

Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture XI

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Lecture XI, Exercise 1.

Recall the definition: $[[X]] := X_a - X_b$.

- (i) $\alpha[[A]] = \alpha(A_a - A_b) = \alpha A_a - \alpha A_b$.
 $[[\alpha A]] = \alpha_a A_a - \alpha_b A_b$.
Therefore, if $\alpha_a = \alpha_b$ ($\Leftrightarrow [[\alpha]] = 0$), $\alpha[[A]] = [[\alpha A]]$.
- (ii) $[[A + B]] = (A_a + B_a) - (A_b + B_b)$
 $= A_a - A_b + B_a - B_b = [[A]] + [[B]]$.
- (iii) $[[AB]] = (A_a B_a) - (A_b B_b)$.
 $[[A]][[B]] = (A_a - A_b)(B_a - B_b) = A_a B_a - A_a B_b - A_b B_a + A_b B_b$.
Therefore $[[AB]] \neq [[A]][[B]]$.
- (iv) $[[A]][[B]] = (A_a - A_b)(B_a - B_b) = (B_a - B_b)(A_a - A_b) = [[B]][[A]]$.
- (v) $[[A]]^2 = (A_a - A_b)^2 = A_a^2 - 2A_a A_b + A_b^2$.
 $[[A^2]] = (A_a^2 - A_b^2)$.
Therefore $[[A]]^2 \neq [[A^2]]$.

Lecture XI, Exercise 2.

From the junction conditions, the velocities on either side of the shock front in terms of the physical state can be written as

$$v_a^2 = \frac{(p_a - p_b)(e_b + p_a)}{(e_a - e_b)(e_a + p_b)} \quad (1)$$

$$v_b^2 = \frac{(p_a - p_b)(e_a + p_b)}{(e_a - e_b)(e_b + p_a)} \quad (2)$$

From these two equations, we can derive following relations

$$\frac{v_a}{v_b} = \frac{e_b + p_a}{e_a + p_b} \quad (3)$$

$$v_a v_b = \frac{p_a - p_b}{e_a - e_b} \quad (4)$$

Now we consider the case of an ultrarelativistic fluid with $p = e/3$ and $c_s = 1/\sqrt{3}$. Then the eq (3) can be written as

$$\frac{v_a}{v_b} = \frac{e_b + e_a/3}{e_a + e_b/3} = \frac{3e_b + e_a}{3e_a + e_b}. \quad (5)$$

Therefore the velocity ahead the shock can be written as

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b} \right) v_b. \quad (6)$$

From the assumption of an ultrarelativistic fluid, eq (4) can be expressed as

$$v_a v_b = \frac{e_a - e_b}{3(e_a - e_b)} = \frac{1}{3}. \quad (7)$$

Therefore the velocity behind the shock can be obtained as

$$v_b = \frac{1}{3v_a}. \quad (8)$$

We put eq (8) into eq (6) and obtain

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b} \right) \frac{1}{3v_a} \quad (9)$$

$$v_a^2 = \frac{1}{3} \left(\frac{3e_b + e_a}{3e_a + e_b} \right). \quad (10)$$

And

$$1 - v_a^2 = 1 - \frac{1}{3} \left(\frac{3e_b + e_a}{3e_a + e_b} \right) = \frac{9e_a + 3e_b - 3e_b - e_a}{9e_a + 3e_b} \quad (11)$$

$$= \frac{8e_a}{9e_a + 3e_b} = \frac{8}{3} \left(\frac{e_a}{3e_a + e_b} \right). \quad (12)$$

Therefore the square of the Lorentz factor relative to the velocity ahead the shock is

$$W_a^2 = \frac{1}{1 - v_a^2} = \frac{3}{8} \left(\frac{3e_a + e_b}{e_a} \right). \quad (13)$$

Similarly using eqs (6) and (8), we can get

$$\frac{1}{3v_b} = \left(\frac{3e_b + e_a}{3e_a + e_b} \right) v_b \quad (14)$$

$$v_b^2 = \frac{3e_a + e_b}{3(3e_b + e_a)}. \quad (15)$$

And

$$1 - v_b^2 = \frac{3(3e_b + e_a) + 3e_a + e_b}{3(3e_b + e_a)} = \frac{8e_b}{3(3e_b + e_a)}. \quad (16)$$

As a result, the square of the Lorentz factor relative to the velocity behind the shock is

$$W_b^2 = \frac{1}{1 - v_b^2} = \frac{3}{8} \frac{(3e_a + e_b)}{e_b}. \quad (17)$$

The relative velocity of the fluid ahead and behind the shock is given by

$$v_{ab} = \frac{v_a - v_b}{1 - v_a v_b} = \sqrt{\frac{(p_a - p_b)(e_a - e_b)}{(e_a + p_b)(e_b + p_a)}}. \quad (18)$$

From the assumption of an ultrarelativistic fluid, the eq (18) can be written as

$$v_{ab} = \sqrt{\frac{(e_a/3 - e_b/3)(e_a - e_b)}{(e_a + e_b/3)(e_b + e_a/3)}} \quad (19)$$

$$= \sqrt{\frac{(e_a - e_b)(3e_a - 3e_b)}{(3e_a + e_b)(3e_b + e_a)}}. \quad (20)$$

And

$$1 - v_{ab}^2 = \frac{(3e_a + e_b)(3e_b + e_a) - (e_a - e_b)(3e_a - 3e_b)}{(3e_a + e_b)(3e_b + e_a)} \quad (21)$$

$$= \frac{9e_a e_b + 3e_a^2 + 3e_b^2 + e_a e_b - (3e_a^2 - 3e_a e_b - 3e_a e_b + 3e_b^2)}{(3e_a + e_b)(3e_b + e_a)} \quad (22)$$

$$= \frac{16e_a e_b}{(3e_a + e_b)(3e_b + e_a)}. \quad (23)$$

Therefore the Lorentz factor square of the relative velocity is

$$W_{ab}^2 = \frac{1}{1 - v_{ab}^2} = \frac{(3e_a + e_b)(3e_b + e_a)}{16e_a e_b} = \frac{4}{9} W_a^2 W_b^2, \quad (24)$$

where

$$W_a^2 W_b^2 = \frac{9}{64} \frac{(3e_a + e_b)(3e_b + e_a)}{e_a e_b}. \quad (25)$$

Lecture XI, Exercise 3.

From eqs (13) and (17),

$$W_a^2 + W_b^2 = \frac{3}{8} \frac{(3e_a + e_b)e_b + (3e_b + e_a)e_a}{e_a e_b} \quad (26)$$

$$= \frac{3(3e_a e_b + e_b^2 + 3e_a e_b + e_a^2)}{8e_a e_b} \quad (27)$$

$$= \frac{3e_a^2 + 18e_a e_b + 3e_b^2}{8e_a e_b}. \quad (28)$$

From eq (24), the square of the Lorentz factor of the relative velocity can be written as

$$W_{ab}^2 = \frac{9e_a e_b + 3e_a^2 + 3e_b^2 + e_a e_b}{16e_a e_b} \quad (29)$$

$$= \frac{3e_a^2 + 10e_a e_b + 3e_b^2}{16e_a e_b}. \quad (30)$$

Finally, from eqs (28) and (30) we deduce that

$$W_a^2 - 2W_{ab}^2 + W_b^2 = \frac{3e_a^2 + 18e_a e_b + 3e_b^2 - (3e_a^2 + 10e_a e_b + 3e_b^2)}{8e_a e_b} \quad (31)$$

$$= \frac{8e_a e_b}{8e_a e_b} \quad (32)$$

$$= 1. \quad (33)$$