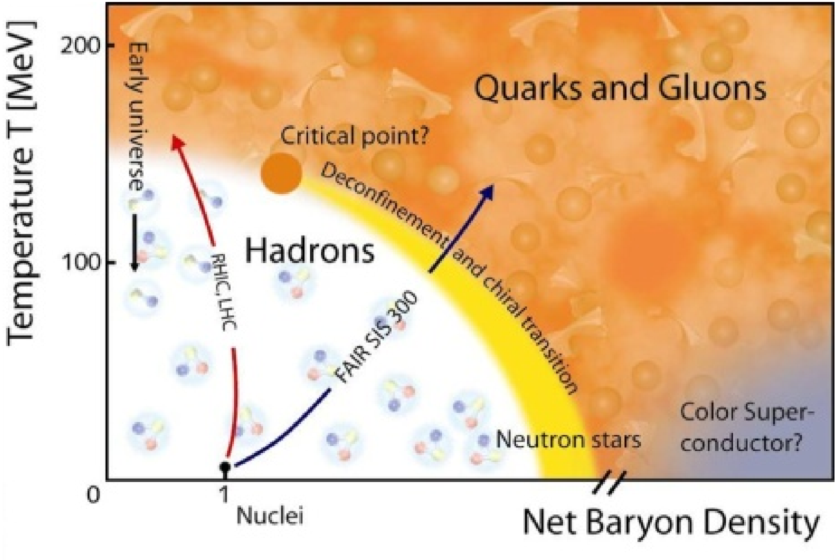
Complex Langevin simulations and the QCD phase diagram

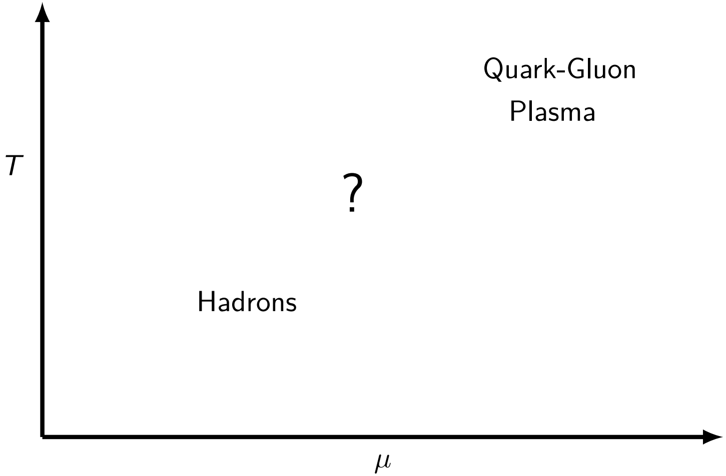
Benjamin Jäger

Outline

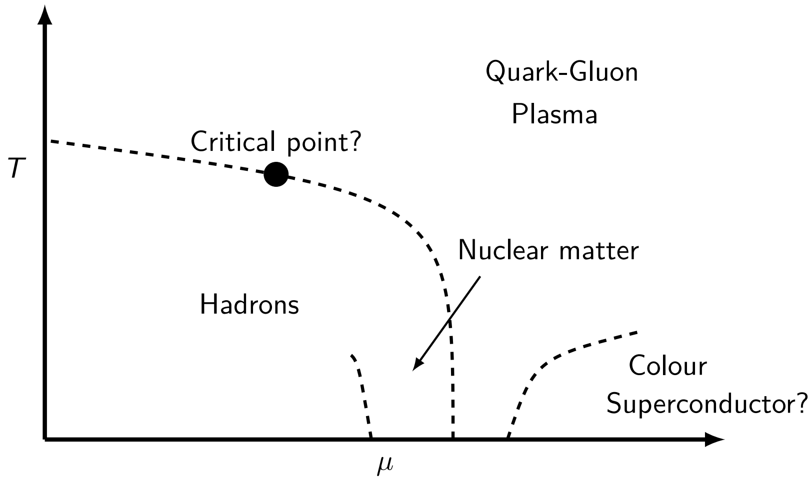
Phase diagram for QCD



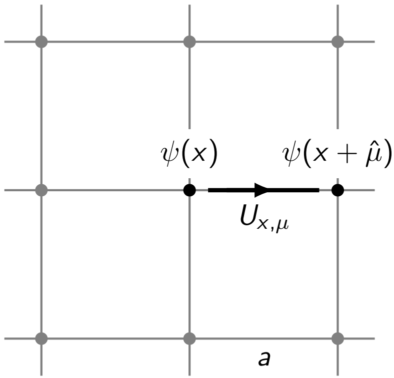
Phase diagram for QCD



Phase diagram for QCD



Overview on Lattice QCD



$U_{x,\mu} \in \mathrm{SU}(3)$

- Discretize Euclidean space-time by a hypercubic lattice Λ
- Quantize QCD using Euclidean path integrals
- Calculate expectation values using Monte Carlo techniques:

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}[U]\, A[U]\, (\det D)^{N_f}\; e^{-S_G[U]} \qquad U_{x,\mu} \in \mathrm{SU}(3)$$

- **Ab initio** method: Only QCD parameters $m_q \to \kappa$, $g_0 \to \beta$.

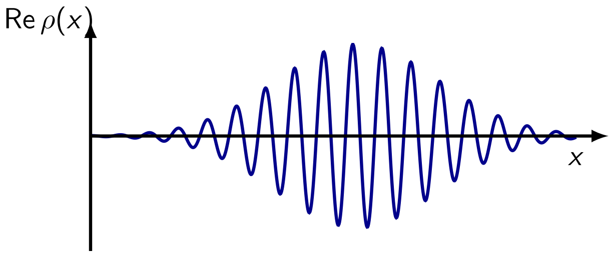
The SIGN problem

- Finite chemical potential $\rightarrow$ SIGN problem

$$(\det D(\mu))^* = \det D(-\mu^*) \;\rightarrow\; \det D(\mu \neq 0) \in \mathbb{C}.$$

- Importance Sampling based Monte Carlo methods fail

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U]\, O(U)\, |\det D|\, e^{i\phi}\, e^{-S_G(U)}$$



How bad is the SIGN problem?!

- Overlap problem

$$\langle O \rangle = \frac{\int \mathcal{D}[U]\, O\, |\det D|\, e^{i\phi}\, e^{-S_G(U)}}{\int \mathcal{D}[U]\, |\det D|\, e^{i\phi}\, e^{-S_G(U)}} = \frac{\left\langle O\, e^{i\phi} \right\rangle_{pq}}{\left\langle e^{i\phi} \right\rangle_{pq}}$$

- The average phase vanishes in the thermodynamic limit: $V \to \infty$

$$\left\langle e^{i\phi} \right\rangle_{pq} = e^{-V \cdot \Delta F} \to 0$$

Alternatives?

Taylor expansion

- Expand around small μ

$$\frac{T_c(\mu)}{T_c(0)} = 1 + c_2 \cdot \left(\frac{\mu}{T_c(0)}\right)^2 + c_4 \cdot \left(\frac{\mu}{T_c(0)}\right)^4 + \ldots$$

- Radius of convergence?

Reweighting

- Absorb phase in the observable

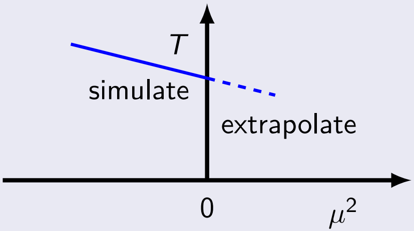
$$\langle O \rangle = \frac{\int \mathcal{D}[U]\, O\, |\det D|\, e^{i\phi}\, e^{-S_G(U)}}{\int \mathcal{D}[U]\, |\det D|\, e^{i\phi}\, e^{-S_G(U)}} = \frac{\left\langle O\, e^{i\phi} \right\rangle_{pq}}{\left\langle e^{i\phi} \right\rangle_{pq}}$$

- Overlap problem

Alternatives?

Analytic continuation

- Continue from imaginary μ to real μ



- How far can one extrapolate?

Many more ...

Here: Complex Langevin

- Stochastic simulation with complexified degrees of freedom

Complex Langevin - Idea

- Consider a simple Gaussian integral

$$Z(a,b) = \int \mathrm{d}x\, \mathrm{e}^{-S}, \text{ with } S = \frac{1}{2}ax^2 + \mathrm{i}bx$$

- Solution: Complete the square or complexify $x \to z = x + \mathrm{i}y$

- Consider **Real action**
- Langevin equation: Stochastic process (Brownian motion)

$$\frac{\partial x}{\partial t} = \frac{\partial S}{\partial x} + \eta(t)$$

where $\langle \eta(t)\eta(t') \rangle = 2\delta(t - t')$ Gaussian white noise

- The associated Fokker-Plank equation

$$\langle O(x) \rangle_\eta = \int \mathrm{d}x\, \rho(x,t)\, O(x)$$

- Stationary solution $\rho(x) = \mathrm{e}^{-S}$

Complex Langevin - Idea

- Consider **Complex action**
- Complexify $x \to z = x + iy$ Langevin equation

$$\frac{\partial x}{\partial t} = \mathrm{Re}\frac{\partial S}{\partial z} + \eta(t) \text{ and } \frac{\partial y}{\partial t} = \mathrm{Im}\frac{\partial S}{\partial z}$$

- The associated Fokker-Plank equation

$$\langle O(x+iy)\rangle_\eta = \int dx\, dy\, \widetilde{\rho}(x+iy)\, O(x+iy)$$

- Convergence:
  - Imaginary direction y is compact (No boundary terms)
  - Action S is holomorphic

$$\Rightarrow \int dx\, dy\, \widetilde{\rho}(x+iy)\, O(x+iy) = \int dx\, \rho(x,t)\, O(x)$$

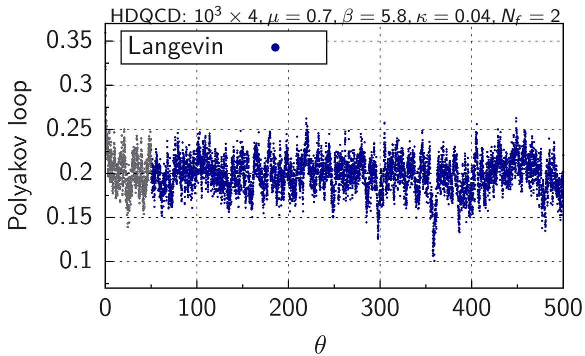
Complex Langevin - Gauge theories

- Complexify degrees of freedom $SU(3) \to SL(3,\mathbb{C})$

$$U_{x,\mu} = \exp\left[\mathrm{i}\, a\, \lambda^c \left(A^c_{x,\mu} + \mathrm{i}\, B^c_{x,\mu}\right)\right]$$

- Evolve links according (1st order) Langevin equation

$$U_{x,\mu}(\theta+\varepsilon) = \exp\left[\mathrm{i}\, \lambda^a \left(-\varepsilon\, D^a_{x,\mu}\, S + \sqrt{\varepsilon}\, \eta^a_{x,\mu}\right)\right] U_{x,\mu}(\theta)$$



Complex Langevin simulations

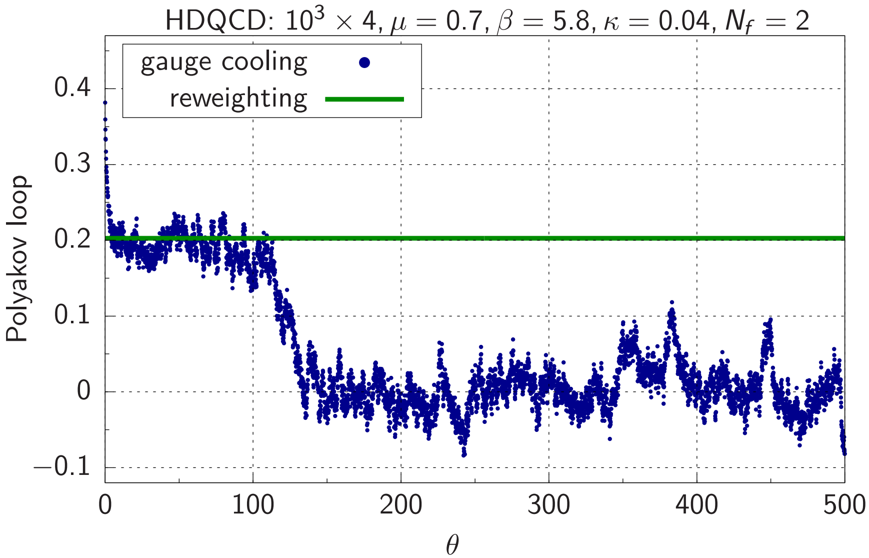
- However, $SL(3,\mathbb{C})$ is not a compact group...
- Convergence $\Leftrightarrow$
  - Action S is holomorphic
  - "Imaginary" direction of $SL(3,\mathbb{C})$ falls off quickly enough
- Measure distance to SU(3) manifold

$$\text{unitnorm} = \mathrm{Tr}\left(U_{x,\mu}\, U^{\dagger}_{x,\mu} - 1\right)^2$$

- Gauge cooling is essential, but sometimes not sufficient...

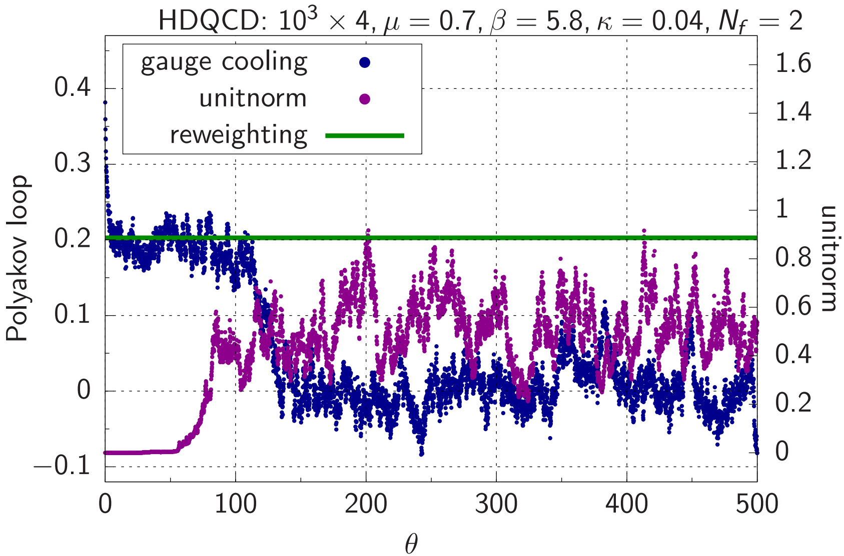
$$U_{x,\mu} \to \Omega_x\, U_{x,\mu}\, \Omega^{-1}_{x+\mu}$$

Gauge cooling



- Tunneling to wrong results.

Gauge cooling



- Tunneling to wrong results, when unitnorm grows too large.

Dynamic stablization

- Adding a trivial force to the Langevin dynamics

$$U_{x,\nu}(\theta+\varepsilon) = \exp\left[\mathrm{i}\lambda^a\left(\varepsilon\, K^a_{x,\nu} + \mathrm{i}\,\varepsilon\,\alpha_{DS}\, M^a_x + \sqrt{\varepsilon}\,\eta^a_{x,\nu}\right)\right] U_{x,\nu}(\theta)$$

where

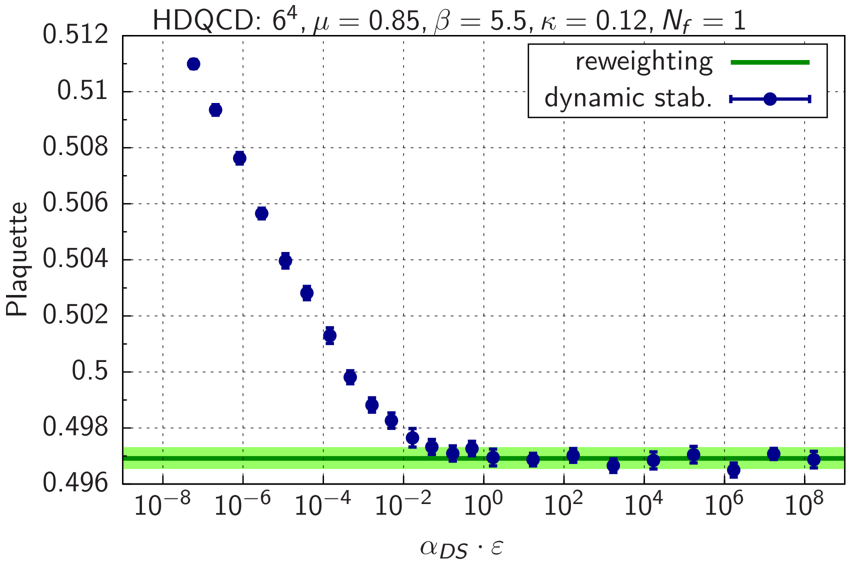
$$M^a_x = \mathrm{i} b^a_x \left(\sum_c b^c_x\, b^c_x\right)^3 \text{ and } b^a_x = \mathrm{Tr}\left[\lambda^a \sum_\nu U_{x,\nu} U^\dagger_{x,\nu}\right].$$

- Expanding the force in terms of gauge fields A and B

$$M^a_x \sim a^7 \left(\overline{B}^c_y\, \overline{B}^c_y\right)^3 \overline{B}^a_x + \mathcal{O}(a^8)\,.$$

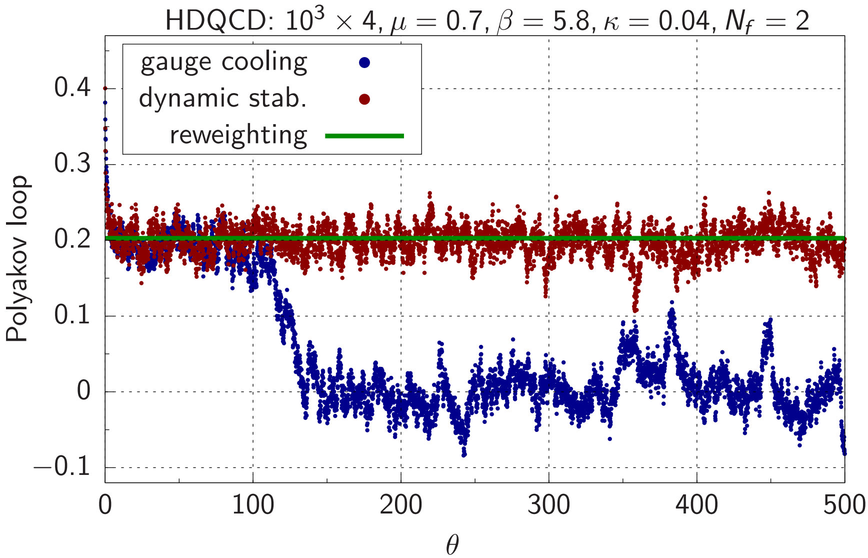
- Dynamic stabilization is numerically cheap and can be combined with gauge cooling (Here: 1 step)

Dynamic stablization



- For sufficient large α_{DS} we find agreement with reweighting.

Dynamic stablization

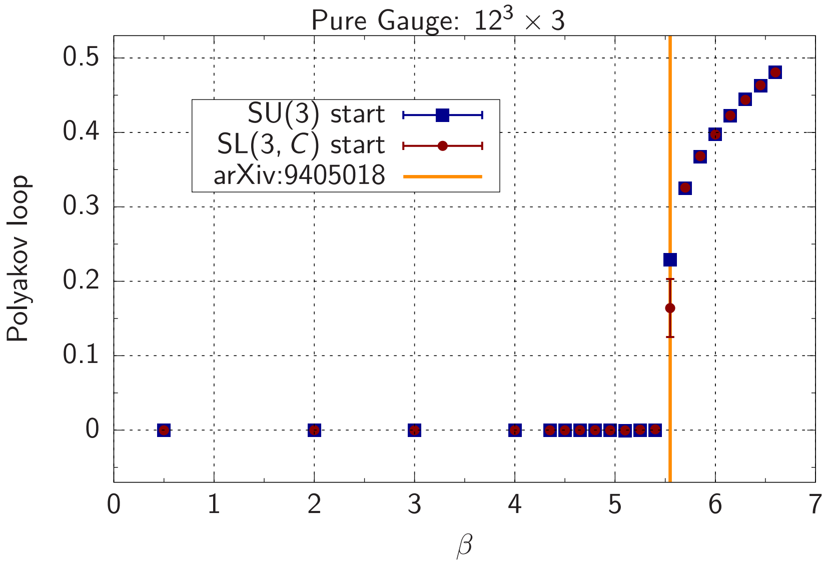


- Improved stability using dynamic stablization

Results

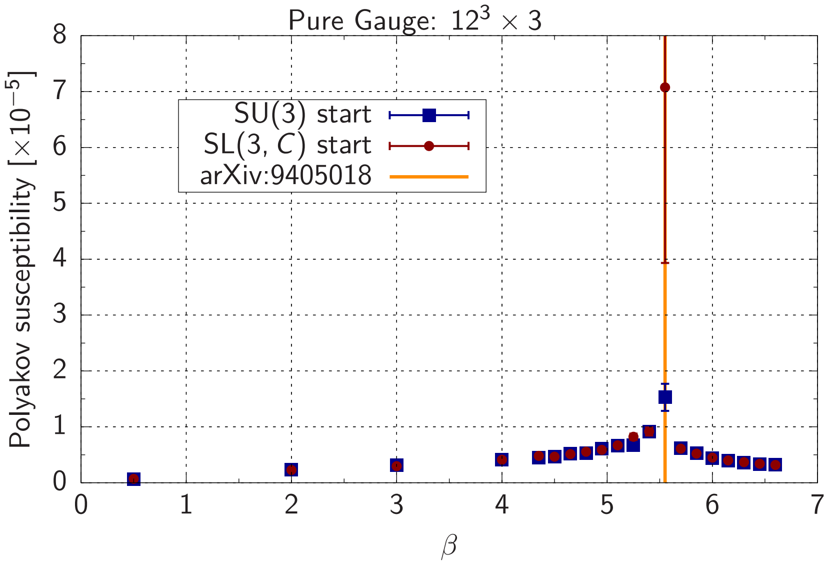
- **Pure Yang-Mills** $\Leftrightarrow$ No sign problem
  - Checking Langevin simulations
  - Compare to standard Hybrid Monte Carlo methods
- **Heavy dense QCD** (HDQCD)
  - Heavy and dense approximation of QCD
  - Has a sign problem and phase transitions
  - Numerical cheap :)
- **XY model with finite** μ
  - Toy model for QCD
- **Full (Staggered) QCD**
  - Including light dynamical fermions
  - Very preliminary results

Pure Yang-Mills



- The correct transition is obtained, even for $SL(3,\mathbb{C})$ start.

Pure Yang-Mills



- The correct transition is obtained, even for $SL(3, \mathbb{C})$ start.

Heavy Dense QCD

- **Here: QCD in the limit of heavy quarks (HDQCD).**
  - Fermion determinant simplifies with the Polyakov loops $P_{\vec{x}}$ and $P_{\vec{x}}^{-1}$ as

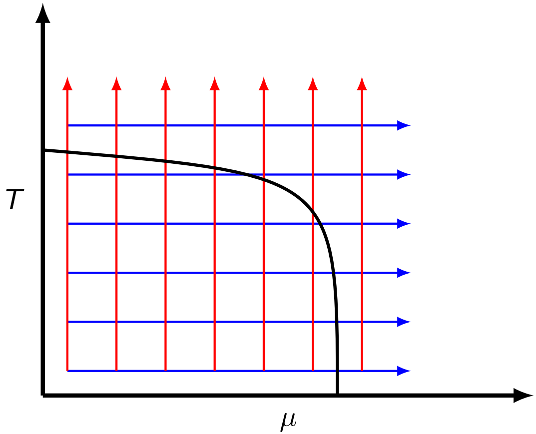
$$\det D(\mu) = \prod_{\vec{x}} \det\left(1 + C\, P_{\vec{x}}\right)^2 \det\left(1 + C'\, P_{\vec{x}}^{-1}\right)^2 ,$$

where the Polyakov loop $P_{\vec{x}}$ is defined as

$$P_{\vec{x}} = \frac{1}{V} \sum_t U_0(\vec{x}, t)$$

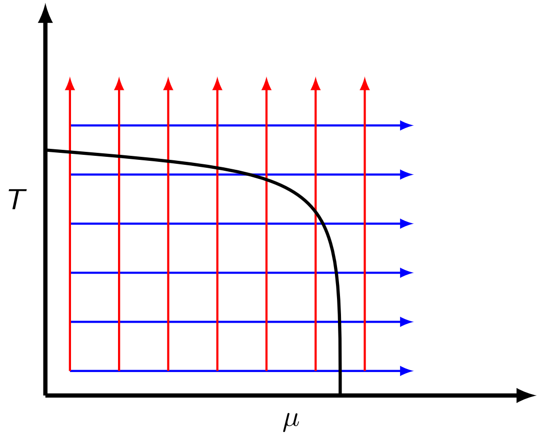
  - For the gluonic part, we use the full Wilson gauge action.
  - Map out the phase diagram for HDQCD.
  - Expected transition: $\mu_c^0 = -\ln(2\kappa)$

Heavy Dense QCD



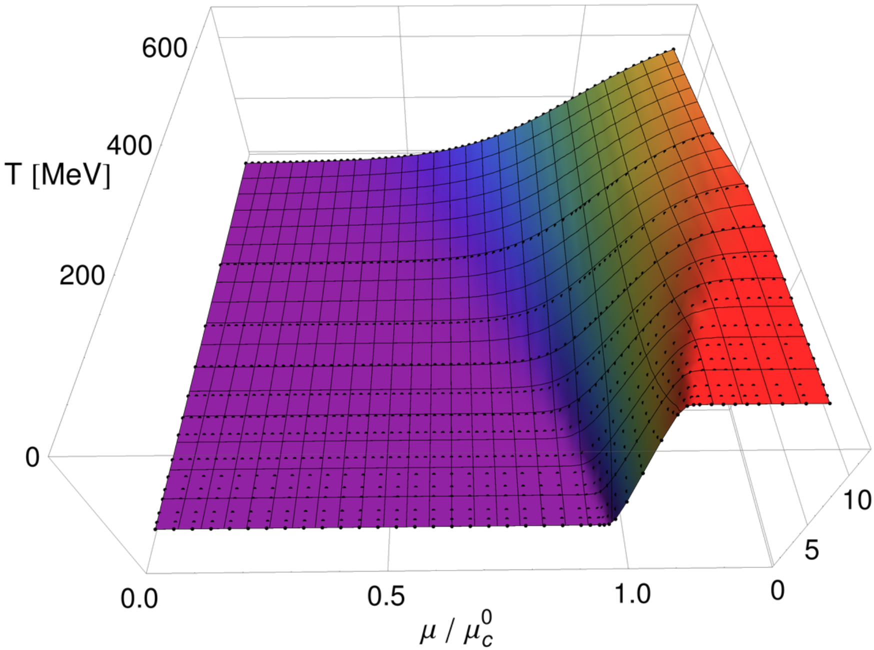
- Strategy:
  - Determine μ-transition in Fermion density
  - Determine T-transition in Polyakov loop

Heavy Dense QCD



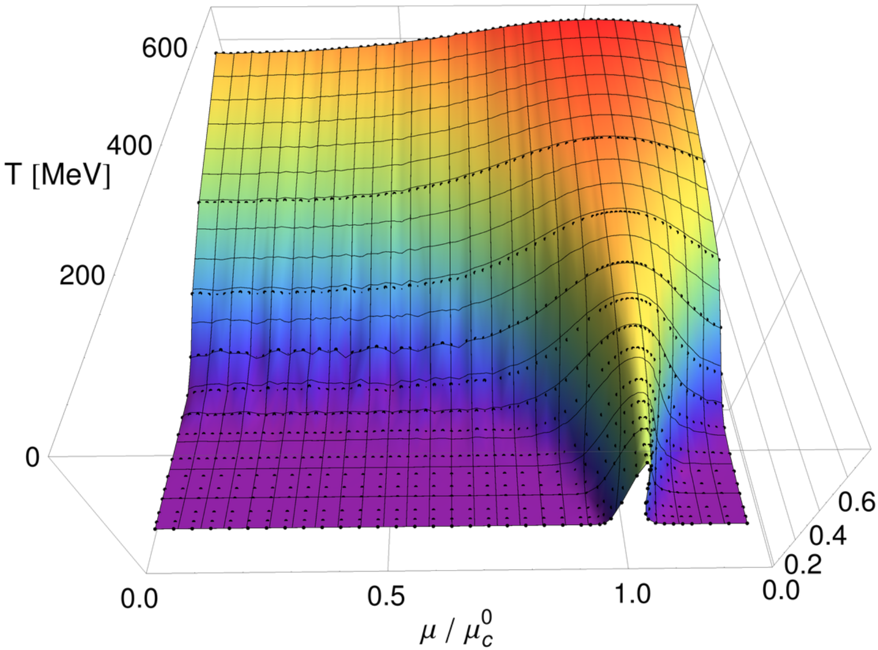
$\beta = 5.8$ $\kappa = 0.04$	$V = 6^3, 8^3, 10^3$ $N_f = 2$	$a \sim 0.15$ fm $\mu_c^0 = 2.53$
N_τ T [MeV]	28 - 2 48 - 671	

Heavy Dense QCD



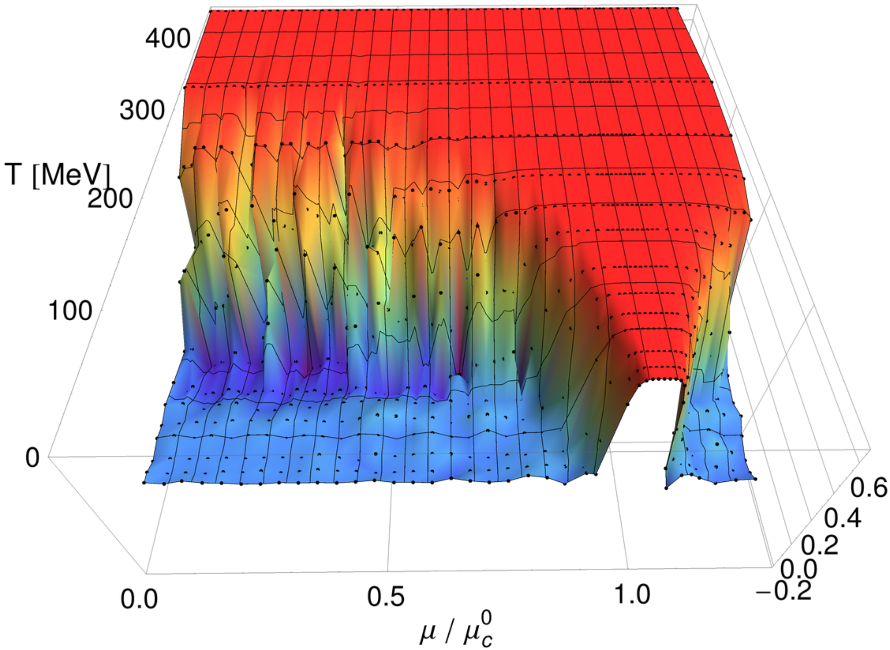
- Fermion density: $n = \frac{1}{N_t} \frac{\partial \ln Z}{\partial \mu}$

Heavy Dense QCD



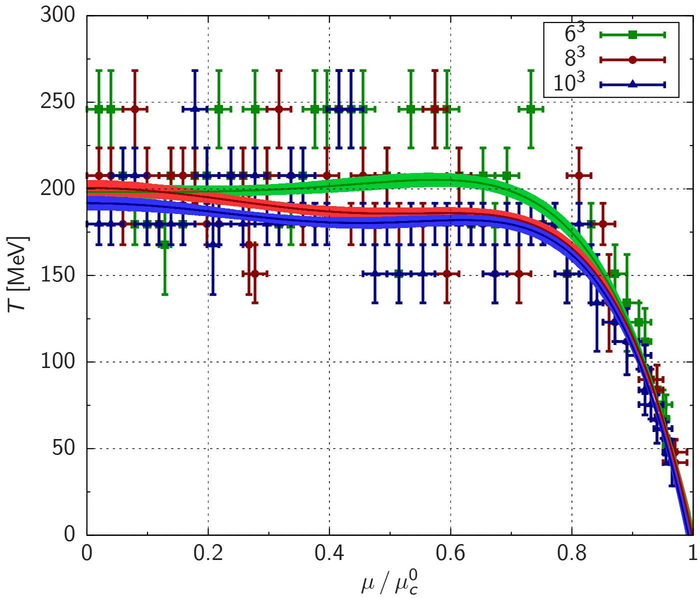
- Polyakov loop

Heavy Dense QCD



- Binder cumulant of the Polyakov loop $B = 1 - \frac{\langle P^4 \rangle}{3 \langle P^2 \rangle^2}$

Heavy Dense QCD



- Fit the phase boundary using $T_c(\mu) = \sum_k b_k \left(1 - \mu/\mu_c\right)^k$

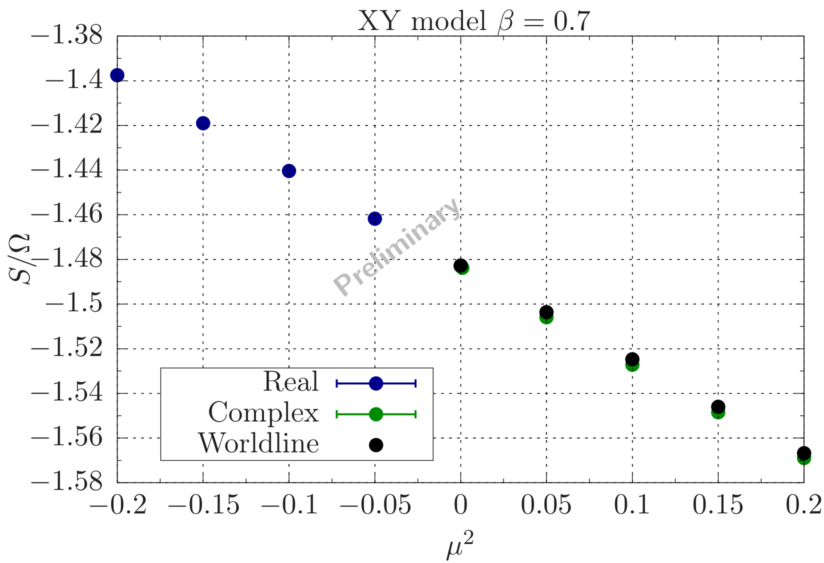
XY model with finite μ

[Aarts, James, 2005]

- **XY model with finite μ**
  - Toy model to test Complex Langevin
  - Has a phase transition at $\beta_c \sim 0.45$
  - Ordered phase: $\beta > \beta_c$, disordered phase: $\beta < \beta_c$

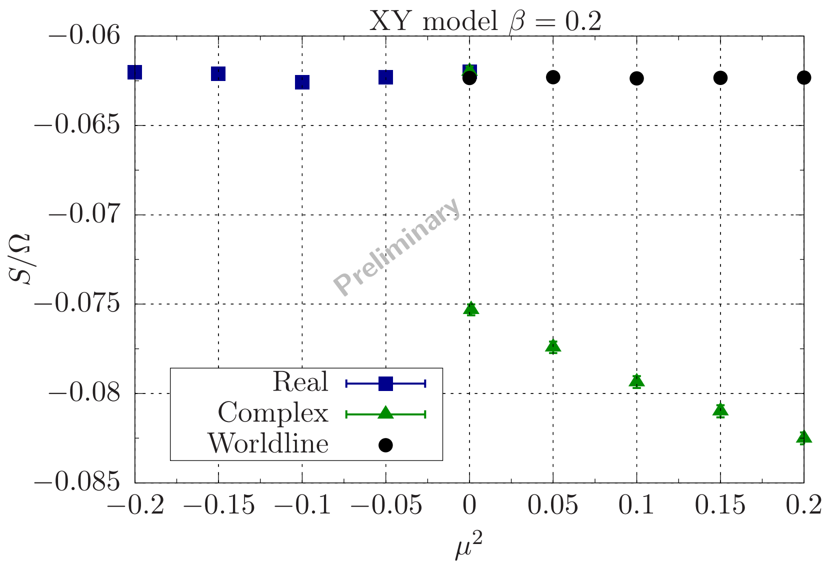
$$S = -\beta \sum_x \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}} - i\mu\delta_{\nu,0}\right)$$

XY model with finite μ



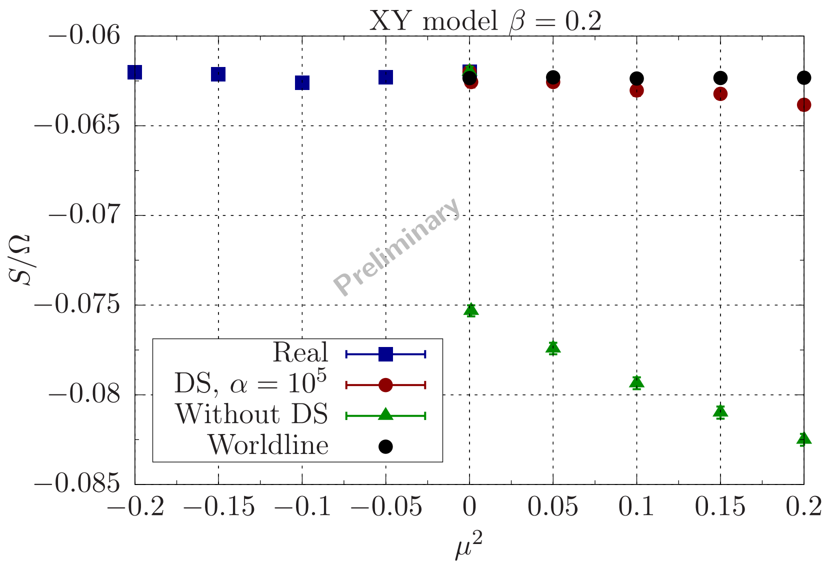
- In the ordered phase Complex Langevin agrees with (dual) Worldline formulation

XY model with finite μ



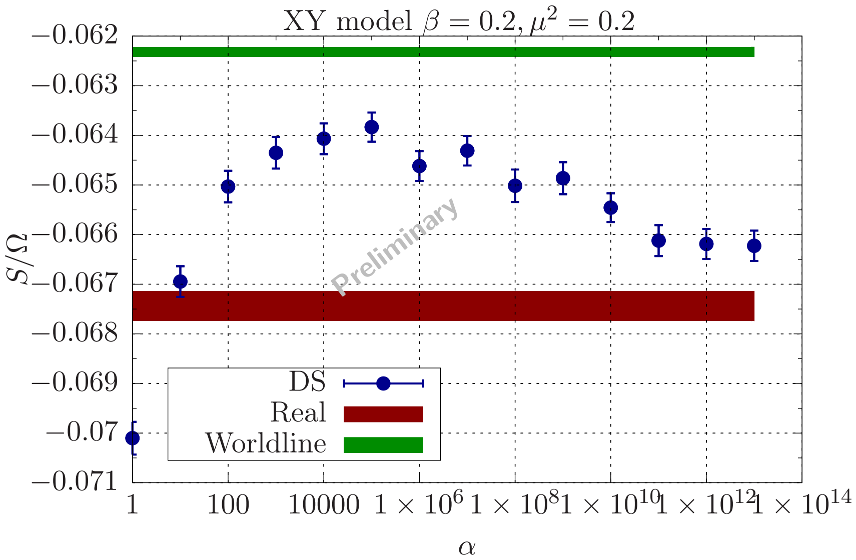
- In the disordered phase Complex Langevin fails spectacularly

XY model with finite μ



- Adding dynamic stabilization

XY model with finite μ



- For large μ^2 a discrepancy remains

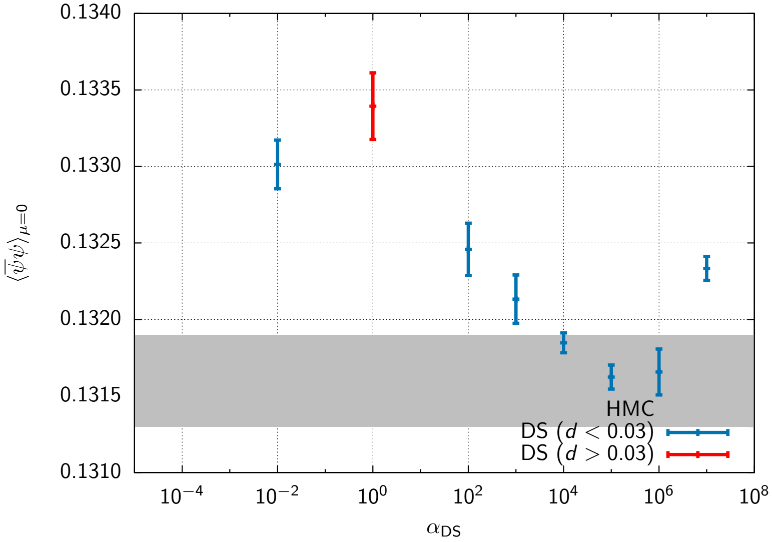
Naïve Staggered QCD

- **Here: QCD using unimproved Staggered quarks.**
  - Setup: $\beta = 5.6$, $am_q = 0.025$
  - Every Langevin update needs inversion of the Dirac operator...

$$D^a_{x,\mu} S = D^a_{x,\mu} S_{YM} - \frac{N_f}{4} \mathrm{Tr}\left[M^{-1} D^a_{x,\mu} M\right]$$

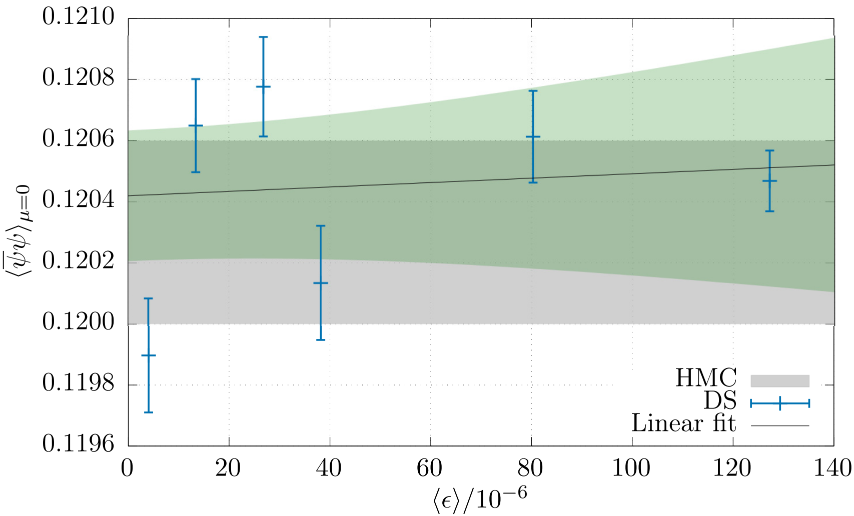
  - Comparison @ $\mu = 0$ with HMC

Naïve Staggered QCD



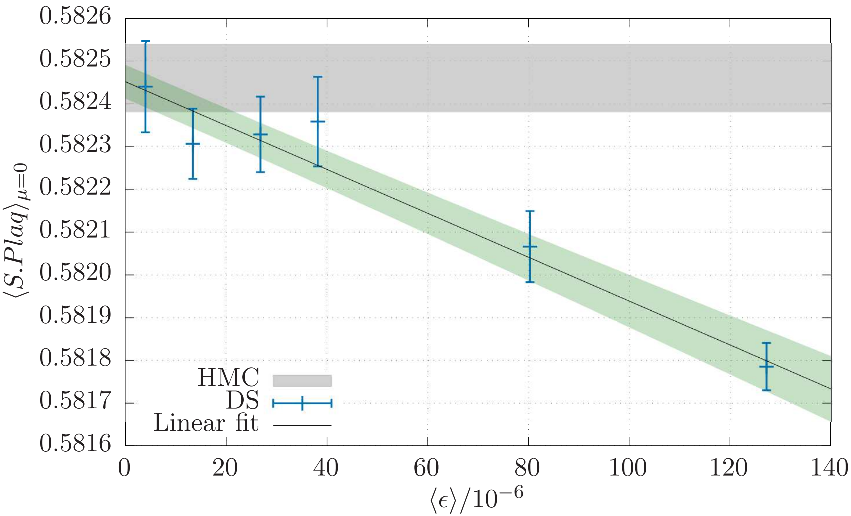
- Comparison @ $\mu = 0$ with HMC: 6^4

Naïve Staggered QCD



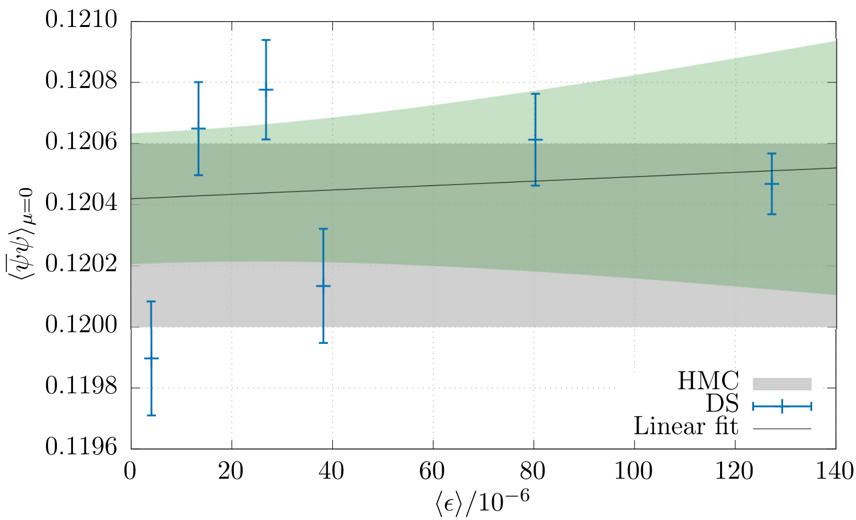
- Comparison @ $\mu = 0$ with HMC: 6^4 - Stepsize extrapolation

Naïve Staggered QCD



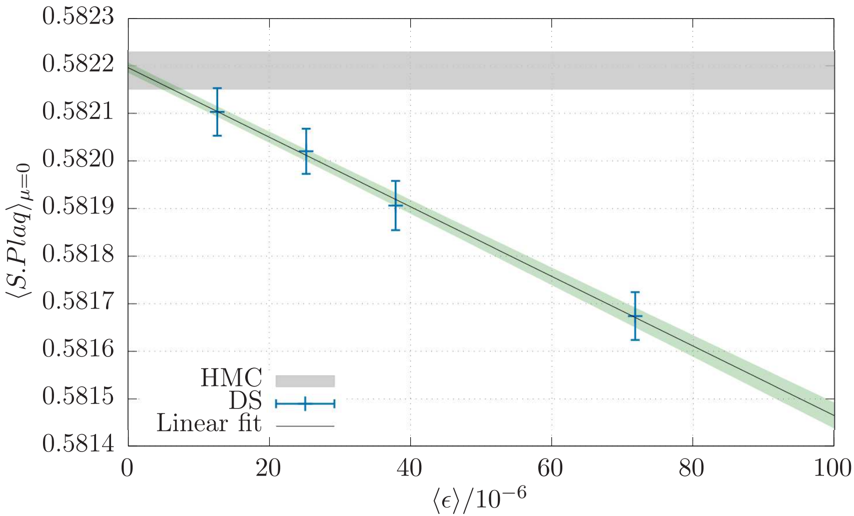
- Comparison @ $\mu = 0$ with HMC: 6^4 - Stepsize extrapolation

Naïve Staggered QCD



- Comparison @ $\mu = 0$ with HMC: 8^4 - Stepsize extrapolation

Naïve Staggered QCD



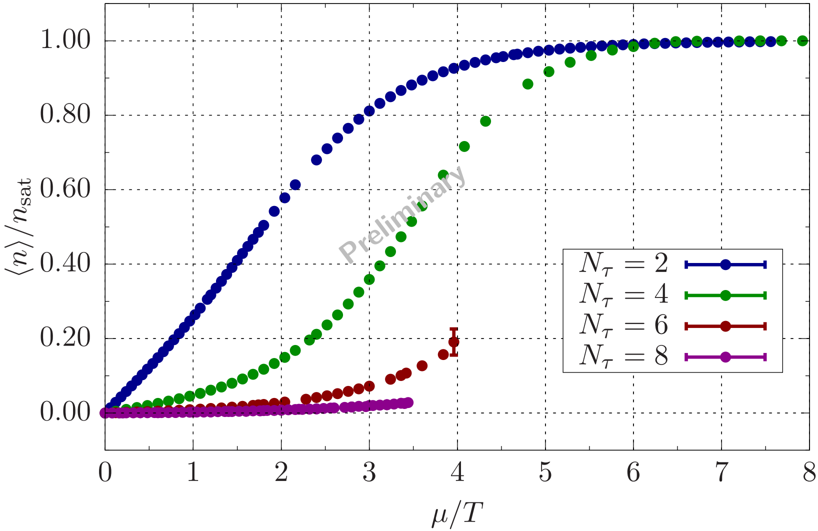
- Comparison @ $\mu = 0$ with HMC: 8^4 - Stepsize extrapolation

Naïve Staggered QCD

- **Comparison @ $\mu = 0$ with HMC.**

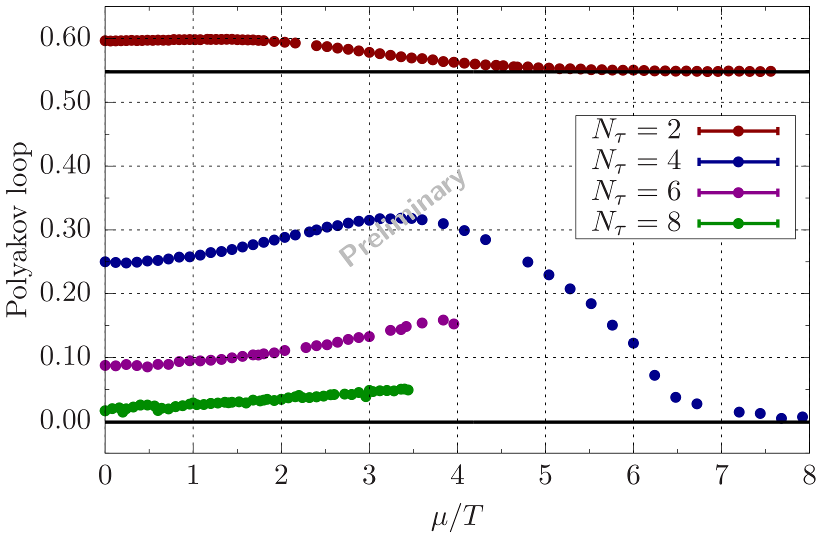
Volume	Plaquette		$\bar{\psi}\psi$	
	HMC	CL	HMC	CL
6^4	0.58246(8)	0.582452(4)	0.1203(3)	0.12042(2)
8^4	0.58219(4)	0.582196(1)	0.1316(3)	0.1319(2)
10^4	0.58200(5)	0.58201(4)	0.1372(3)	0.1370(6)
12^4	0.58196(6)	0.58195(2)	0.1414(4)	0.1409(3)

Naïve Staggered QCD



- Fermion density

Naïve Staggered QCD

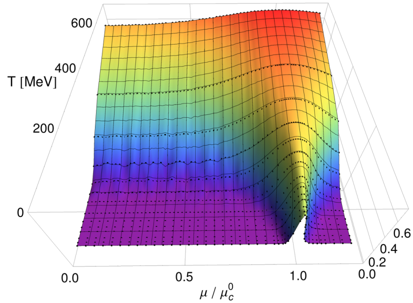


- Polyakov loop

Future work

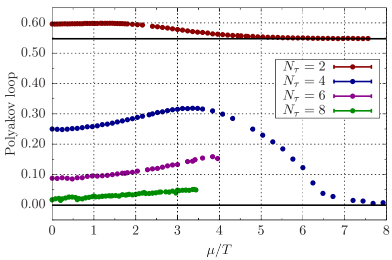
Conclusion

- Complex Langevin simulation can be used to study the QCD phase diagram
- Dynamical stabilisation improves convergence
- Work on the convergence, especially around μ_c.



Future work

- Start proper Full QCD simulations to identify phase structure of QCD.
- A lot of work to be done!



Thank you for your attention!