Nonlinear Dynamics and Complex Systems

Exercise No. 3

Investigation of one-dimensional discrete maps

A discrete map is defined by the iteration procedure

$$x_{n+1} = f(x_n) \qquad n = 0, 1, ..., \infty$$

A few typical examples, defined over the unit interval $0 \leq x \leq 1$, are

$$\begin{aligned} \text{Logistic map} \quad f(x) &= rx(1-x) \qquad (\text{with } 0 \leq r \leq 4) \\ \text{Sine map} \quad f(x) &= q\sin(\pi x) \qquad (\text{with } 0 \leq q \leq 1) \\ \text{Tent map} \quad f(x) &= \begin{cases} 2tx & (x \leq 1/2) \\ 2t(1-x) & (x \geq 1/2) \end{cases} \qquad (\text{with } 0 \leq t \leq 1) \end{aligned}$$

Use the program Chaos for Java by Brian Davies to investigate the following questions.

1. The Feigenbaum cascade

The window Bifurcation Diagrams displays the distribution of finals states of the iteration as a function of the system parameter. With increasing parameter a series of *period doubling bifurcations* is observed.
Note the *self-similarity* property of the branches of the bifurcation cascade!
In 1978 Mitchell Feigenbaum discovered, that this cascade follows a simple scaling law

$$r_n \simeq r_\infty - \frac{K}{\delta^n} \qquad \text{for} \quad n \to \infty$$

with constant values K, r_∞, and δ.
a) Determine the value of the Feigenbaum constant δ to 4 significant digits from the series of bifurcation points r_n of the logistic map. The values of r_n can be read off from the display by zooming in.
b) Repeat this calculation for the Sine map and compare the results.

2. Lyapunov exponents

Use the window Lyapunov exponents to analyze the logistic map. Confirm that the value of σ approaches zero (from below) at each bifurcation point.
Observe that in the *chaotic region* with positive Lyapunov exponent there are many narrow "islands of stability" where σ drops below zero. This investigation is severely limited by

numerical accuracy. Actually, there should be an infinite number of such (very narrow) regions.
Compare the results for the *tent map* with that for the logistic map.

3. Fourier analysis

Periodic behaviour of a discrete series x_k, where $k = 0, \ldots, N$, can be detected by its Fourier decomposition

$$x_k = \sum_{m=0}^{N/2} A_{\nu_m} \cos\bigl(2\pi\nu_m k - \varphi_{\nu_m}\bigr)$$

with the frequency $\nu_m = m/N$ taking values between 0 and 1/2. A_{ν_m} and φ_{ν_m} are the amplitude and phase coefficients.
The window FOURIER ANALYSIS displays the Fourier amplitudes as a function of frequency on a logarithmic scale.
Study the shape of the Fourier spectrum for the logistic map in the region of periodic orbits and confirm the occurrence of period doubling at the bifurcation points determined in part 1. Observe how the calculated spectrum changes if (a) the initial transient stage is not discarded, (b) the number of sample points is not an integer multiple of the period.
Compare the Fourier spectrum in the periodic and in the in the chaotic parameter regions. What happens at $r = 3.83$?

4. Fractal dimension

To define the *capacity dimension* d_c of a set A of points in an n-dimensional euclidean space, the space is covered by a grid of n-dimensional (hyper-)cubes of side length ϵ. Then the number $N(\epsilon)$ of these boxes is counted which contains at least one point from the set A. If in the limit $\epsilon \to 0$ the number $N(\epsilon)$ scales inversely with a power of ϵ then the exponent is called the capacity dimension:

$$N(\epsilon) \sim K\epsilon^{-d_c}$$

with a constant K. Equivalently, for a sequence of box lengths ϵ_n approaching zero for $n \to \infty$ the capacity dimension is

$$d_c = \lim_{n\to\infty} \frac{\ln\left(N_{n+1}/N_n\right)}{\ln\left(\epsilon_n/\epsilon_{n+1}\right)}$$

where $N_n = N(\epsilon_n)$.

Use the window ITERATIONS(2D) to estimate the *capacity dimension of the attractor of the logistic map*. Investigate the values $r = 3.56$, $r = 3.57$, and $r_\infty = 3.5699456$. *Hint:*

The box counting procedure is implemented in "Chaos for Java" only for two-dimensional iterative maps. However, the program offers the choice "Controlled logistic map". This is a two-dimensional extension of the logistic map which reduces to the one-dimensional case if the second control parameter is set to the value $b = 0$. The button  can be used to obtain $N(\epsilon)$ for a given box dimension ϵ (defined throuth "Options").