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Nonlinear Dynamics and Complex Systems

Solutions No. 3

One-dimensional discrete maps

1. The Feigenbaum cascade

According to M. Feigenbaum, the positions of period-doubling parameter values should satisfy the scaling law $r_n \simeq r_\infty - \frac{K}{\delta^n}$. To find the parameters δ and r_∞ one has to determine a sequence of values r_n numerically. From these one can evaluate the series of approximate values

$$\delta_n = \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \qquad \text{and} \qquad r_{\infty,n} = \frac{r_n r_{n-2} - r_{n-1}^2}{r_n - 2r_{n-1} + r_{n-2}}$$

If the scaling relation holds, these values should converge to constant values $\delta = \lim_{n\to\infty} \delta_n$ and $r_\infty = \lim_{n\to\infty} r_{\infty,n}$. Using ‘Chaos for Java’ we find the approximate values

Logistic map				Sine map			
n	r_n	δ_n	$r_{\infty,n}$	n	q_n	δ_n	$q_{\infty,n}$
1	3	-	-	1	0.71996083	-	-
2	3.44948936	-	-	2	0.83326622	-	-
3	3.54409013	4.75143	3.5693073	3	0.85860898	4.47092	0.8659104
4	3.56440714	4.65624	3.5699640	4	0.86408412	4.62870	0.8655930
5	3.56875935	4.66821	3.5699458	5	0.86525892	4.66049	0.8655799
6	3.56969160	4.66850	3.5699453	6	0.86551065	4.66691	0.8655792
7	3.56989125	4.66942	3.5699427	7	0.86556458	4.66772	0.8655794
8	3.56993401	4.66908	3.5699535	8	0.86557613	4.66926	0.8655805

The exact value of the Feigenbaum constant should be $\delta = 4.669201609\ldots$.

2. Lyapunov exponents

The Lyapunov exponent of the one-dimensional discrete map is defined as the average value of the logarithm of the stability coefficients $|f'(x_l)|$ for a given orbit x_l, i.e.

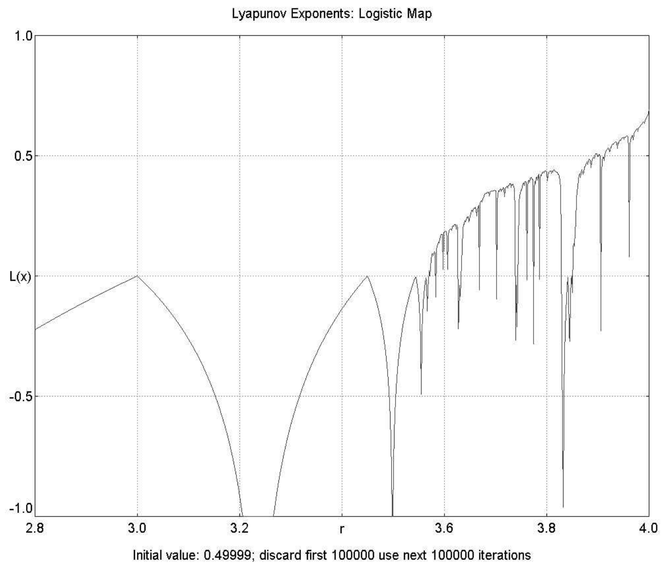
$$\sigma = \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \ln |f'(x_l)|$$

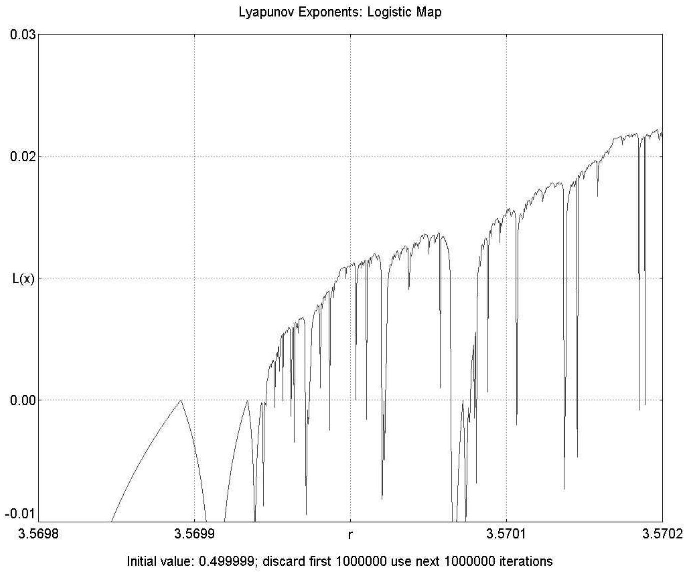
The *tent map* has the very special property that the derivative of the function $f(t)$ is constant

$$f'(x) = \begin{cases} 2t & (x < 1/2) \\ -2t & (x > 1/2) \end{cases}$$

so that $|f'(x)| = 2t$ everywhere (except at $x = 1/2$ where the derivative is not defined). Therefore the Lyapunov exponent is given by the simple function $\sigma = \ln 2t$. Neighboring trajectories for $t < 1/2$ converge and for $t > 1/2$ diverge.

In contrast, for the logistic map (and other examples with a curved function $f(x)$) the dependence of σ on the parameter r is highly complex.

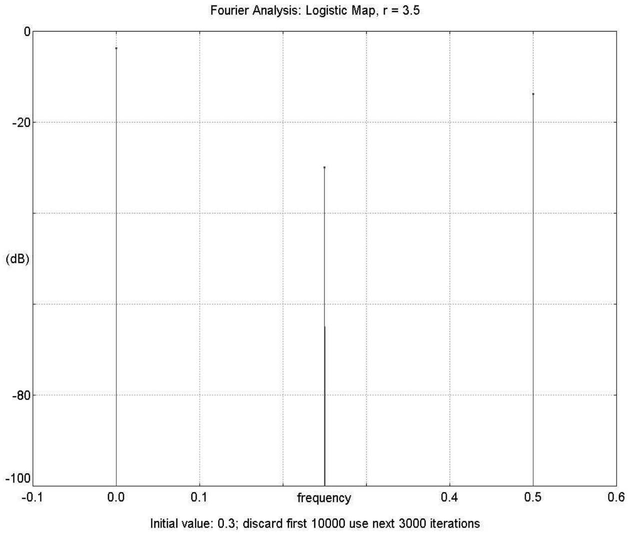


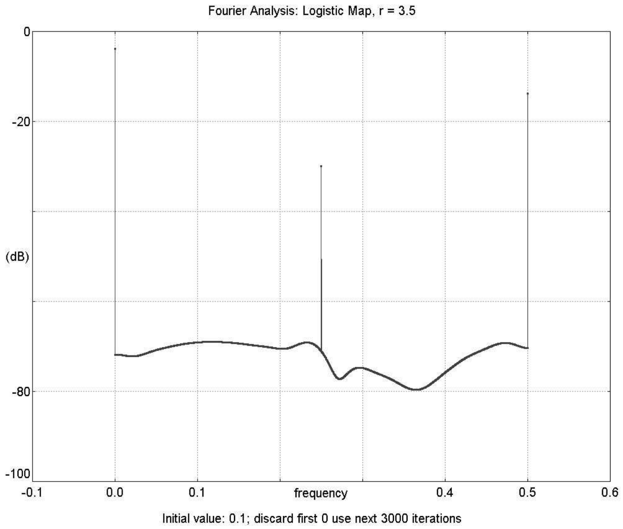


The numerical calculation of the Lyapunov exponent finds only a small number of the 'islands of stability' $\sigma < 0$ embedded in the chaotic regions. In fact there exists an infinite number of them, but their sizes are so narrow that most of them are not observed numerically. There even is an infinite number of 'superstable' points with $\sigma = -\infty$, caused by the fact that one point of the orbit hits the value $x_l = 1/2$ where $f'(1/2) = 0$ so that $\ln(f'(1/2)) = -\infty$.

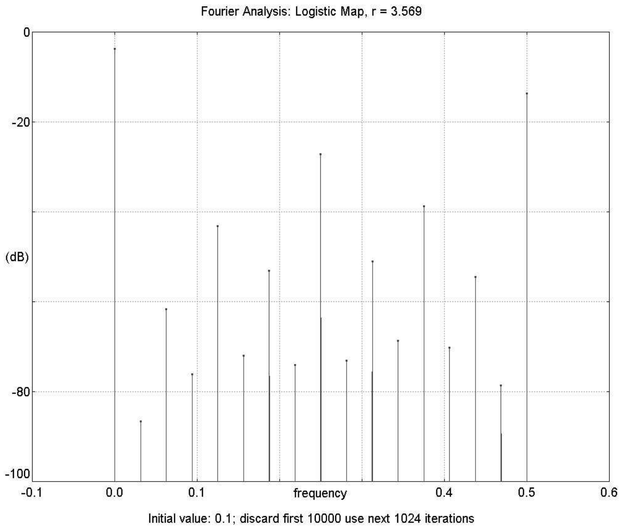
3. Fourier analysis

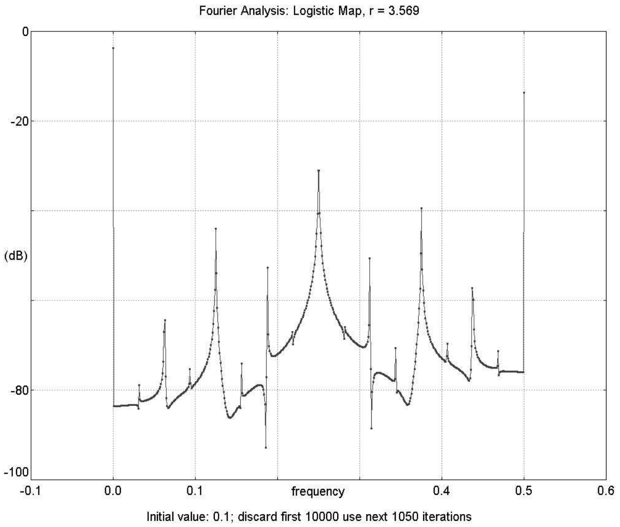
For a periodic orbit with period length p only the Fourier coefficients belonging to the frequency $1/p$ and its harmonics should be present. E.g., for $r = 3.5$ in the second step of the bifurcation cascade the period is $p = 4$ and we expect frequencies of $\nu = 0$ (describing the constant mean value), $\nu = 0.25$ and $\nu = 0.5$, which is confirmed by the calculation using the periodic attractor. However, if the initial part of the trajectory is not discarded we find a small 'continuum contribution' of other frequencies which are caused by the nonperiodic initial transient region of the orbit.



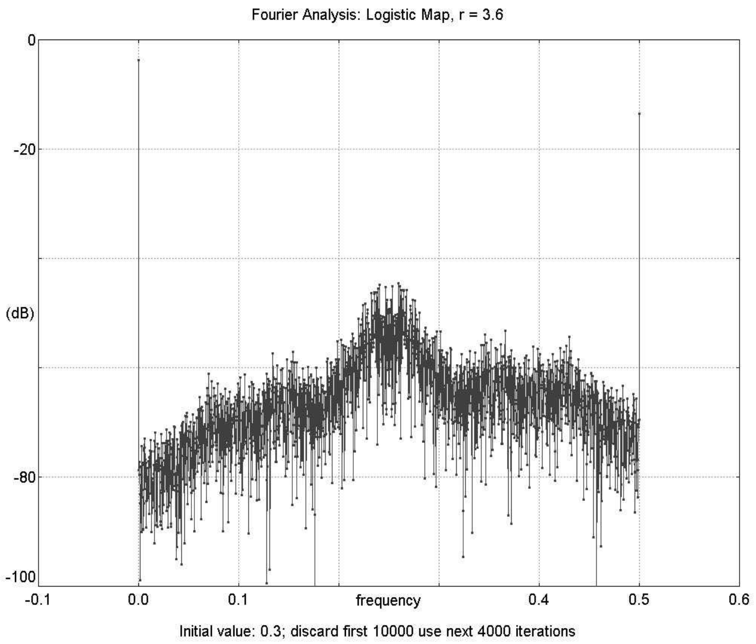


The periodicity of an orbit is easily spotted by looking at the Fourier coefficients. However, artefacts are produced if the sampling interval does not contain an integer multiple of the oscillations. This is demonstrated below for the example $r = 3.569$ which leads to a period $p = 32$. We compare the results obtained with $N = 1024 = 32 \cdot 32$ (left) and $N = 1050$ (right).





In the chaotic region all Fourier components are nonzero and the plot looks somewhat as if it contained ‘white noise’. This is no artefact of the numerics but signals the nonperiodic nature of the orbit. See the example for $r = 3.6$ below. The orbit has a strong period-2 component but all the other frequencies also are present.



At $r = 3.83$ a stable $p = 3$ orbit is found which at about $r = 3.8415$ bifurcates into a $p = 6$ orbit.

4. Fractal dimension

The software “Chaos for Java” has the built-in capacity to determine fractal dimensions for two-dimensional discrete iterated maps. One option is the ‘controlled logistic map’ defined by

$$\begin{aligned} x_{n+1} &= r x_n (1 - x_n) - b x_n + y_n \ , \\ y_{n+1} &= b x_n \ . \end{aligned}$$

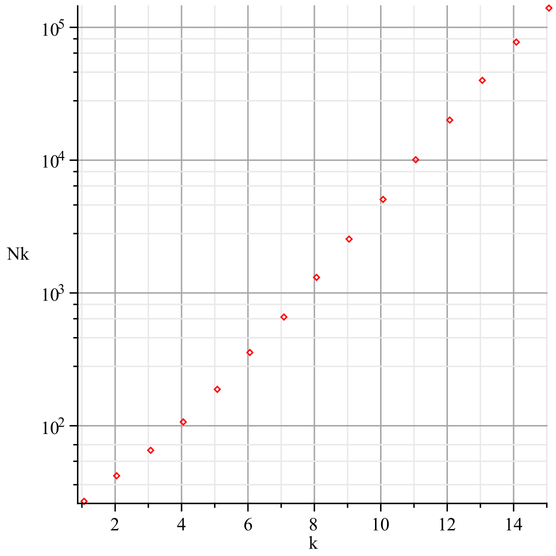
Obviously this reduces to the ordinary one-dimensional logistic map for the parameter choice $b = 0$.

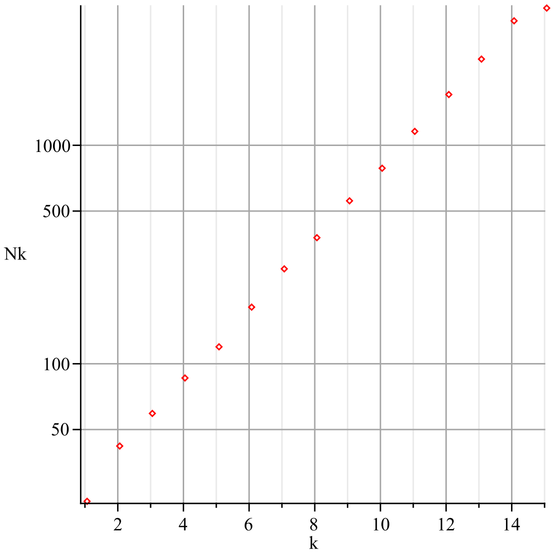
The calculation of the dimension d_c for the attractor of the logistic map with parameter $r = 3.56$ is trivial: Since the orbit is *periodic* (with period 8) only a fixed number of boxes $N(\epsilon) = 8$ get filled which implies a scaling with $d_c = 0$ (no ϵ-dependence).

The following table shows results for the parameters $r = 3.57$ and $r = 3.569945672 \simeq r_\infty$. Orbits of length $N = 10^6$ were used, discarding the first 10^4 points. The value for the box length was halved in each step according to $\epsilon_k = \dfrac{0.004}{2^k}$ for $k = 0 \ldots 14$. Counting the number of boxes N_k which contain at least one point leads to a series of approximations for the fractal (capacity) dimension

$$d_{c,k} = \frac{\ln(N_{k+1}/N_k)}{\ln 2}$$

$r = 3.57$			$r = 3.569945672$		
k	N_k	$d_{c,k}$	k	N_k	$d_{c,k}$
0	28	0.6517	0	24	0.8416
1	44	0.6276	1	43	0.5050
2	68	0.7200	2	61	0.5291
3	112	0.8147	3	88	0.4833
4	197	0.8938	4	123	0.5888
5	366	0.9108	5	185	0.5879
6	688	0.9680	6	278	0.4845
7	1346	0.9863	7	389	0.5559
8	2666	0.9899	8	572	0.4998
9	5296	0.9957	9	809	0.5460
10	10561	0.9957	10	1181	0.5676
11	21061	0.9848	11	1750	0.5372
12	41689	0.9404	12	2540	0.5812
13	80044	0.8585	13	3799	0.1929
14	144971	-	14	4343	-





In both cases the points for N_k quite nicely follow a straight line in the doubly logarithmic plot. This demonstrates the validity of the scaling behaviour $N(\epsilon) \simeq \epsilon^{-d_c}$. A deviation is seen for small values of k where the box size ϵ is too large to resolve the 'fine structure' of the attractor. Also if ϵ becomes very small (large k) deviations from the scaling behaviour can be expected. In fact, it is obvious that for any finite sample length N the number of filled boxes N_k has to go to a constant $N_k \leq N$ in the limit $\epsilon \longrightarrow 0$. A blind application of the formula for the capacity dimension thus will always lead to the result $d_c = 0$ (since a finite set of points always has dimension 0)!

The result for $r = 3.57$ leads to a scaling exponent slightly smaller than one. In view of the numerical uncertainty the result is compatible with $d_c = 1$. It has been shown for *one-dimensional* iterated maps that the chaotic attractors do *not* show fractal geometry *except* *at very special parameter values*, i.e. at the accumulation points of the period-doubling cascades (onset of chaos). The first such accumulation point is $r = r_\infty \simeq 3.569945672$. The numerical result shows that in this case the scaling exponent is considerably smaller than one. Fitting the straight part of the N_k-curve leads to $d_c \simeq 0.535$ which agrees well with the result $d_c \simeq 0.538$ found in the literature.[1]

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[1] P. Grassberger and I. Procaccia: Measuring the strangeness of strange attractors, Physica 9D (1983) 189