Plasma Astrophysics
Chapter 3: Kinetic Theory

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Kinetic Theory

- Single particle description: tenuous plasma with strong external fields, important for gaining insight into physical processes involved
- For a system with a large number of particles it is neither possible nor desirable to determine the motion of every single particle
  => statistical approaches, average macroscopic properties
- Kinetic theory averages out microscopic information to obtain statistical, kinetic equations. No knowledge of individual particle motion is required to describe observable phenomena.

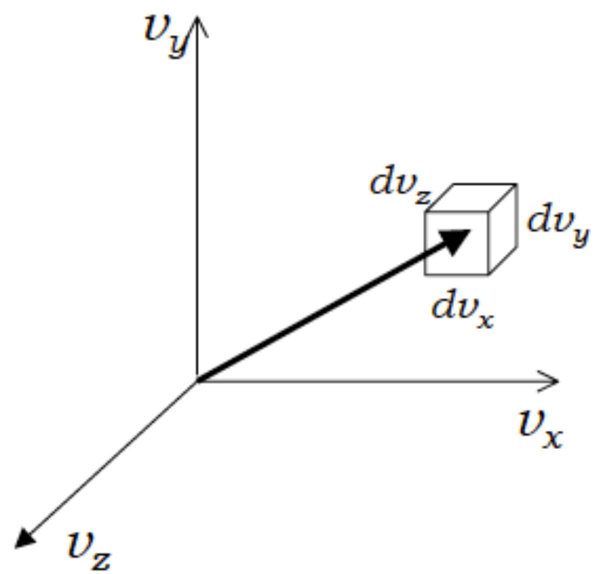
Particle Phase Space

- A particle's dynamical state can be specified using its position and velocity:

$$\boldsymbol{r} = (x, y, z) \quad \text{and} \quad \boldsymbol{v} = (v_x, v_y, v_z)$$

- Combining position and velocity information gives particle's position in phase space:

$$(\boldsymbol{r}, \boldsymbol{v}) = (x, y, z, v_x, v_y, v_z)$$



- The state space for position and momentum (or velocity) is a 6D phase space
- Volume of a small element of velocity space is

$$dv_x dv_y dv_z = d^3v = d\boldsymbol{v}$$

- Volume element in phase space is $d^3r d^3v$

Velocity distribution functions

- Single-particle approach has limited application where collective motion not important. Non-zero electric fields in a plasma generally arise self-consistently, so must consider collective motion of many plasma particles.
- State of plasma described by the *velocity distribution function* :

$$f(x, y, z, v_x, v_y, v_z, t)$$

- Gives the number of particles per unit volume as a position ***r*** and a time *t* with velocity, v_x, v_y, v_z. Has 7 independent variables defining a 6D phase space.
- Number of particles in a phase space volume d^3rd^3v is:

$$dn = f(r, v, t)dxdydzdv_xdv_ydv_z = f(r, v, t)d^3rd^3v$$

- The total number of particles is therefore

$$n = \int_{-\infty}^{\infty} f(r, v, t)d^3rd^3v$$

Moments

- Let $f(x)$ be any function that is defined and positive on an interval $[a, b]$. The moments of this function is defined as

Zeroth moment $$M_0 = \int_b^a f(x)dx$$

First moment $$M_1 = \int_b^a xf(x)dx$$

Second moment $$M_2 = \int_b^a x^2 f(x)dx$$

n^{th} moment $$M_n = \int_b^a x^n f(x)dx$$

Moments (cont.)

- In particular case that distribution is a probability density, $p(x)$, then

$$M_0 = 1$$
$$M_1 = \int_b^a x p(x) dx = \langle x \rangle = \text{mean}(x)$$
$$M_2 = \int_b^a x^2 p(x) dx = \text{variance}(x)$$

- Higher order moments correspond to skewness and kurtosis.

- Skewness: a measure of symmetry or lack of symmetry
- Kurtosis: a measure of whether the data are peaked or flat relative to normal distribution

Moments of distribution function

- Velocity distribution function gives microscopic description of statistical information on particles. However, most important use is in determining macroscopic (i.e., averages) values such as density, current, etc.
- *Zeroth order moment* of $f(\mathbf{r}, \mathbf{v}, t)$ is: $n(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} f(\boldsymbol{r}, \boldsymbol{v}, t) d^3 v$
- *First order moment* is bulk velocity: $\boldsymbol{u} = \frac{1}{n}\int_{-\infty}^{\infty} \boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}, t) d^3 v$
- Charge and current densities of spices (s) can be expressed in using moments:

$$\rho_e = \sum_s q_s n_s \quad \boldsymbol{j} = \sum_s q_s n_s \boldsymbol{u}_s$$

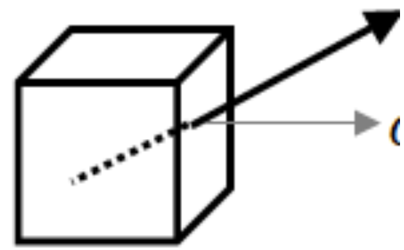
- *Second order moment* relates to kinetic energy

$$\left\langle \frac{1}{2} m v^2 \right\rangle = \frac{1}{n}\int_{-\infty}^{\infty} \frac{1}{2} m \boldsymbol{v}^2 f(\boldsymbol{r}, \boldsymbol{v}, t) d^3 v$$

Derivation of Boltzmann Equation

- Evolution of $f(\boldsymbol{r},\boldsymbol{v},t)$ is described by the *Boltzmann Equation*.
- Consider particles entering and leaving a small volume of space. Since $\boldsymbol{r}$ and $\boldsymbol{v}$ is independent, can treat separately.
- Position: Number of particles leaving d^3r per second through its surface $d\mathbf{S}$ is

$$\int f(\boldsymbol{r},\boldsymbol{v},t)\dot{\boldsymbol{r}}\cdot d\boldsymbol{S} = \int f(\boldsymbol{r},\boldsymbol{v},t)\boldsymbol{v}\cdot d\boldsymbol{S}$$



- Velocity: Number of particles leaving d^3v per second through its surface $d\mathbf{S}_v$ is

$$\int f(\boldsymbol{r},\boldsymbol{v},t)\dot{\boldsymbol{v}}\cdot d\boldsymbol{S}_v = \int f(\boldsymbol{r},\boldsymbol{v},t)\boldsymbol{a}\cdot d\boldsymbol{S}_v$$

- So the net number of particles leaving the phase space volume d^3rd^3v is

$$\int f(\boldsymbol{r},\boldsymbol{v},t)\boldsymbol{v}\cdot d\boldsymbol{S}d^3v + \int f(\boldsymbol{r},\boldsymbol{v},t)\boldsymbol{a}\cdot d\boldsymbol{S}_v d^3r$$

Derivation of Boltzmann Equation (cont.)

- The rate of change of particle number in d^3rd^3v is:

$$\frac{\partial}{\partial t}\left[\int f d^3r d^3v\right] = -\left[\int f\boldsymbol{v}\cdot d\boldsymbol{S} d^3v + \int f\boldsymbol{a}\cdot d\boldsymbol{S}_v d^3r\right]$$

- As total number of particles in d^3rd^3v is conserved:

$$\frac{\partial}{\partial t}\left[\int f d^3r d^3v\right] + \left[\int f\boldsymbol{v}\cdot d\boldsymbol{S} d^3v + \int f\boldsymbol{a}\cdot d\boldsymbol{S}_v d^3r\right] = 0$$

- Recall Gauss' Divergence Theorem: $\int_V (\nabla\cdot\boldsymbol{F})dV = \int_S (\boldsymbol{F}\cdot\boldsymbol{n})dS$

- Can change integral over dS to d^3r:

$$\frac{\partial}{\partial t}\left[\int f d^3r d^3v\right] + \left[\int \nabla_r\cdot(f\boldsymbol{v}) d^3r d^3v + \int \nabla_v\cdot(f\boldsymbol{a}) d^3r d^3v\right] = 0$$

or

$$\frac{\partial}{\partial t}\left[\int f d^3r d^3v\right] + \left[\int \frac{\partial}{\partial r}\cdot(f\boldsymbol{v}) d^3r d^3v + \int \frac{\partial}{\partial v}\cdot(f\boldsymbol{a}) d^3r d^3v\right] = 0$$

Derivation of Boltzmann Equation (cont.)

- The phase space volume can be arbitrarily small, such that integrals are constant within the volume. Therefore we have

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot (f\boldsymbol{v}) + \frac{\partial}{\partial \boldsymbol{v}} \cdot (f\boldsymbol{a}) = 0$$

- But since $\boldsymbol{r}$ and $\boldsymbol{v}$ are independent variables, we can take $\boldsymbol{v}$ outside $d/d\boldsymbol{r}$ and similarly for $\boldsymbol{a}$. Then we can write

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0$$

- Replacing $\boldsymbol{a}=\boldsymbol{F}/m$, we have

$$\boxed{\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0} \qquad (3.1)$$

- This is the *collisionless Boltzmann equation*. Can be used in hot plasma where collisions can be neglected

Vlasov equation

- Previous equation written in terms of generalized force. For plasmas, Lorentz force is of interest, so

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{q}{m}[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0$$

- This is called the *Vlasov equation*. Can also be written as

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \qquad (3.2)$$

- This is one of the most important and widely used equations in kinetic theory of plasmas.

- Maxwell's equations for ***E*** and ***B*** and the Vlasov equation represent a complete set of self-consistent equations.

Convective derivative in phase space

- Distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ depends on 7 independent variables. Total time derivative of f is:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial f}{\partial v_x}\frac{\partial v_x}{\partial t} + \frac{\partial f}{\partial v_y}\frac{\partial v_y}{\partial t} + \frac{\partial f}{\partial v_z}\frac{\partial v_z}{\partial t}$$

- This can be written as $\frac{df}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}}$

- To appreciate meaning of this equation, consider $f = f(\boldsymbol{r}, t)$:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} \equiv \frac{Df}{Dt}$$

- Called the *convective derivative* or *Lagrangian derivative*. Second term gives change in f measured by an observed moving in the fluid frame.

Phase space evolution

- A plasma particle's state (***r***, ***v***) evolves in phase space. In absence of collisions, points move along continuous curves and *f* obeys the continuity equation: $$\frac{\partial f}{\partial t} + \nabla_{r,v} \cdot [(\dot{\boldsymbol{r}}, \dot{\boldsymbol{v}})f] = 0$$
- Called *Liouville equation*
- The Liouville equation describes the time evolution of the phase space distribution function. Liouvilles' theorem states that flows in phase space are incompressible.
- In Cartesian coordinates, equation reduces to

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot (f\dot{\boldsymbol{r}}) + \frac{\partial}{\partial \boldsymbol{v}} \cdot (f\dot{\boldsymbol{v}}) = 0$$

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0$$

- Which is in form of the collisionless Boltzmann equation. The Bolzmann and Vlasov equations follow from Liouville equation.

Collisional Boltzmann and Vlasov equations

- In the presence of collisions, the Boltzmann equation can be written

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{\boldsymbol{F}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

where the term on the right is the time rate of change of f due to collisions. This is the *collisional Boltzmann equation*.

- Similarly, the Vlasov equation can be written

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

- This is *the collisional Vlasov equation*. Describes change in particle distribution due to short-range interactions.

- When there are collisions with neutral atoms: $\left(\frac{\partial f}{\partial t}\right)_{coll} \approx \frac{f_n - f}{\tau}$ where f_n is the neutral atom distribution function, and τ is the collision time. Called Krook collision model

Kinetic description of plasma

- Kinetic description of plasma is highly applicable treatment for collisionless plasma (wave-particle interaction, collision-less shock, particle acceleration)
- But evaluation of a 6D distribution function is difficult: analytical solutions of a kinetic equation are rare and numerical are expensive.
- Astrophysical application:
  - Dark matter evolution in cosmological simulation
  - Neutrino transport in core-collapse supernova simulation
  - Stellar interior (equation of state)
  - Collisionless shock (supernova blast wave)
  - Particle acceleration (astrophysical shock)

Moments of Bolzmann -Vlasov equation

- Under certain assumptions not necessary to obtain actual distribution function if only interested in the macroscopic values.
- Instead of solving Boltzmann or Vlasov equation for distribution function and integrating, can take integrals over *collisional Boltzmann-Vlasov equation* and solve for the quantities of interest.

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll} \quad (3.3)$$

- Called taking the moments of Boltzmann-Vlasov equation
- Resulting equations known as the macroscopic transport equations, and form the foundation of plasma fluid theory.
- Results in derivation of the equations of magnetohydrodynamics (MHD).

Zeroth-order moment: continuity equation

- Lowest order moment obtained by integrating eq. (3.3):

$$\int \frac{\partial f}{\partial t} d\boldsymbol{v} + \int \boldsymbol{v} \cdot \nabla f d\boldsymbol{v} + \int \frac{q}{m}[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} = \int \left(\frac{\partial f}{\partial t}\right)_{coll} d\boldsymbol{v}$$

- The first-term gives: $$\int \frac{\partial f}{\partial t} d\boldsymbol{v} = \frac{\partial}{\partial t} \int f d\boldsymbol{v} = \frac{\partial n}{\partial t} \qquad (3.4)$$

- Since ***v*** and ***r*** are independent, ***v*** is not effected by gradient operator:

$$\int \boldsymbol{v} \cdot \nabla f d\boldsymbol{v} = \nabla \cdot \int \boldsymbol{v} f d\boldsymbol{v}$$

- From previous one, the first order moment of distribution function is

$$\boldsymbol{u} = \frac{1}{n} \int \boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}, t) d\boldsymbol{v}$$

- Therefore, $$\int \boldsymbol{v} \cdot \nabla f d\boldsymbol{v} = \nabla \cdot (n\boldsymbol{u}) \qquad (3.5)$$

Zeroth-order moment: continuity equation (cont.)

- For the third term, consider ***E*** and ***B*** separately. ***E*** term vanishes as

$$\int \boldsymbol{E} \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} = \int \frac{\partial}{\partial \boldsymbol{v}} \cdot (f\boldsymbol{E}) d\boldsymbol{v} = \int f\boldsymbol{E} \cdot d\boldsymbol{S} = 0 \qquad (3.6a)$$

where using Gauss' divergence theorem in velocity space. The surface area of velocity space goes as v^2. As $v \Rightarrow \infty$, $f \Rightarrow 0$ more quickly than $S \Rightarrow \infty$ (i.e., f typically goes as $1/v^2$. A Maxwellian goes as $e^{-v^{\wedge}2}$). Integral to v = infinity goes to zero.

- Using vector identity, $\nabla \cdot (a\boldsymbol{A}) = \boldsymbol{A} \cdot \nabla a + a \nabla \cdot \boldsymbol{A}$. The ***v*** x ***B*** term is

$$\begin{aligned} \int (\boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} &= \int \frac{\partial}{\partial \boldsymbol{v}} \cdot (f\boldsymbol{v} \times \boldsymbol{B}) d\boldsymbol{v} - \int f \frac{\partial}{\partial \boldsymbol{v}} \cdot (\boldsymbol{v} \times \boldsymbol{B}) d\boldsymbol{v} \\ &= \int f(\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{S} - \int f \frac{\partial}{\partial \boldsymbol{v}} \cdot (\boldsymbol{v} \times \boldsymbol{B}) d\boldsymbol{v} = 0 \end{aligned} \qquad (3.6b)$$

- The first term on right again vanishes as $f \Rightarrow 0$ more quickly than $S \Rightarrow \infty$. The second term vanishes as ***v*** x ***B*** is perpendicular to $d/d\boldsymbol{v}$

Zeroth-order moment: continuity equation (cont.2)

- Last-term is on right-hand side of eq. (3.3) :

$$\int \left(\frac{\partial f}{\partial t}\right)_{coll} d\boldsymbol{v} = \frac{\partial}{\partial t}\left[\int f d\boldsymbol{v}\right] = 0 \qquad (3.7)$$

- This assumes that the total number of particles remains constant as collisions proceed.
- Combing eq. (3.4)-(3.7) yields the *equation of continuity*

$$\boxed{\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{u}) = 0} \qquad (3.8)$$

- First-term represents rate of change of particle concentration within a volume, second-term represents the divergence of particles of the flow of particles out of the volume.
- Eq (3.8) is the first of *the equations of magnetohydrodynamics* (MHD). Eq (3.8) is a continuity equation for mass or charge transport if we multiply m or q.

First-order of moment: momentum transport

- Re-write eq.(3.3) :

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

- Next moment of the Boltzmann equation is obtained by multiplying Eq (3.3) by ***mv*** and integrating over ***dv***.

$$m\int \boldsymbol{v}\frac{\partial f}{\partial t}d\boldsymbol{v}+m\int \boldsymbol{v}(\boldsymbol{v}\cdot\nabla)f\,d\boldsymbol{v}+q\int \boldsymbol{v}[\boldsymbol{E}+(\boldsymbol{v}\times\boldsymbol{B})]\cdot\frac{\partial f}{\partial \boldsymbol{v}}d\boldsymbol{v} = m\int \left(\frac{\partial f}{\partial t}\right)_{coll} d\boldsymbol{v} \tag{3.9}$$

- The right-hand side is the change of the momentum due to collisions and will be given the term $\boldsymbol{P}_{ij}$.
- The first-term gives

$$\begin{aligned} m\int \boldsymbol{v}\frac{\partial f}{\partial t}d\boldsymbol{v} &= m\frac{\partial}{\partial t}\int \boldsymbol{v}f\,d\boldsymbol{v} \\ &= m\frac{\partial (n\boldsymbol{u})}{\partial t} \end{aligned} \tag{3.10}$$

First-order of moment: momentum transport (cont.)

- Next consider third-term:

$$\int \boldsymbol{v}[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \frac{\partial f}{\partial \boldsymbol{v}} dv = \int \frac{\partial}{\partial \boldsymbol{v}} \cdot [f\boldsymbol{v}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})] d\boldsymbol{v} - \int f\boldsymbol{v} \frac{\partial}{\partial \boldsymbol{v}} \cdot (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) d\boldsymbol{v}$$
$$- \int f(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{v}} \boldsymbol{v} d\boldsymbol{v}$$

- The first and second to integrals on the right vanishes for same reasons as before. Therefore have,

$$q \int \boldsymbol{v}[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \frac{\partial f}{\partial \boldsymbol{v}} dv = -q \int f(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) d\boldsymbol{v}$$
$$= -qn(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \qquad (3.11)$$

- To evaluate second-term of eq.(3.9) , use that $\boldsymbol{v}$ does not depend on gradient operator:

$$\int \boldsymbol{v}(\boldsymbol{v} \cdot \nabla) f d\boldsymbol{v} = \int \nabla \cdot (f\boldsymbol{v}\boldsymbol{v}) d\boldsymbol{v}$$
$$= \nabla \cdot \int f\boldsymbol{v}\boldsymbol{v} d\boldsymbol{v}$$

First-order of moment: momentum transport (cont.)

- Since the average of a quantity is $1/n$ times its weighted integral over $\boldsymbol{v}$, we have

$$\nabla \cdot \int f\boldsymbol{v}\boldsymbol{v}d\boldsymbol{v} = \nabla \cdot (n\langle \boldsymbol{v}\boldsymbol{v}\rangle)$$

- Now separate $\boldsymbol{v}$ into average fluid velocity $\boldsymbol{u}$ and a thermal velocity $\boldsymbol{w}$:

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$$

- Since $\boldsymbol{u}$ is already averaged, so we have

$$\nabla \cdot (n\langle \boldsymbol{v}\boldsymbol{v}\rangle) = \nabla \cdot (n\boldsymbol{u}\boldsymbol{u}) + \nabla \cdot (n\langle \boldsymbol{w}\boldsymbol{w}\rangle) + 2\nabla \cdot (n\boldsymbol{u}\langle \boldsymbol{w}\rangle) \qquad (3.12)$$

- The average thermal velocity is zero => $\langle \boldsymbol{w} \rangle = 0$ and

$$\boldsymbol{P} = mn\langle \boldsymbol{w}\boldsymbol{w}\rangle \qquad (3.13)$$

is the *stress tensor*.

- $\boldsymbol{P}$ is a measure of the thermal motion in a fluid. If all particles moved with same steady velocity $\boldsymbol{v}$, then $\boldsymbol{w} = 0$ and thus $\boldsymbol{P} = 0$ (i.e., a cold plasma).

First-order of moment: momentum transport (cont.3)

- Remaining term in Eq (3.9) can be written

$$\nabla \cdot (n\boldsymbol{u}\boldsymbol{u}) = \boldsymbol{u}\nabla \cdot (n\boldsymbol{u}) + n(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \qquad (3.14)$$

- Correcting eq. (3.10), (3.11), (3. 13), and (3.14), we have

$$m\frac{\partial}{\partial t}(n\boldsymbol{u}) + m\boldsymbol{u}\nabla \cdot (n\boldsymbol{u}) + mn(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \cdot \boldsymbol{P} - qn(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{P}_{ij}$$

- Combing first two terms (using cont. eq.), we obtain the *fluid equation of motion*:

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] = qn(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) - \nabla \cdot \boldsymbol{P} + \boldsymbol{P}_{ij} \qquad (3.15)$$

- This describes flow of momentum – also called *momentum transport equation*.
- Eq (3.15) is a statement of conservation of momentum and represents force balance on components of plasma. On right are the Lorentz force, pressure, and collisions

Summary of moments of Vlasov equation

- Equations of MHD and multi-fluid theory are obtained by taking the moments of the Vlasov equation, corresponding to mass, momentum and energy.

$$\int (\text{Vlasov equation}) d\boldsymbol{v} \Rightarrow \text{conservation of mass}$$

$$\int (\text{Vlasov equation}) \boldsymbol{v} d\boldsymbol{v} \Rightarrow \text{conservation of momentum}$$

$$\int (\text{Vlasov equation}) \boldsymbol{v}^2/2 d\boldsymbol{v} \Rightarrow \text{conservation of energy}$$

- Zeroth moment of the Vlasov equation results in the MHD mass continuity equation (eq. 3.8).
- First moment of the Vlasov equation gives the MHD momentum equation (eq. 3.15)
- Second moment of the Vlasov equation give the MHD energy transport equation