}\partial_\theta\mu_2\partial_\theta(\psi+\nu)+e^{\mu_3-\mu_2}\left[\partial_r(\psi-\mu_2)\partial_r\psi+\partial_r^2\psi\right] \\
&+e^{\mu_2-\mu_3}\left[\partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2+\partial_\theta^2\mu_2\right]+e^{\mu_3-\mu_2}\left[\partial_r(\mu_2-\mu_3)\partial_r\mu_3+\partial_r^2\mu_3\right]\Big\}
\end{aligned} \tag{59}$$

$$\begin{aligned}
R_{33} =& {R^0}_{303}+{R^1}_{313}+{R^2}_{323} \\
=& - e^{-\mu_2-\mu_3}\Big\{ e^{\mu_2-\mu_3}\left[\partial_\theta(\nu-\mu_3)\partial_\theta\nu+\partial_\theta^2\nu+\partial_\theta(\psi-\mu_3)\partial_\theta\psi+\partial_\theta^2\psi\right]-(\partial_\theta\omega)^2\frac{1}{2}e^{2\psi-2\nu+\mu_2-\mu_3} \\
&+e^{\mu_3-\mu_2}\left(\partial_r\psi+\partial_r\nu\right)\partial_r\mu_3+e^{\mu_2-\mu_3}\left[\partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2+\partial_\theta^2\mu_2\right]+e^{\mu_3-\mu_2}\left[\partial_r(\mu_2-\mu_3)\partial_r\mu_3+\partial_r^2\mu_3\right]\Big\}
\end{aligned} \tag{60}$$

$$\begin{aligned}
R_{23} =& {R^0}_{203}+{R^1}_{213} \\
=& - e^{-\mu_2-\mu_3}\Big\{\partial_\theta(\nu-\mu_2)\partial_r\nu+\partial_r\partial_\theta\left(\nu+\psi\right)-\partial_\theta\omega\partial_r\omega\frac{1}{2}e^{2\psi-2\nu}-\partial_\theta\left(\nu+\psi\right)\partial_r\mu_3+\partial_\theta(\psi-\mu_2)\partial_r\psi\Big\}
\end{aligned} \tag{61}$$

Lastly we need the Ricci scalar, which can be computed by considering the contraction of the Ricci tensor. Note that this is quite easy as we are still working in the tetrad basis and thus can pull indices with the Minkowski metric (still in the $-+++$ convention to be consistent).

$$\begin{aligned}
R =& {R^a}_a \\
=& {R^0}_0+{R^1}_1+{R^2}_2+{R^3}_3 \\
=& -R_{00}+R_{11}+R_{22}+R_{33} \\
=& 2e^{-\mu_2-\mu_3}\Big\{\frac{1}{4}e^{2\psi-2\nu}\left[e^{\mu_2-\mu_3}(\partial_\theta\omega)^2+e^{\mu_3-\mu_2}(\partial_r\omega)^2\right] \\
&-e^{\mu_3-\mu_2}\left[\partial_r(\psi+\nu+\mu_3-\mu_2)\partial_r\nu+\partial_r^2\nu+\partial_r(\psi+\mu_3-\mu_2)\partial_r\psi+\partial_r^2\psi+\partial_r(\mu_2-\mu_3)\partial_r\mu_3+\partial_r^2\mu_3\right] \\
&-e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\nu+\mu_2-\mu_3)\partial_\theta\nu+\partial_\theta^2\nu+\partial_\theta(\psi+\mu_2-\mu_3)\partial_\theta\psi+\partial_\theta^2\psi+\partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2+\partial_\theta^2\mu_2\right]\Big\}\,.
\end{aligned} \tag{62}$$

This enables us finally to write down the components of the Einstein tensor

$$G_{ab}=R_{ab}-\frac{R}{2}\eta_{ab}\,, \tag{63}$$

where the metric is Minkowskian once again due to the employed tetrad frame.
Thus we obtain

$$\begin{aligned}
G_{00} =& e^{-\mu_2-\mu_3}\Big\{-\frac{1}{4}e^{2\psi-2\nu}\left[e^{\mu_3-\mu_2}(\partial_r\omega)^2+e^{\mu_2-\mu_3}(\partial_\theta\omega)^2\right] \\
&-e^{\mu_3-\mu_2}\left[\partial_r(\psi+\mu_3-\mu_2)\partial_r\psi+\partial_r^2\psi+\partial_r(\mu_2-\mu_3)\partial_r\mu_3+\partial_r^2\mu_3\right] \\
&-e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\mu_2-\mu_3)\partial_\theta\psi+\partial_\theta^2\psi+\partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2+\partial_\theta^2\mu_2\right]\Big\}
\end{aligned} \tag{64}$$

$$\begin{aligned}
G_{11} =& - e^{-\mu_2-\mu_3}\Big\{\frac{3}{4}e^{2\psi-2\nu}\left[e^{\mu_3-\mu_2}(\partial_r\omega)^2+e^{\mu_2-\mu_3}(\partial_\theta\omega)^2\right] \\
&-e^{\mu_3-\mu_2}\left[\partial_r(\nu+\mu_3-\mu_2)\partial_r\nu+\partial_r^2\nu+\partial_r(\mu_2-\mu_3)\partial_r\mu_3+\partial_r^2\mu_3\right] \\
&-e^{\mu_2-\mu_3}\left[\partial_\theta(\nu+\mu_2-\mu_3)\partial_\theta\nu+\partial_\theta^2\nu+\partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2+\partial_\theta^2\mu_2\right]\Big\}
\end{aligned} \tag{65}$$

$$G_{22} = -e^{-\mu_2-\mu_3}\left\{\frac{1}{4}e^{2\psi-2\nu}\left[e^{\mu_2-\mu_3}(\partial_\theta\omega)^2 - e^{\mu_3-\mu_2}(\partial_r\omega)^2\right] - e^{\mu_3-\mu_2}\left[\partial_r(\psi+\mu_3)\partial_r\nu + \partial_r\mu_3\partial_r\psi\right]\right.$$
$$\left. - e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\nu-\mu_3)\partial_\theta\nu + \partial_\theta^2(\nu+\psi) + \partial_\theta(\psi-\mu_3)\partial_\theta\psi\right]\right\} \tag{66}$$

$$G_{33} = -e^{-\mu_2-\mu_3}\left\{\frac{1}{4}e^{2\psi-2\nu}\left[e^{\mu_3-\mu_2}(\partial_r\omega)^2 - e^{\mu_2-\mu_3}(\partial_\theta\omega)^2\right] - e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\mu_2)\partial_\theta\nu + \partial_\theta\mu_2\partial_\theta\psi\right]\right.$$
$$\left. - e^{\mu_3-\mu_2}\left[\partial_r(\psi+\nu-\mu_2)\partial_r\nu + \partial_r^2(\nu+\psi) + \partial_r(\psi-\mu_2)\partial_r\psi\right]\right\} . \tag{67}$$

3 The Einstein field equations

Having now all these objects at our disposal, we can finally confirm the starting equations of Ref. [1]; for this we define

$$X := e^{\mu_3-\mu_2}(\partial_r\omega)^2 + e^{\mu_2-\mu_3}(\partial_\theta\omega)^2 \tag{68}$$
$$\beta := \psi + \nu \tag{69}$$

and employ that in vacuum

$$R_{\mu\nu} = G_{\mu\nu} = 0 ,$$

leading to

$$\frac{1}{2}e^{2\psi-2\nu}X = e^{\mu_3-\mu_2}\left[\partial_r(\psi+\nu+\mu_3-\mu_2)\partial_r\nu + \partial_r^2\nu\right] + e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\nu+\mu_2-\mu_3)\partial_\theta\nu + \partial_\theta^2\nu\right] \tag{70}$$
$$-\frac{1}{2}e^{2\psi-2\nu}X = e^{\mu_3-\mu_2}\left[\partial_r(\psi+\nu+\mu_3-\mu_2)\partial_r\psi + \partial_r^2\psi\right] + e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\nu+\mu_2-\mu_3)\partial_\theta\psi + \partial_\theta^2\psi\right] \tag{71}$$
$$-\frac{1}{4}e^{2\psi-2\nu}X = e^{\mu_3-\mu_2}\left[\partial_r(\psi+\mu_3-\mu_2)\partial_r\psi + \partial_r^2\psi + \partial_r(\mu_2-\mu_3)\partial_r\mu_3 + \partial_r^2\mu_3\right]$$
$$+ e^{\mu_2-\mu_3}\left[\partial_\theta(\psi+\mu_2-\mu_3)\partial_\theta\psi + \partial_\theta^2\psi + \partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2 + \partial_\theta^2\mu_2\right] \tag{72}$$
$$\frac{3}{4}e^{2\psi-2\nu}X = e^{\mu_3-\mu_2}\left[\partial_r(\nu+\mu_3-\mu_2)\partial_r\nu + \partial_r^2\nu + \partial_r(\mu_2-\mu_3)\partial_r\mu_3 + \partial_r^2\mu_3\right]$$
$$+ e^{\mu_2-\mu_3}\left[\partial_\theta(\nu+\mu_2-\mu_3)\partial_\theta\nu + \partial_\theta^2\nu + \partial_\theta(\mu_2-\mu_3)\partial_\theta\mu_2 + \partial_\theta^2\mu_2\right] \tag{73}$$
$$0 = \partial_r\left(e^{3\psi-\nu+\mu_3-\mu_2}\partial_r\omega\right) + \partial_\theta\left(e^{3\psi-\nu+\mu_2-\mu_3}\partial_\theta\omega\right) . \tag{74}$$

Also we can take the difference of G_{22} and G_{33} to obtain

$$0 = \frac{1}{2}e^{2\psi-2\nu}\left[e^{\mu_2-\mu_3}(\partial_\theta\omega)^2 - e^{\mu_3-\mu_2}(\partial_r\omega)^2\right]$$
$$- e^{\mu_2-\mu_3}\left[\partial_\theta(\nu-\mu_2-\mu_3)\partial_\theta\nu + \partial_\theta^2\beta + \partial_\theta(\psi-\mu_2-\mu_3)\partial_\theta\psi\right]$$
$$+ e^{\mu_3-\mu_2}\left[\partial_r(\nu-\mu_2-\mu_3)\partial_r\nu + \partial_r^2\beta + \partial_r(\psi-\mu_2-\mu_3)\partial_r\psi\right] , \tag{75}$$

which can be rewritten as

$$2e^{-\beta}\left[\partial_r\left(e^{\beta+\mu_3-\mu_2}\partial_r\beta\right) - \partial_\theta\left(e^{\beta+\mu_2-\mu_3}\partial_\theta\beta\right)\right] = 4e^{\mu_3-\mu_2}\left(\partial_r\beta\partial_r\mu_3 + \partial_r\psi\partial_r\nu\right) - 4e^{\mu_2-\mu_3}\left(\partial_\theta\beta\partial_\theta\mu_2 + \partial_\theta\psi\partial_\theta\nu\right)$$
$$+ e^{2\psi-2\nu}\left[e^{\mu_3-\mu_2}(\partial_r\omega)^2 - e^{\mu_2-\mu_3}(\partial_\theta\omega)^2\right] . \tag{76}$$

Equations 70 and 71 can be rewritten as

$$\frac{1}{2}e^{3\psi-\nu}X = \partial_r\left(e^{\psi+\nu+\mu_3-\mu_2}\partial_r\nu\right) + \partial_\theta\left(e^{\psi+\nu+\mu_2-\mu_3}\partial_\theta\nu\right) \tag{77}$$
$$-\frac{1}{2}e^{3\psi-\nu}X = \partial_r\left(e^{\psi+\nu+\mu_3-\mu_2}\partial_r\psi\right) + \partial_\theta\left(e^{\psi+\nu+\mu_2-\mu_3}\partial_\theta\psi\right) . \tag{78}$$

The sum and difference of these two gives

$$0 = \partial_r \left(e^{\beta+\mu_3-\mu_2} \partial_r \beta \right) + \partial_\theta \left(e^{\beta+\mu_2-\mu_3} \partial_\theta \beta \right) \tag{79}$$

$$e^{3\psi-\nu} X = \partial_r \left[e^{\beta+\mu_3-\mu_2} \partial_r (\nu - \psi) \right] + \partial_\theta \left[e^{\beta+\mu_2-\mu_3} \partial_\theta (\nu - \psi) \right] \ . \tag{80}$$

Noticing that we can rewrite Eq. 74 as

$$e^{3\psi-\nu} X = \partial_r \left(e^{3\psi-\nu+\mu_3-\mu_2} \omega \partial_r \omega \right) + \partial_\theta \left(e^{3\psi-\nu+\mu_2-\mu_3} \omega \partial_\theta \omega \right) \ , \tag{81}$$

we can insert this result into Eq. 80 and obtain

$$0 = \partial_r \left[e^{3\psi-\nu+\mu_3-\mu_2} \left(e^{2\nu-2\psi} \partial_r (\psi - \nu) + \frac{1}{2} \partial_r \omega^2 \right) \right] + \partial_\theta \left[e^{3\psi-\nu+\mu_2-\mu_3} \left(e^{2\nu-2\psi} \partial_\theta (\psi - \nu) + \frac{1}{2} \partial_\theta \omega^2 \right) \right] \ ,$$

which, after defining

$$\chi := e^{\nu-\psi} \ ,$$

reads

$$0 = \partial_r \left[e^{3\psi-\nu+\mu_3-\mu_2} \partial_r (\chi^2 - \omega^2) \right] + \partial_\theta \left[e^{3\psi-\nu+\mu_2-\mu_3} \partial_\theta (\chi^2 - \omega^2) \right] \ . \tag{82}$$

Comparing the expression above with Eq. 74, we notice that the quantities ω and $\chi^2 - \omega^2$ are determined by the same equation.

4 Null surface and gauge fixing

Our problem, for a vanishing angular velocity, should reduce ot the Schwarzschild metric, which features an event horizon spanned by the Killing vector ∂_t. This event horizon constitutes a so-called *null surface*, as the normal vectors to this surface are null-vectors.
Now in our problem (as it should reduce to the Schwarzschild case in the limit of vanishing rotation) it is reasonable to assume that there is a smooth null surface spanned by the two Killing vectors ∂_t and ∂_ϕ as well.
As done in Ref. [1], we denote the equation determining the surface by

$$N(r, \theta) = 0 \ .$$

Since the gradient of this equation gives the respective normal vector, the condition that the equation above describes a null surface is given by

$$g^{ij} \partial_i N \partial_j N = 0 \ , \tag{83}$$

which (for our Ansatz) reduces to

$$e^{2\mu_3-2\mu_2} (\partial_r N)^2 + (\partial_\theta N)^2 = 0 \ . \tag{84}$$

Now, as we are free to impose any diffeomorphism due to the gauge freedom, we choose

$$e^{2\mu_3-2\mu_2} = \Delta(r) \ , \tag{85}$$

where $\Delta(r)$ is an unspecified function of r only. This implies that on the event horizon $\Delta(r) = 0$ holds (this is because the term $\partial_\theta N$ should vanish separately, and thus N should not depend on θ at all).
In order for the null surface to be spanned by ∂_t and ∂_ϕ, the determinant of the induced $(t, \phi)-$submetric has to vanish[2] (intuitively, this is because a null surface should have a vanishing two-volume (i.e. surface), and in the resulting integral the invariant volume element contains the determinant of this submetric). From Eq. 3 we can compute this determinant and arrive at

$$e^{\beta} = 0 \ , \tag{86}$$

which must hold when $\Delta = 0$. Next we suppose that, in line with the conditions above, e^{β} is separable in r and θ as

$$e^{\beta} = \sqrt{\Delta(r)} f(\theta) \ , \tag{87}$$

where $f(\theta)$ is as of now not specified.
Inserting these expressions for e^{β} and $e^{2\mu_3 - 2\mu_2}$ into Eq. 79, we obtain

$$0 = \underbrace{\partial_r \left(\sqrt{\Delta} \partial_r \sqrt{\Delta} \right)}_{\frac{1}{2}\partial_r^2 \Delta} + \frac{\partial_\theta^2 f}{f} \ . \tag{88}$$

One solution (where the space of solutions is restricted by e.g. the requirement of convexity of the horizon) of this equation is given by $\partial_r^2 \Delta = 2$ and $f(\theta) = \sin(\theta)$. Introducing two constants M and a (which will later be tied to the mass and angular momentum of the black hole), we can thus write Δ as

$$\Delta(r) = r^2 + a^2 - 2Mr \ . \tag{89}$$

5 Reducing the field equations

Using our solutions

$$e^{\mu_3 - \mu_2} = \sqrt{\Delta} \ , \qquad e^{\beta} = \sqrt{\Delta} \sin(\theta) \tag{90}$$

and substituting

$$\mu := \cos(\theta) \ , \qquad \delta := 1 - \mu^2 = \sin^2(\theta) \ ,$$

we can rewrite Eqs. 74 and 80 as

$$0 = \partial_r \left(e^{2\psi - 2\nu} \Delta \partial_r \omega \right) + \partial_\mu \left(e^{2\psi - 2\nu} \delta \partial_\mu \omega \right) \tag{91}$$

$$e^{2\psi - 2\nu} \left[\Delta (\partial_r \omega)^2 + \delta (\partial_\mu \omega)^2 \right] = \partial_r \left[\Delta \partial_r (\nu - \psi) \right] + \partial_\mu \left[\delta \partial_\mu (\nu - \psi) \right] \ , \tag{92}$$

where we used that

$$X = \sqrt{\Delta} (\partial_r \omega)^2 + \frac{\delta}{\sqrt{\Delta}} (\partial_\mu \omega)^2 \ .$$

These equations can be expressed as

$$\begin{aligned} 0 &= \partial_r \left(\frac{\Delta}{\chi^2} \partial_r \omega \right) + \partial_\mu \left(\frac{\delta}{\chi^2} \partial_\mu \omega \right) \\ \iff \quad 2 \left(\Delta \partial_r \chi \partial_r \omega + \delta \partial_\mu \chi \partial_\mu \omega \right) &= \chi \left[\partial_r \left(\Delta \partial_r \omega \right) + \partial_\mu \left(\delta \partial_\mu \omega \right) \right] \end{aligned} \tag{93}$$

$$\begin{aligned} \frac{1}{\chi^2} \left[\Delta (\partial_r \omega)^2 + \delta (\partial_\mu \omega)^2 \right] &= \partial_r \left(\frac{\Delta}{\chi} \partial_r \chi \right) + \partial_\mu \left(\frac{\delta}{\chi} \partial_\mu \chi \right) \\ \iff \quad \Delta (\partial_r \omega)^2 + \delta (\partial_\mu \omega)^2 + \Delta (\partial_r \chi)^2 + \delta (\partial_\mu \chi)^2 &= \chi \left[\partial_r \left(\Delta \partial_r \chi \right) + \partial_\mu \left(\delta \partial_\mu \chi \right) \right] \ , \end{aligned} \tag{94}$$

where we employed the definition of χ from earlier.
Next we define

$$\mathcal{X} := \chi + \omega \ , \qquad \mathcal{Y} := \chi - \omega \ , \tag{95}$$

allowing us to rewrite Eqs. 93 and 94 as

$$\begin{aligned} &2 \left[\Delta (\partial_r \mathcal{X})^2 - \Delta (\partial_r \mathcal{Y})^2 + \delta (\partial_\mu \mathcal{X})^2 - \delta (\partial_\mu \mathcal{Y})^2 \right] \\ =& (\mathcal{X} + \mathcal{Y}) \left[\partial_r \left(\Delta \partial_r \mathcal{X} \right) - \partial_r \left(\Delta \partial_r \mathcal{Y} \right) + \partial_\mu \left(\delta \partial_\mu \mathcal{X} \right) - \partial_\mu \left(\delta \partial_\mu \mathcal{Y} \right) \right] \end{aligned} \tag{96}$$

$$\begin{aligned} &2 \left[\Delta (\partial_r \mathcal{X})^2 + \Delta (\partial_r \mathcal{Y})^2 + \delta (\partial_\mu \mathcal{X})^2 + \delta (\partial_\mu \mathcal{Y})^2 \right] \\ =& (\mathcal{X} + \mathcal{Y}) \left[\partial_r \left(\Delta \partial_r \mathcal{X} \right) + \partial_r \left(\Delta \partial_r \mathcal{Y} \right) + \partial_\mu \left(\delta \partial_\mu \mathcal{X} \right) + \partial_\mu \left(\delta \partial_\mu \mathcal{Y} \right) \right] \ . \end{aligned} \tag{97}$$

Adding and subtracting these two equations, we end up with a pair of symmetric expressions:

$$\Delta(\partial_r\mathcal{X})^2+\delta(\partial_\mu\mathcal{X})^2=\frac{1}{2}(\mathcal{X}+\mathcal{Y})\left[\partial_r\left(\Delta\partial_r\mathcal{X}\right)+\partial_\mu\left(\delta\partial_\mu\mathcal{X}\right)\right] \tag{98}$$

$$\Delta(\partial_r\mathcal{Y})^2+\delta(\partial_\mu\mathcal{Y})^2=\frac{1}{2}(\mathcal{X}+\mathcal{Y})\left[\partial_r\left(\Delta\partial_r\mathcal{Y}\right)+\partial_\mu\left(\delta\partial_\mu\mathcal{Y}\right)\right]\ . \tag{99}$$

We should note that, with these definitions, Eq. 76 can be expressed as

$$\begin{aligned}
&\frac{2}{\sqrt{\Delta}}\left[\partial_r\left(\sqrt{\Delta}\partial_r\sqrt{\Delta}\right)-\partial_\mu\left(\sqrt{\delta}\partial_\mu\sqrt{\delta}\right)\right]\\
&=4\sqrt{\Delta}\left(\frac{1}{\sqrt{\Delta\delta}}\partial_r\sqrt{\Delta\delta}\partial_r\mu_3+\partial_r\psi\partial_r\nu\right)-4\frac{\delta}{\sqrt{\Delta}}\left(\frac{1}{\sqrt{\Delta\delta}}\partial_\mu\sqrt{\Delta\delta}\partial_\mu\mu_2+\partial_\mu\psi\partial_\mu\nu\right)\\
&\quad+\frac{1}{\chi^2}\left[\sqrt{\Delta}(\partial_r\omega)^2-\frac{\delta}{\sqrt{\Delta}}(\partial_\mu\omega)^2\right]\\
\iff\quad&\frac{2}{\sqrt{\Delta}}\left[(\partial_r\sqrt{\Delta})^2+\sqrt{\Delta}\partial_r^2\sqrt{\Delta}-(\partial_\mu\sqrt{\delta})^2-\sqrt{\delta}\partial_\mu^2\sqrt{\delta}\right]\\
&=4\sqrt{\Delta}\left(\frac{1}{\sqrt{\Delta\delta}}\partial_r\sqrt{\Delta\delta}\partial_r\mu_3+\partial_r\psi\partial_r\nu\right)-4\frac{\delta}{\sqrt{\Delta}}\left(\frac{1}{\sqrt{\Delta\delta}}\partial_\mu\sqrt{\Delta\delta}\partial_\mu\mu_2+\partial_\mu\psi\partial_\mu\nu\right)\\
&\quad-\frac{1}{\chi^2\sqrt{\Delta}}\left\{\Delta\left[\partial_r\mathcal{X}\partial_r\mathcal{Y}-\chi^2(\partial_r(\nu-\psi))^2\right]+\delta\left[\partial_\mu\mathcal{X}\partial_\mu\mathcal{Y}-\chi^2(\partial_\mu(\nu-\psi))^2\right]\right\}\\
\iff\quad&\frac{2}{\sqrt{\Delta}}\left[(\partial_r\sqrt{\Delta})^2+\sqrt{\Delta}\partial_r^2\sqrt{\Delta}-(\partial_\mu\sqrt{\delta})^2-\sqrt{\delta}\partial_\mu^2\sqrt{\delta}\right]\\
&=4\partial_r\sqrt{\Delta}\partial_r\mu_3-4\sqrt{\frac{\delta}{\Delta}}\partial_\mu\sqrt{\delta}\partial_\mu\mu_2+\frac{1}{\sqrt{\Delta}}\left[(\partial_r\sqrt{\Delta})^2-(\partial_\mu\sqrt{\delta})^2\right]-\frac{1}{\chi^2\sqrt{\Delta}}\left[\Delta\partial_r\mathcal{X}\partial_r\mathcal{Y}-\delta\partial_\mu\mathcal{X}\partial_\mu\mathcal{Y}\right]\\
\iff\quad&2\left[(\partial_r\sqrt{\Delta})^2+\sqrt{\Delta}\partial_r^2\sqrt{\Delta}-(\partial_\mu\sqrt{\delta})^2-\sqrt{\delta}\partial_\mu^2\sqrt{\delta}\right]\\
&=4(r-M)\partial_r\mu_3+4\mu\partial_\mu\mu_2+\left[(\partial_r\sqrt{\Delta})^2-(\partial_\mu\sqrt{\delta})^2\right]-\frac{1}{\chi^2}\left[\Delta\partial_r\mathcal{X}\partial_r\mathcal{Y}-\delta\partial_\mu\mathcal{X}\partial_\mu\mathcal{Y}\right]\\
\iff\quad&2\left[(r-M)\partial_r(\mu_3-\mu_2)+\sqrt{\Delta}\partial_r^2\sqrt{\Delta}-(\partial_\mu\sqrt{\delta})^2-\sqrt{\delta}\partial_\mu^2\sqrt{\delta}\right]\\
&=4(r-M)\partial_r\mu_3+4\mu\partial_\mu\mu_2+2\mu\partial_\mu(\mu_3-\mu_2)+\left[(\partial_r\sqrt{\Delta})^2-(\partial_\mu\sqrt{\delta})^2\right]-\frac{1}{\chi^2}\left[\Delta\partial_r\mathcal{X}\partial_r\mathcal{Y}-\delta\partial_\mu\mathcal{X}\partial_\mu\mathcal{Y}\right]\\
\iff\quad&0=2(r-M)\partial_r\left(\mu_2+\mu_3\right)+2\mu\partial_\mu\left(\mu_2+\mu_3\right)-\frac{1}{\chi^2}\left[\Delta\partial_r\mathcal{X}\partial_r\mathcal{Y}-\delta\partial_\mu\mathcal{X}\partial_\mu\mathcal{Y}\right]\\
&\quad+(\partial_r\sqrt{\Delta})^2-2\sqrt{\Delta}\partial_r^2\sqrt{\Delta}+(\partial_\mu\sqrt{\delta})^2+2\sqrt{\delta}\partial_\mu^2\sqrt{\delta}\\
\iff\quad&0=2(r-M)\partial_r\left(\mu_2+\mu_3\right)+2\mu\partial_\mu\left(\mu_2+\mu_3\right)-\frac{1}{\chi^2}\left[\Delta\partial_r\mathcal{X}\partial_r\mathcal{Y}-\delta\partial_\mu\mathcal{X}\partial_\mu\mathcal{Y}\right]\\
&\quad+(\partial_r\sqrt{\Delta})^2-2\sqrt{\Delta}\partial_r^2\sqrt{\Delta}-1-\frac{1}{\delta}\\
\iff\quad&0=2\left[(r-M)\partial_r+\mu\partial_\mu\right]\left(\mu_2+\mu_3\right)-\frac{1}{\chi^2}\left[\Delta\partial_r\mathcal{X}\partial_r\mathcal{Y}-\delta\partial_\mu\mathcal{X}\partial_\mu\mathcal{Y}\right]+3\frac{M^2-a^2}{\Delta}-\frac{1}{\delta}\ .
\end{aligned} \tag{100}$$

In this series of „elementary reductions“ (cf. Ref. [1], p. 410), we employed that

$$\partial_r\sqrt{\Delta}=\partial_r(\mu_3-\mu_2)\sqrt{\Delta}=\frac{1}{\sqrt{\Delta}}(r-M)$$

and

$$2\partial_\mu(\mu_3-\mu_2)=\frac{\partial_\mu\Delta}{\Delta}=0\ .$$

The remaining task is now as follows: Solve Eqs. 98 and 99 to get expressions for $\mathcal{X}, \mathcal{Y}$ and thus for χ, ω, after which Eq. 100 may be solved for $\mu_2 + \mu_3$. Then the metric, which after our calculations so far can already be described through

$$ds^2 = \sqrt{\Delta\delta}\left[-\chi dt^2 + \frac{1}{\chi}\left(d\phi - \omega dt\right)^2\right] + \frac{e^{\mu_2+\mu_3}}{\sqrt{\Delta}}\left(dr^2 + \Delta d\theta^2\right) \ , \tag{101}$$

will be fully specified.

6 Deriving and solving Ernst's equation

In order to solve Eqs. 98 and 99, we are now going to rewrite them into a single complex expression that is called Ernst's equation.
For this we define

$$f := \mathcal{X}\mathcal{Y}e^{2\psi} = \sqrt{\Delta\delta}\frac{\chi^2 - \omega^2}{\chi} \ , \qquad \mathcal{W} := \frac{\omega}{\mathcal{X}\mathcal{Y}} = \frac{\omega}{\chi^2 - \omega^2}$$

Since we know from Eq. 82 that ω and $\chi^2 - \omega^2 = \mathcal{X}\mathcal{Y}$ satisfy the same equation, it holds that

$$\partial_r\left(\frac{f^2}{\Delta}\partial_r\mathcal{W}\right) + \partial_\theta\left(\frac{f^2}{\delta}\partial_\theta\mathcal{W}\right) = 0 \ . \tag{102}$$

This can be seen by expanding the expression:

$$\begin{aligned}
&\partial_r\left(\frac{f^2}{\delta}\partial_r\mathcal{W}\right) + \partial_\mu\left(\frac{f^2}{\Delta}\partial_\mu\mathcal{W}\right) \\
=&\partial_r\left[\frac{\Delta}{\chi^2}(\mathcal{X}\mathcal{Y})^2\partial_r\mathcal{W}\right] + \partial_\mu\left[\frac{\delta}{\chi^2}(\mathcal{X}\mathcal{Y})^2\partial_\mu\mathcal{W}\right] \\
=&\partial_r\left[\frac{\Delta}{\chi^2}\left(\mathcal{X}\mathcal{Y}\partial_r\omega - \omega\partial_r(\mathcal{X}\mathcal{Y})\right)\right] + \partial_\mu\left[\frac{\delta}{\chi^2}\left(\mathcal{X}\mathcal{Y}\partial_\mu\omega - \omega\partial_\mu(\mathcal{X}\mathcal{Y})\right)\right] \\
=&\mathcal{X}\mathcal{Y}\partial_r\left[\frac{\Delta}{\chi^2}\partial_r\omega\right] - \omega\partial_r\left[\frac{\Delta}{\chi^2}\partial_r(\mathcal{X}\mathcal{Y})\right] + \mathcal{X}\mathcal{Y}\partial_\mu\left[\frac{\delta}{\chi^2}\partial_\mu\omega\right] - \omega\partial_\mu\left[\frac{\delta}{\chi^2}\partial_\mu(\mathcal{X}\mathcal{Y})\right] \\
=&0 \ ,
\end{aligned} \tag{103}$$

where Eq. 91 was used at the end. This implies that $\mathcal{W}$ may be derived from a potential g, where

$$\partial_r g = \frac{f^2}{\Delta}\partial_\mu\mathcal{W} \ , \qquad \partial_\mu g = -\frac{f^2}{\delta}\partial_r\mathcal{W} \ ,$$

such that

$$\nabla \times \nabla g = 0 \tag{104}$$

in the two-dimensional $r, \mu-$subspace. The potential g itself obviously satisfies

$$\partial_r\left(\frac{\Delta}{f^2}\partial_r g\right) + \partial_\mu\left(\frac{\delta}{f^2}\partial_\mu g\right) = 0 \ , \tag{105}$$

which can be rewritten as

$$f\left[\partial_r\left(\Delta\partial_r g\right) + \partial_\mu\left(\delta\partial_\mu g\right)\right] = 2\Delta\partial_r g\partial_r f + 2\delta\partial_\mu g\partial_\mu f \ . \tag{106}$$

Along the same lines, we can see that

$$\partial_r\left(\frac{\Delta}{f}\partial_r f\right) + \partial_\mu\left(\frac{\delta}{f}\partial_\mu f\right) = -f^2\left[\frac{1}{\delta}(\partial_r\mathcal{W})^2 + \frac{1}{\Delta}(\partial_\mu\mathcal{W})^2\right] \tag{107}$$

holds, which can be expanded to

$$f\left[\partial_r\left(\Delta\partial_r f\right)+\partial_\mu\left(\delta\partial_\mu f\right)\right]=\Delta(\partial_r f)^2+\delta(\partial_\mu f)^2-\Delta(\partial_r g)^2-\delta(\partial_\mu g)^2\ . \tag{108}$$

Now we introduce the complex variable

$$Z:=f+ig\ ,$$

through the use of which we can combine Eqs. 106 and 108 into

$$\Re(Z)\left[\partial_r\left(\Delta\partial_r Z\right)+\partial_\mu\left(\delta\partial_\mu Z\right)\right]=\Delta(\partial_r Z)^2+\delta(\partial_\mu Z)^2\ . \tag{109}$$

Finally introducing the transformation

$$Z=\frac{1+\mathcal{E}}{1-\mathcal{E}}=\frac{1-\mathcal{E}\bar{\mathcal{E}}}{|1-\mathcal{E}|^2}+\frac{\mathcal{E}-\bar{\mathcal{E}}}{|1-\mathcal{E}|^2}\ , \tag{110}$$

we find

$$\begin{aligned}
&\frac{1-\mathcal{E}\bar{\mathcal{E}}}{|1-\mathcal{E}|^2}\left[\frac{2}{(1-\mathcal{E})^2}\partial_r(\Delta\partial_r\mathcal{E})+\frac{2}{(1-\mathcal{E})^2}\partial_\mu(\delta\partial_\mu\mathcal{E})+\frac{4\Delta}{(1-\mathcal{E})^3}(\partial_r\mathcal{E})^2+\frac{4\delta}{(1-\mathcal{E})^3}(\partial_\mu\mathcal{E})^2\right]\\
&=\frac{4\Delta}{(1-\mathcal{E})^4}(\partial_r\mathcal{E})^2+\frac{4\delta}{(1-\mathcal{E})^4}(\partial_\mu\mathcal{E})^2\\
\iff\quad &(1-\mathcal{E}\bar{\mathcal{E}})\left[\frac{2}{(1-\mathcal{E})^2}\partial_r(\Delta\partial_r\mathcal{E})+\frac{2}{(1-\mathcal{E})^2}\partial_\mu(\delta\partial_\mu\mathcal{E})\right]\\
&=-\bar{\mathcal{E}}(1-\mathcal{E})\left[\frac{4\Delta}{(1-\mathcal{E})^3}(\partial_r\mathcal{E})^2+\frac{4\delta}{(1-\mathcal{E})^3}(\partial_\mu\mathcal{E})^2\right]\\
\iff\quad &(1-\mathcal{E}\bar{\mathcal{E}})\left[\partial_r(\Delta\partial_r\mathcal{E})+\partial_\mu(\delta\partial_\mu\mathcal{E})\right]=-\bar{\mathcal{E}}\left[2\Delta(\partial_r\mathcal{E})^2+2\delta(\partial_\mu\mathcal{E})^2\right]\ .
\end{aligned}$$

This expression, when using the new variable

$$\eta^2:=\frac{(r-M)^2}{M^2-a^2}=\frac{\Delta}{M^2-a^2}+1\ ,$$

can be rewritten as

$$(1-\mathcal{E}\bar{\mathcal{E}})\left\{\partial_\eta[(\eta^2-1)\partial_\eta\mathcal{E}]+\partial_\mu[(1-\mu^2)\partial_\mu\mathcal{E}]\right\}=-\bar{\mathcal{E}}\left[2(\eta^2-1)(\partial_\eta\mathcal{E})^2+2(1-\mu^2)(\partial_\mu\mathcal{E})^2\right]\ , \tag{111}$$

which constitutes Ernst's equation.
This equation is solved by

$$\mathcal{E}=-p\eta-iq\mu\ ,\ p^2+q^2=1\ ,$$

as can be directly verified. In order to do this we firstly evaluate the right-hand side of Eq. 111:

$$\begin{aligned}
&(1-\mathcal{E}\bar{\mathcal{E}})\left\{\partial_\eta[(\eta^2-1)\partial_\eta\mathcal{E}]+\partial_\mu[(1-\mu^2)\partial_\mu\mathcal{E}]\right\}\\
=&(1-p^2\eta^2-q^2\mu^2)\left\{\partial_\eta[(\eta^2-1)(-p)]+\partial_\mu[(1-\mu^2)(-iq)]\right\}\\
=&(1-p^2\eta^2-q^2\mu^2)\left(-2\eta p+2iq\mu\right)\\
=&2\left[-\eta p+iq\mu+\eta^3p^3-ip^2q\eta^2\mu+\eta q^2p\mu^2-iq^3\mu^3\right]\ .
\end{aligned}$$

The right-hand side gives

$$\begin{aligned}
&-\bar{\mathcal{E}}\left[2(\eta^2-1)(\partial_\eta\mathcal{E})^2+2(1-\mu^2)(\partial_\mu\mathcal{E})^2\right]\\
=&(p\eta-iq\mu)\left[2(\eta^2-1)p^2+2(1-\mu^2)(-q^2)\right]\\
=&2\left[p^3\eta^3-p^3\eta-iq\eta^2p^2\mu+iqp^2\mu-q^2p\eta+iq^3\mu+q^2p\eta\mu^2-iq^3\mu^3\right]
\end{aligned}$$

$$=2\left[p^3\eta^3 - p\eta\underbrace{(p^2+q^2)}_{=1} - iq\eta^2p^2\mu + iq\underbrace{(p^2+q^2)}_{=1}\mu + q^2p\eta\mu^2 - iq^3\mu^3\right] ,$$

such that the proposed solution does indeed solve Ernst's equation.
Thus we can split up the complex variable Z into its real and imaginary parts

$$Z = Z_1 + iZ_2$$

and obtain

$$Z_1 = \Re(Z) = \frac{1-\mathcal{E}\bar{\mathcal{E}}}{|1-\mathcal{E}|^2} = \frac{1-p^2\eta^2-q^2\mu^2}{(1+p\eta)^2+q^2\mu^2} = -\frac{p^2(\eta^2-1)-q^2(1-\mu^2)}{(1+p\eta)^2+q^2\mu^2} \tag{112}$$

$$Z_2 = \Im(Z) = \frac{\mathcal{E}-\bar{\mathcal{E}}}{|1-\mathcal{E}|^2} = -\frac{2q\mu}{(1+p\eta)^2+q^2\mu^2} . \tag{113}$$

Since the equation of motion for Z, Eq. 109, is invariant under a change $Z \to -Z$, we will use the negative of the solutions above.
Resubstituting the variable η, we arrive at

$$Z_1 = \frac{\Delta - \frac{q^2}{p^2}(M^2-a^2)\delta}{\left[p^{-1}\sqrt{M^2-a^2}+r-M\right]^2 + (M^2-a^2)\frac{q^2}{p^2}\mu^2} \tag{114}$$

$$Z_2 = \frac{2\frac{q}{p^2}(M^2-a^2)\mu}{\left[p^{-1}\sqrt{M^2-a^2}+r-M\right]^2 + (M^2-a^2)\frac{q^2}{p^2}\mu^2} . \tag{115}$$

Next we choose

$$p = \frac{\sqrt{M^2-a^2}}{M} , \qquad q = \frac{a}{M} ,$$

which manifestly fulfils $p^2+q^2=1$ and leads us to

$$Z_1 = \frac{\Delta - a^2\delta}{r^2+\mu^2a^2} \tag{116}$$

$$Z_2 = \frac{2aM\mu}{r^2+a^2\mu^2} . \tag{117}$$

Introducing the variable

$$\rho^2 := r^2 + a^2\mu^2$$

and reverting back to the functions f and g, we obtain

$$f = (\chi^2-\omega^2)e^{2\psi} = e^{2\nu} - \omega^2 e^{2\psi} = \frac{\Delta - a^2\delta}{\rho^2} \tag{118}$$

$$g = \frac{2aM\mu}{\rho^2} . \tag{119}$$

Employing the defining relations of g, we find the constraints

$$\partial_r g = -\frac{4raM\mu}{\rho^4} = \frac{(\Delta-a^2\delta)^2}{\Delta\rho^4}\partial_\mu \mathcal{W} \tag{120}$$

$$\partial_\mu g = \frac{2aM(\rho^2-a^2\mu^2)}{\rho^4} = -\frac{(\Delta-a^2\delta)^2}{\delta\rho^4}\partial_r \mathcal{W} , \tag{121}$$

which are fulfilled by

$$\mathcal{W} = \frac{\omega}{\chi^2 - \omega^2} = \frac{2aMr\delta}{\Delta - a^2\delta} \ . \tag{122}$$

Combining the expressions for f and $\mathcal{W}$, we find

$$\omega e^{2\psi} = \frac{2aMr\delta}{\rho^2} \ . \tag{123}$$

We can combine the expression above with Eq. 118 to obtain

$$\frac{\Delta - a^2\delta}{\rho^2} e^{2\psi} = e^{\beta} - \omega^2 e^{4\psi} = \frac{\Delta\delta\rho^4 - 4a^2M^2r^2\delta^2}{\rho^4} \ , \tag{124}$$

where we used our solution for e^{β}. These expressions for ω and $e^{2\psi}$ are solved by

$$e^{2\psi} = \frac{\delta\Sigma^2}{\rho^2} \tag{125}$$

$$\omega = \frac{2aMr}{\Sigma^2} \ , \tag{126}$$

where

$$\Sigma^2 = \frac{\rho^4\Delta - 4a^2M^2r^2\delta}{\Delta - a^2\delta} = (r^2 + a^2)^2 - a^2\Delta\delta \ .$$

Furthermore, we find the solution for $e^{2\nu}$ by considering the identity

$$e^{2\nu} = e^{2\beta - 2\psi} \ ,$$

yielding

$$e^{2\nu} = \frac{\rho^2\Delta}{\Sigma^2} \ . \tag{127}$$

In the final step we can find the solution for $\mu_2 + \mu_3$, for which we first need to find the objects $\mathcal{X}$ and $\mathcal{Y}$:

$$\mathcal{X} = \frac{\Delta - a^2\delta}{\sqrt{\delta}\left[\rho^2\sqrt{\Delta} - 2aMr\sqrt{\delta}\right]} = \frac{\sqrt{\Delta} + a\sqrt{\delta}}{\sqrt{\delta}\left[r^2 + a^2 + a\sqrt{\Delta\delta}\right]} \tag{128}$$

$$\mathcal{Y} = \frac{\Delta - a^2\delta}{\sqrt{\delta}\left[\rho^2\sqrt{\Delta} + 2aMr\sqrt{\delta}\right]} = \frac{\sqrt{\Delta} - a\sqrt{\delta}}{\sqrt{\delta}\left[r^2 + a^2 - a\sqrt{\Delta\delta}\right]} \ . \tag{129}$$

Here we used the decomposition

$$\rho^4\Delta - 4a^2M^2r^2\delta = \left(\rho^2\sqrt{\Delta} - 2aMr\sqrt{\delta}\right)\left(\rho^2\sqrt{\Delta} + 2aMr\sqrt{\delta}\right) \ .$$

The derivatives of these quantities can be computed to

$$\partial_r\mathcal{X} = \frac{(r - M)\rho^2 - (\sqrt{\Delta} + a\sqrt{\delta})2r\sqrt{\Delta}}{\sqrt{\Delta\delta}\left[r^2 + a^2 + a\sqrt{\Delta\delta}\right]^2} \ , \qquad \partial_\mu\mathcal{X} = \frac{\mu\sqrt{\Delta}\left[r^2 + a^2 + a^2\delta + 2a\sqrt{\Delta\delta}\right]}{\delta^{\frac{3}{2}}\left[r^2 + a^2 + a\sqrt{\Delta\delta}\right]^2} \ ,$$

$$\partial_r\mathcal{Y} = \frac{(r - M)\rho^2 - (\sqrt{\Delta} - a\sqrt{\delta})2r\sqrt{\Delta}}{\sqrt{\Delta\delta}\left[r^2 + a^2 - a\sqrt{\Delta\delta}\right]^2} \ , \qquad \partial_\mu\mathcal{Y} = \frac{\mu\sqrt{\Delta}\left[r^2 + a^2 + a^2\delta - 2a\sqrt{\Delta\delta}\right]}{\delta^{\frac{3}{2}}\left[r^2 + a^2 - a\sqrt{\Delta\delta}\right]^2} \ .$$

These results can now be inserted into Eq. 100 to obtain

$$\left[(r - M)\partial_r + \mu\partial_\mu\right](\mu_2 + \mu_3) = 2 - \frac{(r - M)^2}{\Delta} - 2\frac{rM}{\rho^2} \ , \tag{130}$$

which is solved (as is easily checked) by

$$e^{\mu_2+\mu_3} = \frac{\rho^2}{\sqrt{\Delta}} \, . \tag{131}$$

With this result we are finally done; the line element (reverting back to the $(+---)$ convention) is now given by

$$ds^2 = \frac{\rho^2}{\Sigma^2}\Delta dt^2 - \frac{\Sigma^2}{\rho^2}\sin^2(\theta)\left(d\phi - \frac{2aMr}{\Sigma^2}dt\right)^2 - \frac{\rho^2}{\Delta}\left(dr^2 + \Delta d\theta^2\right) \, . \tag{132}$$

In matrix notation, we have

$$(g_{\mu\nu}) = \begin{pmatrix} 1-\frac{2Mr}{\rho^2} & 0 & 0 & \frac{2aMr}{\rho^2}\sin^2(\theta) \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{2aMr}{\rho^2}\sin^2(\theta) & 0 & 0 & -\left[r^2+a^2+\frac{2Ma^2r\sin^2(\theta)}{\rho^2}\right]\sin^2(\theta) \end{pmatrix} , \tag{133}$$

$$(g^{\mu\nu}) = \begin{pmatrix} \frac{\Sigma^2}{\rho^2\Delta} & 0 & 0 & \frac{2aMr}{\rho^2\Delta} \\ 0 & -\frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho^2} & 0 \\ \frac{2aMr}{\rho^2\Delta} & 0 & 0 & -\frac{\Delta-a^2\sin^2(\theta)}{\rho^2\Delta\sin^2(\theta)} \end{pmatrix} . \tag{134}$$

7 Properties of the Kerr metric

Having, after a lot of work, finally having established the Kerr metric, we want to take a look at some of its properties.

7.1 Asymptotic properties

Firstly, when letting $a \to 0$, we obtain

$$(g_{\mu\nu}) = \begin{pmatrix} 1-\frac{2M}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix} , \tag{135}$$

which is the usual Schwarzschild metric. Thus the parameter M has to be identified with the mass of the black hole and $r_S = 2M$ (in natural units) is the Schwarzschild radius.
When letting r go to infinity, we obtain (up to order $\mathcal{O}(r^{-3})$)

$$(g_{\mu\nu}) = \begin{pmatrix} 1-\frac{2M}{r} & 0 & 0 & \frac{2aM}{r}\sin^2(\theta) \\ 0 & -\frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ \frac{2aM}{r}\sin^2(\theta) & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix} . \tag{136}$$

Manifestly, this object constitutes the Schwarzschild metric with a $d\phi dt$ cross term, which should correspond to the rotation of the body in question. We observe that for $a \to 0$, this term of course vanishes. Furthermore, if we exclude all orders smaller than $\mathcal{O}(1)$, we are left with the usual Minkowski metric in spherical coordinates, thus letting us conclude that the Kerr metric is asymptotically flat.
If we calculate the rate of rotation $\mathbf{\Omega}$ for Eq. 136 (cf. [4]), we find, when identifying $a = \frac{J}{M}$ (with J the total angular momentum of the body) that

$$\mathbf{\Omega} = \frac{3(\mathbf{J}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{J}}{r^3} , \tag{137}$$

in agreement with the known Lense-Thirring effect, which constitutes a weak-field approximation of the exact Kerr solution.
Finally, for $M \to 0$, we find

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{r^2+a^2\cos^2(\theta)}{r^2+a^2} & 0 & 0 \\ 0 & 0 & -[r^2+a^2\cos^2(\theta)] & 0 \\ 0 & 0 & 0 & -\left(r^2+a^2\right)\sin^2(\theta) \end{pmatrix} , \tag{138}$$

which is the flat Minkowski metric in ellipsoidal coordinates.

7.2 Special surfaces

As was the case with the Schwarzschild metric, we are facing some interesting behaviours in Eq. 135, which we will now take a closer look at.
The first one occurs when $\Delta = 0$, which translates to

$$r_{H,\pm} = \frac{r_S}{2} \pm \sqrt{\left(\frac{r_S}{2}\right)^2 - a^2} . \tag{139}$$

This makes it apparent that the one event horizon from the Schwarzschild metric at r_S (for $a = 0$) splits up into an inner and an outer event horizon, with the distance between the two increasing with a. Notice that for a sufficiently large, Δ can *not* become zero at all, implying that there will be no event horizon, thus leading to a *naked singularity*, with „singularity“as it is predicted by GR (we will in the next subsection show the existence of such a singularity in the first place).
Secondly, the tt component of the metric changes sign when $\rho^2 = 2Mr$; this leads to

$$r_{E,\pm} = \frac{r_S}{2} \pm \sqrt{\left(\frac{r_S}{2}\right)^2 - a^2\cos^2(\theta)} . \tag{140}$$

These two surfaces are called the inner and outer *ergospheres*. Since $\cos^2(\theta) \leq 1$, the outer ergosphere lies outside the outer event horizon, touching it at the poles, while the inner ergosphere lies within the inner event horizon. Inside this (outer) ergosphere, every particle on a timelike path has to co-rotate with the central mass.
Due to the respective particles not yet having crossed the event horizon, they may still be ejected from the black hole, having gained energy through the forced co-rotation. Thus the rotating black hole emits highly energetic particles, which is called the *Penrose process* and is one theory to explain gamma-ray bursts.

7.3 Singularities

In order to see where the singularities of the Kerr metric lie, we would have to express the components of the Riemann tensor in terms of our solution; this however, we will not do right now, because this was enough work already, but refer to Ref. [2], where it becomes clear that the components of the Riemann tensor only diverge for $\rho = 0$. This in turn can only occur for $r = 0$ and $\theta = \frac{\pi}{2}$ (Here we also can see that the inner ergosphere touches the singularity in the equatorial plane). In order to clarify the meaning of this singularity we firstly change to new time and angular variables

$$\begin{aligned} du =& dt - \frac{r^2+a^2}{\Delta} dr \\ d\tilde{\phi} =& d\phi - \frac{a}{\Delta} dr . \end{aligned}$$

Hereafter (and after a few steps in Ref. [2]) we introduce

$$dx^0 = du + dr$$

$$
\begin{aligned}
x &= \left[r\cos(\tilde{\phi}) + a\sin(\tilde{\phi})\right]\cos(\theta) \\
y &= \left[r\sin(\tilde{\phi}) - a\cos(\tilde{\phi})\right]\cos(\theta) \\
z &= r\cos(\theta) \\
x^2 + y^2 &= (r^2 + a^2)\sin^2(\theta)\ ,
\end{aligned}
$$

after which the line element can be brought into the following form:

$$
ds^2 = (dx^0)^2 - dx^2 - dy^2 - dz^2 - \frac{2Mr^3}{r^4 + a^2 z^2}\left\{dx^0 - \frac{1}{r^2 + a^2}\left[r\left(xdx + ydy\right) + a\left(xdy - ydx\right)\right] - \frac{z}{r}dz\right\}^2\ .
$$

The advantage of this form lies in the fact that it reduces to a cartesian coordinate system in the limit of $M \to 0$, thus removing the degeneracy of spherical (or ellipsoidal) coordinates at $r = 0$.
Here we can clearly see that the point of the singularity $r = 0, \theta = \frac{\pi}{2}$ corresponds to

$$
x^2 + y^2 = a^2 \tag{141}
$$

in the equatorial plane ($z = 0$). Thus the Kerr metric actually features a *ring singularity* with radius a, that of course shrinks to a point in the limit of vanishing rotation.
As a final remark, the existence of a ring singularity makes it obvious that we are (somehow) allowed to continue our solution to negative values of r. This entails many interesting consequences, which we will however not deal with here and refer to e.g. Ref. [5].

References

[1] S. Chandrasekhar *The Kerr Metric and Stationary Axisymmetric Gravitational Fields* Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 358, No. 1695 (Jan. 16, 1978), pp. 405-420
url: `https://www.jstor.org/stable/79479`

[2] S. Chandrasekhar *The Mathematical Theory of Black Holes* Oxford University Press, 1983

[3] T. Eguchi, P. B. Gilkey, A. J. Hanson *Graviation, gauge theories and differential geometry* PHYSICS REPORTS (Review Section of Physics Letters) 66. No. 6 (1980) 213—393

[4] C. Chakraborty *Lense-Thirring Precession in Strong Gravitational Fields*
url: `https://indico.cern.ch/event/336103/contributions/786924/attachments/1203051/1813620/chakraborty_proc.pdf`

[5] W. Israel *Source of the Kerr Metric* Phys. Rev. D 2, 641 (1970)