Plasma Astrophysics
Chapter 5: Waves in Plasma

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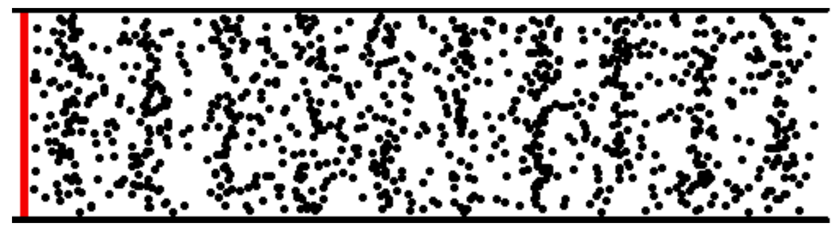
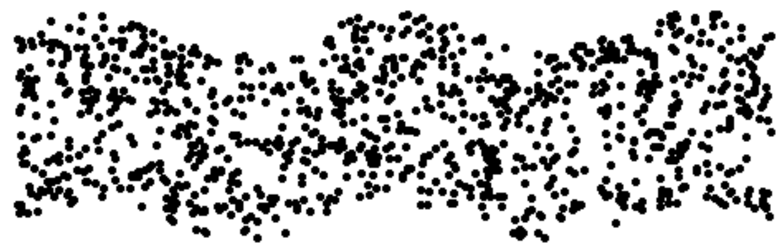
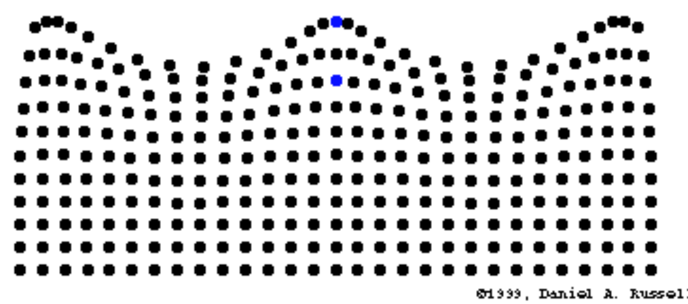
What defines a wave?

- Mechanical Example:
  - Sound, string, water
- Energy transfer
- Restoring forces:
  - Pressure, tension, gravity
- Characteristics:
  - Wave speed
  - Motion of medium
  - Direction of propagation
- Dispersion relation – very important







Simple wave representation

- For plane waves propagating with wave vector $\boldsymbol{k} = (k_x, k_y, k_z)$ and angular frequency ω, [where $\boldsymbol{r} = (x, y, z)$ is the position vector]

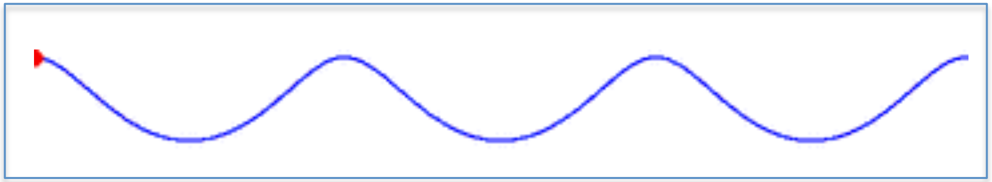
$$U = C_U \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$$

- And for propagation in only the x-direction

$$U = C_U \exp[i(kx - \omega t)]$$

- Constant phase is maintained for a point on the wave when,

$$\frac{d}{dt}(kx - \omega t) = 0$$



$$\frac{d(kx)}{dt} - \omega = 0$$

$$\boxed{\frac{dx}{dt} = \frac{\omega}{k} = v_p}$$

Phase speed

Wave group speed

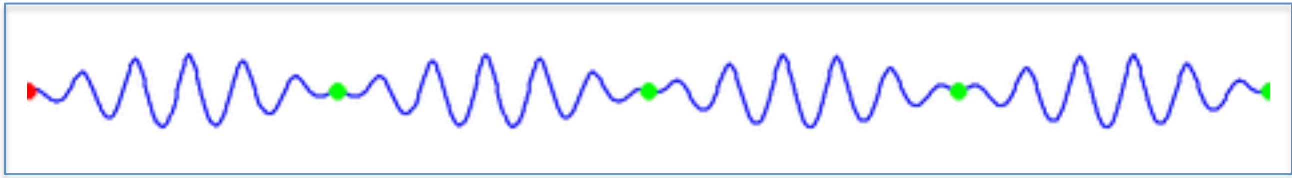
- Phase speed is not the rate of information (i.e., energy) transfer
- Group speed is similarly defined, but for constant phase on a modulated wave envelope,

$$U \propto \exp[i(\Delta k x - \Delta \omega t)]$$

- Giving,

$$\frac{d}{dt}(\Delta k x - \Delta \omega t) = 0$$

$$\frac{dx}{dt} = \frac{\Delta \omega}{\Delta k}$$



$$\lim_{\Delta \omega \to 0} \left(\frac{\Delta \omega}{\Delta k} \right) = \boxed{\frac{d\omega}{dk} = v_g}$$

group speed

Wave dispersion relation

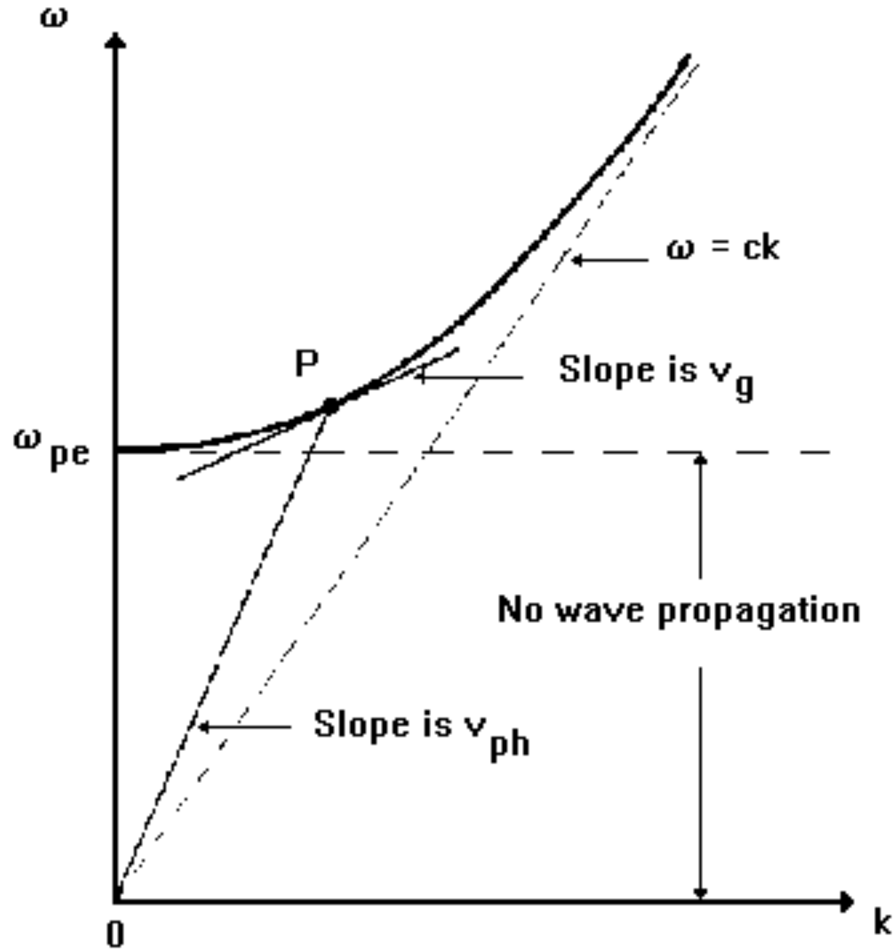
- Everything is contained in *dispersion relation*,

$$\omega = \omega(k)$$

- k often complex, but wave propagate only for,

$$\mathcal{R}(\omega(k)) \neq 0$$

- Dispersion relation indicates cutoffs and resonances



What makes plasma waves

Plasma properties

- Gas-like → Fluid equations
- charged → Electromagnetic equations
- Magnetic field → Electromagnetic equations

- Fluid equations
  - mass continuity
  - equation of motion
  - energy equation
  - ideal equation of state

- Electromagnetic equations
  - Maxwell's equations
  - induction equation
  - Ohm's law

Fluid equation

- Mass continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$
- Using vector identity, it expands as:

$$\boxed{\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho + \rho \nabla \cdot \boldsymbol{v} = 0}$$

- Consider gravity force ($\rho \boldsymbol{g}$), equation of Motion:

$$\boxed{\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} + \rho \boldsymbol{g}}$$

- Ideal Equation of state: $$\boxed{p = \frac{R}{\mu}\rho T}$$

$R = k_B/m_i$

μ is mean atomic weight

- Energy (entropy) equation: $$\boxed{\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = -L}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$$

Convective time derivative

- L represents all energy losses. Only consider adiabatic case (L=0)

Modified energy equation

Apply change rule

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{1}{\rho^\gamma}\frac{Dp}{Dt} - \frac{\gamma p}{\rho^{\gamma+1}}\frac{D\rho}{Dt} = 0$$

Expand D/Dt

$$\frac{1}{\rho^\gamma}\frac{\partial p}{\partial t} + \frac{1}{\rho^\gamma}(\boldsymbol{v}\cdot\nabla)p - \frac{\gamma p}{\rho^{\gamma+1}}\frac{\partial \rho}{\partial t} - \frac{\gamma p}{\rho^{\gamma+1}}(\boldsymbol{v}\cdot\nabla)\rho = 0$$

$$\frac{\partial p}{\partial t} + (\boldsymbol{v}\cdot\nabla)p - \frac{\gamma p}{\rho}\left[\frac{\partial \rho}{\partial t} + (\boldsymbol{v}\cdot\nabla)\rho\right] = 0$$

But $-\left[\frac{\partial \rho}{\partial t} + (\boldsymbol{v}\cdot\nabla)\rho\right] = \rho\nabla\cdot\boldsymbol{v}$ from mass continuity equation,

$$\frac{\partial p}{\partial t} + (\boldsymbol{v}\cdot\nabla)p + \frac{\gamma p}{\rho}(\rho\nabla\cdot\boldsymbol{v}) = 0$$

$$\boxed{\frac{\partial p}{\partial t} + (\boldsymbol{v}\cdot\nabla)p = -\gamma p\nabla\cdot\boldsymbol{v}}$$

Electromagnetic equation

Ampere's law: $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$

Solenoidal constraints: $\nabla \cdot \boldsymbol{B} = 0$

Faraday's law: $\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}$

Gauss's law: $\nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\epsilon_0}$

Ohm's law: $\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

Induction equation: $\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$

diffusivity term is ignored

Wave assumptions

- Wave amplitudes are small
  $\Rightarrow$ allows for linearization of MHD equations
- Basic state is a static equilibrium

Equation of motion: $$0 = -\nabla p_0 + \boldsymbol{J}_0 \times \boldsymbol{B}_0 + \rho_0 \boldsymbol{g} \quad \text{(A)}$$

Solenoidal constraints: $$\nabla \cdot \boldsymbol{B}_0 = 0 \quad \text{(B)}$$

Ideal equation of state: $$p_0 = \frac{R}{\mu} \rho_0 T_0 \quad \text{(C)}$$

- Quantities X_0 and $\boldsymbol{X}_0$ are the initial equilibrium state
- Not necessary to static

Wave perturbations

- After wave initiation,

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{r}, t) \\ \boldsymbol{v} &= \boldsymbol{v}_0 + \boldsymbol{v}_1(\boldsymbol{r}, t) \\ \rho &= \rho_0 + \rho_1(\boldsymbol{r}, t) \\ p &= p_0 + p_1(\boldsymbol{r}, t) \\ T &= T_0 + T_1(\boldsymbol{r}, t) \end{aligned}$$

  – $\boldsymbol{X}$ and X are perturbed quantities
  – $\boldsymbol{X}_1$ and X_1 are applied perturbation ($\ll \boldsymbol{X}_0$ and X_0 quantities)

- Static initial condition:
  – $\boldsymbol{v}_0 = 0$, $\boldsymbol{v} = \boldsymbol{v}_1(\boldsymbol{r}, t)$
  – Initial quantities are time independent $\dfrac{\partial \boldsymbol{X}_0}{\partial t} = 0, \ \dfrac{\partial X_0}{\partial t} = 0,$

MHD linearization

- Put perturbed quantities into MHD equations and neglect products of small terms (i.e., $X_1 Y_1$)
- Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v}_1) + \nabla \cdot (\rho_1 \boldsymbol{v}_1) = 0$$

- But with $\frac{\partial \rho_0}{\partial t} = 0$ and dropping $X_1 Y_1$ terms,

$$\boxed{\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v}_1) = 0}$$

MHD linearization (cont.)

- Equation of motion:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \boldsymbol{J}\times\boldsymbol{B} + \rho\boldsymbol{g}$$

$$\rho_0\frac{\partial \boldsymbol{v}_1}{\partial t} + \rho_1\frac{\partial \boldsymbol{v}_1}{\partial t} + \rho_0(\boldsymbol{v}_1\cdot\nabla)\boldsymbol{v}_1 + \rho_1(\boldsymbol{v}_1\cdot\nabla)\boldsymbol{v}_1 =$$
$$-\nabla p_0 - \nabla p_1 + \boldsymbol{J}_0\times\boldsymbol{B}_0 + \boldsymbol{J}_0\times\boldsymbol{B}_1 + \boldsymbol{J}_1\times\boldsymbol{B}_0 + \boldsymbol{J}_1\times\boldsymbol{B}_1 + \rho_0\boldsymbol{g} + \rho_1\boldsymbol{g}$$

- Neglecting X_1Y_1 terms and substituting for $\boldsymbol{J}$,

$$\rho_0\frac{\partial \boldsymbol{v}_1}{\partial t} = -\nabla p_1 + \frac{\nabla\times\boldsymbol{B}_0}{\mu_0}\times\boldsymbol{B}_1 + \frac{\nabla\times\boldsymbol{B}_1}{\mu_0}\times\boldsymbol{B}_0 + \rho_1\boldsymbol{g} + (-\nabla p_0 + \boldsymbol{J}_0\times\boldsymbol{B}_0 + \rho_0\boldsymbol{g})$$

- But, $-\nabla p_0 + \boldsymbol{J}_0\times\boldsymbol{B}_0 + \rho_0\boldsymbol{g} = 0$

$$\rho_0\frac{\partial \boldsymbol{v}_1}{\partial t} = -\nabla p_1 + \frac{\nabla\times\boldsymbol{B}_0}{\mu_0}\times\boldsymbol{B}_1 + \frac{\nabla\times\boldsymbol{B}_1}{\mu_0}\times\boldsymbol{B}_0 + \rho_1\boldsymbol{g}$$

MHD linearization (cont.)

- Adiabatic energy equation: $\dfrac{\partial p}{\partial t} + (\boldsymbol{v} \cdot \nabla)p = -\gamma p \nabla \cdot \boldsymbol{v}$

$$\frac{\partial p_0}{\partial t} + \frac{\partial p_1}{\partial t} + (\boldsymbol{v}_1 \cdot \nabla)p_0 + (\boldsymbol{v}_1 \cdot \nabla)p_1 = -\gamma p_0 \nabla \cdot \boldsymbol{v}_1 - \gamma p_1 \nabla \cdot \boldsymbol{v}_1$$

- But $\dfrac{\partial p_0}{\partial t} = 0$ and dropping $X_1 Y_1$ terms,

$$\boxed{\frac{\partial p_1}{\partial t} + (\boldsymbol{v}_1 \cdot \nabla)p_0 = -\gamma p_0 \nabla \cdot \boldsymbol{v}_1}$$

- Induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

$$\frac{\partial \boldsymbol{B}_0}{\partial t} + \frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) + \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_1)$$

- But $\dfrac{\partial \boldsymbol{B}_0}{\partial t} = 0$ and dropping $X_1 Y_1$ terms,

$$\boxed{\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0)}$$

MHD linearization (cont.)

- Ideal equation of state: $p = \frac{R}{\mu}\rho T$

$$p_0 + p_1 = \frac{R}{\mu}\rho_0 T_0 + \frac{R}{\mu}\rho_1 T_0 + \frac{R}{\mu}\rho_0 T_1 + \frac{R}{\mu}\rho_1 T_1$$

- But $p_0 = \frac{R}{\mu}\rho_0 T_0$ and dropping $X_1 Y_1$ terms,

$$\boxed{p_1 = \frac{R}{\mu}\rho_1 T_0 + \frac{R}{\mu}\rho_0 T_1}$$

- Solenoidal constraints: $\nabla \cdot \boldsymbol{B} = 0$

$$\nabla \cdot \boldsymbol{B}_0 + \nabla \cdot \boldsymbol{B}_1 = 0$$

- But with $\nabla \cdot \boldsymbol{B}_0 = 0$

$$\boxed{\nabla \cdot \boldsymbol{B}_1 = 0}$$

Summary of linearized MHD equations

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v}_1) = 0 \qquad (5.1)$$

$$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} = -\nabla p_1 + \frac{\nabla \times \boldsymbol{B}_0}{\mu_0} \times \boldsymbol{B}_1 + \frac{\nabla \times \boldsymbol{B}_1}{\mu_0} \times \boldsymbol{B}_0 + \rho_1 \boldsymbol{g} \qquad (5.2)$$

$$\frac{\partial p_1}{\partial t} + (\boldsymbol{v}_1 \cdot \nabla) p_0 = -\gamma p_0 \nabla \cdot \boldsymbol{v}_1 \qquad (5.3)$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) \qquad (5.4)$$

$$p_1 = \frac{R}{\mu} \rho_1 T_0 + \frac{R}{\mu} \rho_0 T_1 \qquad (5.5)$$

$$\nabla \cdot \boldsymbol{B}_1 = 0 \qquad (5.6)$$

Simple wave solutions

- Looking for plane waves of form,

$$U = C_U \exp\left[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right]$$

- with angular frequency ω, wave vector $\boldsymbol{k} = (k_x, k_y, k_z)$, with position vector $\boldsymbol{r}$ = (x, y, z). Note, $k = 2\pi/\lambda$.

$$\boldsymbol{k}\cdot\boldsymbol{r} = k_x x + k_y y + k_z z$$

$$\boldsymbol{k}\cdot\boldsymbol{k} = k^2 = k_x^2 + k_y^2 + k_z^2$$

- Useful solutions for Fourier analysis since,

$$\frac{\partial}{\partial t} \rightarrow -i\omega, \quad \frac{\partial^2}{\partial t^2} \rightarrow -\omega^2, \quad \frac{\partial}{\partial x} \rightarrow -ik_x, \quad \frac{\partial^2}{\partial x^2} \rightarrow -k_x^2$$

$$\nabla \rightarrow i\boldsymbol{k}, \quad \nabla\cdot \rightarrow i\boldsymbol{k}\cdot, \quad \nabla\times \rightarrow i\boldsymbol{k}\times$$

Acoustic (pressure) wave equations

- Ignore magnetic field and gravity (i.e., $\boldsymbol{B} = \boldsymbol{g} = 0$)
- Assume homogeneous medium
- From equilibrium (A), $\nabla p_0 = 0$ and $p_0 = const.$
- From simplicity, $\rho_0 = const.$

- Linearized equations reduce to:

(5.1) $$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{v}_1 = 0 \quad \rightarrow \quad -i\omega\rho_1 + i\rho_0(\boldsymbol{k} \cdot \boldsymbol{v}_1) = 0$$ (5.7)

(5.2) $$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} = -\nabla p_1 \quad \rightarrow \quad -i\omega\rho_0\boldsymbol{v}_1 = -i\boldsymbol{k}p_1$$ (5.8)

(5.3) $$\frac{\partial p_1}{\partial t} = -\gamma p_0 \nabla \cdot \boldsymbol{v}_1 \quad \rightarrow \quad -i\omega p_1 = -i\gamma p_0(\boldsymbol{k} \cdot \boldsymbol{v}_1)$$ (5.9)

Acoustic wave properties

- From (5.8)

$$\boldsymbol{v}_1 = \left(\frac{p_1}{\omega \rho_0}\right) \boldsymbol{k} \qquad (5.10)$$

  - $\boldsymbol{v}_1$ parallel to $\boldsymbol{k}$
  - particle motion along propagation direction (longitudinal)

- Also
- From (5.7)

$$\frac{\rho_1}{\rho_0} = \frac{(\boldsymbol{k} \cdot \boldsymbol{v}_1)}{\omega} \qquad (5.11)$$

- From (5.9)

$$p_1 = \gamma p_0 \frac{(\boldsymbol{k} \cdot \boldsymbol{v}_1)}{\omega} = \frac{\gamma p_0 \rho_1}{\rho_0}$$

- Defining the sound speed

$$\boxed{c_s^2 = \frac{\gamma p_0}{\rho_0}}$$

$$p_1 = c_s^2 \rho_1 \qquad (5.12)$$

  - for $\boldsymbol{k} \cdot \boldsymbol{v}_1 \neq 0$, then ρ_1 and $p_1 \neq 0$ (compressive)

Acoustic dispersion relation

- Taking scalar product with $\boldsymbol{k}$ to eq(5.10),

$$\boldsymbol{k}\cdot\boldsymbol{v}_1 = \left(\frac{p_1}{\omega\rho_0}\right)\boldsymbol{k}\cdot\boldsymbol{k} = \left(\frac{p_1}{\omega\rho_0}\right)k^2$$

- Rearranging eq(5.11) and (5.12),

$$(\boldsymbol{k}\cdot\boldsymbol{v}_1) = \frac{\omega\rho_1}{\rho_0}$$

$$\rho_1 = \frac{p_1}{c_s^2}$$

- Substitute, $(\boldsymbol{k}\cdot\boldsymbol{v}_1) = \dfrac{\omega p_1}{c_s^2\rho_0}$
- Equating $\boldsymbol{k}\cdot\boldsymbol{v}_1$,

$$\left(\frac{p_1}{\omega\rho_0}\right)k^2 = \frac{\omega p_1}{c_s^2\rho_0}$$

$$\boxed{\omega^2 = k^2 c_s^2} \qquad (5.13)$$ *Dispersion relation*

Acoustic phase and group speeds

- *Phase speed*:
  - From eq (5.13): $\dfrac{\omega}{k} = \pm c_s$

$$\boldsymbol{v}_p = v_p \boldsymbol{k}' = \pm c_s \boldsymbol{k}'$$

- *Group velocity*:

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z} \right)$$

  - From eq (5.13): $\omega^2 = c_s^2 (k_x^2 + k_y^2 + k_z^2)$
  - Differentiating, $2\omega \dfrac{\partial \omega}{\partial \boldsymbol{k}} = c_s^2 (2k_x, 2k_y, 2k_z)$

$$\frac{\partial \omega}{\partial \boldsymbol{k}} = \frac{c_s^2}{\omega} (k_x, k_y, k_z)$$

$$\boldsymbol{v}_g = c_s^2 \frac{k}{\omega} \boldsymbol{k}' = \pm c_s \boldsymbol{k}'$$

Acoustic wave complications

- Consider hydrostatic equilibrium, $\nabla p_0 = -\rho_0 \boldsymbol{g}$

$$p_0(z) = p_0(0)\exp\left(-z/H\right)$$
$$\rho_0(z) = \rho_0(0)\exp\left(-z/H\right)$$

- Where *H* is the pressure scale height,

$$H = \frac{p_0}{\rho_0 g} = \frac{RT}{g}$$

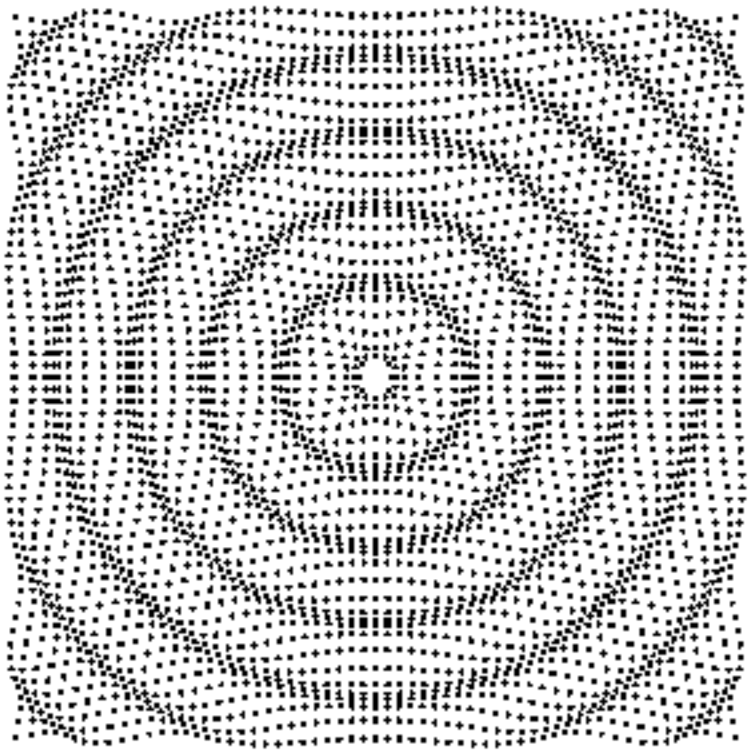
- Pressure variations follow

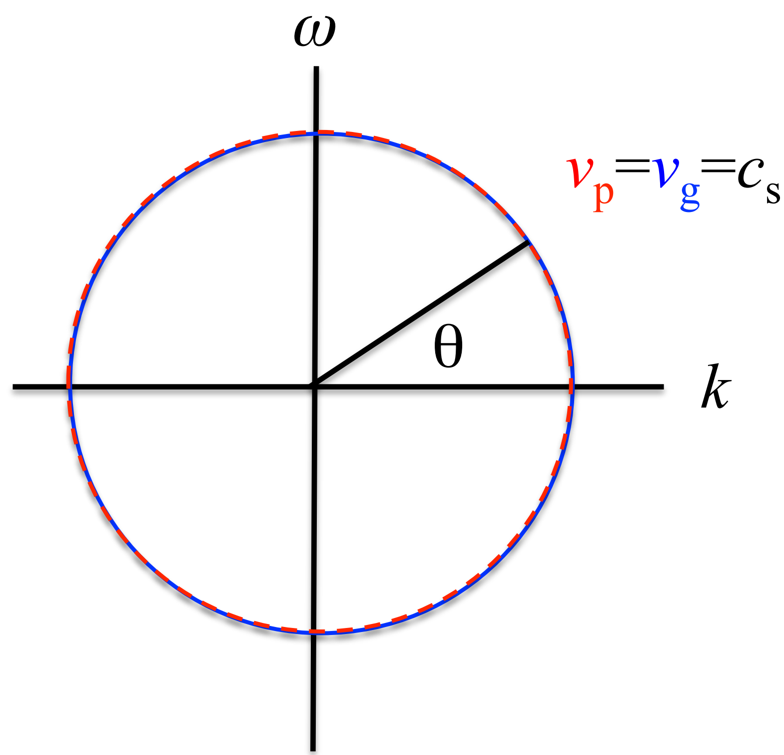
$$\frac{\partial^2 Q}{\partial t^2} - c_s^2(z)\frac{\partial^2 Q}{\partial z^2} + \Omega_s^2(z)Q = 0$$
$$\omega^2 = k_z^2 c_s^2 + \Omega_s^2$$

- Real solutions (propagation) for $k_z > 0$

$$\omega > \Omega_s = \omega_{ac} = \frac{c_s}{2H}$$

Acoustic wave summary

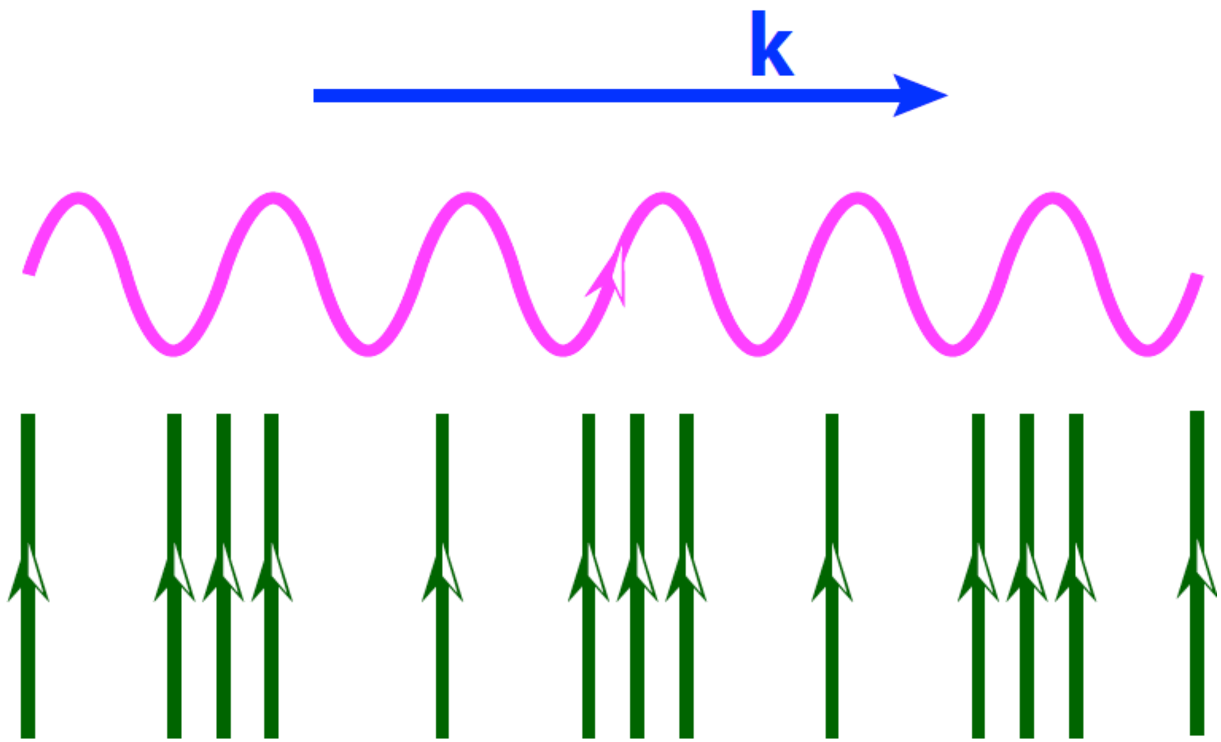




- Restoring force: pressure
- Directionality: isotropic

- Phase speed: c_s
- Group speed: c_s

Waves in magnetic field



- There are two type of propagating waves in magnetic field
- Because magnetic field has two forces, magnetic tension and magnetic pressure.
- Both forces are coming ***J*** x ***B*** force

Alfven wave equations

- Ignore pressure and gravity (i.e., $p_0 = \boldsymbol{g} = 0$)
  - From equilibrium (A), $0 = \mu_0(\boldsymbol{J}_0 \times \boldsymbol{B}_0) = (\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_0$
  - Assume no pressure variations, $p_1 = \rho_1 = 0$
  - Assume uniform equilibrium field distribution, $\boldsymbol{B}_0 = B_0\hat{z}$

- Linearized equations reduce to:

(5.1) $$\nabla \cdot \boldsymbol{v}_1 = 0 \;\rightarrow\; i(\boldsymbol{k} \cdot \boldsymbol{v}_1) = 0$$ (5.14)

(5.2) $$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} = \frac{(\nabla \times \boldsymbol{B}_1)}{\mu_0} \times \boldsymbol{B}_0 \;\rightarrow\; i\omega\rho_0\boldsymbol{v}_1 = \frac{(i\boldsymbol{k} \times \boldsymbol{B}_1)}{\mu_0} \times \boldsymbol{B}_0$$ (5.15)

(5.4) $$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) \;\rightarrow\; -i\omega\boldsymbol{B}_1 = i\boldsymbol{k} \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0)$$ (5.16)

(5.6) $$\nabla \cdot \boldsymbol{B}_1 \;\rightarrow\; i\boldsymbol{k} \cdot \boldsymbol{B}_1 = 0$$ (5.17)

(shear) Alfven wave properties

- From eq (5.1), $\nabla \cdot \boldsymbol{v}_1 = 0$
- no divergent/convergent motions (incompressible)
- From eq (5.14), $\boldsymbol{k} \cdot \boldsymbol{v}_1 \equiv k v_1 \cos\theta_{kv_1} = 0$

$$\theta_{kv_1} = 90^\circ$$

- $\boldsymbol{v}_1$ at right angles to $\boldsymbol{k}$ (transverse)
- Taking scalar product with $\boldsymbol{B}_0$,
- From eq (5.15),

$$-\omega\rho_0 \boldsymbol{v}_1 \cdot \boldsymbol{B}_0 = \frac{(\boldsymbol{k} \times \boldsymbol{B}_1)}{\mu_0} \times \boldsymbol{B}_0 \cdot \boldsymbol{B}_0 = 0 \qquad (\nabla \cdot \boldsymbol{B}_0 = 0)$$

$$\boldsymbol{v}_1 \cdot \boldsymbol{B}_0 \equiv v_1 B_0 \cos\theta_{v_1 B} = 0$$

$$\theta_{v_1 B} = 90^\circ \qquad (5.18)$$

- $\boldsymbol{v}_1$ at right angles to $\boldsymbol{B}_0$ (perpendicular)

(Shear) Alfven wave properties (cont.)

- Expand eq (5.16) using standard vector identity,

$$\begin{aligned} -\omega \boldsymbol{B}_1 &= \boldsymbol{k} \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) \\ &= (\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{v}_1 - (\boldsymbol{k} \cdot \boldsymbol{v}_1)\boldsymbol{B}_0 \end{aligned}$$

- But $(\boldsymbol{k} \cdot \boldsymbol{v}_1) = 0$ from eq (5.14),

$$-\omega \boldsymbol{B}_1 = (\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{v}_1 \qquad (5.19)$$

- Taking scalar product with $\boldsymbol{B}_0$,

$$-\omega \boldsymbol{B}_1 \cdot \boldsymbol{B}_0 = (\boldsymbol{k} \cdot \boldsymbol{B}_0)(\boldsymbol{v}_1 \cdot \boldsymbol{B}_0)$$

- But $(\boldsymbol{v}_1 \cdot \boldsymbol{B}_0) = 0$ from eq (5.18),

$$\boldsymbol{B}_1 \cdot \boldsymbol{B}_0 \equiv B_0 B_1 \cos\theta_{B_0 B_1} = 0 \qquad (5.20)$$

$$\cos\theta_{B_0 B_1} = 90^\circ$$

- $\boldsymbol{B}_1$ at right angles to $\boldsymbol{B}_0$ (perpendicular)

(Shear) Alfven dispersion relation

- Multiply eq (5.16) by ω and substitute for $\boldsymbol{v}_1$ from eq (5.15),

$$\omega^2 \boldsymbol{B}_1 = \frac{1}{\mu_0 \rho_0} \boldsymbol{k} \times \{[(\boldsymbol{k} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0] \times \boldsymbol{B}_0\} \qquad (5.21)$$

- Expanding inner triple vector product,

$$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = (\boldsymbol{C} \cdot \boldsymbol{A})\boldsymbol{B} - (\boldsymbol{C} \cdot \boldsymbol{B})\boldsymbol{A}$$
$$(\boldsymbol{k} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 = (\boldsymbol{B}_0 \cdot \boldsymbol{k})\boldsymbol{B}_1 - (\boldsymbol{B}_0 \cdot \boldsymbol{B}_1)\boldsymbol{k}$$

- But $(\boldsymbol{B}_0 \cdot \boldsymbol{B}_1) = 0$ from eq (5.20),

$$\begin{aligned} \boldsymbol{k} \times \{[(\boldsymbol{k} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0] \times \boldsymbol{B}_0\} &= \boldsymbol{k} \times \{[(\boldsymbol{B}_0 \cdot \boldsymbol{k})\boldsymbol{B}_1] \times \boldsymbol{B}_0\} \\ &= (\boldsymbol{k} \cdot \boldsymbol{B}_0)[(\boldsymbol{B}_0 \cdot \boldsymbol{k})\boldsymbol{B}_1] - (\boldsymbol{k} \cdot [(\boldsymbol{B}_0 \cdot \boldsymbol{k})\boldsymbol{B}_1])\boldsymbol{B}_0 \\ &= (\boldsymbol{k} \cdot \boldsymbol{B}_0)^2 \boldsymbol{B}_1 - (\boldsymbol{k} \cdot \boldsymbol{B}_1)(\boldsymbol{B}_0 \cdot \boldsymbol{k})\boldsymbol{B}_0 \end{aligned}$$

- And $(\boldsymbol{k} \cdot \boldsymbol{B}_1) = 0$ from eq (5.17),

$$\boldsymbol{k} \times \{[(\boldsymbol{k} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0] \times \boldsymbol{B}_0\} = (\boldsymbol{k} \cdot \boldsymbol{B}_0)^2 \boldsymbol{B}_1$$

(Shear) Alfven dispersion relation (cont.)

- From eq (5.21), $\omega^2 \boldsymbol{B}_1 = \dfrac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2}{\mu_0 \rho_0} \boldsymbol{B}_1$

$$\omega^2 = \frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2}{\mu_0 \rho_0} \qquad (5.22)$$

- Recall that $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{z}}$ and $(\boldsymbol{k} \cdot \hat{\boldsymbol{z}}) = k_z = k \cos \theta_{kB_0}$,

$$\omega^2 = \frac{(\boldsymbol{k} \cdot \hat{\boldsymbol{z}})^2 B_0^2}{\mu_0 \rho_0} = \frac{(k \cos \theta_{kB_0})^2 B_0^2}{\mu_0 \rho_0}$$

- Defining *the Alfven speed*,

$$\boxed{v_A^2 = \frac{B_0^2}{\mu_0 \rho_0}}$$

- Dispersion relation is $\boxed{\omega^2 = (k \cos \theta_{kB_0})^2 v_A^2}$ (5.23)

(Shear) Alfven phase and group speeds

- Shear Alfven waves are anisotropic
  - $(\boldsymbol{k} \cdot \boldsymbol{B}_0)$ term in eq (5.22), the generalized dispersion relation
- Phase speed:

From eq (5.23)

$$\frac{\omega}{k} = \pm v_A \cos\theta_{kB_0} = v_p$$

- Group velocity:

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z} \right)$$

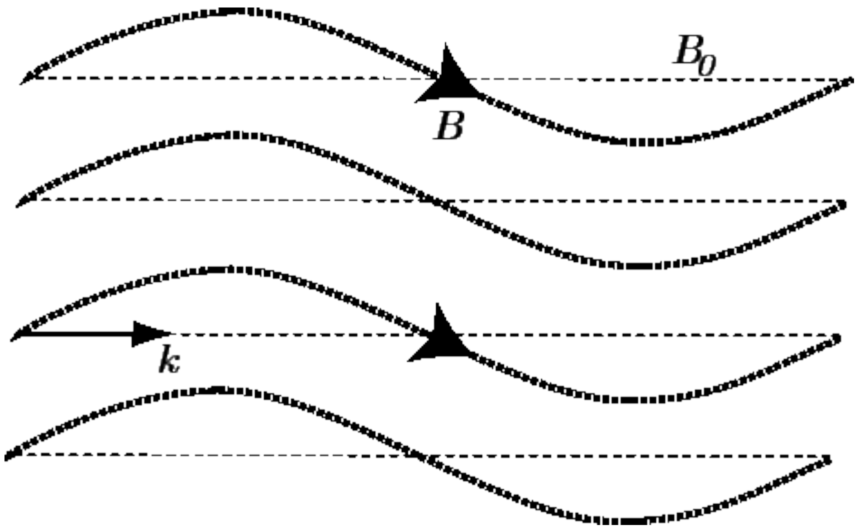
From eq (5.23)

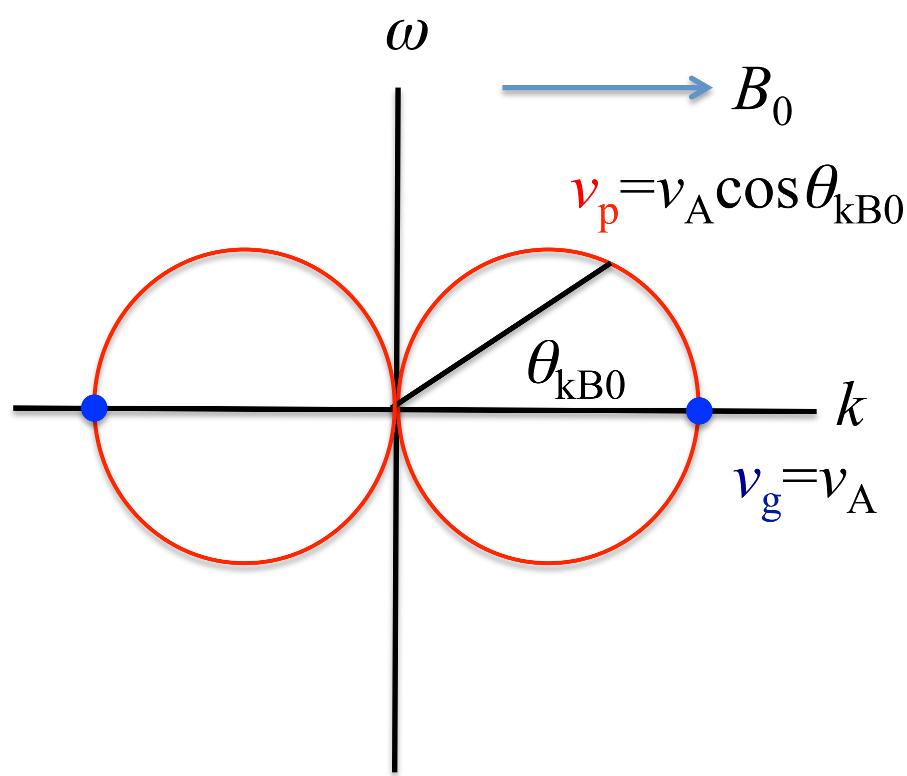
$$\begin{aligned} \omega &= \pm v_A k \cos\theta_{kB_0} \\ &= \pm v_A k_z \end{aligned}$$

- Differentiating:

$$\frac{\partial \omega}{\partial \boldsymbol{k}} = \pm v_A \hat{\boldsymbol{z}} = v_g$$

(shear) Alfven wave summary



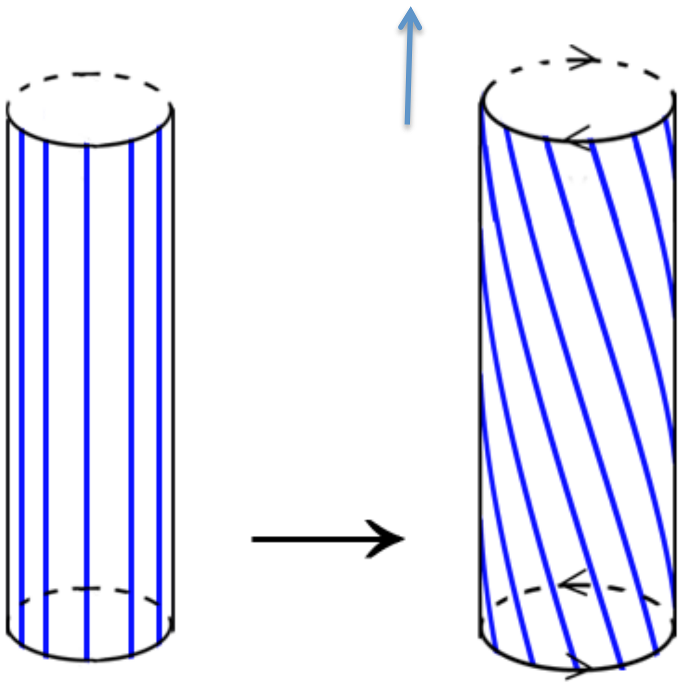


- Restoring force: B-field tension
- Directionality: anisotropic

- Phase speed: $v_A \cos\theta_{kB0}$
- Group speed: v_A

Torsional Alfven wave

- In cylindrically symmetric geometry with an axial field (B_z), there exist waves which posses only azimuthal component
- Such wave as known as torsional Alfven wave
- Torsional Alfven wave propagates with $v_p = v_A$ along axial magnetic field



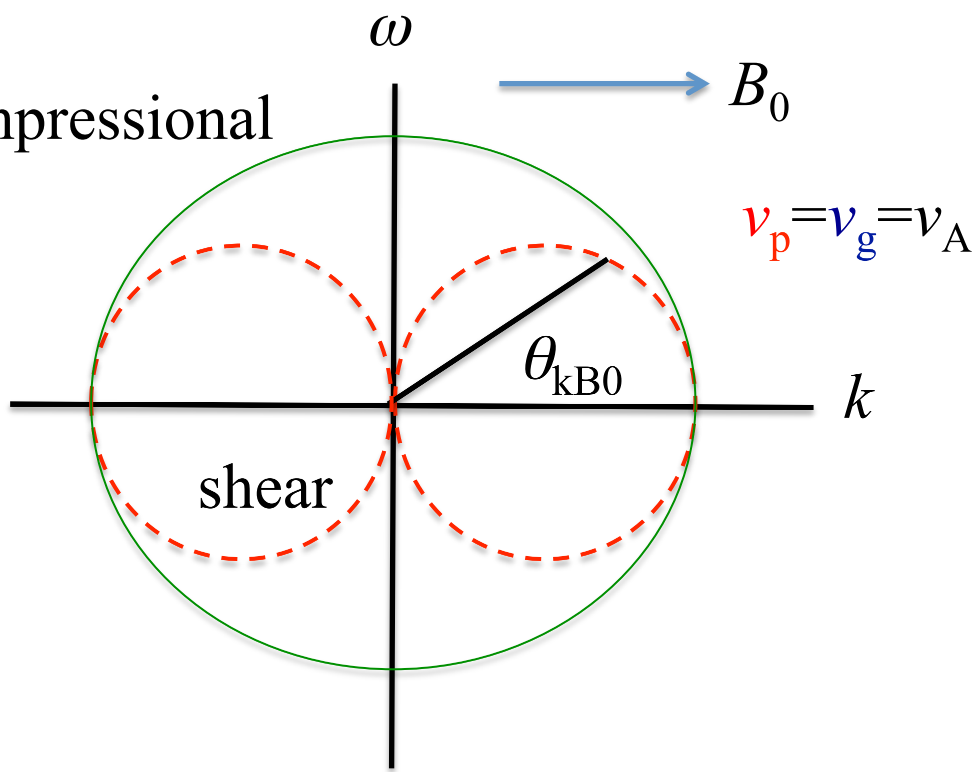
Compressional Alfven wave

- In shear Alfven wave, we assume incompressible ($\nabla \cdot \boldsymbol{v}_1 = 0$).
- If we consider compression (by magnetic pressure), we obtain another solution of Alfven wave. This is called *compressional Alfven wave*
- Dispersion relation is

$$\omega = k v_A$$

- The phase velocity and group velocity is $v_p = v_g = v_A$
- Compressional Alfven wave is isotropic
- If $\theta_{kB_0} = \pi/2$ (perpendicular direction against $\boldsymbol{B}_0$) , $\boldsymbol{v}_1 \parallel \boldsymbol{k}$. So it is compression wave
- If $\theta_{kB_0} = 0$ (parallel direction to $\boldsymbol{B}_0$), compressional wave is matched with shear Alfven wave (not compressional)

Compressional Alfven wave summary

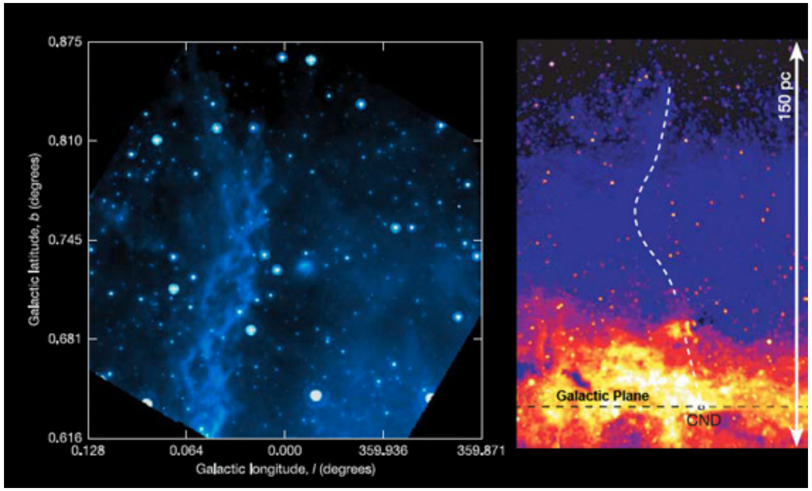


- Restoring force: B-field tension & magnetic pressure
- Directionality: isotropic

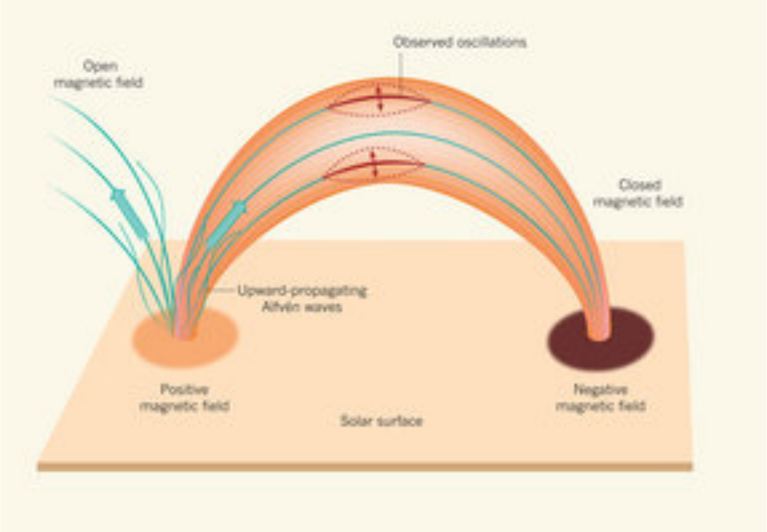
Alfven wave example

Alfven wave in solar corona (Hinode, Ca II H spectral line)

Double helix nebula in the galaxy (IR)



Movie here



Magnetoacoustic wave equation

- Ignore gravity (i.e., $\boldsymbol{g} = 0$), consider compressible (gas pressure & magnetic pressure)
  - assume uniform equilibrium field distribution, $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$

- Linearized equations reduce to, $\hat{\boldsymbol{B}}_0 \equiv B_0/\boldsymbol{B}_0$

$$\frac{\omega^2 \boldsymbol{v}_1}{v_A^2} = k^2 cos^2(\theta_{kB_0})\boldsymbol{v}_1 - (\boldsymbol{k}\cdot\boldsymbol{v}_1)k\cos(\theta_{kB_0})\hat{\boldsymbol{B}}_0 + \left[\left(1+\frac{c_s^2}{v_A^2}\right)(\boldsymbol{k}\cdot\boldsymbol{v}_1) - k\cos(\theta_{kB_0})(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{v}_1)\right]\boldsymbol{k} \quad (5.24)$$

- with resulting dispersion relation,

$$\omega^4 - \omega^2 k^2 (c_s^2 + v_A^2) + c_s^2 v_A^2 k^4 \cos^2\theta_{kB_0} = 0$$

Derivation of dispersion relation for magnetoacoustic wave

- First Eq(5.24) * $\boldsymbol{k}$ & Eq (5.24) * $\boldsymbol{B}_0$ (dot product)

- From these two equations, deleted $(\boldsymbol{v}_1 \cdot \boldsymbol{k})$ & $(\boldsymbol{v}_1 \cdot \hat{\boldsymbol{B}}_0)$

Magnetoacoustic wave properties

- *Phase velocities*:

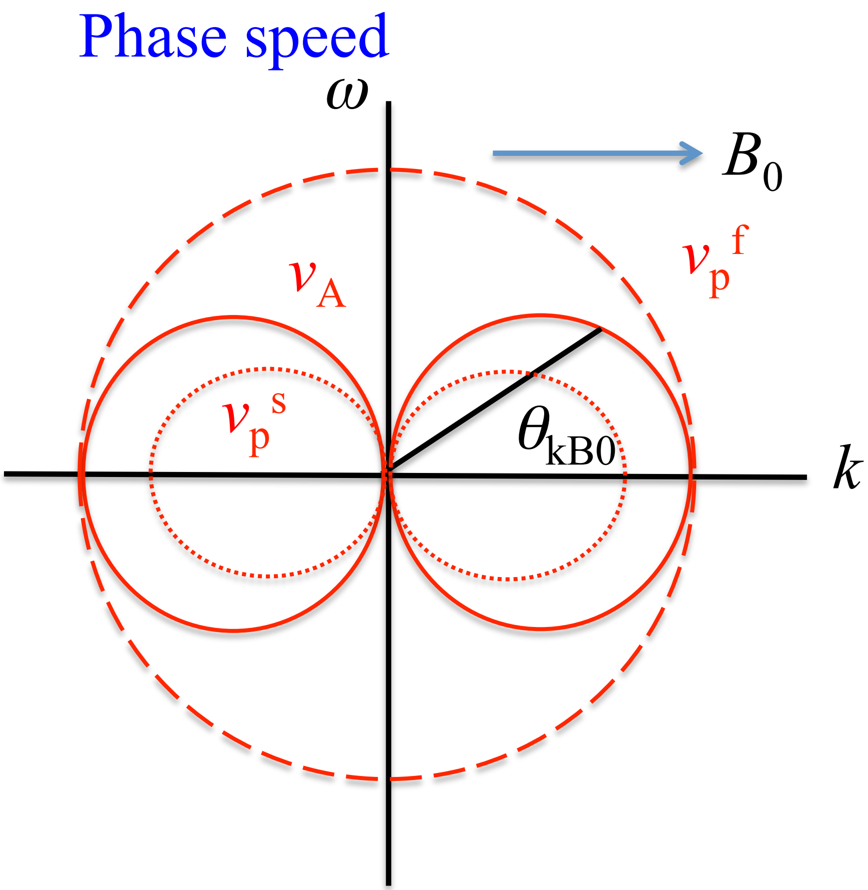
$$\frac{\omega^2}{k^2} = v_f^2 = \frac{(c_s^2 + v_A^2) + \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2(\theta_{kB_0})}}{2}$$

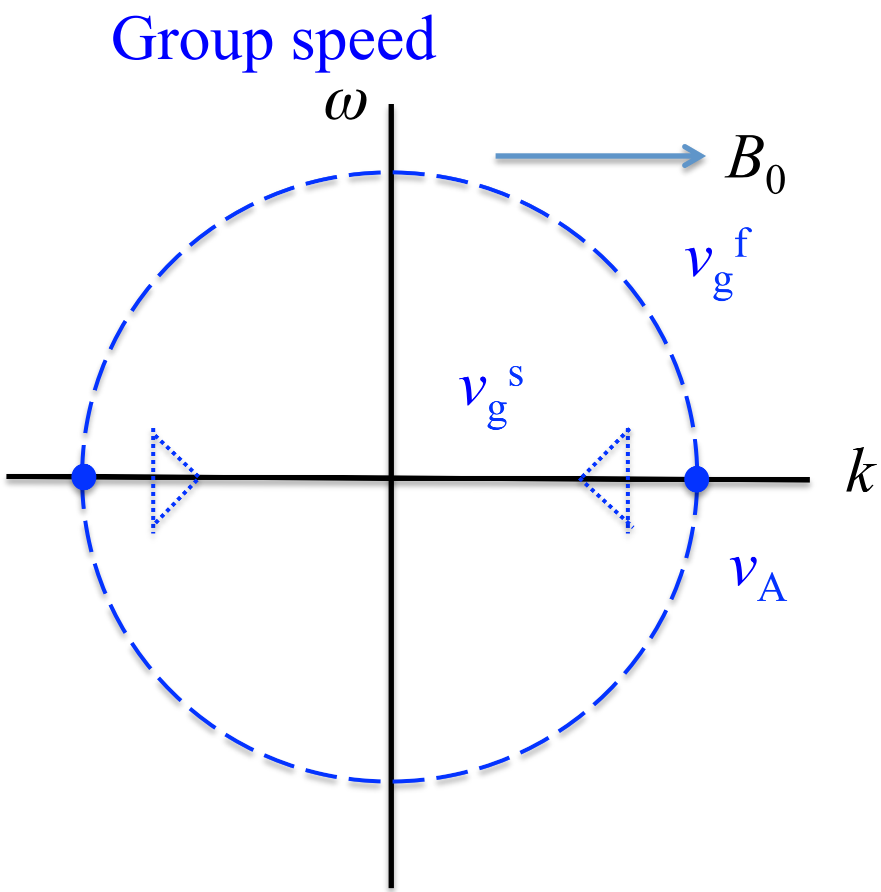
$$v_s^2 = \frac{(c_s^2 + v_A^2) - \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2(\theta_{kB_0})}}{2}$$

Wave Mode	Propagation	Low-beta	High-beta
Alfven	Along B_0	Magnetic tension	
Fast	isotropic	Magnetic pressure	Gas pressure
Slow	Roughly along B_0	Gas pressure	Magnetic tension

Magnetoacoustic wave phase and group speeds

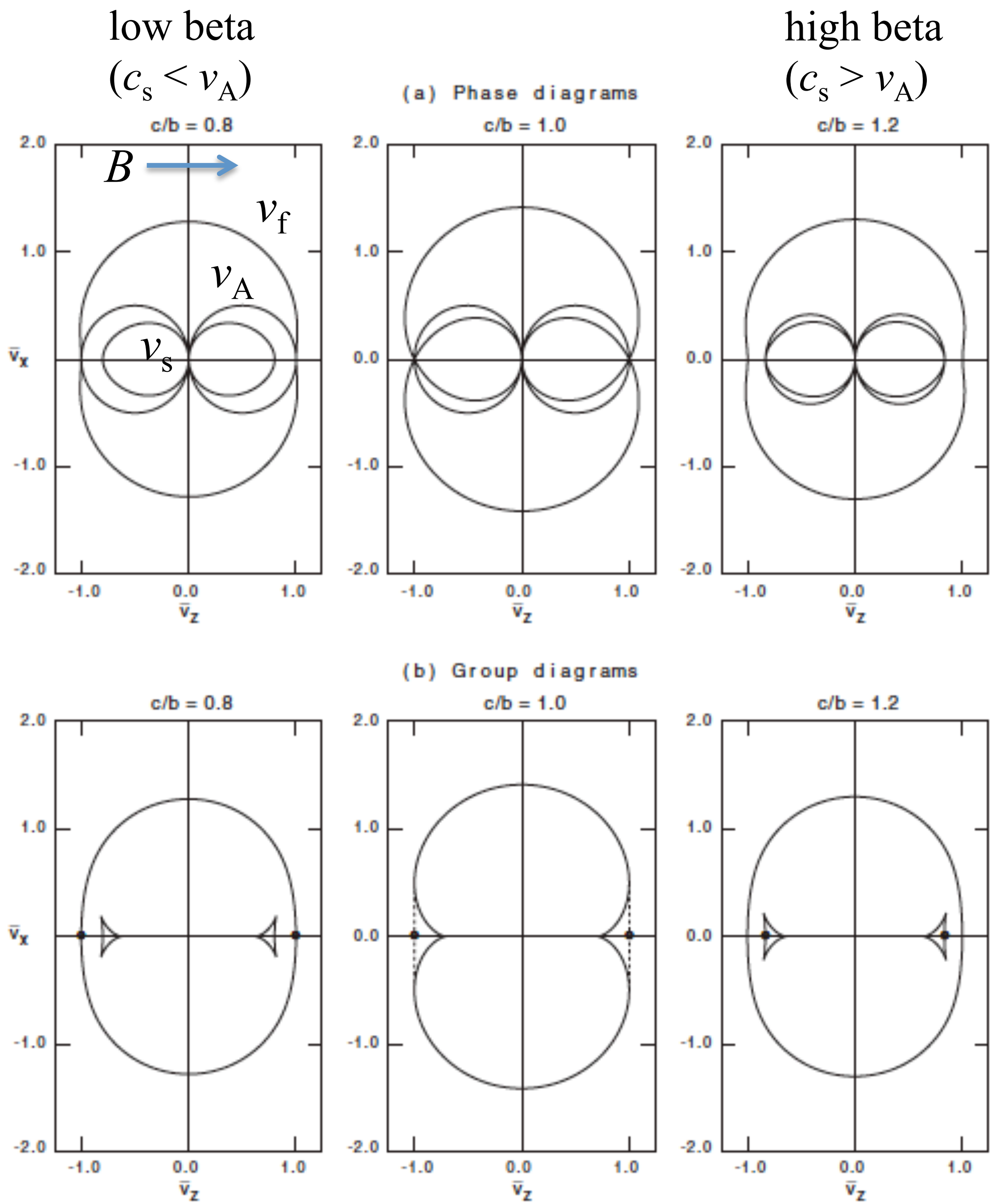
For low-beta case ($c_s < v_A$)





Magnetoacoustic wave phase and group speeds (cont.)

- Low beta case:
  - Fast mode propagates at Alfven speed
  - Slow mode ~ 1D sound wave guided by field
- High beta case:
  - Fast mode behaves like sound wave (restoring force is magnetic pressure)
  - Slow mode propagates at Alfven speed



Summary 1

- Acoustic waves
  - particle motion along *k* direction (longitudinal)
  - phase and group speeds are c_s in all directions (isotropic)

- (Shear) Alfven waves
  - particle motion at right angles to *k* direction (transverse)
  - *B* perturbation at right angles to *k* direction (perpendicular)
  - phase speed varies as $v_A \cos \theta_{kB0}$ (anisotropic)
  - group speed is v_A along *B* direction (anisotropic)

- Magnetoacoustic waves
- Alfven – as above
- Fast – gas and *B* pressure in phase, also isotropic
- Slow – gas and *B* pressure out of phase, also anisotropic

Sound speed and Alfven speed

- Typical velocity of Sound wave and Alfven wave in the universe

Sound wave

- When γ=5/3, m=0.5m_i, μ=0.5 (fully ionized hydrogen gas),

$$c_s \simeq 1.66 \times 10^4 T_0^{1/2} \ (\mathrm{cm/s})$$

$$p = nk_B T = \frac{\rho}{\mu m_p} T$$

- T_0~10^4 (stellar atmosphere) => c_s ~ 16 km/s
- T_0~10^8 (cluster of galaxies) => c_s ~ 1.6 x 10^3 km/s

Alfven wave

$$v_A = 2.8 \times 10^5 \left(\frac{B}{1\mu\mathrm{G}}\right) \left(\frac{n_0}{1\mathrm{cm}^{-3}}\right)^{-1/2} (\mathrm{cm/s})$$

Waves in gravitational field

- Next, we consider the wave propagating in the gravitational field.
- Such waves, we called *gravity wave* (not gravitational wave)

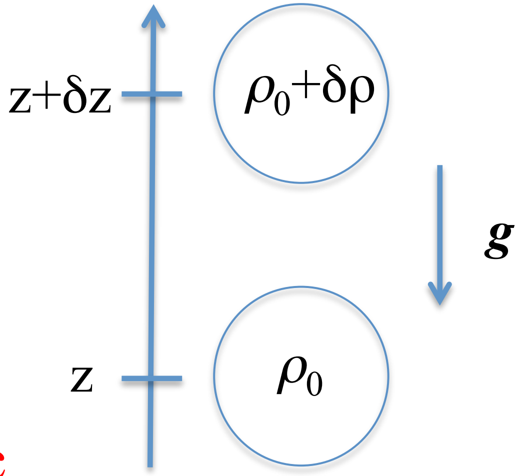
- Internal gravity wave
- Acoustic gravity wave

Internal gravity wave

- Consider a blob of plasma, which displaced vertically a distance δz from equilibrium
- Assumption:

  (1) Remains in pressure equilibrium with its surrounding

  (2) Density changes inside the blob are adiabatic



- At original height z, the blob are in equilibrium balance between pressure gradient and gravity

$$\frac{dp_0}{dz} = -\rho_0 g \qquad (5.25)$$

Internal gravity wave (cont.)

- Outside the blob the pressure and density at height $z+\delta z$ are $p_0+\delta p_0$ & $\rho_0+\delta\rho_0$, by eq (5.25),

$$\delta p_0 = -\rho_0 g \delta z, \quad \delta\rho_0 = \frac{d\rho_0}{dz}\delta z \qquad (5.26)$$

- Inside the blob the pressure and density at height $z+\delta z$ are $p_0+\delta p$ & $\rho_0+\delta\rho$, by assumption (1),

$$\delta p = \delta p_0 = -\rho_0 g \delta z \qquad (5.27)$$

- Assumption (2) means that, as the blob rises, its pressure and density obey $p/\rho^\gamma =$ const, So that $\delta p = c_s^2 \delta\rho$, from eq (5.27) internal density change as

$$\delta\rho = -\frac{\rho_0 g \delta z}{c_s^2} \qquad (5.28)$$

Internal gravity wave (cont.)

- Since the new density inside the blob differs from the ambient density at its new height, the blob experiences a buoyancy force
- From eq (5.26) & (5.28),

$$g(\delta\rho_0 - \delta\rho) = -N^2\rho_0\delta z \quad (5.29)$$

$$N^2 = -g\left(\frac{1}{\rho_0}\frac{d\rho_0}{dz} + \frac{g}{c_s^2}\right)$$

Brunt-Vaisala frequency

- An alternative expression is obtained by eq(5.25) & adiabatic EoS ($p_0=\rho_0 RT_0/\mu$, p_0/ρ_0^γ=const):

$$N^2 = -\frac{g}{T_0}\left[\frac{dT_0}{dz} + \left(\frac{dT}{dz}\right)_{ad}\right]$$

- where $\left(\frac{dT}{dz}\right)_{ad} = -(\gamma - 1)\frac{T_0 g}{c_s^2}$

Internal gravity wave (cont.)

- In general, N varies with height z but, in particular case when the equilibrium temperature (T_0) is uniform (no dependence on height),

$$N^2 = \frac{(\gamma - 1)g^2}{c_s^2}$$

- In the presence of a horizontal magnetic field, Brunt-Vaisala frequency is increased to

$$N^2 = -g\left(\frac{1}{\rho_0}\frac{d\rho_0}{dz} + \frac{g}{c_s^2 + v_A^2}\right)$$

- Or in case of uniform temperature

$$N^2 = \frac{g^2}{c_s^2}\left(\gamma - \frac{c_s^2}{c_s^2 + v_A^2}\right)$$

Internal gravity wave (cont.)

- If the only resultant force acting on the plasma blob is due to buoyancy, (eq. 5.29), the equation of motion becomes

$$\rho_0 \frac{d^2(\delta z)}{dt^2} = -N^2 \rho_0 \delta z \qquad (5.30)$$

- When $N^2 > 0$, this is simple harmonic motion with frequency $\omega = N$
- So that the temperature decreases with height more slowly than adiabatic (= isothermal)

$$-\frac{dT_0}{dz} < -\left(\frac{dT}{dz}\right)_{ad}$$

Schwarzschild criterion for convective stability

- If temperature decreases with height faster than adiabatic, the condition $N^2 > 0$ is violated => solution of eq(5.30) is exponentially growing (*convective instability*)
- The region of the solar interior where this is so is convection zone
- (Using entropy, we can also discuss this criterion)

Internal gravity wave (cont.)

- The simple harmonic motion leads to expect the existence of gravity waves when $N^2 > 0$ due to the tendency for plasma to oscillate slowly with frequency N
- Linearize equation:

$$\omega^2 \boldsymbol{v}_1 = c_s^2 \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{v}_1) + i(\gamma-1)g(\boldsymbol{k}\cdot\boldsymbol{v}_1)\hat{\boldsymbol{z}} + igkv_{1z}$$

- Taking scalar product with $\boldsymbol{k}$ and \hat{$\boldsymbol{z}$} in turn and gathering together terms in v_{1z} and $\boldsymbol{k}*\boldsymbol{v}_1$,

$$\begin{aligned} igk^2 v_{1z} &= (\boldsymbol{k}\cdot\boldsymbol{v}_1)\{\omega^2 - c_s^2k^2 - i(\gamma-1)gk_z\} \\ (\omega^2 - igk_z)v_{1z} &= (\boldsymbol{k}\cdot\boldsymbol{v}_1)\{c_s^2k_z + i(\gamma-1)g\} \end{aligned}$$

- Then an elimination of $(\boldsymbol{k}*\boldsymbol{v}_1)/v_{1z}$

$$(\omega^2 - igk_z)\{\omega^2 - c_s^2k^2 - i(\gamma-1)gk_z\} = igk^2\{c_s^2k_z + i(\gamma-1)g\} \tag{5.31}$$

Internal gravity wave (cont.)

- The object is to seek waves with a frequency of the order of Brunt-Vaisala frequency (*N*) and much slower than that of sound waves, so

$$\omega \approx g/c_s \ll kc_s$$

- The wavelength is much smaller than a scale-height,. Eq (5.31) reduces to

$$\omega^2 c_s^2 \approx (\gamma - 1)g^2(1 - k_z^2/k^2)$$

- $\theta_g = \cos^{-1}(k_z/k)$: the inclination between the propagation direction and z-axis
- The dispersion relation (temperature is uniform) is

$$\omega = N \sin \theta_g$$

Internal gravity wave

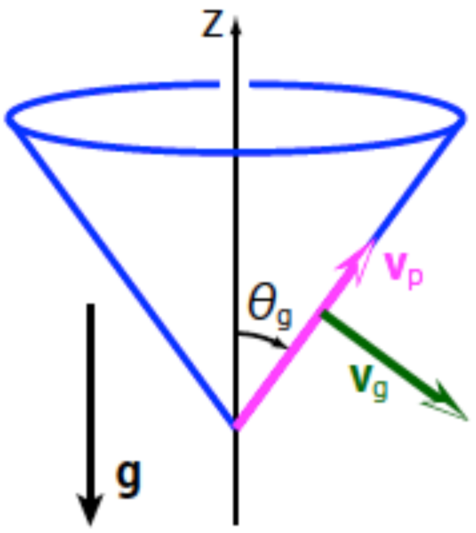
where

$$N^2 = \frac{(\gamma - 1)g^2}{c_s^2}$$

- Typical value for N^{-1} is 50s. So the gravity mode tends to be rather slow by comparison with other wave

Properties of internal gravity wave

- Phase speed: $v_p = \dfrac{\omega}{k} = \dfrac{N}{k}\sin\theta_g$



- They propagate along two cones with angle θ_g (not propagate in vertical direction)
- z-component of group velocity:

$$v_{gz} = \frac{\partial\omega}{\partial k_z} = -\frac{\omega k_z}{k^2}$$

- A group of upwind propagating wave carries energy downward (negative direction)
- group velocity is in a direction perpendicular to the surface of the cone with angle θ_g

Acoustic-gravity wave

- Consider propagation of sound (acoustic) wave in gravitational field (consider compressibility and buoyancy forces are present together)
- Using linearize equation is the same as internal gravity wave:

$$\omega^2 \boldsymbol{v}_1 = c_s^2 \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{v}_1) + i(\gamma - 1)g(\boldsymbol{k} \cdot \boldsymbol{v}_1)\hat{\boldsymbol{z}} + igkv_{1z}$$

- We consider $\boldsymbol{k}=(k_x, k_z)$ and $\boldsymbol{v}_1=(v_x, v_z)$
- After some calculation, we get dispersion relation

$$\omega^4 - \left\{ k_x^2 + \left(k_z + i\frac{\gamma g}{2c_s^2} \right)^2 + \frac{\gamma^2 g^2}{4c_s^4} \right\} c_s^2 \omega^2 + (\gamma - 1)g^2 k_x^2 = 0$$

Acoustic-gravity wave (cont.)

- We define $N^2 = \frac{(\gamma - 1)g^2}{c_s^2}, \quad N_s^2 = \frac{\gamma^2 g^2}{4c_s^2},$

$$\boldsymbol{k}' = \boldsymbol{k} + i\frac{N_s}{c_s}\hat{z}, \quad \sin^2\theta_g = 1 - \frac{k_z'^2}{k'^2} = \frac{k_x^2}{k'^2}$$

- The dispersion relation is rewritten as

$$\omega^4 - \left(k'^2 + \frac{N_s^2}{c_s^2}\right)c_s^2\omega^2 + N^2 c_s^2 k'^2 \sin^2\theta_g' = 0$$

- When γ=2, N_s=N. But this is not realistic. When γ=5/3, $N_s \simeq 1.02N$ So usually $N_s \geq N$
- When $\omega^2 \ll k'^2 c_s^2$, $\omega \simeq N \sin^2\theta_g'$. This is *internal gravity mode* (*g-mode*).
- When $\omega^2 \gg N$, $\omega \simeq k' c_s$. This is *acoustic wave mode* (*p-mode*).

Acoustic-gravity wave (cont.)

- When this wave propagates perpendicular direction ($\theta'_g = 0$),

$$\omega^2 = N_s^2 + k'^2 c_s^2$$

- Therefore, p-mode only exists when $\omega > N_s$

- If acoustic-gravity wave propagates not perpendicular direction, there are two solution ($k'^2 > 0$, $\omega^2 > 0$)
- Dispersion relation is

$$\omega^2 = \frac{1}{2}\left(k'^2 c_s^2 + N_s^2 \pm \sqrt{(k'^2 c_s^2 + N_s^2)^2 - 4N^2 c_s^2 k'^2 \sin^2\theta'_g}\right)$$

- From this, the solutions are $\omega < N \sin\theta'_g$ or $\omega > N_s$

Acoustic-gravity wave (cont.)

- higher frequency mode ($\omega > N_s$) is usually p-mode
  but group velocity is $v_g < c_s$ even though phase velocity is $v_p > c_s$
- In the limit of $\omega \rightarrow N_s$, $v_p \rightarrow \infty$ and $v_g \rightarrow 0$

- Lower frequency mode ($\omega < N \sin\theta'_g$) is usually g-mode
  and phase velocity is $v_p < c_s$
- In the limit of $\omega \rightarrow N \sin\theta'_g$, $v_p \rightarrow 0$

- The wave with the frequency between N_s and $N \sin\theta'_g$ does not propagate (decays in short distance), called evanescent
- If k' is purely imaginary, the standing wave can exist in this frequency. But no energy can be propagated.

Acoustic-gravity wave (cont.)

- Next, we investigate the wave which propagates perpendicular direction against k_x and ω ($\theta'_g = \pi/2$)
- The dispersion relation is

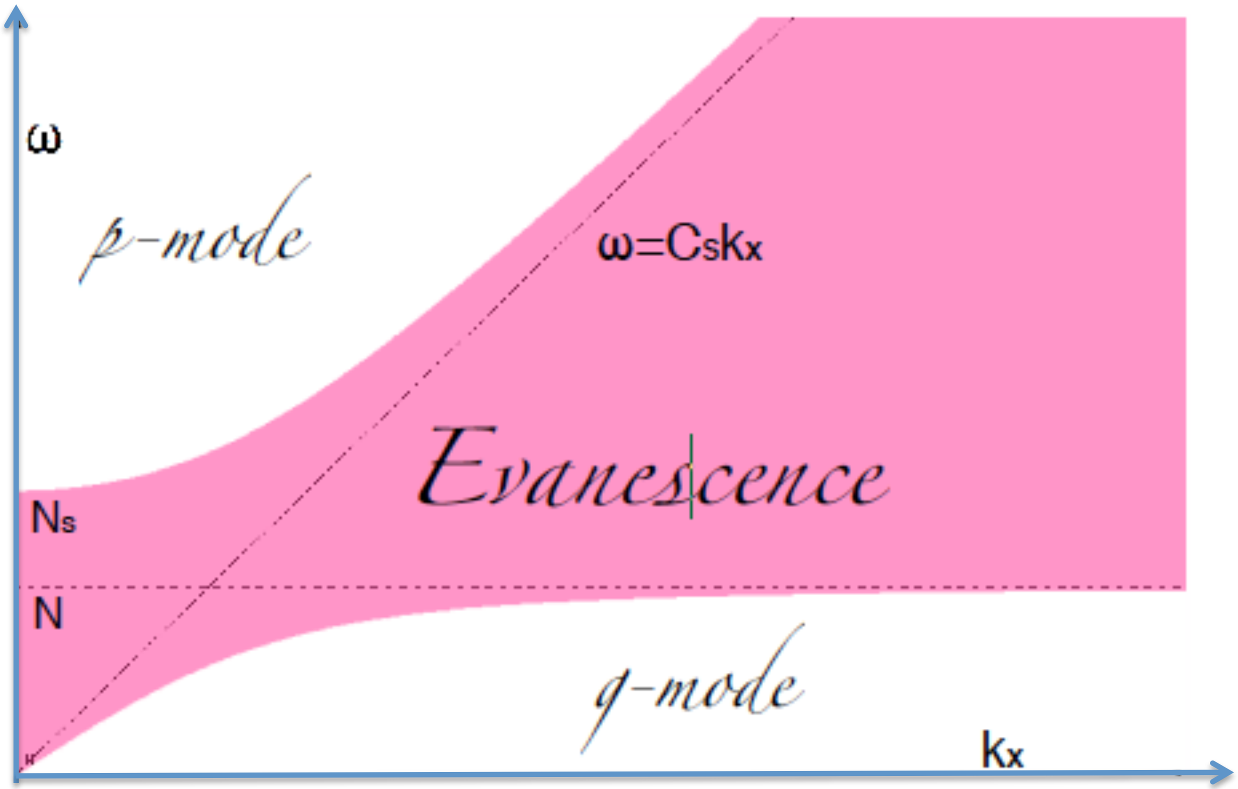
$$\omega^4 - \left(k_x^2 + k_z'^2 + \frac{N_s^2}{c_s^2}\right) c_s^2\omega^2 + N^2 c_s^2 k_x^2 = 0$$

$$\rightarrow \omega^2(\omega^2 - N_s^2) - (\omega^2 - N^2)c_s^2 k_x^2 = k_z'^2 c_s^2 \omega^2$$

- From this, the two solution is $k_x^2 > 0$, $\omega^2 > 0$
  Therefore the condition for $k'_z > 0$ is

$$\omega^2(\omega^2 - N_s^2) > (\omega^2 - N^2)c_s^2 k_x^2$$

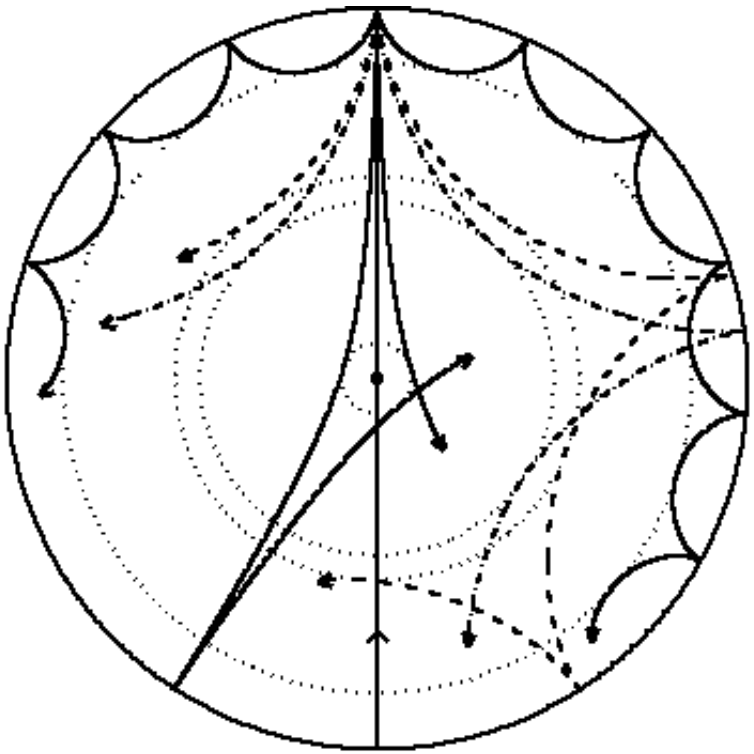
Acoustic-gravity wave (cont.)

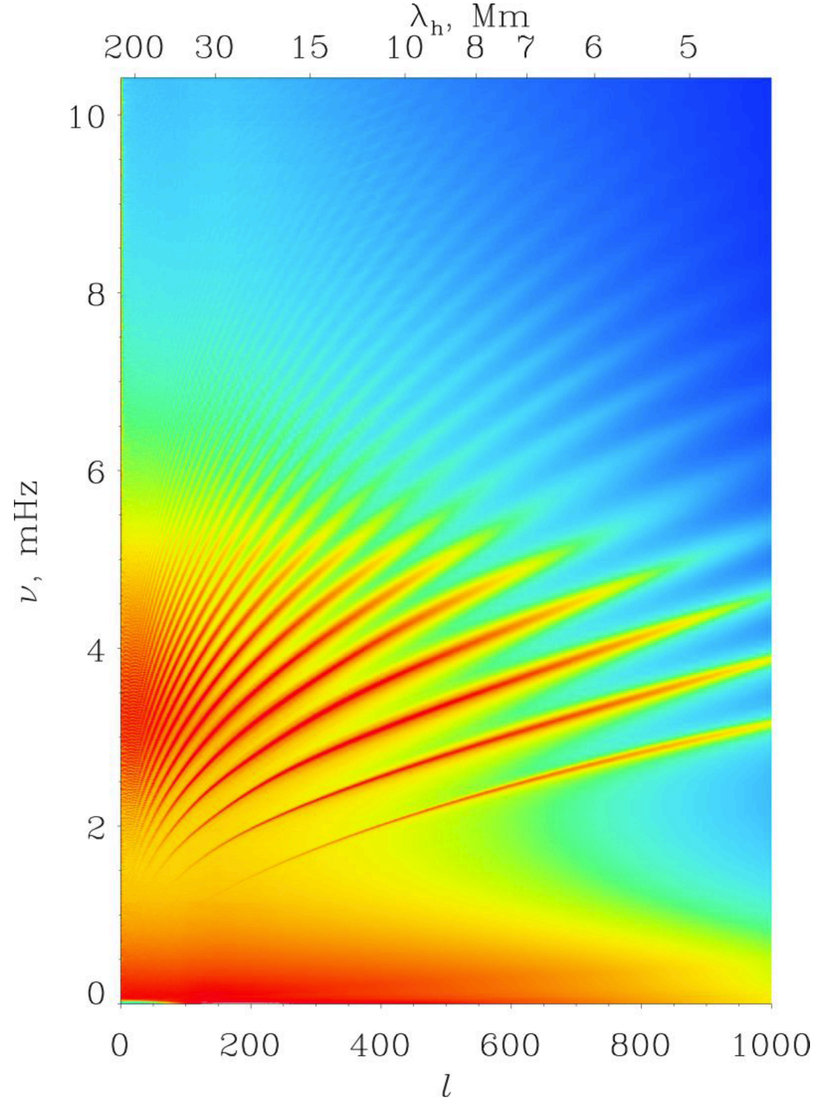


- This condition divides the ω - k_x plane. This figure sometimes referred as a diagnostic diagram

Acoustic gravity wave example

Helioseismology





Summary 2

- We have two type of waves propagating in the gravitational field.
  - g-mode (internal gravity wave) restoring by buoyancy force
  - p-mode (acoustic wave) restoring by pressure
- Between these two modes, evanescent region exists.

Exercise 2-3

Derivation of dispersion relation of magneto-acoustic waves

From linearized equation:

$$\frac{\omega^2 \boldsymbol{v}_1}{v_A^2} = k^2 \cos^2(\theta_{kB_0}) \boldsymbol{v}_1 - (\boldsymbol{k} \cdot \boldsymbol{v}_1) k \cos(\theta_{kB_0}) \hat{\boldsymbol{B}}_0 + \left[\left(1 + \frac{c_s^2}{v_A^2} \right) (\boldsymbol{k} \cdot \boldsymbol{v}_1) - k \cos(\theta_{kB_0}) (\hat{\boldsymbol{B}}_0 \cdot \boldsymbol{v}_1) \right] \boldsymbol{k}$$

Exercise 2-3 (cont.)

- Dot product of $\boldsymbol{k}$ $$\boldsymbol{k}\cdot\hat{\boldsymbol{B}}_0 = k\cos(\theta_{kB_0})$$

$$\Rightarrow \frac{\omega^2\boldsymbol{v}_1}{v_A^2}\cdot\boldsymbol{k} = k^2\cos^2(\theta_{kB_0})\boldsymbol{v}_1\cdot\boldsymbol{k} - \underline{k\cos(\theta_{kB_0})(\boldsymbol{k}\cdot\boldsymbol{v}_1)(\boldsymbol{k}\cdot\hat{\boldsymbol{B}}_0)}$$

$$+\left[\left(1+\frac{c_s^2}{v_A^2}\right)(\boldsymbol{k}\cdot\boldsymbol{v}_1)\underline{-k\cos(\theta_{kB_0})(\hat{\boldsymbol{B}}_0\cdot\boldsymbol{v}_1)}\right]k^2$$

$$\Rightarrow \quad = \left[k^2\cos^2(\theta_{kB_0}) + k^2\left(1+\frac{c_s^2}{v_A^2}\right)\underline{-k^2\cos^2(\theta_{kB_0})}\right](\boldsymbol{k}\cdot\boldsymbol{v}_1)$$

$$\underline{-k^3\cos(\theta_{kB_0})(\boldsymbol{v}_1\cdot\hat{\boldsymbol{B}}_0)}$$

$$= k^2\left(1+\frac{c_s^2}{v_A^2}\right)(\boldsymbol{k}\cdot\boldsymbol{v}_1) - k^3\cos(\theta_{kB_0})(\boldsymbol{v}_1\cdot\hat{\boldsymbol{B}}_0)$$

$$\left[\frac{\omega^2}{v_A^2} - k^2\left(1+\frac{c_s^2}{v_A^2}\right)\right](\boldsymbol{k}\cdot\boldsymbol{v}_1) = k^3\cos(\theta_{kB_0})(\boldsymbol{v}_1\cdot\hat{\boldsymbol{B}}_0) \qquad (4)$$

Exercise 2-3 (cont.)

- Dot product of $\hat{\boldsymbol{B}}_0$ =>

$$\boldsymbol{k}\cdot\hat{\boldsymbol{B}}_0 = k\cos(\theta_{kB_0})$$

$$\begin{aligned}\frac{\omega^2\boldsymbol{v}_1}{v_A^2}\cdot\hat{\boldsymbol{B}}_0 &= \underline{k^2\cos^2(\theta_{kB_0})\boldsymbol{v}_1\cdot\hat{\boldsymbol{B}}_0} - (\boldsymbol{k}\cdot\boldsymbol{v}_1)k\cos(\theta_{kB_0}) \\ &+ \left[\left(1+\frac{c_s^2}{v_A^2}\right)(\boldsymbol{k}\cdot\boldsymbol{v}_1) \underline{- k\cos(\theta_{kB_0})(\boldsymbol{v}_1\cdot\hat{\boldsymbol{B}}_0)}\right]\boldsymbol{k}\cdot\hat{\boldsymbol{B}}_0 \\ &= \left(1+\frac{c_s^2}{v_A^2}\right)k\cos(\theta_{kB_0})(\boldsymbol{k}\cdot\boldsymbol{v}_1) - k\cos(\theta_{kB_0})(\boldsymbol{k}\cdot\boldsymbol{v}_1) \\ &= \left(\frac{c_s^2}{v_A^2}\right)k\cos(\theta_{kB_0})(\boldsymbol{k}\cdot\boldsymbol{v}_1) \quad (5)\end{aligned}$$

Exercise 2-3 (cont.)

- From eq(4) & (5), vanish $(\boldsymbol{k} \cdot \boldsymbol{v}_1)$ and $(\boldsymbol{v}_1 \cdot \hat{\boldsymbol{B}}_0)$

$$\left[\frac{\omega^2}{v_A^2} - k^2 \left(1 + \frac{c_s^2}{v_A^2}\right)\right] \frac{v_A^2}{c_s^2 k \cos(\theta_{kB_0})} = k^3 \cos(\theta_{kB_0}) \frac{v_A^2}{\omega^2}$$

=> $\omega^4 - k^2(c_s^2 + v_A^2)\omega^2 - c_s^2 v_A^2 k^4 \cos^2(\theta_{kB_0}) = 0$