

Beyond second order theories of relativistic dissipative fluids

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Flow and dissipation in ultrarelativistic Heavy Ion Collisions
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- Development of causal theories of relativistic dissipative fluid dynamics for heavy ion collisions is an important achievement.
- It has inspired many authors to apply its methodology in describing observables from heavy ion collision experiments and to compare to transport models
- AM & D.H. Rischke, U. Heinz & H. Song, D. Teaney & Dusling, P. Romatschke & M. Luzum, R. Baier, S. Baas, M. Bleicher, P. Huovinen & D. Molnar, T. Kodama & T. Koide, S. Pratt & J. Vredevoogd, C. Greiner, A. El, Z. Xu, H. Niemi, & E. Molnar, I. Bouras, G. Torrieri, G. Denicol, M. Martinez, H. Peterson, T. Hirano, C. Nonaka, A. Monnai, R. Lacey ,... String Theorists

- causal theories are mainly furnished by imposing relativity and entropy principles up to second order in dissipative fluxes
- we refrained from exploiting subsequent orders because the methodology is long and cumbersome; also because the contribution of these higher order terms to the solution of the problem at hand might be negligible.
- However, the exploitation of these higher order terms in dissipative fluxes is desirable for the following reasons

- ... a second order theory is necessary to link more closely the relativistic case to the non-relativistic one; we are happy just to know that the principle of relativity is not violated
- ... imposing the conditions up to order N we get the N -th order equations. These equations depends also on the lower order terms. Thus the conditions to the solution at higher order constrain the lower order expressions.
- we will present the results at third order because it is at this order were all three main dissipative fluxes for a single fluid component couple and this coupling is very interesting to study in detail

Notation and conventions

- The metric tensor $g^{\alpha\beta} = \text{diag}(1, -1, -1, -1)$;
- The four velocity of the fluid is u^α and we have $u^\alpha u_\alpha = c^2 = 1$;
- Projection tensor is $\Delta^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ and we have $u_\beta \Delta^{\alpha\beta} = 0$.
- The comoving time derivative: $D \equiv u^\alpha \partial_\alpha$.
- The comoving space derivative: $\nabla^\alpha \equiv \Delta^{\alpha\beta} \partial_\beta$.
- Parenthesis around some indices denote symmetrization :

$$A_{(\mu} B_{\nu)} = \frac{1}{2} (A_\mu B_\nu + A_\nu B_\mu), \quad A_{[\mu} B_{\nu]} = \frac{1}{2} (A_\mu B_\nu - A_\nu B_\mu).$$

- The angular brackets around two indices denote skew-symmetrization: a tensor that is symmetric, traceless, and orthogonal to the fluid velocity:

$$S^{\langle\alpha\beta\rangle} = \left(\Delta_\mu^\alpha \Delta_\nu^\beta - \frac{1}{3} \Delta^{\alpha\beta} \Delta_{\mu\nu} \right) S^{\mu\nu}$$

14-Fields Theory of Relativistic Dissipative Fluid Dynamics :

The 14 fields are $p(n, \varepsilon)$, Π , u^α , q^α , $\pi^{\langle\alpha\beta\rangle}$ and they are governed by the following fields equations

$$\begin{aligned}\partial_\alpha N^\alpha &= 0 \\ \partial_\alpha T^{\alpha\beta} &= 0 \\ \partial_\gamma F^{\alpha\beta\gamma} &= P^{\alpha\beta}\end{aligned}$$

For all thermodynamic processes the entropy principle holds

$$\partial_\alpha S^\alpha \geq 0$$

Representation of primary variables

$$\begin{aligned}N^\alpha &= nu^\alpha \\T^{\alpha\beta} &= \varepsilon u^\alpha u^\beta - (p + \Pi)\Delta^{\alpha\beta} + 2q^{(\alpha} u^{\beta)} + \pi^{\langle\alpha\beta\rangle} \\P^{\alpha\beta} &= \mathcal{P}_\Pi \Pi (\Delta^{\alpha\beta} - 3u^\alpha u^\beta) + 2\mathcal{P}_q q^{(\alpha} u^{\beta)} + \mathcal{P}_\pi \pi^{\langle\alpha\beta\rangle},\end{aligned}$$

Using of u^α and $\Delta^{\alpha\beta}$ we can express local rest frame quantities in terms of N^α and $T^{\alpha\beta}$

$$\begin{aligned}n &= \left(N^\alpha N_\alpha\right)^{1/2}, \quad u^\alpha = \frac{N^\alpha}{n}, \quad \varepsilon = u_\alpha u_\beta T^{\alpha\beta} \\p + \Pi &= -\frac{1}{3}\Delta_{\alpha\beta} T^{\alpha\beta}, \quad q^\alpha = \Delta_\mu^\alpha u_\nu T^{\mu\nu} \\ \pi^{\langle\alpha\beta\rangle} &= \left(\Delta_\mu^\alpha \Delta_\nu^\beta - \frac{1}{3}\Delta^{\alpha\beta} \Delta_{\mu\nu}\right) T^{\mu\nu}\end{aligned}$$

The coefficients \mathcal{P}_F are related to transport coefficients for corresponding flux F

Zeroth order: Equilibrium

The equilibrium values of N^α , $T^{\alpha\beta}$, $F^{\alpha\beta\gamma}$ and S^α are calculated as moments of the Jütner-type equilibrium distribution function

$$\begin{aligned}F^{\alpha\beta\gamma} &= \frac{1}{2}\mathcal{F}_1^0 g^{(\alpha\beta} u^{\gamma)} + \frac{1}{2}\mathcal{F}_2^0 \left(g^{(\alpha\beta} u^{\gamma)} - 2u^\alpha u^\beta u^\gamma \right) \\S^\alpha &= S_1^0 u^\alpha\end{aligned}$$

where here and below the calligraphic coefficients are functions of (ε, n) and thus determined by the equation of state.

$$\begin{aligned}F^{\alpha\beta\gamma(1)} &= \mathcal{F}_1^1 \Pi \left(\Delta^{(\alpha\beta} u^{\gamma)} - u^\alpha u^\beta u^\gamma \right) \\ &\quad + \mathcal{F}_2^1 \left(\Delta^{(\alpha\beta} q^{\gamma)} - 5u^{(\alpha} u^\beta q^{\gamma)} \right) \\ &\quad + \mathcal{F}_3^1 \pi^{(\langle\alpha\beta\rangle} u^{\gamma)} \\ S^\alpha(1) &= \mathcal{S}_1^1 \Pi u^\alpha + \mathcal{S}_2^1 q^\alpha\end{aligned}$$

Second order

$$\begin{aligned}
 F^{\alpha\beta\gamma(2)} = & \mathcal{F}_1^2 \Pi^2 \left(\Delta^{(\alpha\beta} u^{\gamma)} - u^\alpha u^\beta u^\gamma \right) \\
 & + \mathcal{F}_2^2 \left(q^\nu q_\nu \Delta^{(\alpha\beta} u^{\gamma)} - 3u^{(\alpha} q^\beta q^{\gamma)} \right) \\
 & + \mathcal{F}_3^2 q^\nu q_\nu \left(\Delta^{(\alpha\beta} u^{\gamma)} - u^\alpha u^\beta u^\gamma \right) \\
 & + \mathcal{F}_4^2 \left(3u^\alpha \pi^{2\langle\beta\gamma\rangle} - \pi^{2\langle\nu\nu\rangle} u^\alpha u^\beta u^\gamma \right) \\
 & + \mathcal{F}_5^2 \pi^{2\langle\nu\nu\rangle} \left(\Delta^{(\alpha\beta} u^{\gamma)} - u^\alpha u^\beta u^\gamma \right) \\
 & + \mathcal{F}_6^2 \left(q^{(\alpha} \pi^{\langle\beta\gamma\rangle)} - 2u^{(\alpha} u^\beta \pi^{\langle\gamma\rangle\nu)} q_\nu \right) \\
 & + \mathcal{F}_7^2 \left(\Delta^{(\alpha\beta} \pi^{\langle\gamma\rangle\nu)} q_\nu - 5u^\alpha u^\beta \pi^{\langle\gamma\rangle\nu)} q_\nu \right) \\
 & + \mathcal{F}_8^2 \Pi u^{(\alpha} \pi^{\langle\beta\gamma\rangle)} + \mathcal{F}_9^2 \Pi \left(\Delta^{(\alpha\beta} q^{\gamma)} - 5q^{(\alpha} u^\beta u^{\gamma)} \right)
 \end{aligned}$$

Second order

$$\begin{aligned} S^{\alpha(2)} = & \left(S_1^2 \Pi^2 - S_2^2 q^\nu q_\nu + S_3^2 \pi^{2\langle\nu\nu\rangle} \right) u^\alpha \\ & + S_4^2 \Pi q^\alpha + S_5^2 \pi^{\langle\alpha\nu\rangle} q_\nu \end{aligned}$$

$$\begin{aligned} S^{\alpha(3)} = & \left(S_1^3 \Pi^3 - S_2^3 \Pi q_\nu q^\nu + S_3^3 \Pi \pi^{2\langle\nu\nu\rangle} + S_4^3 q_\nu q_\mu \pi^{\langle\mu\nu\rangle} \right. \\ & \left. + S_5^3 \pi^{3\langle\nu\nu\rangle} \right) u^\alpha \\ & + \left(S_6^3 \Pi^2 + S_7^3 q_\nu q^\nu + S_8^3 \pi^{2\langle\nu\nu\rangle} \right) q^\alpha \\ & + S_9^3 \Pi \pi^{\langle\alpha\nu\rangle} q_\nu + S_{10}^3 \pi^{2\langle\alpha\nu\rangle} q_\nu \end{aligned}$$

Dissipative fluxes: Zeroth order: Equilibrium

$$\begin{aligned}\Pi &= \Pi_{Eq.} = 0 \\ q^\alpha &= q_{Eq.}^\alpha = 0 \\ \pi^{\langle\alpha\beta\rangle} &= \pi_{Eq.}^{\langle\alpha\beta\rangle} = 0\end{aligned}$$

Dissipative fluxes: First order

$$\begin{aligned}\Pi^{(1)} &= \Pi_E = -\zeta \nabla_\alpha u^\alpha \\ q^\alpha{}^{(1)} &= q_E^\alpha = \kappa T \Delta^{\alpha\mu} \left(\frac{\nabla_\alpha T}{T} - a_\alpha \right) \\ \pi^{\langle\alpha\beta\rangle}{}^{(1)} &= \pi_E^{\langle\alpha\beta\rangle} = 2\eta \Delta^{\alpha\mu} \Delta^{\beta\nu} \nabla_{\langle\alpha} u_{\beta\rangle}\end{aligned}$$

Dissipative fluxes: Second order

$$\begin{aligned} \Pi^{(2)} &= \Pi_{MIS} = -\zeta \left[2S_1^2 \dot{\Pi} + S_4^2 \nabla_\alpha q^\alpha \right] \\ &\quad - \zeta \left[\Pi (\dot{S}_1^2 + S_1^2 \nabla_\alpha u^\alpha) + q^\alpha (\nabla_\alpha S_4^2 - S_4^2 a_\alpha) \right] \end{aligned}$$

$$\begin{aligned} q^\mu{}^{(2)} &= q_{MIS}^\mu = \kappa T \Delta^{\alpha\mu} \left[2S_2^2 \dot{q}_\alpha + S_4^2 \nabla_\alpha \Pi + S_5^2 \nabla^\beta \pi_{\langle\alpha\beta\rangle} \right] \\ &\quad + \kappa T \Delta^{\alpha\mu} \left[q_\alpha (\dot{S}_2^2 + S_2^2 \nabla_\nu u^\nu) + \Pi (\nabla_\alpha S_4^2 - S_4^2 a_\alpha) \right. \\ &\quad \left. + \pi_{\langle\alpha\beta\rangle} (\nabla^\beta S_5^2 - S_5^2 a^\beta) \right] \end{aligned}$$

$$\begin{aligned} \pi^{\langle\mu\nu\rangle}{}^{(2)} &= \pi_{MIS}^{\langle\mu\nu\rangle} = 2\eta \Delta^{\alpha\mu} \Delta^{\beta\nu} \left[2S_3^2 \dot{\pi}_{\langle\alpha\beta\rangle} + S_5^2 \nabla_{\langle\alpha} q_{\beta\rangle} \right] \\ &\quad + 2\eta \Delta^{\alpha\mu} \Delta^{\beta\nu} \left[\pi_{\langle\alpha\beta\rangle} (\dot{S}_3^2 + S_3^2 \nabla_\lambda u^\lambda) \right. \\ &\quad \left. + q_{\langle\alpha} (\nabla_{\beta\rangle} S_5^2 - S_5^2 a_{\beta\rangle}) \right] \end{aligned}$$

Dissipative fluxes: Third order: Bulk equation

$$\begin{aligned}
 \Pi^{(3)} = & -\zeta \left[3\mathcal{S}_1^3 \dot{\Pi} + 2\mathcal{S}_2^3 \dot{q}_\lambda q^\lambda + 2\mathcal{S}_3^3 \dot{\pi}^{\langle\alpha\beta\rangle} \pi^{\langle\alpha\beta\rangle} \right. \\
 & \left. + \mathcal{S}_6^3 (\Pi \nabla_\alpha q^\alpha + q^\alpha \nabla_\alpha \Pi) + \mathcal{S}_9^3 (\pi^{\langle\alpha\beta\rangle} \nabla_\alpha q_\beta + q_\beta \nabla_\alpha \pi^{\langle\alpha\beta\rangle}) \right] \\
 & -\zeta \left[\Pi^2 (\dot{\mathcal{S}}_1^3 + \mathcal{S}_1^3 \nabla_\alpha u^\alpha) - q^\alpha q_\alpha (\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3 \nabla_\alpha u^\alpha) \right. \\
 & \left. + \pi^{2\langle\alpha\beta\rangle} (\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3 \nabla_\alpha u^\alpha) \right. \\
 & \left. + \Pi q^\alpha (\nabla_\alpha \mathcal{S}_6^3 - \mathcal{S}_6^3 a_\alpha) + \pi^{\langle\alpha\beta\rangle} q_\beta (\nabla_\alpha \mathcal{S}_9^3 - \mathcal{S}_9^3 a_\alpha) \right]
 \end{aligned}$$

Dissipative fluxes: Third order: Heat equation

$$\begin{aligned}
 q^\mu{}^{(3)} = & \kappa T \Delta^{\alpha\mu} \left[-S_2^3 (2\Pi \dot{q}_\alpha + q_\alpha \dot{\Pi}) + S_4^3 (2\dot{q}^\beta \pi_{\langle\alpha\beta\rangle} + q^\beta \dot{\pi}_{\langle\alpha\beta\rangle}) \right. \\
 & + 2S_6^3 \Pi \nabla_\alpha \Pi - 2S_7^3 q^\beta \nabla_\alpha q_\beta + S_9^3 (\Pi \nabla^\beta \pi_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle} \nabla^\beta \Pi) \\
 & \left. + 2S_{10}^3 \pi_{\langle\beta\nu\rangle} \nabla_\alpha \pi^{\langle\beta\nu\rangle} \right] \\
 & + \kappa T \Delta^{\alpha\mu} \left[\Pi q_\alpha (\dot{S}_2^3 + S_2^3 \nabla_\nu u^\nu) + q^\beta \pi_{\langle\alpha\beta\rangle} (\dot{S}_4^3 + S_4^3 \nabla_\nu u^\nu) \right. \\
 & + (\Pi^2 \nabla_\alpha S_6^3 - q^\lambda q_\lambda \nabla_\alpha S_7^3 + \pi^{2\langle\lambda\lambda\rangle} \nabla_\alpha S_8^3) \\
 & + \Pi \pi_{\langle\alpha\beta\rangle} (\nabla^\beta S_9^3 - S_9^3 a^\beta) + \pi_{\langle\alpha\beta\rangle}^2 (\nabla^\beta S_{10}^3 - S_{10}^3 a^\beta) \\
 & \left. + S_7^3 q_\alpha q^\lambda a_\lambda \right]
 \end{aligned}$$

Dissipative fluxes: Third order: Shear equation

$$\begin{aligned}
 \pi^{\langle\mu\nu\rangle(3)} = & 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\left[\mathcal{S}_3^3(2\Pi\dot{\pi}_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle}\dot{\Pi}) + 2\mathcal{S}_4^3\dot{q}_{\langle\alpha}q_{\beta\rangle}\right. \\
 & + 3\mathcal{S}_5^3\dot{\pi}_{\langle\alpha\lambda\rangle}\pi_{\beta\rangle}^{\langle\lambda} + \mathcal{S}_8^3\pi_{\langle\alpha\beta\rangle}\nabla_{\lambda}q^{\lambda} + \mathcal{S}_9^3(\Pi\nabla_{\langle\alpha}q_{\beta\rangle} + q_{\langle\alpha}\nabla_{\beta\rangle}\Pi) \\
 & \left. + \mathcal{S}_{10}^3(q_{\beta}\nabla^{\lambda}\pi_{\langle\alpha\rangle\lambda} + \pi_{\langle\lambda\alpha\rangle}\nabla^{\lambda}q_{\beta})\right] \\
 & + 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\left[\Pi\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3\nabla_{\lambda}u^{\lambda})\right. \\
 & + q_{\langle\alpha}q_{\beta\rangle}(\dot{\mathcal{S}}_4^3 + \mathcal{S}_4^3\nabla_{\lambda}u^{\lambda}) + \pi_{\langle\alpha\lambda\rangle}\pi_{\beta\rangle}^{\langle\lambda}(\dot{\mathcal{S}}_5^3 + \mathcal{S}_5^3\nabla_{\lambda}u^{\lambda}) \\
 & + \pi_{\langle\alpha\beta\rangle}q^{\lambda}(\nabla_{\lambda}\mathcal{S}_8^3 - \mathcal{S}_8^3a_{\lambda}) + \Pi q_{\langle\beta}(\nabla_{\alpha}\mathcal{S}_9^3 - \mathcal{S}_9^3a_{\alpha}) \\
 & \left. + \pi_{\langle\alpha\lambda}q^{\lambda}(\nabla_{\beta}\mathcal{S}_{10}^3 - \mathcal{S}_{10}^3a_{\beta})\right]
 \end{aligned}$$

Conclusions

- We can extend causal theories of relativistic dissipative fluid dynamics to N -th order and here we have reported the results up to 3rd order with respect to entropy 4-current
- The equations are now highly coupled via space gradients and time gradients of both equilibrium quantities and dissipative fluxes
- The equations depends also on the lower order terms
- It is desirable to include higher order and in particular third order corrections (where the marriage of the three dissipative fluxes takes place)