Nonlinear Dynamics and Complex Systems

The van der Pol oscillator

In 1927 the Dutch engineer Balthazar van der Pol at the Philips Research Institute, Eindhoven, investigated the following differential equation to describe an electronic (radio tube) oscillator

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0 \quad , \quad \alpha > 0 \; . \tag{1}$$

This is a famous example for a simple nonlinear dynamical system possessing a *stable limit cycle*. Independently of the initial condition the solution quickly approaches a unique periodic function. Its shape and period depend on the parameter α, the amplitude always is 2.

1. Investigation of the fixed point(s)

The standard first-order form of the van der Pol equation is

$$\dot{x} = y \; , \tag{2}$$

$$\dot{y} = -x - \alpha(x^2 - 1)y \; . \tag{3}$$

There is a single fixed point $\vec{x}^* = (0,0)^{\mathrm{T}}$. The corresponding Jacobian matrix is

$$\underline{M} = \begin{pmatrix} 0 & 1 \\ -1 - 2\alpha x y & -\alpha(x^2 - 1) \end{pmatrix}_{\vec{x}^*} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}$$

This leads to $s = \frac{1}{2}\mathrm{Tr}\underline{M} = \frac{\alpha}{2}$ and $d = \det\underline{M} = 1$. The characteristic values are $\lambda_{1,2} = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 1}$. Thus we find an *unstable node (repeller)* for $\alpha > 2$ and a *spiral repeller* for $\alpha < 2$.

2. The van der Pol equation in polar coordinates

The limiting case of weak nonlinearity, $\alpha \ll 1$, can be best studied by using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Differentiation leads to

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \; , \tag{4}$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \; . \tag{5}$$

An equation for $\dot{r}$ can be obtained by combining (4) $\cdot \cos\theta +$ (5) $\cdot \sin\theta$ and then using (2) and (3):

$$\begin{aligned} \dot{r} &= \dot{x}\cos\theta + \dot{y}\sin\theta \\ &= y\cos\theta + [-x - \alpha(x^2 - 1)y]\sin\theta \\ &= r\sin\theta\cos\theta + [-r\cos\theta - \alpha(r^2\cos^2\theta - 1)r\sin\theta]\sin\theta \\ &= -\alpha(r^2\cos^2\theta - 1)r\sin^2\theta \; . \end{aligned}$$

Similarly the combination $-(4)\cdot\sin\theta+(5)\cdot\cos\theta$ leads to

$$\begin{aligned} r\dot{\theta} &= -\dot{x}\sin\theta+\dot{y}\cos\theta \\ &= -y\sin\theta+[-x-\alpha(x^2-1)y]\cos\theta \\ &= -r\sin^2\theta+[-r\cos\theta-\alpha(r^2\cos^2\theta-1)r\sin\theta]\cos\theta \\ &= -r-\alpha(r^2\cos^2\theta-1)r\sin\theta\cos\theta\ . \end{aligned}$$

Thus the equations of motion for the radial and angular coordinates read

$$\dot{r} = -\alpha(r^2\cos^2\theta-1)r\sin^2\theta\ , \tag{6}$$

$$\dot{\theta} = -1-\alpha(r^2\cos^2\theta-1)\sin\theta\cos\theta\ . \tag{7}$$

3. The limit of weak nonlinearity $\alpha \ll 1$

In the linear limit $\alpha=0$ eqs. (6) and (7) reduce to $\dot{r}=0$ and $\dot{\theta}=-1$ so that $r_0(t)=\text{const}$ and $\theta_0(t)=-t$ (circular motion with constant angular velocity). For nonzero but small $\alpha \ll 1$ the function $r(t)$ will slowly deviate from r_0. This can be approximated by *time-averaging* the r.h.s. over the (fast moving) phase angle $\theta \simeq \theta_0$, keeping r fixed to a constant mean value $\overline{r}$ over one period of oscillation $T_0 \simeq 2\pi$. The required averages (denoted by angular brackets) of the angular functions are easily found:

$$\langle\cos\theta\sin\theta\rangle=0 \quad , \quad \langle\cos^3\theta\sin\theta\rangle=0\ ,$$
$$\langle\sin^2\theta\rangle=\frac{1}{2} \quad , \quad \langle\cos^2\theta\sin^2\theta\rangle=\frac{1}{8}\ .$$

This leads to an approximate equation of motion for the time-averaged coordinates

$$\begin{aligned} \dot{\overline{r}} &= -\alpha\Big(\overline{r}^2\frac{1}{8}-\frac{1}{2}\Big)\overline{r}=\frac{\alpha}{2}\Big(1-\frac{1}{4}\overline{r}^2\Big)\overline{r}\equiv f(\overline{r})\ , \\ \dot{\overline{\theta}} &= -1\ . \end{aligned}$$

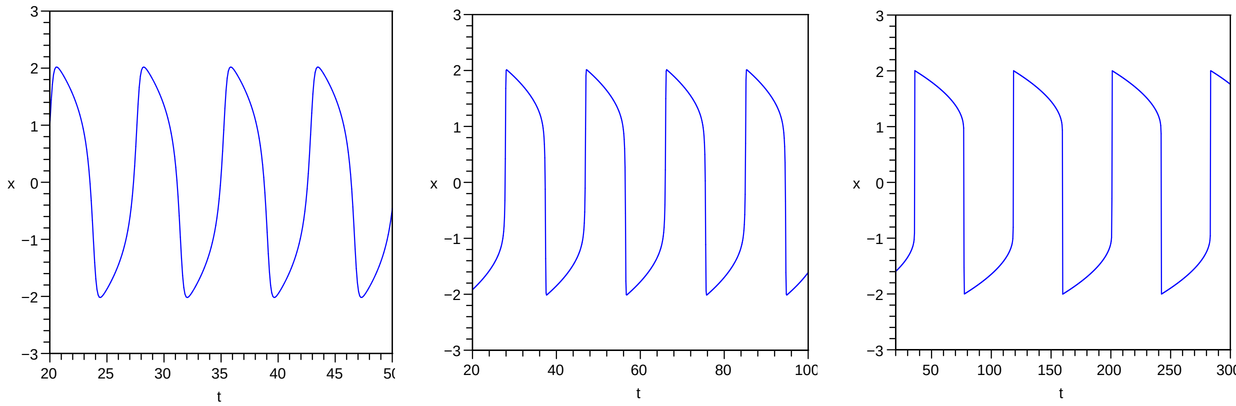
Thus the angular velocity remains constant while the radial variable undergoes a simple *one-dimensional* flow. The corresponding fixed points $f(\overline{r}^*)=0$ are $\overline{r}_1^*=0$ and $\overline{r}_2^*=2$. The stability analysis leads to

$$\frac{df}{d\overline{r}}=\frac{\alpha}{2}\Big(1-\frac{3}{4}\overline{r}^2\Big) \quad \longrightarrow \quad \begin{cases} f'(\overline{r}_1^*)=\frac{\alpha}{2}>0 & \text{instable} \\ f'(\overline{r}_2^*)=-\alpha<0 & \text{stable} \end{cases} \tag{8}$$

Thus the point $r=2$ is an attractor and in two dimensions the corresponding circle is a *stable limit cycle*. The fixed point is approached according to $r(t)=2+(r(0)-2)e^{-\alpha t}$.

4. The limit of strong nonlinearity $\alpha \gg 1$

Numerical solution of the differential equation shows periodic oscillations with amplitude 2 which become increasingly distorted with growing α. The period of oscillation is found to increase about linearly with the parameter α.



Trajectories $x(t)$ *for* $\alpha = 2$ *(left),* $\alpha = 10$ *(middle), and* $\alpha = 50$ *(right). Note the different timescales.*

One finds that for large α the motion alternates between "quiet" periods in which the value of $|x(t)|$ slowly drops from 2 to 1 and rapid, nearly discontinuous jumps. Such type of behaviour is called a "relaxation oscillation".
This problem is best analyzed by using a different set of differential equations:

$$\dot{x} = \alpha(z - g(x)) \quad \text{with} \quad g(x) = -x + \frac{1}{3}x^3 \, , \tag{9}$$

$$\dot{z} = -\frac{1}{\alpha}x \, . \tag{10}$$

Differentiation of (9) and insertion of (10) yields

$$\ddot{x} = \alpha\Big(\dot{z} + \dot{x} - \frac{1}{3}3x^2\dot{x}\Big) = -x + \alpha(1 - x^2)\dot{x}$$

which coincides with van der Pol's equation.
The phase flow can be analyzed by inspecting the slope of the function $z(x)$:

$$\frac{dz}{dx} = \frac{dz/dt}{dx/dt} = -\frac{1}{\alpha^2}\frac{x}{z - g(x)} \tag{11}$$

or

$$(z - g(x))\frac{dz}{dx} = -\frac{x}{\alpha^2} \, . \tag{12}$$

This reveals the advantage of using the modified system of equations, since now a *small parameter* $1/\alpha$ is present. For large α the r.h.s. is nearly zero which requires that either $z = g(x)$ or $\frac{dz}{dx} = 0$. The cubic parabola $z = g(x)$ divides the phase plane into two halves. Because of (9) in the upper half, $z > g(x)$, the trajectories move horizontally (fast) to the right until they hit this curve. Similarly for $z < g(x)$ the motion is horizontally to the left. Subsequently the system slowly moves (slightly shifted) along the curve $g(x)$, upwards if $x < 0$ or downwards if $x > 0$. The boundary curve is a cubic polynomial with a maximum at $x = -1, z = +2/3$ and a minimum at $x = +1, z = -2/3$.

When the maximum at B is reached, the boundary curve can no longer be followed because the sign of $\dot{z}$ cannot change but the curve $g(x)$ bends down. Therefore the trajectory is forced to switch over to the second type of motion ($dz/dx = 0$) and horizontally "jumps" to the other branch of the boundary curve $g(x)$ at C. Because of (9) the value of x changes fast during this part of the cycle. Then this sequence of events repeats itself in the opposite direction. The full limit cycle thus consists of the closed curve ABCD.
The *period length* T is determined by the "slow sections" AB and CD. We find

$$
\begin{aligned}
T_{AB} &= \int_A^B dt = \int_{-2}^{-1} dx \, \frac{dz}{dx}\frac{dt}{dz} = \int_{-2}^{-1} dx \, \frac{dg}{dx}\Big(-\frac{\alpha}{x}\Big) = \alpha \int_{-2}^{-1} dx \, (1-x^2)\frac{1}{x} \\
&= \alpha\Big(\ln|x| - \frac{1}{2}x^2\Big)\Big|_{-2}^{-1} = \alpha\Big(\ln 1 - \ln 2 - \frac{1}{2} + 2\Big) = \alpha\Big(\frac{3}{2} - \ln 2\Big) \, .
\end{aligned}
$$

For symmetry reasons $T_{CD} = T_{AB}$ while T_{BC} and T_{DA} are of the order $1/\alpha$ and thus can be neglected. The total oscillation period therefore is approximately

$$
T \simeq (3 - 2\ln 2)\alpha \simeq 1.6137\alpha \quad . \tag{13}
$$