# Plasma Astrophysics
# Chapter 7-2: Instabilities II


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(last week)

- What is the instability? (How to analyze the instability)
- Rayleigh-Taylor instability
- Kelvin-Helmholtz instability

(today)

- Parker instability
- Magneto-rotational instability
- Jeans instability
- Current-driven instability

# Magnetic buoyancy

- Convective motion in fluid is driven by thermal buoyancy
- In compressible plasma, we have another important buoyancy, i.e., magnetic buoyancy

- Consider isolated flux tube (density $\rho_i$, gas pressure $p_i$ and magnetic pressure $p_m$) embedded in nonmagnetized plasma (density $\rho_e$, gas pressure $p_e$ ) under uniform gravity $g$
- Assume: isothermal, i.e., the temperature $T_i = T_e = T$
- Assume: tube is thin, i.e., the radius of flux tube is much smaller than local pressure scale height

$$H = \frac{p}{\rho g} = \frac{RT}{g} \qquad pV = nRT \rightarrow p = \rho RT$$

# Magnetic buoyancy (cont.)

- Equilibrium configuration of magnetic flux tube is determined by the balance of total pressure

$$p_i + B^2/2\mu_0 = p_e$$

- Then, from EoS of $p = \rho RT$, the density inside the tube ($\rho_i$) becomes smaller than the density outside the tube ($\rho_e$)
- Therefore

$$\rho_e - \rho_i = \frac{B^2}{2\mu_0 RT} \; (> 0)$$

- Hence the tube suffers the buoyancy force

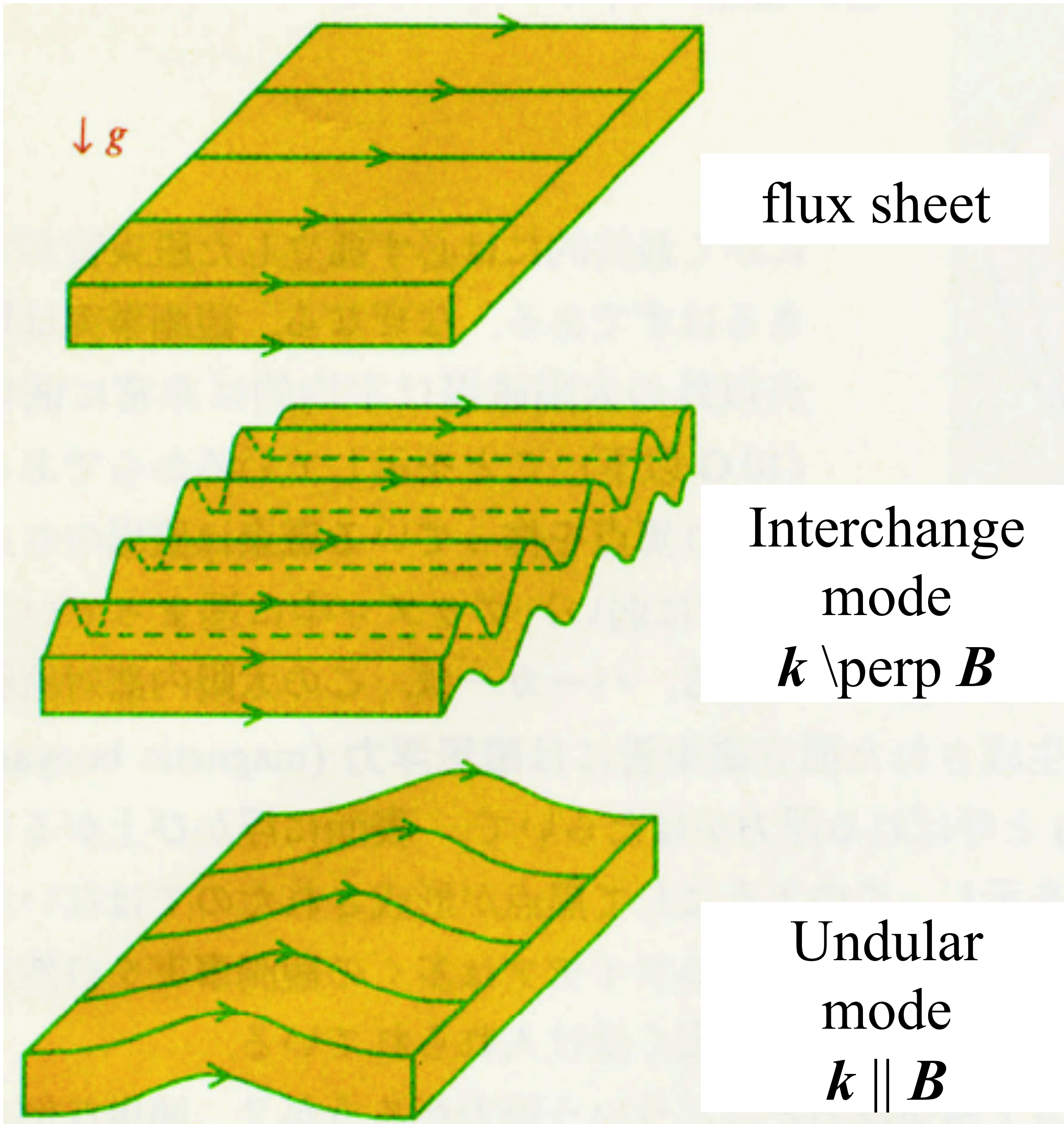
$$\Delta\rho g = \frac{B^2}{2\mu_0 H}$$

- This is called *magnetic buoyancy*
- This is fundamental force to raise the flux tube to the surface of the Sun, stars and accretion disk (galactic gas disk)

# Magnetic buoyancy (cont.)

- A horizontal isothermal, isolated flux tube cannot be in equilibrium.
- On the other hand, a 2D isothermal flux sheet can be in equilibrium
- Even in this case, the sheet often becomes unstable because of magnetic buoyancy
- There are two kinds of *magnetic buoyancy instability*
- *Interchange mode* : $\boldsymbol{k} \perp \boldsymbol{B}$
  - Necessary condition: $$\frac{d}{dz}\left(\frac{B}{\rho}\right) < 0$$
  - wavelength: arbitrary $\lambda$
  - Flute instability, magnetic RT instability (Kruskal-Schwarzschild instability)
- *Undular mode*: $\boldsymbol{k} \parallel \boldsymbol{B}$
  - Necessary condition: $\frac{d}{dz} B < 0$
  - Wavelength: $\lambda > \lambda_c \sim 10H$
  - Ballooning instability, Parker instability

# Magnetic buoyancy (cont.)

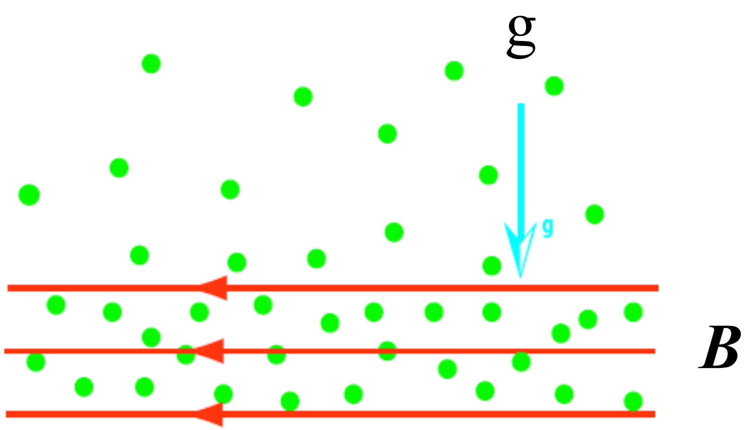


# Parker instability

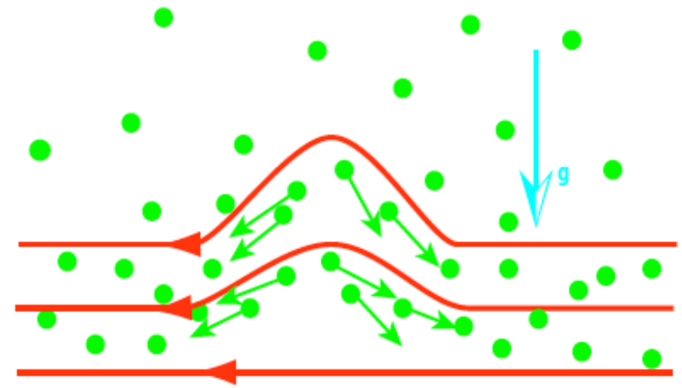
- Parker (1966) emphasized an importance of magnetic buoyancy instability (undular mode) in the Galactic disk (including cosmic-ray pressure effect) and explain the formation of interstellar cloud complexes.

- Hence, in astrophysics, magnetic buoyancy instability (undular mode) is usually called *Parker instability*

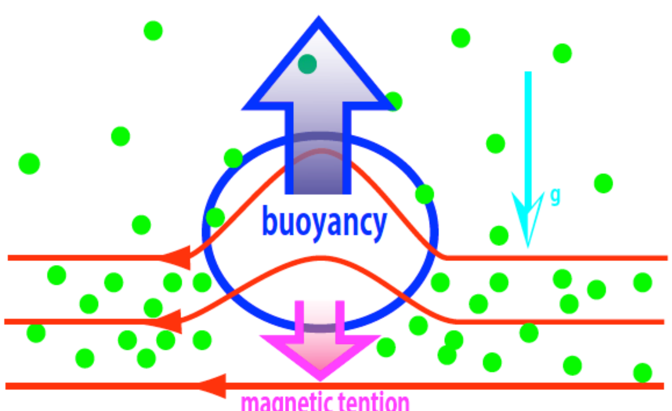
# Parker instability (cont.)




Magnetic field lift-up from equilibrium state




Plasma falls down along bending magnetic field lines




Top region becomes more lighter.
Then buoyancy force is working more (magnetic field lift-up more)
= growth of instability

# Parker instability (cont.)

- Here, drive instability condition
- Consider : a horizontal flux sheet in magneto-static equilibrium with gravity ( $\rho = \rho(z)$ and $\boldsymbol{B} = B_x(z)\hat{x}$ )

- Assume: isothermal and plasma beta $\beta = 2\mu_0 p / B^2$ = constant
- From isothermal condition (assume $\gamma$=1),

$$c_s^2 = \frac{dp}{d\rho} = \frac{p}{\rho} = const$$

# Parker instability (cont.)

- From pressure balance (to gravity) in z-direction,

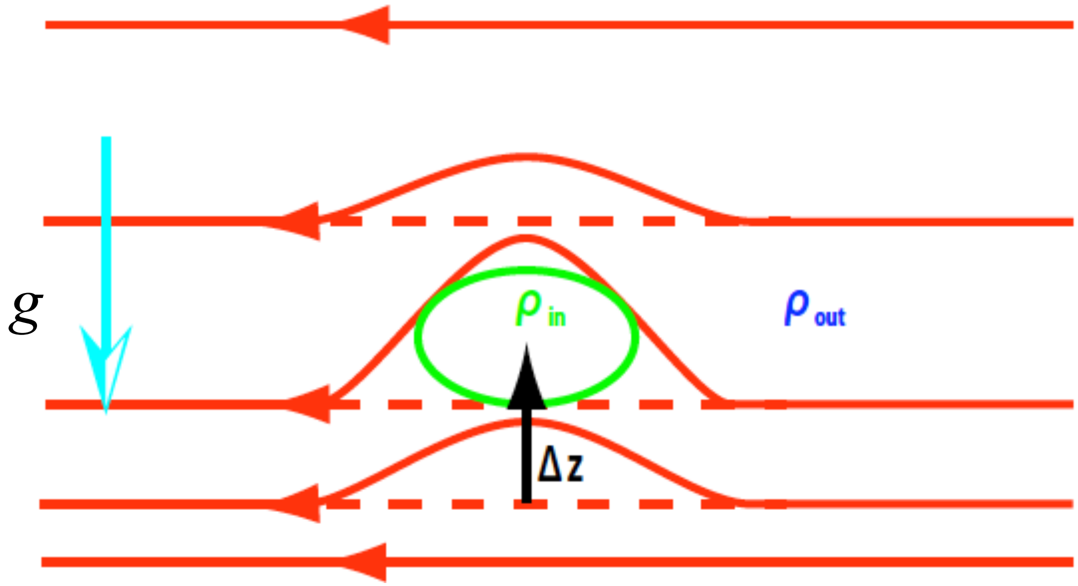
$$\frac{d}{dz}\left(p + \frac{B^2}{2\mu_0}\right) = -\rho g \rightarrow (1+\beta^{-1})c_s^2 \frac{d\rho}{dz} = -\rho g$$

- When there is no magnetic field ($\beta$=0),
- $\rho = \rho_0 \exp(-z/H)$ : hydrostatic equilibrium
- where $H$ is scale height: $H = p/\rho g = c_s^2/g$
- When the magnetic field is exist,

$$\rho = \rho_0 \exp(-z/H')$$

- Where $H' = (1+\beta^{-1})H$
- In the magnetic field supported disk, the plasma is located higher region than hydrostatic disk case.

# Parker instability (cont.)




- From equilibrium state, magnetic field lifts up $\Delta z$ (<< 1).
- Plasma lift-up time-scale is much longer than sound crossing time scale. So maintains pressure balance everywhere.
- After lift-up, plasma inside the bent magnetic field can move along field lines. Therefore magnetic pressure inside bent magnetic field does not change.

# Parker instability (cont.)

- Calculate density variance between the inside and outside of the bent magnetic field

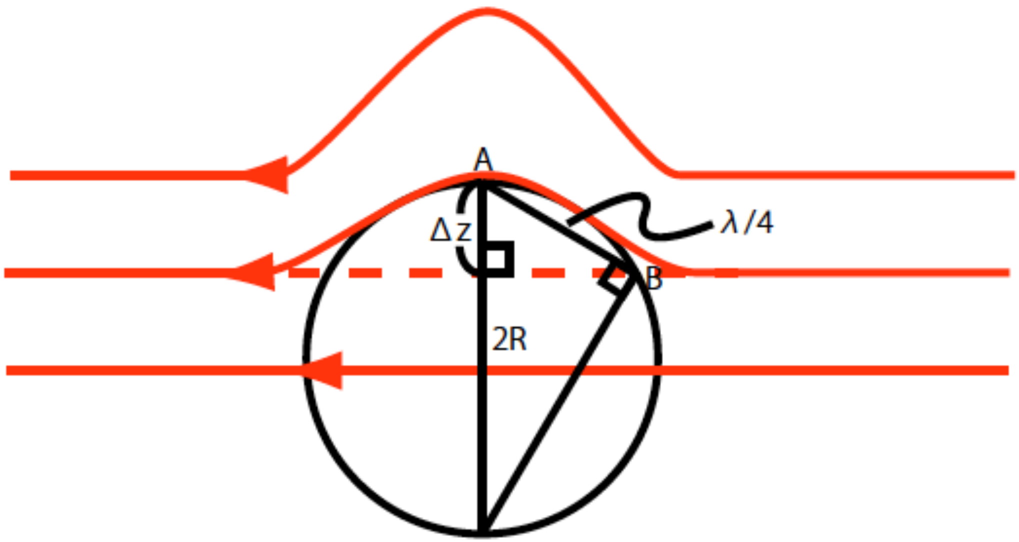
$$\begin{aligned}\Delta\rho &= \rho_{in}(\Delta z) - \rho_{out}(\Delta z) = \rho_0(e^{-\Delta z/H} - e^{-\Delta z/H'}) \\ &\simeq \rho_0\left(\frac{\Delta z}{H'} - \frac{\Delta z}{H}\right) < 0\end{aligned}$$

- It shows that inside the bent magnetic field becomes lighter
- Plasma inside the bent magnetic field has buoyancy force (+z direction)

$$F_{buoyancy} = -\Delta\rho g = \rho_0 g \Delta z(1/H - 1/H') \qquad (7.43)$$

- On the other hand, due to bent magnetic field, there is magnetic tension force in –z direction

# Parker instability (cont.)




- Here curvature radius is $R$, the magnetic tension is estimated as

$$F_{tension} \simeq \frac{B^2}{\mu R} \quad (7.44)$$

- Consider triangle in the circle R, the relation between $R$ and $\Delta z$ is

$$\frac{\lambda}{4} : \Delta z \simeq 2R : \frac{\lambda}{4} \rightarrow R \simeq \frac{\lambda^2}{32\Delta z} \quad (7.45)$$

- Here $\lambda$ is perturbed wavelength (distance AB ~ $\lambda/4$)

# Parker instability (cont.)

- If $F_{\text{buoyancy}} > F_{\text{tension}}$, plasma inside the bent magnetic field continuously lift-up. It means that it is *unstable*
- From eq (7.43), (7.44), (7.45), critical wavelength is

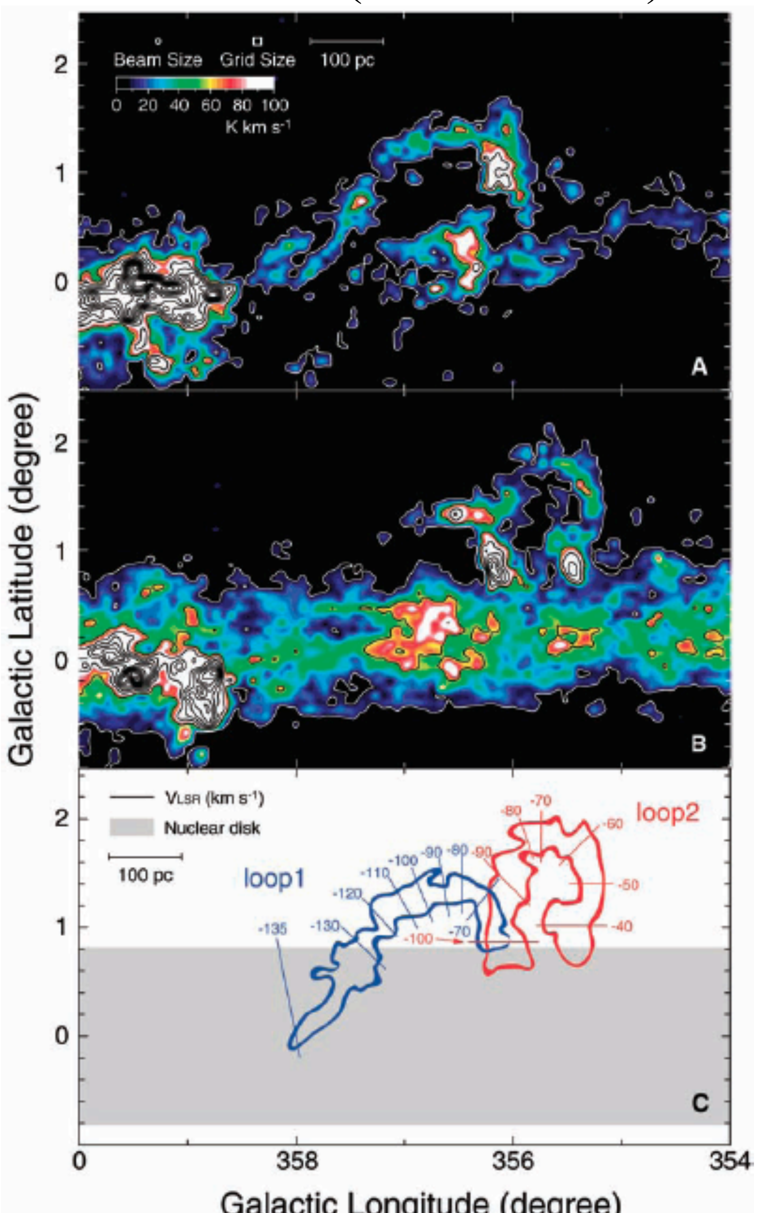
$$\lambda > \lambda_c = 8\sqrt{(1+\beta^{-1})}H$$

- Therefore if the magnetic field is perturbed the wavelength enough longer than scale-height, the buoyancy overcomes magnetic tension and gravity then inside plasma continuously lift-up (unstable).
- This is so-called *Parker instability*
- Growth rate is roughly estimated by

$$\gamma \sim v_A/\lambda \sim 0.1 v_A/H \ \ (\beta \sim 1)$$

# Parker instability (cont.)

Molecular Loops in the Galactic Center (radio CO obs)

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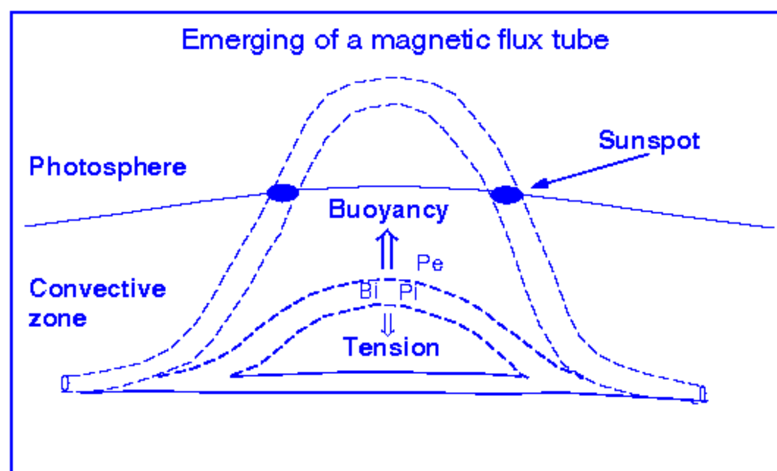
Solar coronal loop（Three year obs., by SDO）



# Parker instability (cont.)

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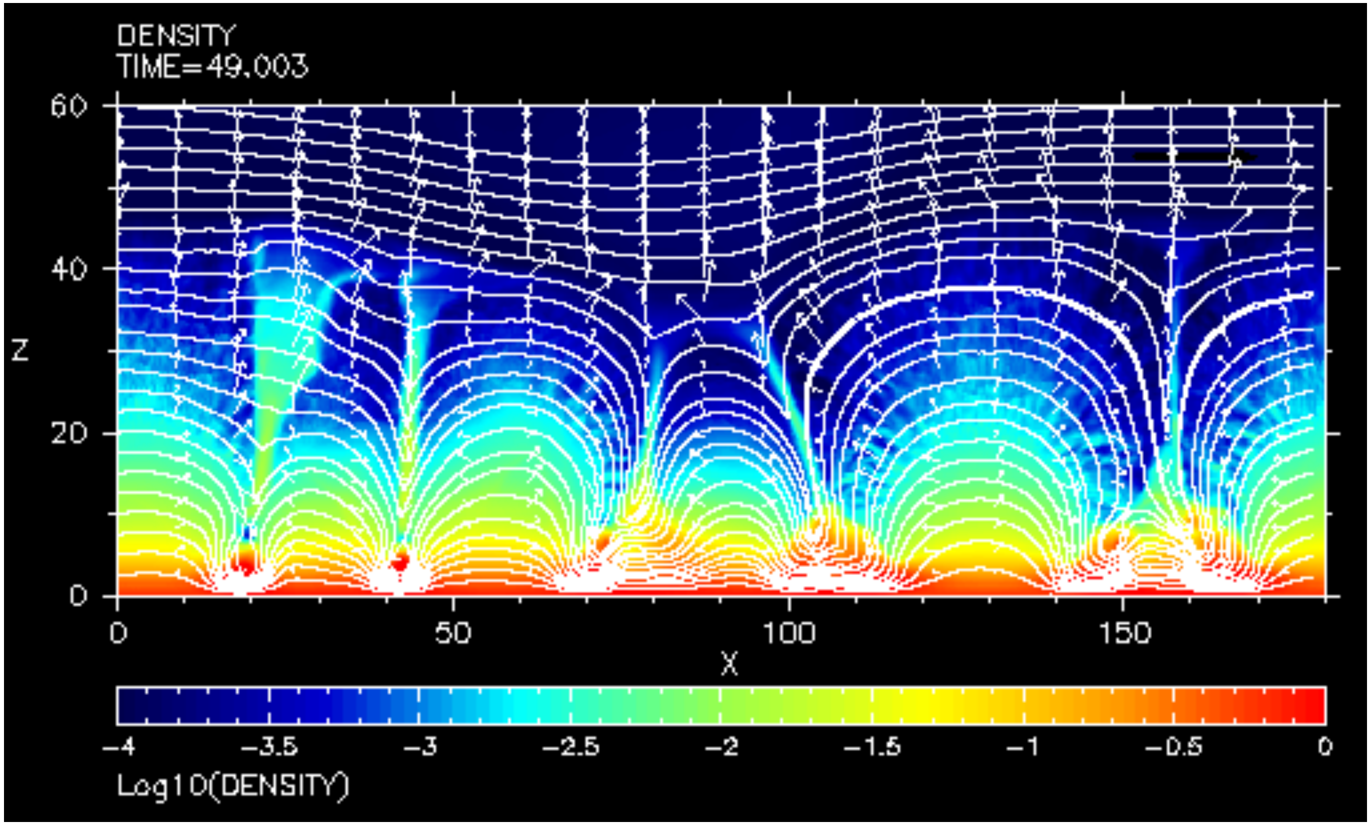
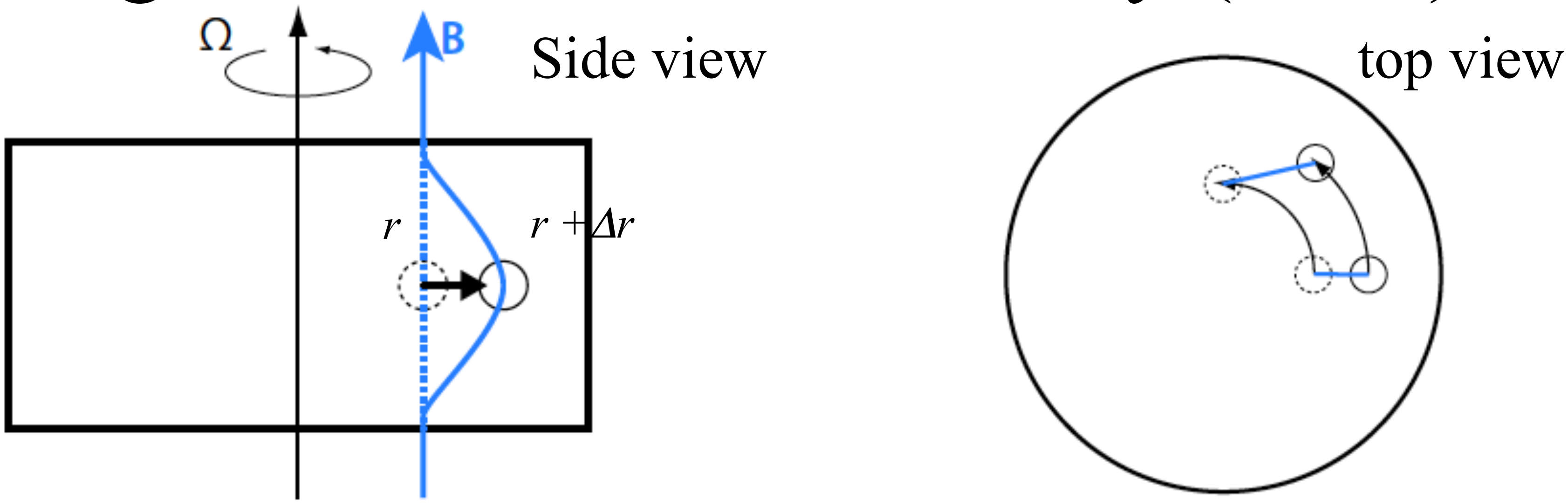
2D MHD simulations of Parker Instability



# Magneto-rotational instability

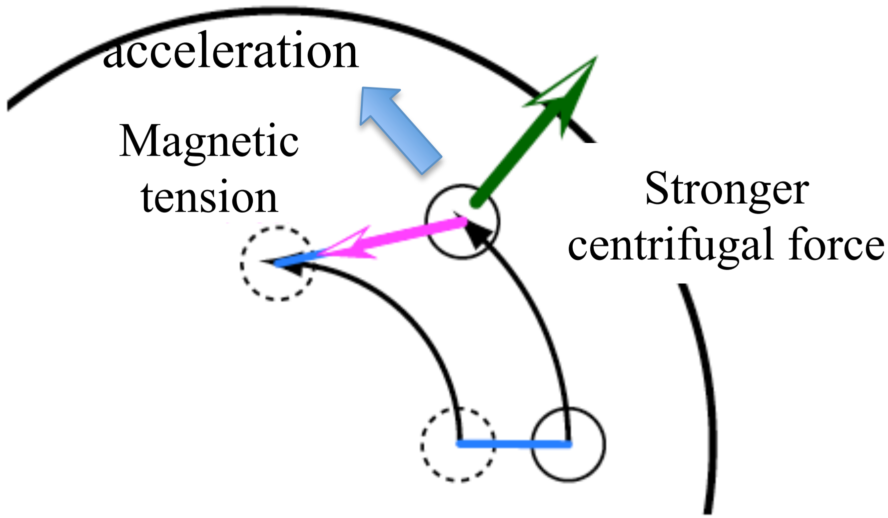
- Important for angular momentum transport in accretion disk
- In the standard theory of accretion disks (Shakura & Sunyaev 1973), the $\alpha$-prescription of viscosity is adopted for radial angular momentum transport
- What phenomenological viscosity parameter $\alpha$?
- From observation of dwarf novae, $\alpha$=0.02 (quiescent) - 0.1 (bursting phase)
- Molecular viscosity: NO (too small)
- Hydrodynamic shear flow instability makes convective turbulence in accretion disk (turbulent viscosity).
  - But in geometrically thin Keplerian disk, $\alpha=O(10^{-3})$
- In MHD model: magnetic stress enhanced turbulent viscosity incurred by fluctuating magnetic field <= generated by *Magnetorotational instability (MRI)* (Balbus & Hawley 1991)

# Magneto-rotational instability (cont.)




- Understanding of MRI through Lagrangian point of view
- Consider differentially rotating plasma disk with vertical magnetic field (penetrate disk) in some gravitational field (stationary)
- Put small radial perturbation in rotating plasma at radius $r$ from rotation axis (angular momentum is conserved) and moves to $r+\Delta r$
- The angular velocity in $r+\Delta r$ is slower than that in $r$. Thus magnetic field is deformed more and magnetic tension is happened.

# Magneto-rotational instability (cont.)

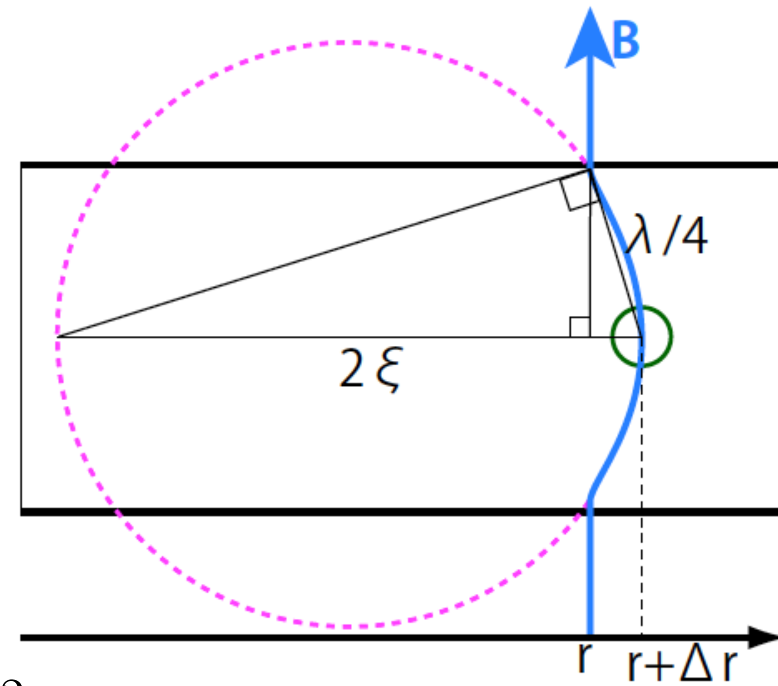



- Due to the magnetic tension, plasma is accelerated to rotational direction.
- The plasma in $r+\Delta r$ tries to rotate with angular velocity at $r$.
- This faster angular velocity makes stronger centrifugal force which is larger than gravitational force.
- Then the plasma is push outward more. Again magnetic field is stretched more and make larger magnetic tension.
- This process is so-called *Magneto-rotational instability (MRI)*.

# Magneto-rotational instability (cont.)

- Rough estimate the instability condition
- Assume Keplerian rotating plasma disk with vertical magnetic field (penetrate disk) in some gravitational field (stationary)
- Put small radial perturbation in rotating plasma at radius $r$ from rotation axis (angular momentum is conserved) and moves to $r+\Delta r$
- Consider radial force balance at $r+\Delta r$
  - Gravity: $\frac{GM}{r^2} \rightarrow \frac{GM}{(r+\Delta r)^2} \simeq \frac{GM}{r^2}\left(1 - 2\frac{\Delta r}{r}\right)$
  - Centrifugal force:
$$r\Omega(r)^2 \rightarrow (r+\Delta r)\Omega(r)^2$$
- Where, the effect of acceleration by magnetic tension in rotational direction is included in centrifugal force

# Magneto-rotational instability (cont.)




- From Keplerian rotation,
  $\Omega(r) = \sqrt{GM/r^3}$
- Radial force including gravity and centrifugal force is
  $$2\frac{GM}{r^3}\Delta r + \Delta r\Omega(r)^2 = 3\Delta r\Omega(r)^2$$
- Next calculate radial force by magnetic tension.
- As shown in figure, the deformation of magnetic field is approximate as a circle with radius ξ
- Magnetic tension is $\dfrac{1}{\mu_0\rho}\dfrac{B^2}{\xi} = \dfrac{v_A^2}{\xi}$

# Magneto-rotational instability (cont.)

- From similarity relation

$$2\xi : \frac{\lambda}{4} = \frac{\lambda}{4} : \Delta r \rightarrow \xi = \frac{\lambda^2}{32\Delta r} = \frac{\pi^2}{8\Delta r}\frac{1}{k^2} \qquad \lambda = 2\pi/k$$

- Using this value, magnetic tension is

$$\frac{8}{\pi^2}\Delta r k^2 v_A^2 \simeq \Delta r k^2 v_A^2$$

- The system is unstable when
  (gravity + centrifugal force) > (magnetic tension in radial direction)
- Therefore, the condition for growing instability is

$$3\Delta\Omega(r)^2 > \Delta r k^2 v_A^2$$

=> $3\Omega^2 > k^2 v_A^2 \rightarrow \boxed{0 < k < \sqrt{3}\frac{\Omega}{v_A}}$

# Magneto-rotational instability (cont.)

- From instability condition, the instability occurs longer than the critical wavelength.

$$\lambda > \lambda_c = \frac{2\pi v_A}{\sqrt{3}\Omega}$$

- This feature is similar to that of Parker instability, i.e., stabilized by magnetic tension force.
- If magnetic field is strong, this instability is stabilized because the critical wavelength $\lambda_c$ exceeds the disk thickness $H$.
- In this case, the critical field strength for stability is

$$B > B_c \sim \frac{\sqrt{\mu_0 \rho} H \Omega}{2\pi}$$

- For growth of MRI, weak magnetic field in the accretion disk is important

# Magneto-rotational instability (cont.)

- Next, we derive dispersion relation of MRI,
- Linearized equations

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\boldsymbol{v} = 0$$

$$\frac{\partial \delta\boldsymbol{v}}{\partial t} + (\delta\boldsymbol{v} \cdot \nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0 \cdot \nabla)\delta\boldsymbol{v} + \frac{1}{\rho_0}\nabla\left(\delta p + \frac{1}{\mu_0}\boldsymbol{B}_0 \cdot \delta\boldsymbol{B}\right) - \frac{\delta\rho}{\rho_0^2}\nabla\left(p_0 + \frac{1}{2\mu_0}B_0^2\right)$$
$$- \frac{1}{\mu_0\rho_0}[(\boldsymbol{B}_0 \cdot \nabla)\delta\boldsymbol{B} + (\delta\boldsymbol{B} \cdot \nabla)\boldsymbol{B}_0] + \frac{\delta\rho}{\mu_0\rho_0^2}(\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{B}_0 = 0$$

$$\nabla \cdot \delta\boldsymbol{B}_0 = 0$$

$$\frac{\partial \delta\boldsymbol{B}}{\partial t} - \nabla \times [\delta\boldsymbol{v} \times \boldsymbol{B}_0 + \boldsymbol{v}_0 \times \delta\boldsymbol{B}]$$

# Magneto-rotational instability (cont.)

- Consider the frame of rotating around z-axis with angular velocity $\Omega(r)$ in cylindrical coordinates ($r$, $\phi$, $z$) i.e., $\boldsymbol{v}_0 = v_{0\phi}\hat{\phi} = r\Omega(r)\hat{\phi}$
- In equilibrium state, gravity and centrifugal force is balanced
- Uniform magnetic field, $\boldsymbol{B} = (0, B_{0\phi}, B_{0z})$
- Perturbation: $\delta \propto e^{-\omega t + ikz}$ , wavenumber $\boldsymbol{k} = (0, 0, k)$
- In detail of calculation, need to use $kr \simeq r/\lambda \gg 1$ from local analysis
- After some manipulations, we get following dispersion relation (for simply use $B_{0\phi}$=0),

$$\omega^2 - k_z^2 v_{Az}^2 = \pm\sqrt{4\Omega^2\omega^2 + (\omega^2 - k_z^2 v_{Az}^2)\frac{d\Omega^2}{d\ln r}}$$

# Magneto-rotational instability (cont.)

- Rigid rotation case
  - From $d\Omega/dr = 0$, the dispersion relation is

$$\left(\frac{\omega}{\Omega}\right)^2 = \left(\frac{k_z v_{Az}}{\Omega}\right)^2 \pm \sqrt{4\left(\frac{\omega}{\Omega}\right)^2}$$

  - There is no solution with $\omega^2 < 0$, therefore rigid rotation disk is stable against MRI

- Keplerian rotation case

$$\Omega^2 = \frac{GM}{r^3} \rightarrow \frac{d\Omega^2}{d\ln r} = -3\Omega^2$$

  - From this, the dispersion relation is

$$\left(\frac{\omega}{\Omega}\right)^2 = \left(\frac{k_z v_{Az}}{\Omega}\right)^2 \pm \sqrt{\left(\frac{\omega}{\Omega}\right)^2 + 3\left(\frac{k_z v_{Az}}{\Omega}\right)^2}$$

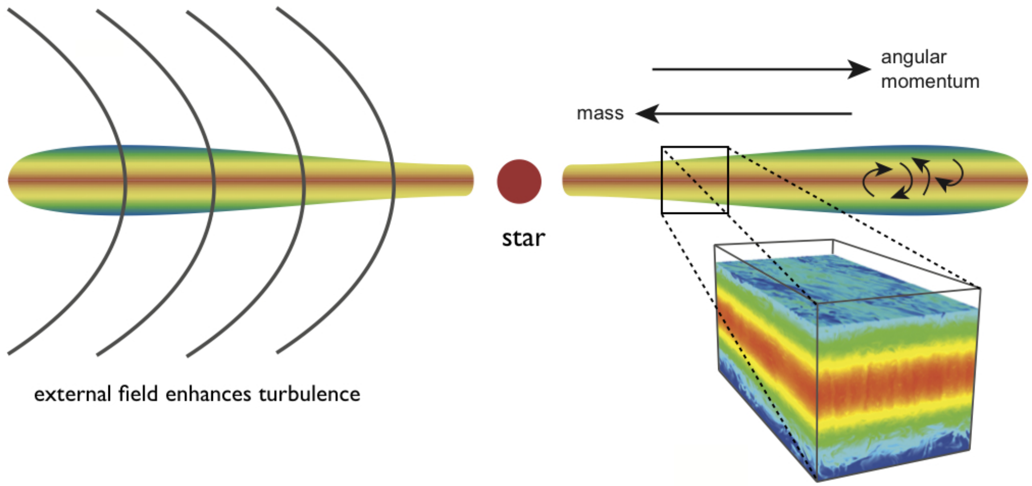
  - When $0 < k < \sqrt{3}\Omega/v_{Az}$, $\omega^2 < 0$. Therefore Keplerian disk is unstable against MRI. And maximum growth rate is $\gamma_{MRI} \sim 0.75\Omega$

# Magneto-rotational instability (cont.)

- In Keplerian disk, growth rate is comparable with $\Omega$, i.e., this instability is fairly fast instability occurring at the rotation time scale of disk
- This instability occurs even if the magnetic field is very weak
- People often neglected the effect of magnetic field in accretion disk simply because magnetic field is very weak in the disk
- But from properties of MRI, we cannot neglect magnetic field any more.

# Magneto-rotational instability (cont.)

3D MHD Simulation in local shearing box




3D MHD in global accretion disk

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# Jeans instability

- In many astrophysical phenomena, gravitational field plays an important role.
- In particular, self-gravity and the associated instability are essential when we consider the formation of various objects (e.g., stars, galaxies, and the clusters of galaxies) due to density fluctuations

- Consider an infinite homogeneous medium at rest
$\rho = \rho_0$=uniform, $p = p_0$=uniform, $\boldsymbol{v} = \boldsymbol{v}_0 = 0$, $\Phi = \Phi_0$
- Here we consider self-gravity of medium but neglect magnetic field and assume adiabatic, $p = K\rho^{\gamma}$

# Jeans instability (cont.)

- Linearized equations

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{v}_1 = 0$$

$$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} = -c_s^2 \nabla \rho_1 - \rho_0 \nabla \Phi_1$$

$$\nabla^2 \Phi_1 = 4\pi G \rho_1$$

$$p_1 = c_s^2 \rho_1$$

- From these equations, we obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0$$

- If $G$=0, this equation expresses the propagation of sound wave in a homogeneous medium.
- In other word, this equation shows that how the propagation of sound wave is modified in self-gravity field

# Jeans instability (cont.)

- If we consider a plane wave and put $\rho_1 \propto e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$ we obtain the dispersion relation

$$\omega^2 = k^2 c_s^2 - 4\pi G \rho_0$$

- $\omega^2$ becomes negative when

$$k < \sqrt{\frac{4\pi G \rho_0}{c_s^2}} \equiv k_J = \frac{2\pi}{\lambda_J}$$

- where $\lambda_J$ is so-called *Jeans wavelength (radius)*.
- In the perturbation with longer wavelength, attracting force from self-gravity overcomes increase of gas pressure then gravitational collapse is occurred (*unstable*)
- This is so-called *Jeans instability*.

# Jeans instability (cont.)

- Compute the mass contained within the Jeans radius (consider as a sphere)

$$M_J \equiv \frac{4\pi}{3}\lambda_J^3\rho_0 = \frac{4\pi}{3}\left(\frac{\pi c_s^2}{G\rho_0}\right)^{3/2}\rho_0 \propto T^{3/2}\rho^{-1/2}$$

- Here $M_J$ is so-called *Jeans Mass*. From this instability, the object with $M > M_J$ is formed.
- Jeans mass is small if temperature is low and density is high.
- Therefore because of the density increase by the cloud contracts, the Jeans wavelength becomes shorter and shorter.
- It means that the Jeans instability is takes place at smaller and smaller scales as the cloud contracts, leading to a fragmentation into many small pieces.

# Jeans instability (cont.)

- In this lecture, we consider the simple model, an infinite homogeneous medium at rest
- If we consider the rotating disk, the Coriolis force is protected to contraction of gas => instability condition is changing (Toomre 1964)

# Jeans instability (cont.)

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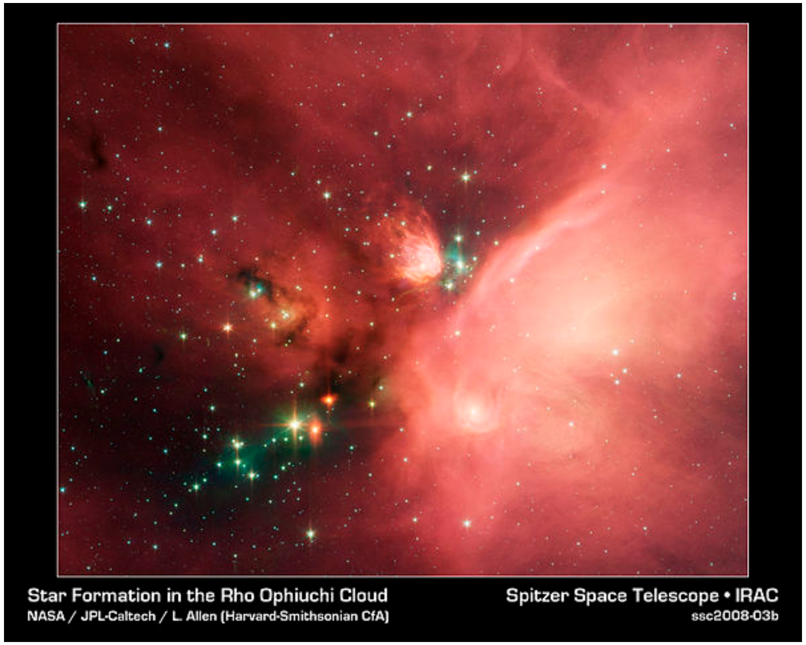
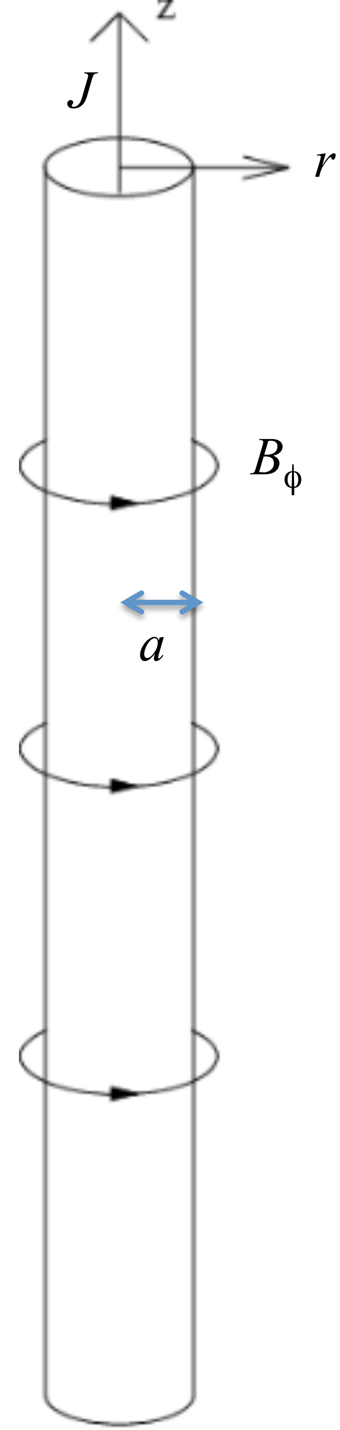
Star formation in molecular cloud




3D SPH simulations of star formation from gas cloud

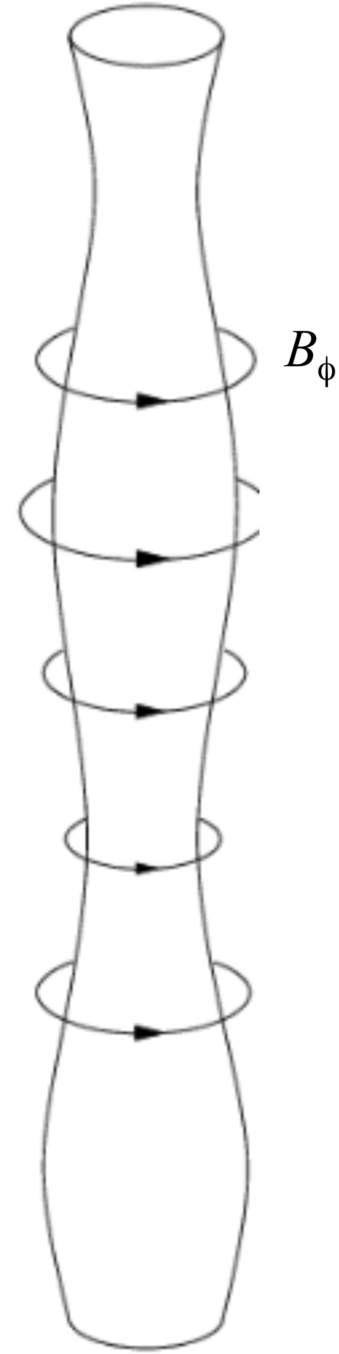
# Current driven (kink) instability

- A linear pinched discharge in the laboratory is a cylindrical plasma column (radius $a$) that is confined (or pinched) by toroidal magnetic field due to current ($J\hat{z}$) flowing along its surface or through its interior
- This configuration is similar to magnetic flux tubes present in the solar atmosphere and astrophysical jets formed from compact rotator.
- So summaries its stability properties here



# Current driven (kink) instability (cont.)

- The radially inwards ***J*** x ***B*** force (magnetic pressure $B^2/2\mu_0$ and magnetic tension $B^2/2\mu_0 r$) is balanced by outwards pressure gradient.
- When plasma (at pressure $p_0$ & density $\rho_0$) contains no magnetic field (interior), the pinch is unstable to the interchange mode (***k*** \perp ***B***), since the confining field is concave to plasma
- The place where it pinched, toroidal field is increases and radius is deceases. Therefore magnetic pressure and tension increase => inward force is no longer balances with gas pressure => perturbation grows
- The place where it bulges out, toroidal field is decreases and radius is increases. Therefore magnetic pressure and tension decrease => perturbation grows



# Current driven (kink) instability (cont.)

- This instability is so-called sausage instability (*m*=0 mode of current-driven instability, sausage mode).
- This instability is unstable in all wavelength (for cylindrical plasma column with toroidal field)
- The growth rate of this instability with the wavenumber $k \approx a^{-1}$

$$\gamma = \frac{\sqrt{2p_0/\rho_0}}{a}$$

- The cylindrical plasma column can be stabilized against the sausage mode by the presence of a large enough axial field ($B_{0z}$)
- The value of toroidal field at the interface is $B_\phi$, the force balance on the interface gives

$$p_0 + \frac{B_{0z}^2}{2\mu_0} = \frac{B_\phi^2}{2\mu_0}$$

# Current driven (kink) instability (cont.)

- The effect of Alfven wave propagating along the axis with speed $v_{Az} = B_{0z}/\sqrt{\mu_0 \rho_0}$ is to modify the dispersion relation to

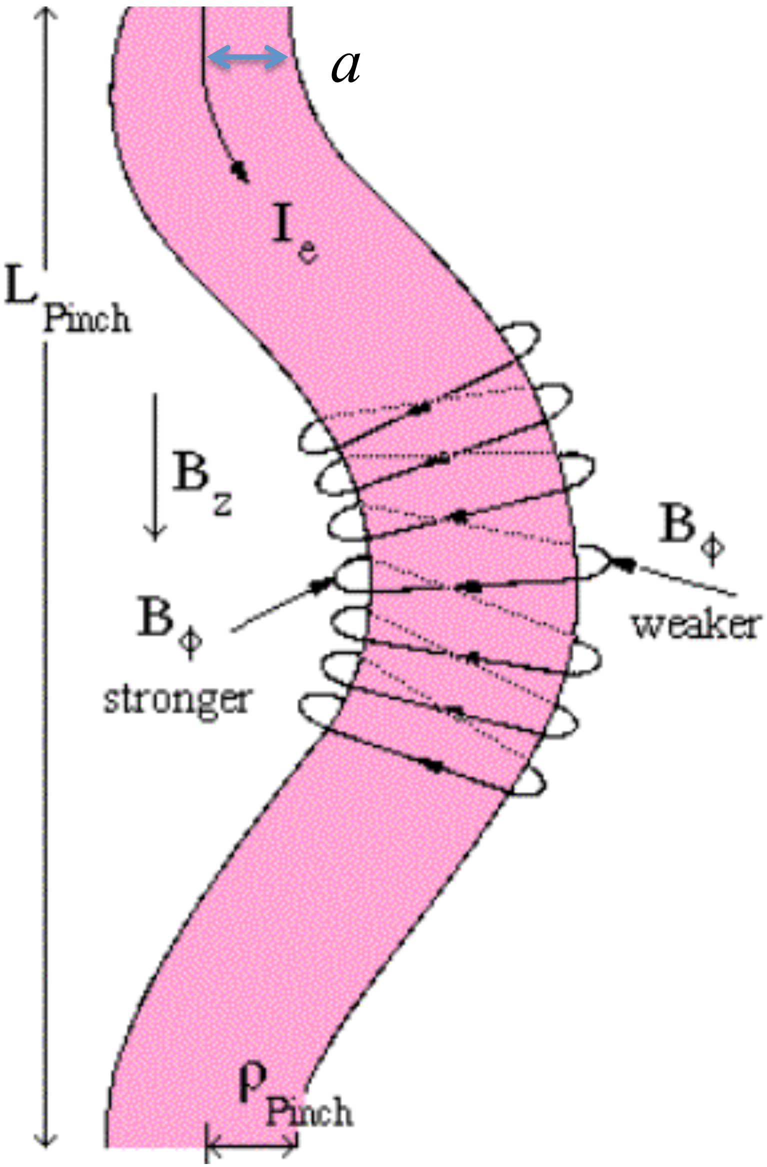
$$\omega^2 = -\frac{2p_0}{\rho_0 a^2} + \frac{B_{0z}^2}{\mu_0 \rho_0 a^2}$$

- This force balance gives stability ($\omega^2 > 0$) when

$$B_{0z}^2 > \frac{1}{2} B_\phi^2$$

# Current driven (kink) instability (cont.)

- Consider the perturbation of kink to cylindrical plasma column
- Inside kinked plasma column, magnetic pressure becomes strong, while outside of the kinked plasma column, magnetic pressure becomes weak => perturbation grows (*unstable*).
- This instability is so-called kink instability (*m*=1 mode of current-driven instability, sausage mode)
- The axial field in cylindrical plasma column also affects the stabilize of this instability (kink-mode)



# Current driven (kink) instability (cont.)

- The condition of stability for kink mode is

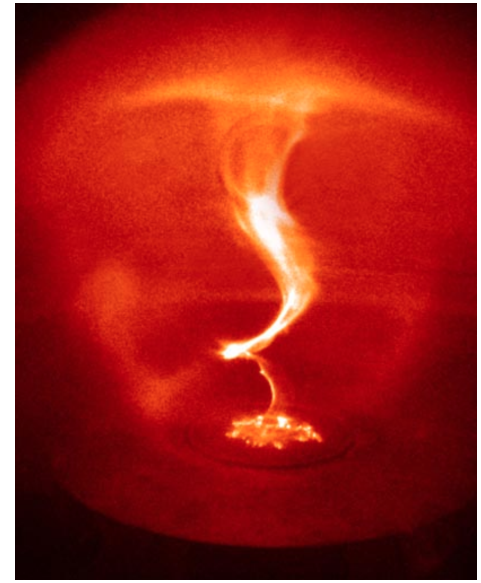
$$\left(\frac{B_\phi}{B_z}\right)^2 < (ka)^2 = \left(\frac{2\pi a}{\lambda}\right)^2$$

*Kruskal-Shafranov criterion*

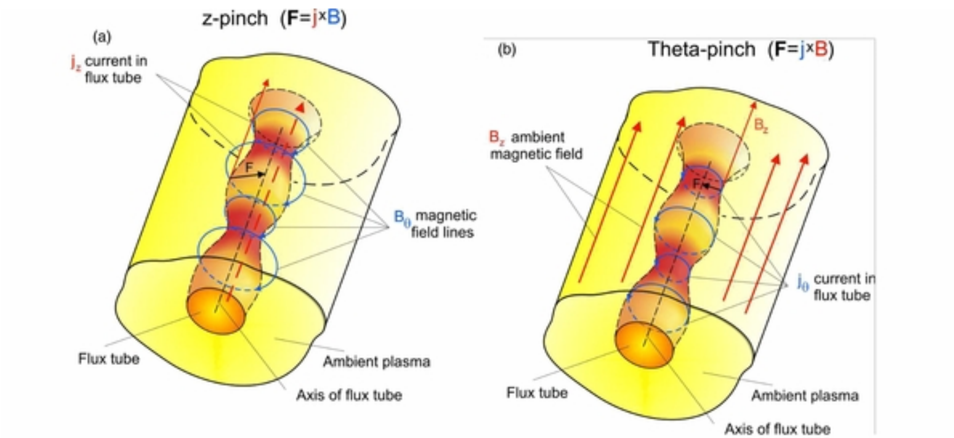
- If the perturbed wavelength is long enough, the plasma column with helical magnetic field is unstable against kink instability.

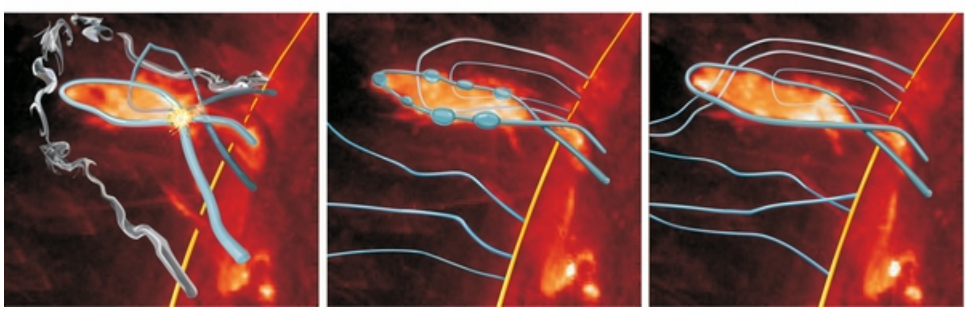
# Current driven (kink) instability (cont.)

Kink instability in laboratory experiment



Sausage pinch instability in solar corona (Obs by SDO)



MHD simulation of kink instability

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# Summary

- There are many potentially growing instabilities in the universe.
- These instabilities are strongly related the dynamics in the universe.
- Important:
  - what system is stable/unstable against instabilities (condition for stable/unstable of instability)
  - What is the time scale of growing instabilities (growth rate). Does it affects the dynamics of system?
- Here not covered…but may be important
  - Thermal instability, radiation (pressure)-driven instability, Richitmeyer-Meshkov instability, Corrugation instability, Tearing instability …