Plasma Astrophysics
Chapter 3.5: Multi-fluid theory of plasma

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Fluid approach to plasmas

- Fluid approach describes bulk properties of plasma. We do not attempt to solve unique trajectories of all particles in plasma. This simplification works very well for majority of plasma.
- Fluid theory follows directly from moments of the Boltzmann equation (previous chapter).
- Each of moments of Boltzmann (Vlasov) equation is a transport equation describing the dynamics of a quantity associated with a given power of ***v***

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{u}) = 0$$

Continuity of mass or charge transport

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla\cdot\boldsymbol{P} + \boldsymbol{P}_{ij}$$

Momentum transport

$$\frac{\partial}{\partial t}\left[n\frac{1}{2}mu^2\right] + \nabla\cdot\left[n\frac{1}{2}m\langle u^2\boldsymbol{u}\rangle\right] - nq\langle\boldsymbol{E}\cdot\boldsymbol{u}\rangle = \frac{m}{2}\int u^2\left(\frac{\partial f}{\partial t}\right)_{coll} d\boldsymbol{u}$$

Energy transport

Fluid motion

- The motion of fluid is described by a vector velocity field $\boldsymbol{v}(\boldsymbol{r})$, (which is mean velocity of all individual particles which make up the fluid at $\boldsymbol{r}$) and particle density $\boldsymbol{n}(\boldsymbol{r})$.
- We discuss the motion of fluid of a *single type* of particle of mass/ charge, m/q, so charge and mass density are qn and mn
- The particle conservation equation (continuity equation):

$$\frac{\partial}{\partial t}n + \nabla \cdot (n\boldsymbol{v}) = 0$$

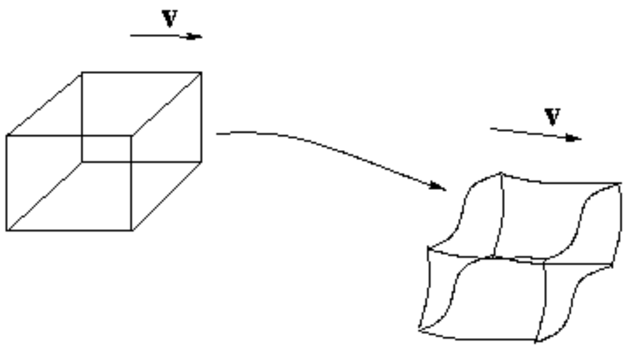
- Expand the $\nabla\cdot$ to get: $\frac{\partial}{\partial t}n + (\boldsymbol{v} \cdot \nabla)n + n\nabla \cdot \boldsymbol{v} = 0$
- Significance is that first two terms are *convective derivative* of n

$$\frac{D}{Dt} \equiv \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$$

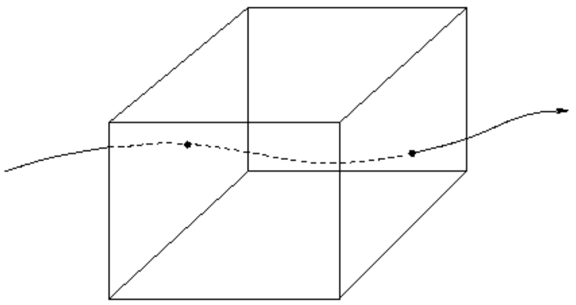
- So continuity equation can be written: $\frac{D}{Dt}n = -n\nabla \cdot \boldsymbol{v}$

Lagrangian & Eulerian viewpoint

- Lagrangian: sit on a fluid element and move with it as fluid moves
- Eulerian: sit at a fixed point in space and watch fluid move through your volume element: identity of fluid in volume continually changing
  - $\partial/\partial t$: rate of change at fixed point (Euler)
  - $D/Dt \equiv \partial/\partial t + \boldsymbol{v} \cdot \nabla$: rate of change at *moving* point (Lagrange)
  - $\boldsymbol{v} \cdot \nabla$: change due to motion



Lagrangian viewpoint



Eulerian viewpoint

Lagrangian & Eulerian viewpoint (cont.)

- Derivation of continuity is Eulerian. From Langrangian view

$$\frac{D}{Dt}n = \frac{d}{dt}\frac{\Delta N}{\Delta V} = -\frac{\Delta N}{\Delta V^2}\frac{d}{dt}\Delta V = -n\frac{1}{\Delta V}\frac{d\Delta V}{dt}$$

- Since total number of particles in volume element (*ΔN*) is constant (we are moving with them)
- Now: $$\begin{aligned}\frac{d}{dt}\Delta V &= \frac{d\Delta x}{dt}\Delta y\Delta z + \frac{d\Delta y}{dt}\Delta z\Delta x + \frac{d\Delta z}{dt}\Delta y\Delta x \\ &= \Delta V\left[\frac{1}{\Delta x}\frac{d\Delta x}{dt} + \frac{1}{\Delta y}\frac{d\Delta y}{dt} + \frac{1}{\Delta z}\frac{d\Delta z}{dt}\right]\end{aligned}$$
- But: $$\frac{d\Delta x}{dt} = v_x(\Delta x/2) - v_x(-\Delta x/2) \simeq \Delta x\frac{\partial v_x}{\partial t}$$
- Therefore, $$\frac{d\Delta V}{dt} = \Delta V\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] = \Delta V\nabla\cdot\boldsymbol{v}$$

so $$\frac{D}{Dt}n = -n\nabla\cdot\boldsymbol{v}$$

($-\nabla\cdot\boldsymbol{v}$ is the rate of (Volume) compression of element)

Cold-Plasma model

- Simplest set of macroscopic equations can be obtained by simplifying the momentum transfer equation and neglect thermal motions of particles.
- Here, set kinetic pressure tensor to zero, i.e., $P = mn \langle ww \rangle = 0$ as $w = 0$ (w is thermal velocity)
- Remaining macroscopic variables n, $\boldsymbol{u}$ are described by

$$\frac{\partial n}{\partial t} = \nabla \cdot (n\boldsymbol{u}) = 0$$

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] = qn(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) + \boldsymbol{P}_{ij}$$

- Collision term $\boldsymbol{P}_{\mathrm{ij}}$ can be approximated by an “effective” collision frequency
- Assumed that collisions cause a rate of decrease in momentum:

$$\boldsymbol{P}_{ij} = -mnv_{eff}\boldsymbol{u}$$

Warm-Plasma model

- Alternative set of macroscopic equation is obtained by truncating energy conservation equation.
- Consider pressure tensor: $\boldsymbol{P} = mn\langle \boldsymbol{w}\boldsymbol{w} \rangle = \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{pmatrix}$
- Components represent transport of momentum. Diagonal elements represent pressure, while off-diagonal represent shearing stresses.
- In warm-plasma model, only consider diagonal pressure elements so

$$\nabla \cdot \boldsymbol{P} = \nabla p$$

- That is, viscous forces are neglected. We then have

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{u}) = 0$$

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] = qn(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) - \nabla p + \boldsymbol{P}_{ij}$$

Warm-Plasma model (cont.)

- The previous system of equations does not form a closed set, since scalar pressure is now a third variable. Usually determined by energy equation.
- If plasma is *isothermal*, assume *equation of state* of form:

$$p = nk_B T \quad \text{and} \quad \nabla p = k_B T \nabla n$$

- Holds for slow time variations, allowing temperatures to reach equilibrium
- If plasma does not exchange energy with its surrounds, assume it is *adiabatic*:

$$pn^{-\gamma} = constant \quad \text{and} \quad \frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}$$

- Where γ is specific heat ratio at constant pressure
  - Isothermal: T=*const.* : γ=1
  - Adiabatic /Isotropic 3 degree of freedom: γ=5/3
  - Adiabatic / 1 degree of freedom: γ=3

Simplified energy equation

- Note, the energy equation can be written

$$\frac{\partial\left[1/2nm\langle w^2\rangle\right]}{\partial t}+\nabla\cdot\left(\frac{1}{2}nm\langle w^2\rangle\boldsymbol{u}\right)+(\boldsymbol{P}\cdot\nabla)\boldsymbol{u}+\nabla\cdot\boldsymbol{q}=\boldsymbol{P}_{ij}$$

- where $\boldsymbol{q}$ is the heat flow vector. For electrons, commonly used approximation for $\boldsymbol{q}$ is $q = K\nabla T$
- where K is thermal *Spitzer conductivity*.
- As average energy of plasma is $1/2m\langle ww\rangle=3/2k_BT$ and using $p=nk_BT$ => $3/2p = 1/2nm\langle ww\rangle$. Energy equation can then be written

$$\frac{\partial(3/2p)}{\partial t}+\nabla\cdot(3/2p\boldsymbol{u})-p\nabla\cdot\boldsymbol{u}+\nabla\cdot\boldsymbol{q}=\boldsymbol{P}_{ij}$$

- The quantity $3/2pu$ represents the flow of energy density at the fluid velocity.

Effect of collisions

- Like particle collisions *do not* change that total momentum (which is averaged over all particles of that species)
- Unlike particle collision *do* exchange momentum between the species.
- Therefore, any quasi-neutral plasma consisted of at least two different species (electrons and ions) is needed to account for another momentum loss (gain) term via collision
- The rate of momentum density loss by species 1 colliding with species 2:

$$\boldsymbol{P}_{12} = \nu_{12} n_1 m_1 (\boldsymbol{v}_1 - \boldsymbol{v}_2)$$

Complete set of two-fluid equations

- Consider plasma of two species; ions and electrons, in which fluid is fully ionised, isotropic and collisionless (& adiabatic). The charge and current densities are

$$\rho_e = n_i q_i + n_e q_e \text{ and } \boldsymbol{J} = n_i q_i \boldsymbol{v}_i + n_e q_e \boldsymbol{v}_e$$

- Using *v*=*u*, complete set of two-fluid equations are then (*j* = *i* or *e*)

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \boldsymbol{v}_j) = 0$$

$$m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla) \boldsymbol{v}_j \right] = -\nabla p_j + q_j n_j (\boldsymbol{E} + \boldsymbol{v}_j \times \boldsymbol{B})$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\epsilon_0}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} \qquad (\varepsilon_0 \mu_0 = 1/c^2)$$

$$p_j n_j^{-\gamma} = const.$$

Complete set of two-fluid equations (cont.)

- Equations still very difficult and complicated mostly because it is *Nonlinear*.
- In some cases, we can get a tractable problem by "linearizing"

Fluid drifts perpendicular to B

- Since a fluid element is composed of many individual particles, expect drifts perpendicular to ***B***. But, the *grad* (*p*) term results in a fluid drift called *diamagnetic drift*.
- Consider momentum equation for each species:

$$mn\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p - qn(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

(1) (2) (3)

- Consider ratio (1) to (3):

$$\frac{(1)}{(3)} \approx \left|\frac{mni\omega v_\perp}{qnv_\perp B}\right| \approx \frac{\omega}{\omega_c}$$

- Here we have used $\partial/\partial t = i\omega$. If only consider slow drifts compared to time-scale of the gyro-frequency, we can set (1) to zero

Fluid drifts perpendicular to B (cont.)

- Therefore, we can write: $0 \approx -qn(\boldsymbol{E} + \boldsymbol{v}_\perp \times \boldsymbol{B}) - \nabla p$
- Where $\boldsymbol{v} \times \boldsymbol{B} = (\boldsymbol{v}_\parallel + \boldsymbol{v}_\perp) \times \boldsymbol{B} = \boldsymbol{v}_\perp \times \boldsymbol{B}$
- Taking cross-product of $\boldsymbol{B}$:

$$0 = -qn[\boldsymbol{E} \times \boldsymbol{B} + (\boldsymbol{v}_\perp \times \boldsymbol{B}) \times \boldsymbol{B}] - \nabla p \times \boldsymbol{B}$$

- Using the identity $(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = (\boldsymbol{C} \cdot \boldsymbol{A})\boldsymbol{B} - (\boldsymbol{C} \cdot \boldsymbol{B})\boldsymbol{A}$
- We can write:

$$0 = -qn[\boldsymbol{E} \times \boldsymbol{B} + (\boldsymbol{v}_\perp \cdot \boldsymbol{B})\boldsymbol{B} - (\boldsymbol{B} \cdot \boldsymbol{B})\boldsymbol{v}_\perp] - \nabla p \times \boldsymbol{B}$$

- As $\boldsymbol{v}_\perp$ is perpendicular to $\boldsymbol{B}$, $\boldsymbol{v}_\perp \cdot \boldsymbol{B} = 0$. Therefore

$$\begin{aligned} \boldsymbol{v}_\perp &= \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{\nabla p \times \boldsymbol{B}}{qnB^2} \\ &= \boldsymbol{v}_E + \boldsymbol{v}_D \end{aligned}$$

Fluid drifts perpendicular to B (cont.)

- In previous equation: $$\boldsymbol{v}_E = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}$$ *E x B drift*

  and $$\boldsymbol{v}_D = -\frac{\nabla p \times \boldsymbol{B}}{qnB^2}$$ *diamagnetic drift*

- The $\boldsymbol{v}_E$ drift is same as for guiding center, but there is now a new drift, called the diamagnetic drift. Is in opposite directions for ions and electrons.
- Consider electrons + single ions, from quasi-neutrality $n_i q_i = -n_e q_e$

$$n_e q_e \boldsymbol{v}_e + n_i q_i \boldsymbol{v}_i = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \underbrace{(n_e q_e + n_i q_i)}_{=0} - \nabla(p_e + p_i) \times \frac{\boldsymbol{B}}{B^2}$$

- Therefore current density: $$\boldsymbol{J} = -\nabla(p_e + p_i) \times \frac{\boldsymbol{B}}{B^2}$$

diamagnetic current

Summary

- Fluid theory follows directly from moments of the Boltzmann equation (Kinetic theory)
- We drive two-fluid (MHD) equations consisting Continuity, Momentum, Energy/ EoS, and Maxwell's equations
- Equations still very difficult and complicated mostly because it is *Nonlinear*.
- Additional drift motion by pressure gradient, diamagnetic drift.
- Using wave properties of multi-species plasma
  - Langmuir wave, Ion(electron)-acoustic wave, ion(electron)-cyclotron wave, Whistler wave, electrostatic wave, electromagnetic plasma wave etc.
  - Two-stream instability
- Astrophysical application
  - partially ionized gas such as interstellar medium, solar atmosphere