Kinetic Derivation of Fluid Dynamics for Mixtures

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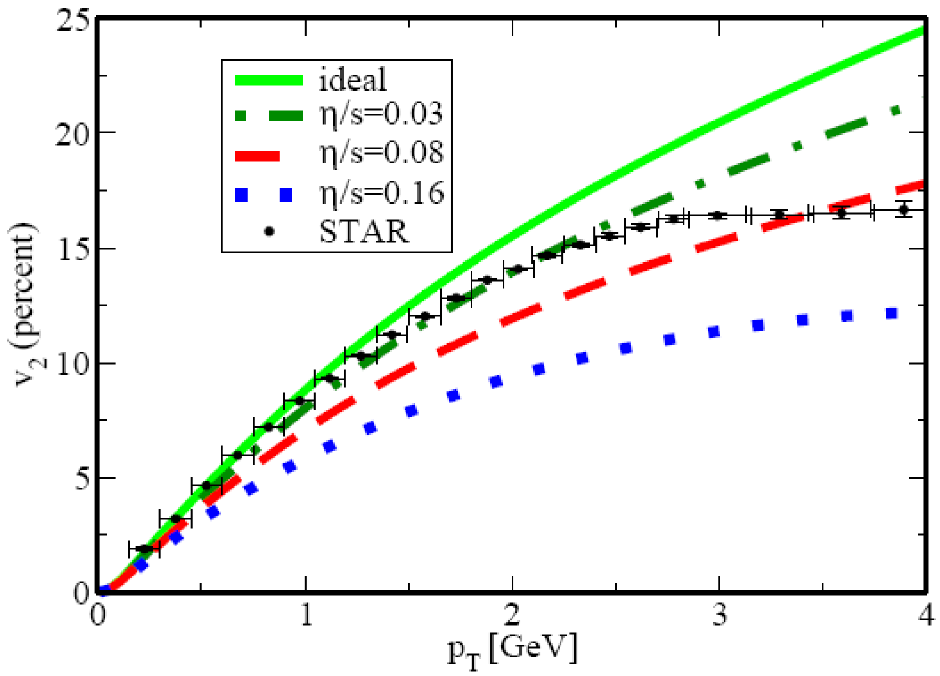
Contents

- Motivation/Introduction
- Approach
- Kinetic Corrections
- Conclusions/Perspectives

Motivation

- The significant effect of the shear viscosity



Romatschke&Romatschke

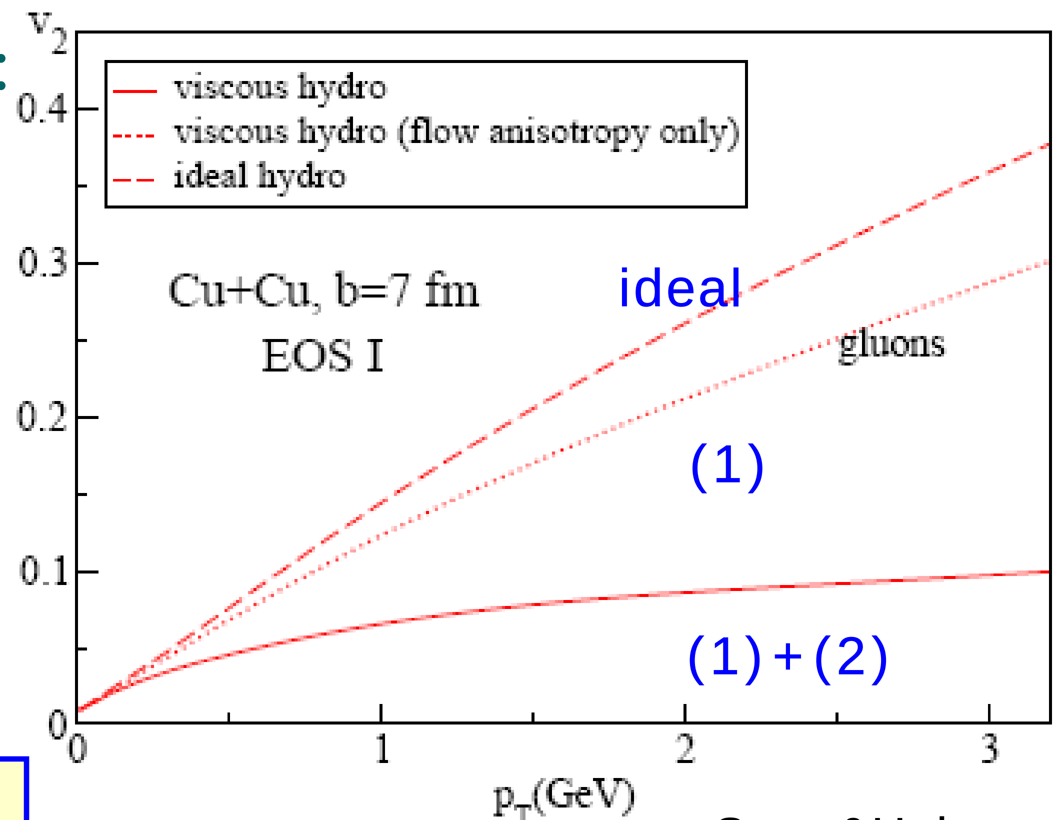
Effects of Viscosity

Two different effects :

1) Flow of the fluid

2) Momentum distribution function

$$f_{(i)} = f_{0(i)} + \delta f_{(i)}$$



Song&Heinz

Several Approaches

De Groot's book

- BGK

Bhatnagar, Gross and Kook (1954)
Marle (1964)

- Chapman-Enskog-Hilbert

Hilbert (1912), Chapman (1916) and Enskog (1917)
Marle (1964)

- Variational Methods

Robinson, Bernstein (1952), Van Leeuwen (1971)

- Moments Method

Maxwell (1867), Grad (1949) , Mintzner(1965)
Chernikov (1963), Stewart (1969), Marle (1966/69)
Israel&Stewart(1978)

Several Approaches

Usually applied in HIC

- Moments Method

Maxwell (1867), Grad (1949) , Mintzner(1965)
Chernikov (1963), Stewart (1969), Marle (1966/69)
Israel&Stewart(1978)

Several Approaches

However, the method was constructed for a single component fluid

Usually applied in HIC

- Moments Method

Maxwell (1867), Grad (1949) , Mintzner(1965)
Chernikov (1963), Stewart (1969), Marle (1966/69)
Israel&Stewart(1978)

What we investigate

- Moments Method for a mixture
- What are the differences ?

Basics

Boltzmann Equation

$$K^{\mu}_{(i)}\partial_{\mu} f_{(i)} = C\left[\{f\}\right]$$

Conserved Currents

$$T^{\mu\nu} = \sum_i T^{\mu\nu}_{(i)} \qquad N^{\mu}_{r} = \sum_i q^{r}_{(i)} N^{\mu}_{(i)}$$

Currents for each particle specie

$$T^{\mu\nu}_{(i)} = \int d\omega_{(i)} K^{\mu}_{(i)} K^{\nu}_{(i)} f_{(i)}$$

$$N^{\mu}_{(i)} = \int d\omega_{(i)} K^{\mu}_{(i)} f_{(i)}$$

$$d\omega_{(i)} \equiv \frac{d^3K}{(2\pi)^3 K^0{}_{(i)}}$$

Basics

The separation

$$K^{\mu}_{(i)} = u^{\mu} E_{(i)} + \Delta^{\mu\alpha} K_{\alpha(i)}$$

$$E_{(i)} \equiv u^{\mu} K_{\mu(i)}$$

General form of the currents

$$T^{\mu\nu}_{(i)} = \varepsilon_{(i)} u^{\mu} u^{\nu} - \Delta^{\mu\nu} (P_{0(i)} + \Pi_{(i)}) + 2 q^{(\mu}_{(i)} u^{\nu)} + \pi^{\mu\nu}_{(i)}$$

$$N^{\mu}_{(i)} = n_{(i)} u^{\mu} + \nu^{\mu}_{(i)}$$

Matching Conditions

$$\varepsilon_{(i)} = \varepsilon_{0(i)} \qquad n_{(i)} = n_{0(i)}$$

Basics

The separation

$$K^{\mu}_{(i)} = u^{\mu} E_{(i)} + \Delta^{\mu\alpha} K_{\alpha(i)}$$

$$E_{(i)} \equiv u^{\mu} K_{\mu(i)}$$

General form of the currents

$$T^{\mu\nu}_{(i)} = \varepsilon_{(i)} u^{\mu} u^{\nu} - \Delta^{\mu\nu} (P_{0(i)} + \Pi_{(i)}) + 2 q^{(\mu}_{(i)} u^{\nu)} + \pi^{\mu\nu}_{(i)}$$

$$N^{\mu}_{(i)} = n_{(i)} u^{\mu} + \nu^{\mu}_{(i)}$$

Bulk

Energy Difusion

Shear

Particle Difusion

for each specie

Irreversible Currents

$$\Pi_{(i)} = -\frac{1}{3} m_{(i)}^2 \int d\omega_{(i)} \delta f_{(i)}$$

$$\pi_{(i)}^{\mu\nu} = \int d\omega_{(i)} \Delta^{\mu\nu\alpha\beta} K_{\alpha(i)} K_{\beta(i)} \delta f_{(i)}$$

$$q_{(i)}^{\mu} = \int d\omega_{(i)} E_{(i)} \Delta^{\mu\nu} K_{\nu(i)} \delta f_{(i)}$$

$$\nu_{(i)}^{\mu} = \int d\omega_{(i)} \Delta^{\mu\nu} K_{\nu(i)} \delta f_{(i)}$$

$$\Delta^{\alpha\beta} = g^{\alpha\beta} - u^{\alpha} u^{\beta}$$

$$\Delta^{\mu\nu\alpha\beta} = \Delta^{<\mu\nu} \Delta^{\alpha\beta>}$$

$\downarrow$

Symmetric

+

Traceless

Local Eq. distribution function

$$f_{0(i)} = \left(\exp(\beta_0 E_{(i)} - \alpha_{0(i)}) \pm 1 \right)^{-1}$$

Deviation

$$\delta f_{(i)} = f_{(i)} - f_{0(i)}$$

Total Currents

$$T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - \Delta^{\mu\nu}(P_0 + \Pi) + 2q^{(\mu} u^{\nu)} + \pi^{\mu\nu}$$

$$N_r^{\mu} = n_r u^{\mu} + \nu_r^{\mu}$$

$$\Pi = \sum_{(i)} \Pi_{(i)}$$

$$\pi^{\mu\nu} = \sum_{(i)} \pi_{(i)}^{\mu\nu}$$

$$q^{\mu} = \sum_{(i)} q_{(i)}^{\mu}$$

$$\nu_r^{\mu} = \sum_{(i)} q_{(i)}^{r} \nu_{(i)}^{\mu}$$

→ Diffusion relative to the net number density

$$T^{\mu\nu} = \sum_i T_{(i)}^{\mu\nu}$$

$$N_r^{\mu} = \sum_i q_{(i)}^{r} N_{(i)}^{\mu}$$

H. GRAD, *Comm. Pure Appl. Math.* **2** (1949), 331

Grad's Method (by Grad)

Expand the distribution function in terms of a complete set

$$f_{(i)} = f_{0(i)} \sum_n \frac{1}{n!} a_{(n)}^{i_1 \dots i_n} H_{(n)}^{i_1 \dots i_n}$$

$H_{(n)}^{i_1 \dots i_n} \rightarrow$ Hermite Polynomials

$$a_{(n)}^{i_1 \dots i_n} = \int d^3 p f H_{(n)}^{i_1 \dots i_n}$$

Third Order Approximation $\rightarrow$ stop at n=3

$a_{(2)}^{i_1 i_2} \quad a_{(3)}^{i_1 i_2 i_3}$ — 20 moments!

13 Moments Approx. (no Bulk)

$$a_{(3)}^{i_1 i_2 i_3} = \frac{1}{5}\left(a_{(3)}^{i_1} \delta^{i_2 i_3} + a_{(3)}^{i_2} \delta^{i_1 i_3} + a_{(3)}^{i_3} \delta^{i_1 i_2} \right)$$

H. GRAD, *Comm. Pure Appl. Math.* **2** (1949), 331

Grad's Method (by Grad)

Expand the distribution function in terms of a complete set

$$f_{(i)} = f_{0(i)} \sum_n \frac{1}{n!} a_{(n)}^{i_1 \ldots i_n} H_{(n)}^{i_1 \ldots i_n}$$

$H_{(n)}^{i_1 \ldots i_n} \rightarrow$ Hermite Polynomials

$$a_{(n)}^{i_1 \ldots i_n} = \int d^3 p f H_{(n)}^{i_1 \ldots i_n}$$

Third Order Ap... p at n=3

$a_{(2)}^{i_1 i_2}$ $a_{(3)}^{i_1 i_2 i_3}$

20 moments!

13 Moments Ap... $\ldots_{(3)} \ldots \frac{}{5}\big(\ldots_{(3)} \ldots^{i_2 i_3} + a_{(3)}^{i_2} \delta^{i_1 i_3} + a_{(3)}^{i_3} \delta^{i_1 i_2} \big)$

(no Bulk)

Not so easy in a relativistic regime

Grad's Method (by IS)

Assume

$$f_{(i)} = \left(\exp(y_{(i)}) \pm 1\right)^{-1}$$

The following expansion is applied

$$y_{(i)} - y_{0(i)} = \varepsilon(E_{(i)}) + \varepsilon_{\mu}(E_{(i)})\Delta^{\mu\nu}K_{\nu(i)} + \varepsilon_{\mu\nu}(E_{(i)})\Delta^{\mu\nu\alpha\beta}K_{\alpha(i)}K_{\beta(i)}$$

Expand in Taylor

$$\varepsilon(E_{(i)}) = \varepsilon_{0(i)} + \varepsilon_{1(i)}E_{(i)} + \varepsilon_{2(i)}E_{(i)}^{2}$$

$$\varepsilon_{\mu}(E_{(i)}) = \varepsilon_{0\mu(i)} + \varepsilon_{1\mu(i)}E_{(i)}$$

$$\varepsilon_{\mu\nu}(E_{(i)}) = \varepsilon_{0\mu\nu}$$

14 Moments Approx.

(finite Bulk)

W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979)

Grad's Method (by IS)

Assume

$$f_{(i)} = \left(\exp(y_{(i)}) \pm 1\right)^{-1}$$

Expand in terms of irreducible tensors

$$y_{(i)} - y_{0(i)} = \ldots (E_{(i)})\Delta^{\mu\nu\alpha\beta} K_{\alpha(i)} K_{\beta(i)}$$

$$y_{(i)} - y_{0(i)} = \varepsilon_{(i)} + \varepsilon_{\mu(i)} K^{\mu}_{(i)} + \varepsilon_{\mu\nu(i)} K^{\mu}_{(i)} K^{\nu}_{(i)}$$

$\varepsilon(\ldots$

$\varepsilon_{\mu}(E_{(i)}) \ldots$

$\varepsilon_{\mu\nu}(E_{(i)}) = \varepsilon_{0\mu\nu}$

...nts Approx.

(...nite Bulk)

Grad's Method (by IS)

The expansion variables are determined using

Matching Cond.

$$u_\mu u_\nu (T^{\mu\nu} - T_0^{\mu\nu}) = 0$$

$$u_\mu (N^\mu - N_0^\mu) = 0$$

Definition of Currents

$$\nu^\mu_{(i)} = \Delta^{\mu\nu} N_{\nu(i)}$$

$$q^\mu_{(i)} = u^\alpha \Delta^{\mu\beta} T_{\alpha\beta(i)}$$

$$\pi^{\mu\nu}_{(i)} = \Delta^{\mu\nu\alpha\beta} T_{\alpha\beta(i)}$$

$$\Pi_{(i)} = -\frac{1}{3}\Delta_{\alpha\beta}\left(T^{\alpha\beta}_{(i)} - T^{\alpha\beta}_{0(i)}\right)$$

$$f_{(i)} = \left(\exp(y_{(i)}) \pm 1\right)^{-1}$$

And expanding

$$f_{(i)} \approx f_{0(i)} + f_{0(i)}(1 \mp f_{0(i)})\left(y_{(i)} - y_{0(i)}\right)$$

Grad's Method (by IS)

$$\begin{aligned}
\epsilon_{(0)} J_{00(i)} + \epsilon_{(1)} J_{10(i)} + \epsilon_{(2)} J_{20(i)} &= -\frac{3\Pi_{(i)}}{m_{(i)}^2}, \\
\epsilon_{(0)} J_{10(i)} + \epsilon_{(1)} J_{20(i)} + \epsilon_{(2)} J_{30(i)} &= 0, \\
\epsilon_{(0)} J_{20(i)} + \epsilon_{(1)} J_{30(i)} + \epsilon_{(2)} J_{40(i)} &= 0,
\end{aligned}$$

$$\begin{aligned}
\Delta^{\mu\nu}\epsilon_{\mu(0)} J_{21(i)} + \Delta^{\mu\nu}\epsilon_{\mu(1)} J_{31(i)} + \Delta^{\mu\nu}\epsilon_{\mu(2)} J_{41(i)} &= -n_{(i)}^{\nu} \\
\Delta^{\mu\nu}\epsilon_{\mu(0)} J_{31(i)} + \Delta^{\mu\nu}\epsilon_{\mu(1)} J_{41(i)} + \Delta^{\mu\nu}\epsilon_{\mu(2)} J_{51(i)} &= -q_{(i)}^{\nu}
\end{aligned}$$

$$2\Delta^{\mu\nu\lambda\sigma}\epsilon_{\lambda\sigma(0)} J_{42(i)} = \pi_{(i)}^{\mu\nu}$$

Definition of L.R.F

We use the Landau picture

$$u_\nu (T^{\mu\nu} - T_0^{\mu\nu}) = 0$$

Then,

$$q_\nu = \sum_i q_{\nu(i)} = 0$$

But,

$$q_{\nu(i)} \neq 0$$

The energy flow for each particle specie cannot be neglected

$$q_{\nu(i)} \qquad \nu_{\nu(i)}$$

Solution

$$f_{(i)}(p) = f_{0(i)}\left[1+(1\mp f_{0(i)})\left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p^{\mu}_{(i)} + \varepsilon_{\mu\nu(i)} p^{\mu}_{(i)} p^{\nu}_{(i)}\right)\right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon^{\mu}_{(i)} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q^{\mu}_{(i)} + D_{2(i)} \nu^{\mu}_{(i)}$$

$$\varepsilon^{\mu\nu}_{(i)} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu}u^{\nu}\right)\Pi_{(i)} + 2B_{1(i)} u^{(\mu} q^{\nu)}_{(i)} + 2B_{3(i)} u^{(\mu} \nu^{\nu)}_{(i)} + B_{2(i)} \pi^{\mu\nu}_{(i)}$$

Solution

$$f_{(i)}(p) = f_{0(i)}\left[1 + (1 \mp f_{0(i)})\left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p^{\mu}_{(i)} + \varepsilon_{\mu\nu(i)} p^{\mu}_{(i)} p^{\nu}_{(i)}\right)\right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon^{\mu}_{(i)} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q^{\mu}_{(i)} + D_{2(i)} \nu^{\mu}_{(i)}$$

$$\varepsilon^{\mu\nu}_{(i)} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu} u^{\nu}\right) \Pi_{(i)} + 2B_{1(i)} u^{(\mu} q^{\nu)}_{(i)} + 2B_{3(i)} u^{(\mu} \nu^{\nu)}_{(i)} + B_{2(i)} \pi^{\mu\nu}_{(i)}$$

$$B_{2(i)} = \frac{1}{2J_{42(i)}}$$ → shear viscosity

$$J_{nq(i)} = \frac{1}{(2q-1)!!} \int d\omega_{(i)} E^{n-2q}_{(i)} K^{2q} f_{0(i)} \left(1 - a f_{0(i)}\right)$$

Solution

$$f_{(i)}(p)=f_{0(i)}\left[1+(1\mp f_{0(i)})\left(\varepsilon_{(i)}+\varepsilon_{\mu(i)}p^{\mu}_{(i)}+\varepsilon_{\mu\nu(i)}p^{\mu}_{(i)}p^{\nu}_{(i)}\right)\right]$$

$$\varepsilon_{(i)}=E_{0(i)}\Pi_{(i)}$$

$$\varepsilon^{\mu}_{(i)}=D_{0(i)}\Pi_{(i)}u^{\mu}+D_{1(i)}q^{\mu}_{(i)}+D_{2(i)}\nu^{\mu}_{(i)}$$

$$\varepsilon^{\mu\nu}_{(i)}=B_{0(i)}\left(\Delta^{\mu\nu}-3u^{\mu}u^{\nu}\right)\Pi_{(i)}+2B_{1(i)}u^{(\mu}q^{\nu)}_{(i)}+2B_{3(i)}u^{(\mu}\nu^{\nu)}_{(i)}+B_{2(i)}\pi^{\mu\nu}_{(i)}$$

$$B_{2(i)}=\frac{1}{2J_{42(i)}}$$ → shear viscosity

Boltzmann Gas

$$B_{2(i)}=\frac{1}{2\left(\varepsilon_{(i)}+P_{(i)}\right)T^{2}}$$

$$J_{nq(i)}=\frac{1}{(2q-1)!!}\int d\omega_{(i)}E^{n-2q}_{(i)}K^{2q}f_{0(i)}\left(1-af_{0(i)}\right)$$

Comment

$$\delta f_{(i)}^{shear} = f_{0(i)} \frac{1}{2(\varepsilon_{(i)} + P_{(i)})T^2} \pi_{\mu\nu(i)} K_{(i)}^{\mu} K_{(i)}^{\nu}$$

Solution

$$\delta f_{(i)}^{shear} = f_{0(i)} \frac{\sum_j \pi_{\mu\nu(j)}}{2T^2 \sum_j (\varepsilon_{(j)} + P_{(j)})} K_{(i)}^{\mu} K_{(i)}^{\nu}$$

used so far

$$\frac{\pi_{\mu\nu(i)}}{\varepsilon_{(i)} + P_{(i)}} \approx \frac{\sum_j \pi_{\mu\nu(j)}}{\sum_j (\varepsilon_{(j)} + P_{(j)})}$$

True ?

Comment

$$\delta f_{(i)}^{shear} = f_{0(i)} \frac{1}{2(\varepsilon_{(i)} + P_{(i)})T^2} \pi_{\mu\nu(i)} K_{(i)}^{\mu} K_{(i)}^{\nu}$$

Solution

$$\delta f_{(i)}^{shear} = f_{0(i)} \frac{\sum_j \pi_{\mu\nu(j)}}{2T^2 \sum_j (\varepsilon_{(j)} + P_{(j)})} K_{(i)}^{\mu} K_{(i)}^{\nu}$$

used so far

$$\frac{\pi_{\mu\nu(i)}}{\varepsilon_{(i)} + P_{(i)}} \approx \frac{\sum_j \pi_{\mu\nu(j)}}{\sum_j (\varepsilon_{(j)} + P_{(j)})}$$

True ?

$$\frac{\pi_{\mu\nu(i)}}{\pi_{\mu\nu}} \approx \frac{\eta_{(i)}}{\eta}$$

Solution

$$f_{(i)}(p) = f_{0(i)}\left[1+(1\mp f_{0(i)})\left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p^{\mu}_{(i)} + \varepsilon_{\mu\nu(i)} p^{\mu}_{(i)} p^{\nu}_{(i)}\right)\right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon^{\mu}_{(i)} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q^{\mu}_{(i)} + D_{2(i)} \nu^{\mu}_{(i)}$$

$$\varepsilon^{\mu\nu}_{(i)} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu}u^{\nu}\right)\Pi_{(i)} + 2B_{1(i)} u^{(\mu} q^{\nu)}_{(i)} + 2B_{3(i)} u^{(\mu} \nu^{\nu)}_{(i)} + B_{2(i)} \pi^{\mu\nu}_{(i)}$$

$$D_{1(i)} = \frac{-J_{31(i)}}{J_{31(i)}J_{31(i)} - J_{41(i)}J_{21(i)}}$$

$$D_{2(i)} = \frac{J_{41(i)}}{J_{31(i)}J_{31(i)} - J_{41(i)}J_{21(i)}}$$

↑ Energy difusion ↓

↑ Particle difusion ↓

$$B_{1(i)} = \frac{J_{21(i)}}{J_{31(i)}J_{31(i)} - J_{41(i)}J_{21(i)}}$$

$$B_{3(i)} = \frac{-J_{31(i)}}{J_{31(i)}J_{31(i)} - J_{41(i)}J_{21(i)}}$$

Solution

$$f_{(i)}(p) = f_{0(i)}\left[1+(1\mp f_{0(i)})\left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p^{\mu}_{(i)} + \varepsilon_{\mu\nu(i)} p^{\mu}_{(i)} p^{\nu}_{(i)}\right)\right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon^{\mu}_{(i)} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q^{\mu}_{(i)} + D_{2(i)} \nu^{\mu}_{(i)}$$

$$\varepsilon^{\mu\nu}_{(i)} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu}u^{\nu}\right)\Pi_{(i)} + 2B_{1(i)} u^{(\mu} q^{\nu)}_{(i)} + 2B_{3(i)} u^{(\mu} \nu^{\nu)}_{(i)} + B_{2(i)} \pi^{\mu\nu}_{(i)}$$

$$D_{0(i)} = -3C_{2(i)} B_{0(i)}, \; E_{0(i)} = -3C_{1(i)} B_{0(i)}$$

$$B_{0(i)} = \frac{-1}{3C_{1(i)} J_{21(i)} + 3C_{2(i)} J_{31(i)} + 3J_{41(i)} + 5J_{42(i)}}$$

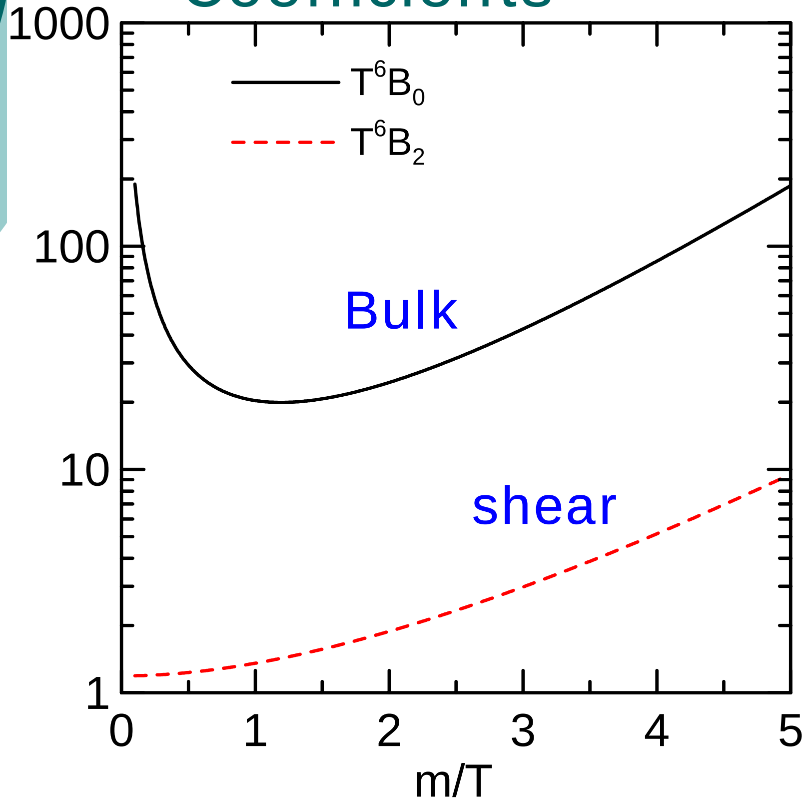
$$C_{1(i)} = -m^2_{(i)} - 4\frac{J_{31(i)} J_{30(i)} - J_{41(i)} J_{20(i)}}{J_{30(i)} J_{10(i)} - J_{20(i)} J_{20(i)}}$$

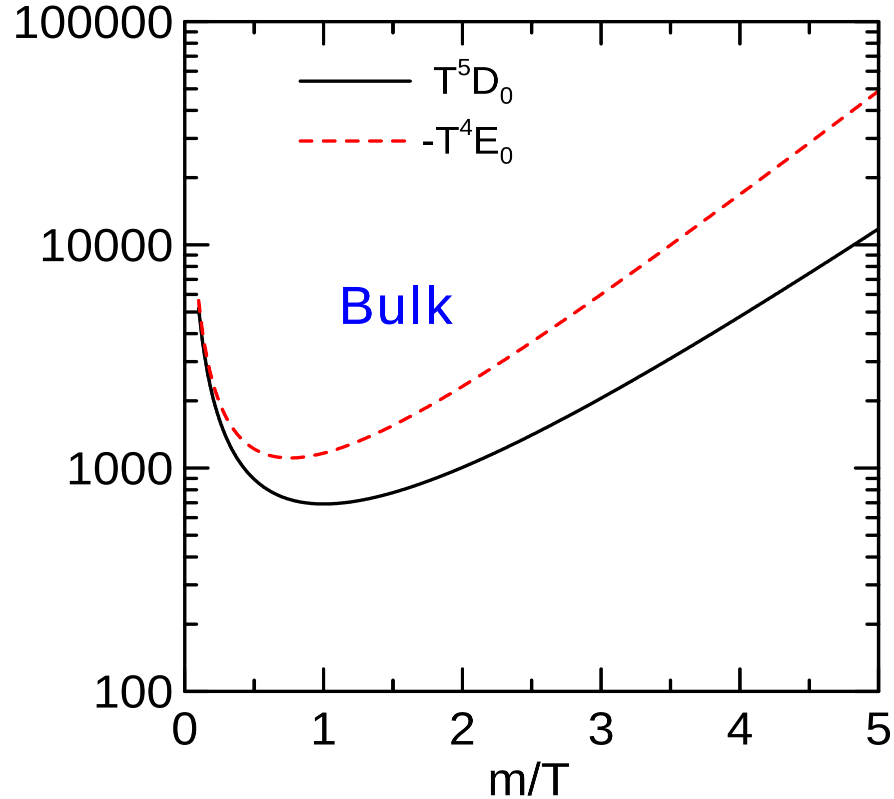
$$C_{2(i)} = 4\frac{J_{31(i)} J_{20(i)} - J_{41(i)} J_{10(i)}}{J_{30(i)} J_{10(i)} - J_{20(i)} J_{20(i)}}$$

bulk viscosity

Grad's Method

Coefficients





$$B_0,\ D_0,\ \mathrm{E}_0 \gg B_2$$

Bulk Viscosity

We obtain (for each specie),

$$\delta f_{(i)}(p) = f_{0(i)}(1 \mp f_{0(i)})\Pi_{(i)}\left(E_{0(i)} + D_{0(i)}E_{p(i)} + B_{0(i)}\left(m_{(i)}^2 - 4E_{p(i)}^2\right)\right)$$

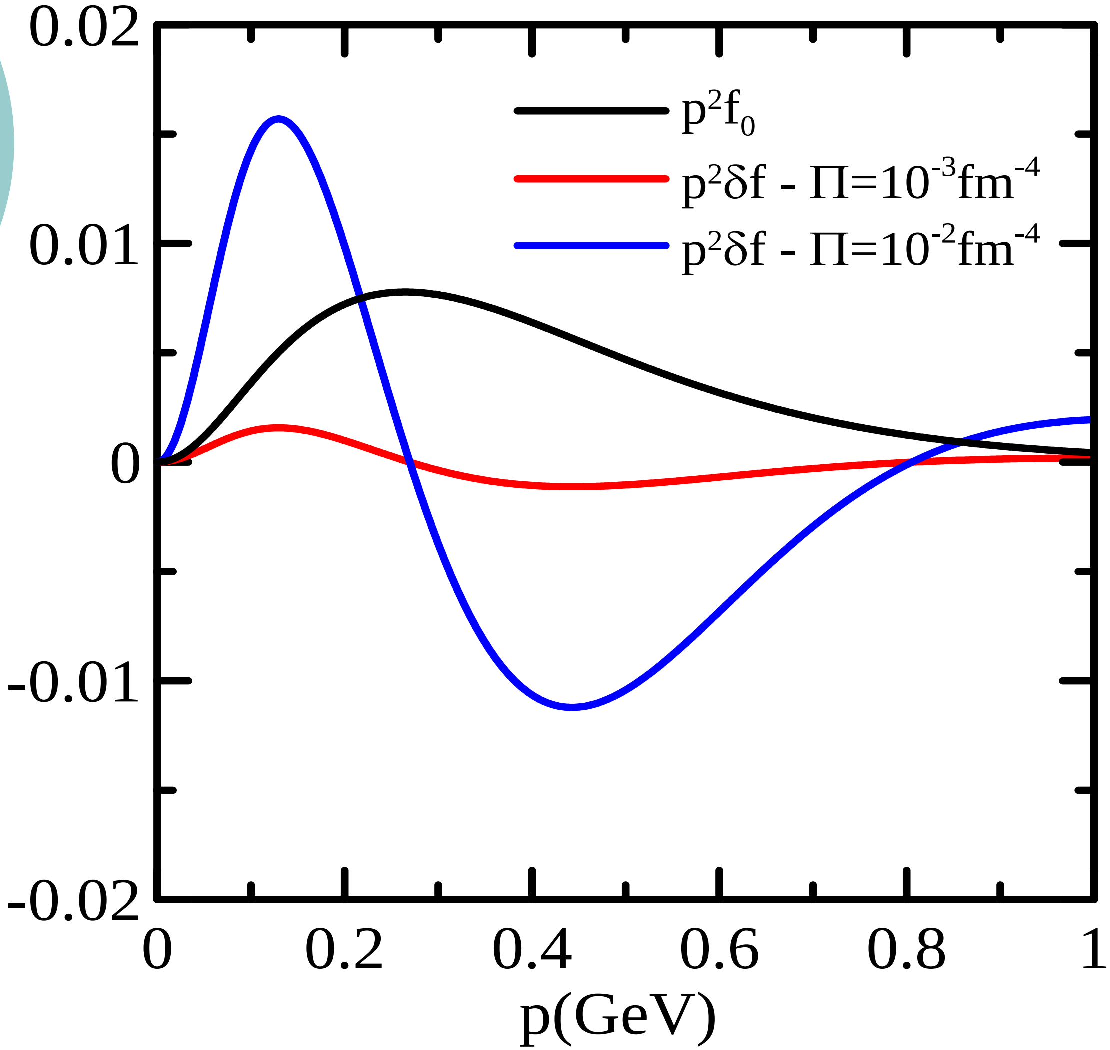
We assume,

$$\frac{\Pi_{(i)}}{\Pi} = \frac{\zeta_{(i)}}{\zeta}$$

with

$$\frac{\zeta_{(i)}}{\zeta} = \frac{s_{(i)}}{s} \sim 0.05$$

Bulk Visc. $p_{(i)}^{2}\delta f_{(i)}$



$$\int d^3p\,\delta f_{(i)} = 0$$

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Trento

Equations – What changes ?

Example - Bulk

$$\frac{d\Pi_{(i)}}{d\tau} + \Pi_{(i)}\nabla_{\alpha}u^{\alpha} + \frac{m_{(i)}^{2}}{3}\int \frac{d^{3}k_{(i)}}{(2\pi)^{3}\,\tau_{(i)}^{2}} C_{(i)}^{(1)}\left[\{f\}\right]$$

$$= -\beta_{(i)}\partial_{\alpha}u^{\alpha} + \frac{d}{d\tau}\left(\frac{\varepsilon_{(i)} - 3P_{(i)}}{3}\right) - \frac{m_{(i)}^{2}\lambda_{\Pi(i)}}{3}\Pi_{(i)}\nabla_{\alpha}u^{\alpha}$$

$$+\frac{m_{(i)}^{2}}{3}\nabla^{\nu}\left(\lambda_{n(i)}n_{\nu(i)} + \lambda_{q(i)}q_{\nu(i)}\right) + \frac{m_{(i)}^{2}\lambda_{\pi(i)}}{3}\pi_{\nu\alpha(i)}\nabla^{\nu}u^{\alpha}$$

Equations – What changes ?

Example - Bulk

$$\frac{d\Pi_{(i)}}{d\tau} + \Pi_{(i)}\nabla_{\alpha}u^{\alpha} + \frac{m_{(i)}^{2}}{3}\int \frac{d^{3}k_{(i)}}{(2\pi)^{3}\,\tau_{(i)}^{2}} C_{(i)}^{(1)}\left[\{f\}\right]$$

$$= \quad -\beta_{(i)}\partial_{\alpha}u^{\alpha} + \frac{d}{d\tau}\left(\frac{\varepsilon_{(i)} - 3P_{(i)}}{3}\right) - \frac{m_{(i)}^{2}\lambda_{\Pi(i)}}{3}\Pi_{(i)}\nabla_{\alpha}u^{\alpha}$$

$$+\frac{m_{(i)}^{2}}{3}\nabla^{\nu}\left(\lambda_{n(i)}n_{\nu(i)} + \lambda_{q(i)}q_{\nu(i)}\right) + \frac{m_{(i)}^{2}\lambda_{\pi(i)}}{3}\pi_{\nu\alpha(i)}\nabla^{\nu}u^{\alpha}$$

Still have to simplify …

Equations – What changes ?

Example - Bulk

$$\frac{d\Pi_{(i)}}{d\tau} + \Pi_{(i)}\nabla_\alpha u^\alpha + \frac{m_{(i)}^2}{3}\int \frac{d^3k_{(i)}}{(2\pi)^3\,\tau_{(i)}^2} C_{(i)}^{(1)}\left[\{f\}\right]$$

$$= -\beta_{(i)}\partial_\alpha u^\alpha + \frac{d}{d\tau}\left(\frac{\varepsilon_{(i)} - 3P_{(i)}}{3}\right) - \frac{m_{(i)}^2\lambda_{\Pi(i)}}{3}\Pi_{(i)}\nabla_\alpha u^\alpha$$

$$+ \frac{m_{(i)}^2}{3}\nabla^\nu\left(\lambda_{n(i)}n_{\nu(i)} + \lambda_{q(i)}q_{\nu(i)}\right) + \frac{m_{(i)}^2\lambda_{\pi(i)}}{3}\pi_{\nu\alpha(i)}\nabla^\nu u^\alpha$$

1) The term $\Lambda_{(i)}\Pi_{(i)} + \sum_j \Lambda_{(i)(j)}\Pi_{(j)}\,.$

Equations – What changes ?

Example - Bulk

Truncated Eq.

$$\frac{d\Pi_{(i)}}{d\tau} + \Lambda_{(i)}\Pi_{(i)} + \sum_j \Lambda_{(i)(j)}\Pi_{(j)} = -\beta_{(i)}\partial_\alpha u^\alpha$$

1) The term $\Lambda_{(i)}\Pi_{(i)} + \sum_j \Lambda_{(i)(j)}\Pi_{(j)}$.

Equations – What changes ?

Example - Bulk

$$\frac{d\Pi_{(i)}}{d\tau}+\Pi_{(i)}\nabla_{\alpha}u^{\alpha}+\frac{m_{(i)}^{2}}{3}\int\frac{d^{3}k_{(i)}}{(2\pi)^{3}\tau_{(i)}^{2}}C_{(i)}^{(1)}\left[\{f\}\right]$$

$$=\quad -\beta_{(i)}\partial_{\alpha}u^{\alpha}+\frac{d}{d\tau}\left(\frac{\varepsilon_{(i)}-3P_{(i)}}{3}\right)-\frac{m_{(i)}^{2}\lambda_{\Pi(i)}}{3}\Pi_{(i)}\nabla_{\alpha}u^{\alpha}$$

$$+\frac{m_{(i)}^{2}}{3}\nabla^{\nu}\left(\lambda_{n(i)}n_{\nu(i)}+\lambda_{q(i)}q_{\nu(i)}\right)+\frac{m_{(i)}^{2}\lambda_{\pi(i)}}{3}\pi_{\nu\alpha(i)}\nabla^{\nu}u^{\alpha}$$

2) Eqs. will depend on → $\Pi_{(i)}, q^{\mu}_{(i)}, \pi^{\mu\nu}_{(i)}, \nu^{\mu}_{(i)}$

⃠ → $\Pi\ , q^{\mu}, \pi^{\mu\nu}, \nu^{\mu}_{r}$ (possible?)

Conclusions

- Corrections to the equilibrim distribution function will depend on the irreversible currents for each particle species

- Same will happen for the equations of motion

Problem is not as simple as it looks