Plasma Astrophysics
Chapter 9: Astrophysical Dynamo

Yosuke Mizuno
Institute of Astronomy
National Tsing-Hua University

Dynamo

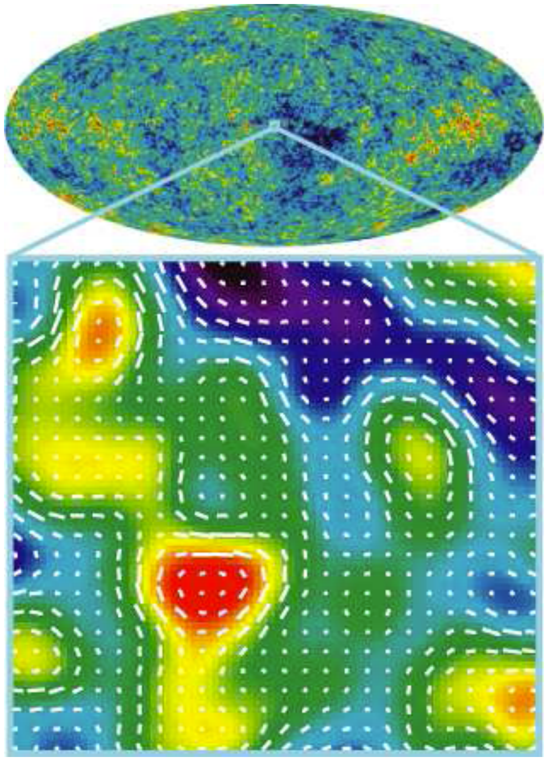
- A process that converts one form of energy into another. e.g., the electric generator (magnetic + kinetic = electric).
- Here, we are dealing with an astrophysical dynamo (electric + kinetic = magnetic)

Magnetic field in the universe

- Astrophysical magnetic fields exist at various scales from those as large as intergalactic space, to those comparably smaller on the size of planets.
- Magnetic fields also exhibit a wide variety of temporal behavior; in some astrophysical systems, they do not vary much on timescales comparable to the lifetime of that system, while in some case, they vary rapidly on very short timescales (e.g., the Sun)
- Their origin and behavior is also different in different systems; however in general, if in any system the magnetic field varies on timescales much shorter than the lifetime of the system, then a dynamo is probably at the origin

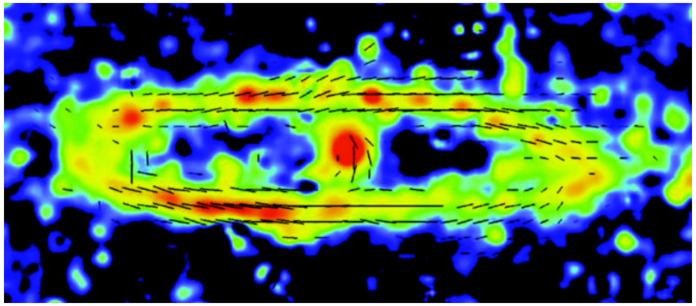
Magnetic field in the universe (cont.)

Cosmological magnetic field

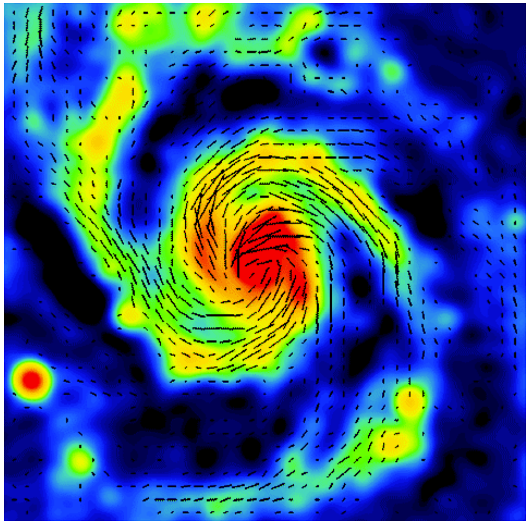


Galactic magnetic field

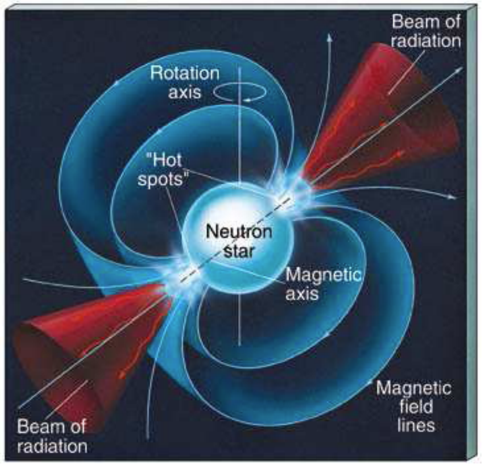
M31



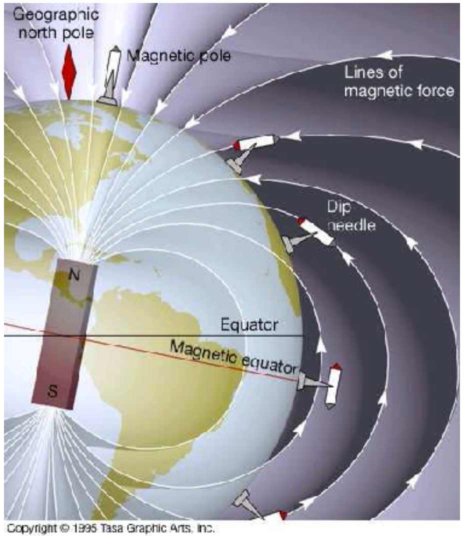
M51



Pulsar (rotating neutron star)



Planetary B-field (Earth)



Solar magnetic field

- The best studied astrophysical object is the Sun

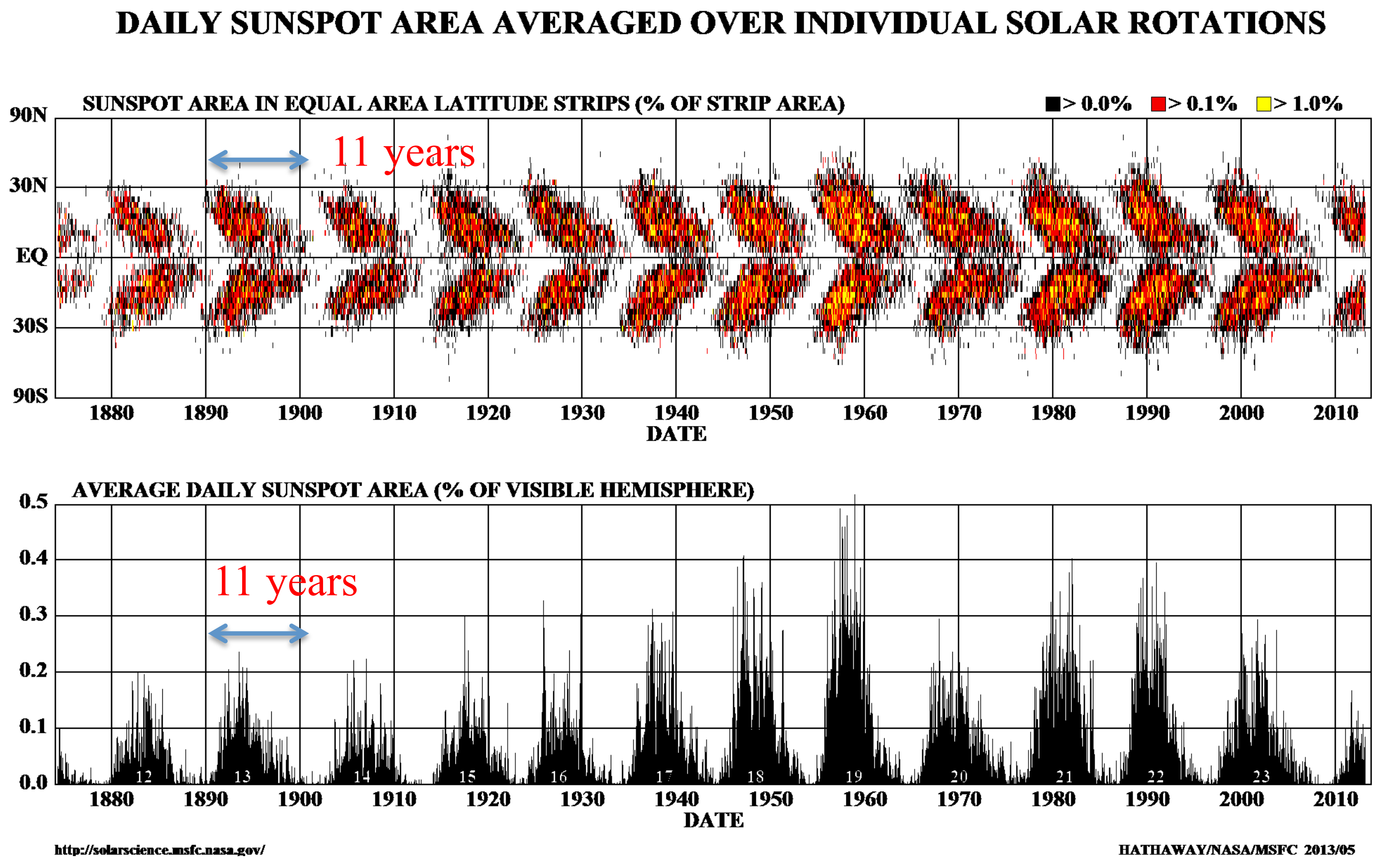
SDO (UV-EUV)

Movie here

Solar cycle

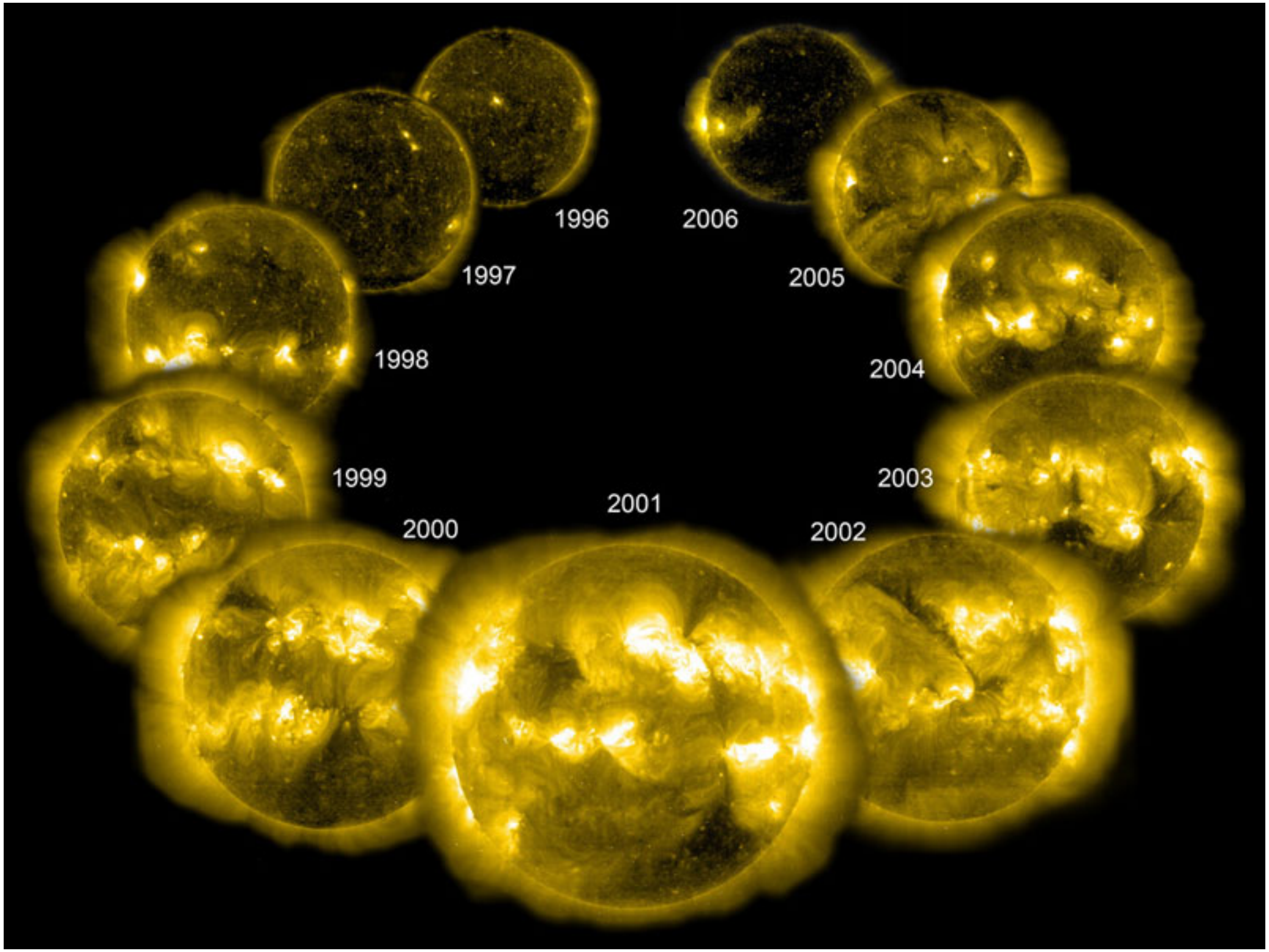
Butterfly diagram

Solar activity is changed ~ 11years



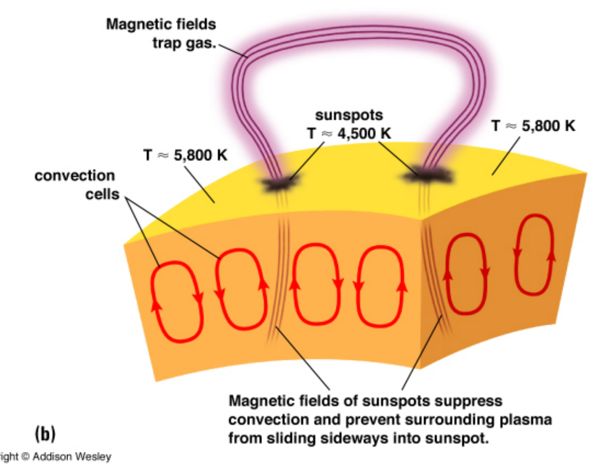
Solar cycle (cont.)

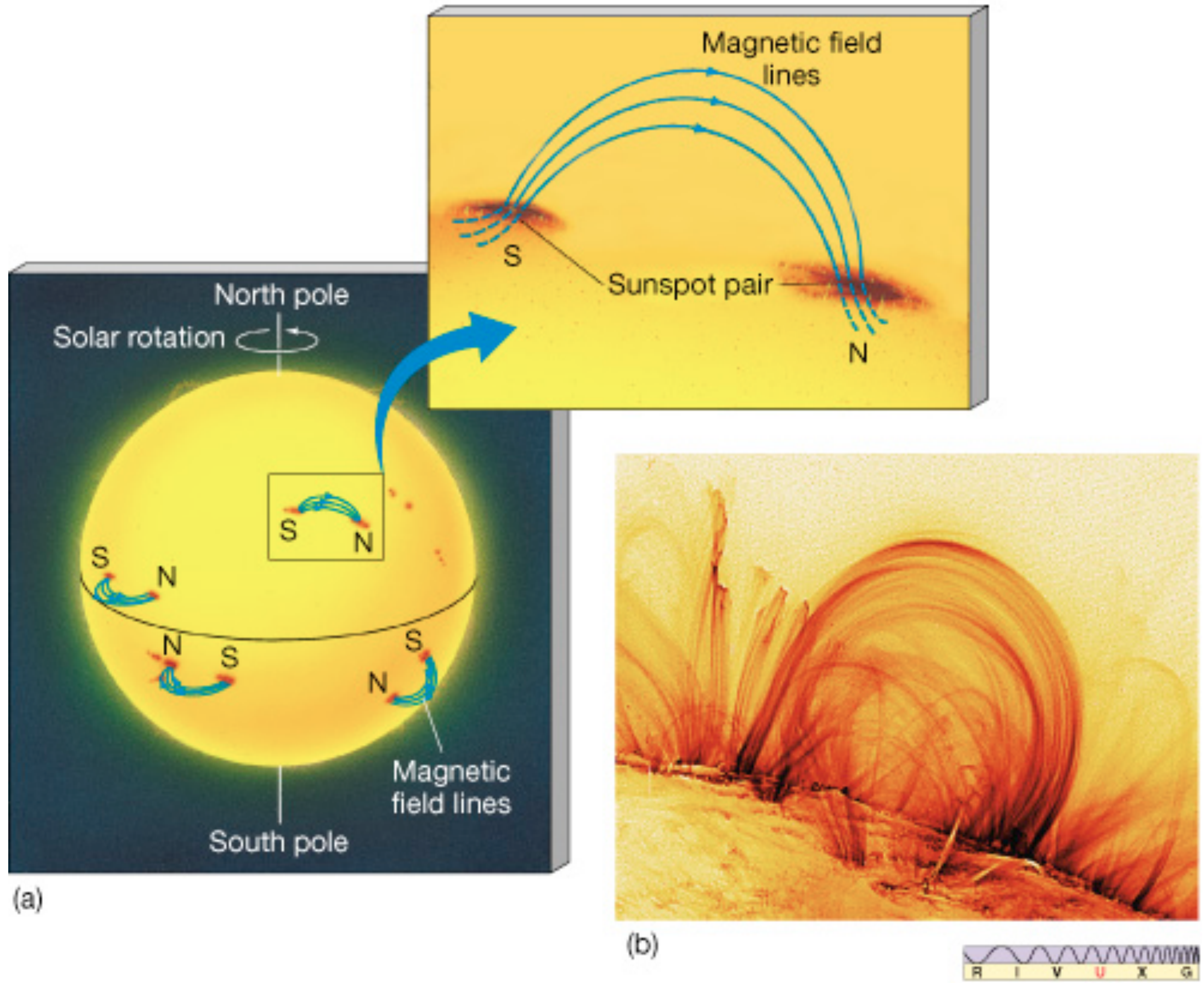
SOHO–EIT (EUV)



Sun spots

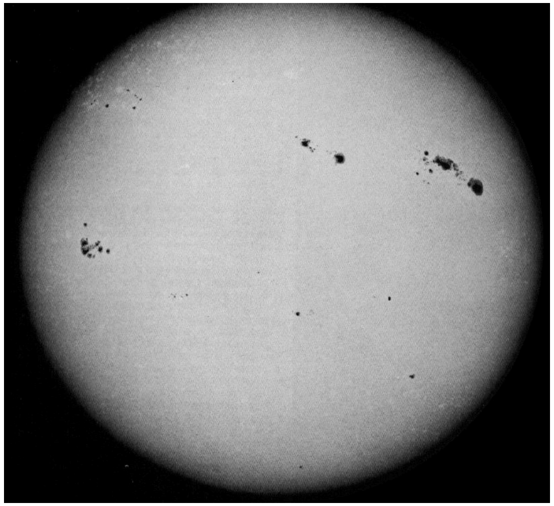
- Sun spots are emerged on solar surface with pair (N-S)



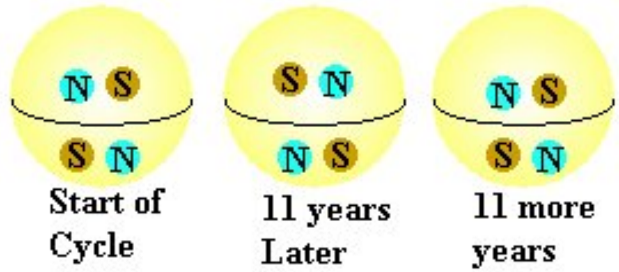


Solar cycle (cont.)

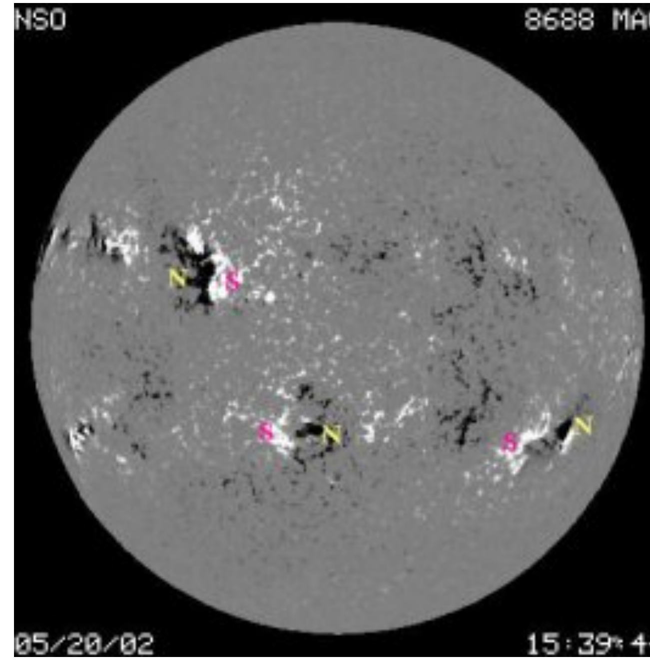
Polarity rules

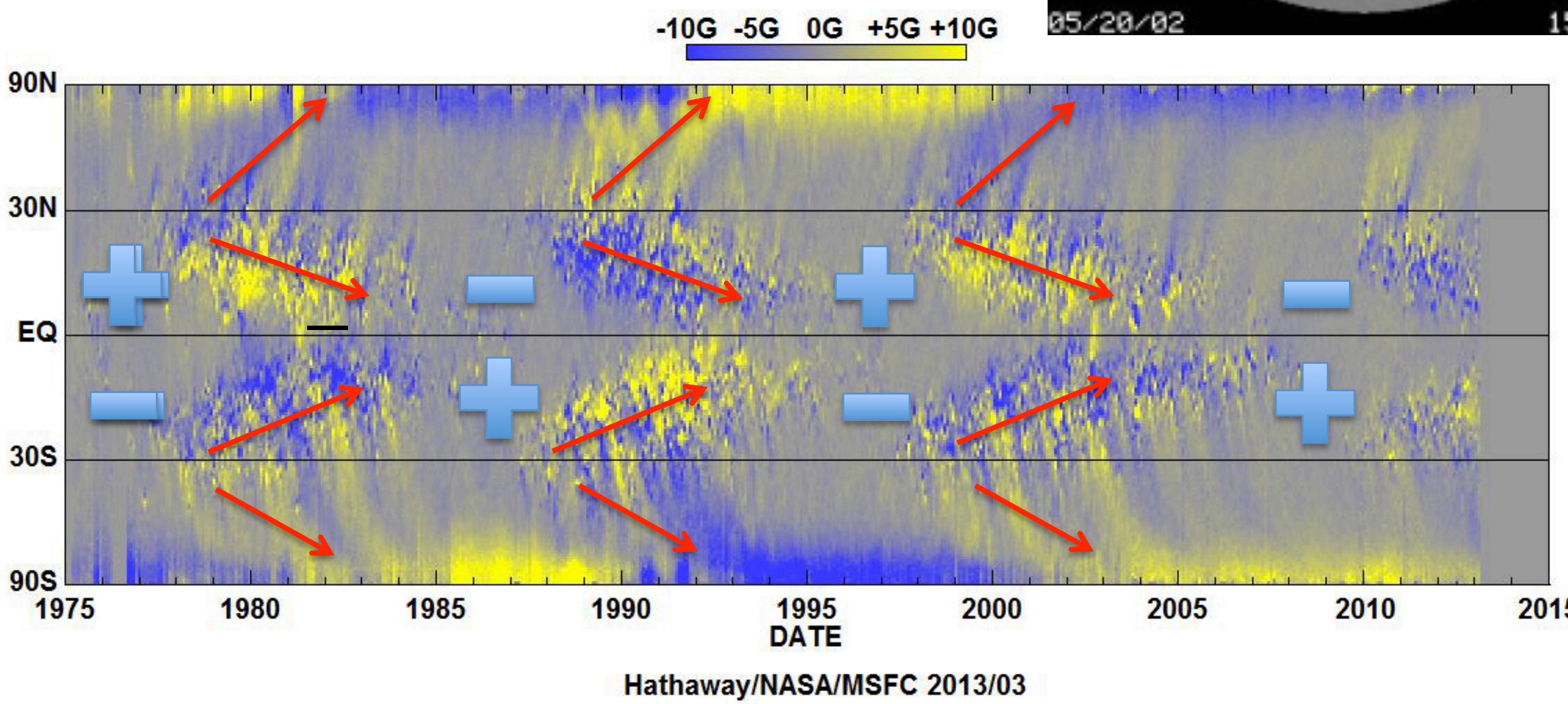


Polarity change ~ 11years

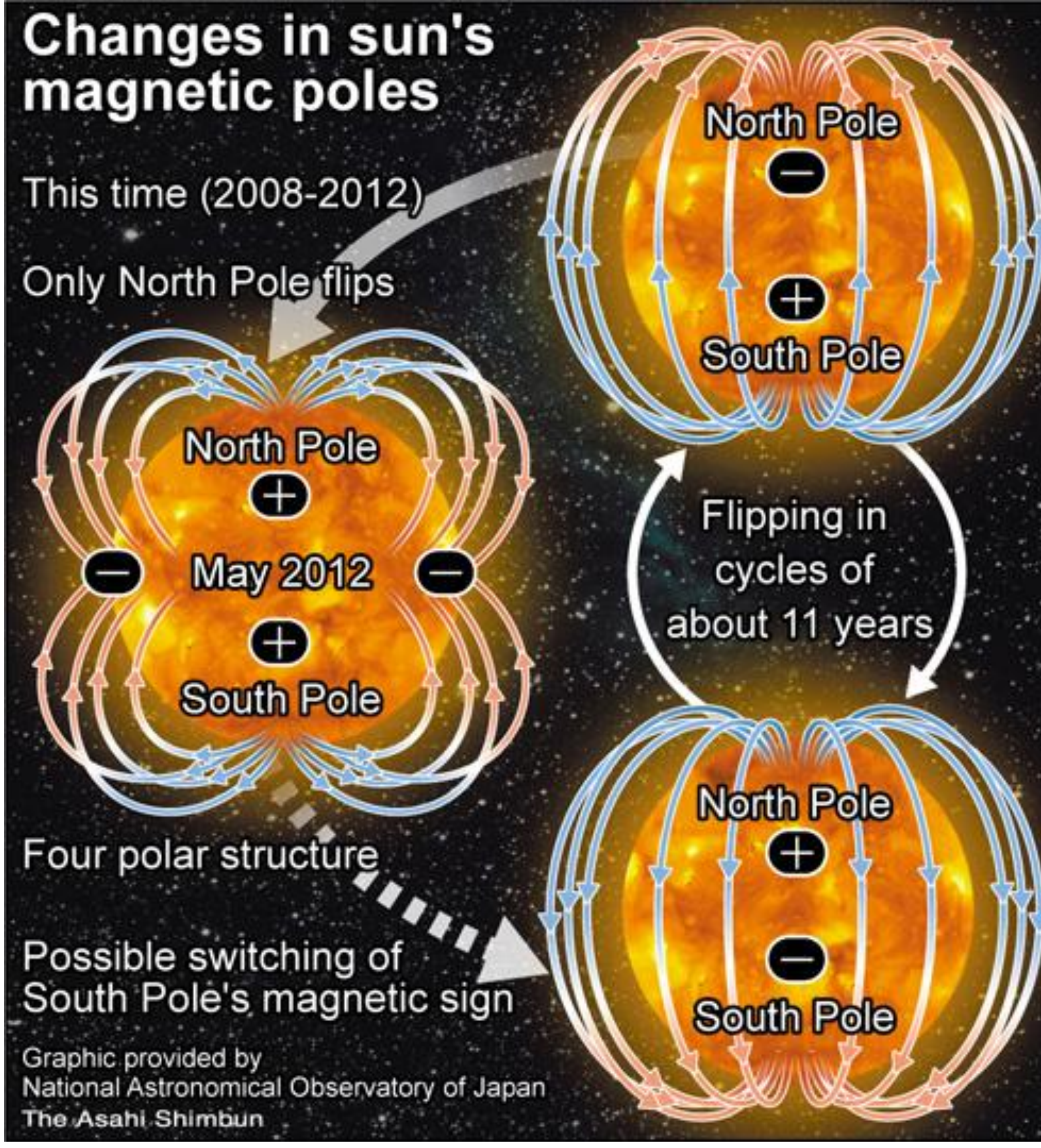


Magnetogram (Zeeman effect)

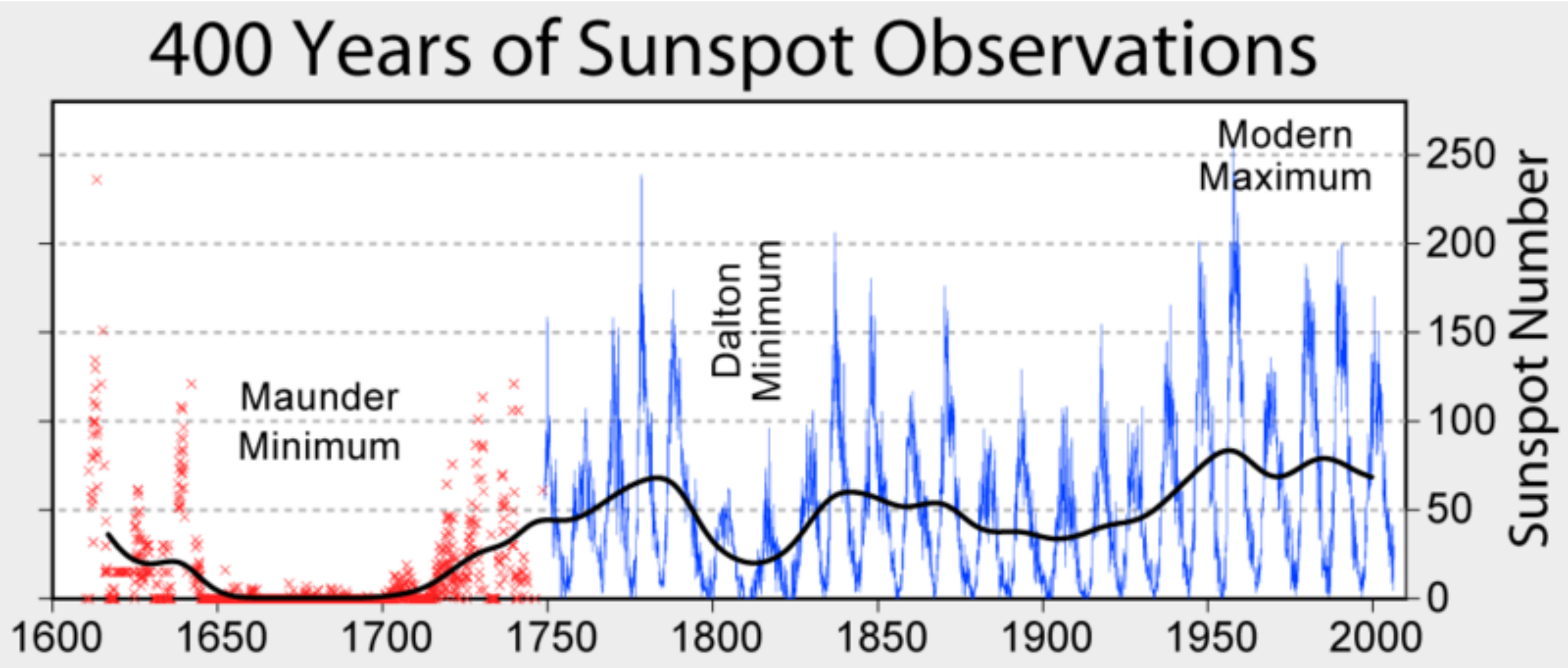




Solar cycle (cont.)



Solar cycle (cont.)



Dynamo hypothesis

- Larmor (1919): Magnetic field of Earth and Sun maintained by self-excited dynamo
- Dynamo: $\boldsymbol{v}$ x $\boldsymbol{B}$ => $\boldsymbol{j}$ => $\boldsymbol{B}$ => $\boldsymbol{v}$ (Faraday, Ampele, & Lorentz) motion of an electrical conductor in an 'inducing' magnetic field => induction of electric current
- Self-excited dynamo: inducing magnetic field created by the electric current (Siemens 1867)
- Example:
  - homopolar (disk) dynamo
  - Homogeneous dynamo (no wires in Earth core or solar convection zone) => complex motion necessary
  - Kinematic ($\boldsymbol{v}$ prescribed, linear)
  - Dynamic ($\boldsymbol{v}$ determined by forces, including Lorentz force, non-linear)

Kinetic and Dynamic Dynamo

- Time evolution of magnetic field is described by induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \qquad (9.1) \qquad \eta = 1/\mu_0 \sigma$$

- If plasma velocity *v* is a given function of *x* and *t*, eq(9.1) becomes a linear equation.
- The dynamo in such a case is called *kinetic dynamo* or *linear dynamo*.
- The kinetic dynamo is applied to the situation where *J* x *B* force is much smaller than other forces.

Kinetic and Dynamic Dynamo (cont.)

- However, in many astrophysical situations, this is not necessarily the case
- We must solve the momentum equation including ***J*** x ***B*** force term

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] = -\nabla p + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \rho \nabla \Phi \quad (9.2)$$

- Since this equation includes the term with ***B***, the velocity ***v*** is a function of ***B***.
- Hence in this case eq (9.1) becomes a nonlinear equation.
- The dynamo process determined by eq (9.1) & (9.2) is called *nonlinear dynamo* or *dynamic dynamo*

Vector potential

- Because $\nabla \cdot \boldsymbol{B} = 0$, we can write

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \ \nabla \cdot \boldsymbol{A} = 0$$

where $\boldsymbol{A}$ is called the vector potential.

- The induction equation in term of $\boldsymbol{A}$ is

$$\frac{\partial \boldsymbol{A}}{\partial t} = (\boldsymbol{v} \times \nabla \times \boldsymbol{A}) + \eta \nabla^2 \boldsymbol{A} + \nabla \phi, \ \nabla^2 \phi = \nabla \cdot (\boldsymbol{v} \times \boldsymbol{B})$$

Axisymmetric field decomposition

- If the magnetic field and the flow are axisymmetric, the field decomposition is

toroidal poloidal

$$\begin{aligned} \boldsymbol{B} &= B\hat{\phi} + \boldsymbol{B}_p = B\hat{\phi} + \nabla \times A\hat{\phi}, \\ \boldsymbol{v} &= R\Omega\hat{\phi} + \boldsymbol{v}_p = R\Omega\hat{\phi} + \nabla \times \frac{\psi}{R}\hat{\phi}, \\ R &= r\sin\theta \end{aligned}$$

(cylindrical)

- The induction equation becomes

poloidal

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{1}{R}(\boldsymbol{v}_p \cdot \nabla)(RA) &= \eta\left(\nabla^2 - \frac{1}{R^2}\right)A, \\ \frac{\partial B}{\partial t} + R(\boldsymbol{v}_p \cdot \nabla)\frac{B}{R} &= \eta\left(\nabla^2 - \frac{1}{R^2}\right)B + R\boldsymbol{B}_p \cdot \nabla\Omega \end{aligned} \quad (9.3)$$

toroidal

Axisymmetric field decomposition (cont.)

- This gives some important insight into the dynamo process.
- $(\boldsymbol{v}_p \cdot \nabla)$: advection term, which movies field around
- $(\nabla^2 - 1/R^2)$: diffusion term, which cannot create field
- The toroidal field (B) can be generated from poloidal field (B_p) through the term, $R\boldsymbol{B}_p \cdot \nabla\Omega$, provided gradients of angular velocity along the field lines
- Poloidal field is stretched out by differential rotation $\nabla\Omega$ to generate toroidal field
- However, the poloidal field (A) itself has no source term, so it will just decay unless we can find a way to sustain it.
- This requires some non-axisymmetric terms to be present.

Cowling's theorem

- Fundamental theorem in the dynamo theory, called *Cowling's theorem (anti-dynamo theorem)*

  "*A steady axisymmetric magnetic field cannot be maintained*"

- Assume: steady axisymmetric magnetic fields have both poloidal (B_r & B_z) and toroidal (B_ϕ) components in cylindrical geometry
- Consider Ohm's law

$$\frac{\boldsymbol{j}}{\sigma} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$$

- From assumption of axisymmetry, we have a circle path C where $B_r = B_z = 0$ (neutral point, see figure)

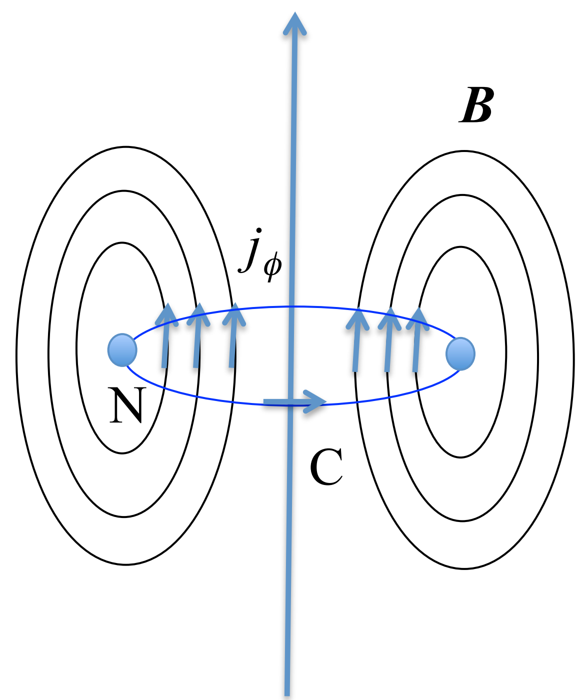
Cowling's theorem (cont.)

- Integrating Eq (9.3) on C,

$$\oint_c \frac{\boldsymbol{j}}{\sigma} \cdot ds = \oint_c \boldsymbol{E} \cdot ds + \oint_c (\boldsymbol{v} \times \boldsymbol{B}) \cdot ds$$

- Using Stoke's theorem,

$$\oint_c \frac{j_\phi}{\sigma} ds = \int_S (\nabla \times \boldsymbol{E}) \cdot dS + \oint_c (\boldsymbol{v} \times \boldsymbol{B})_\phi \cdot ds$$



- The first-term in R.H.S. becomes 0, because from steady assumption
  $\nabla \times \boldsymbol{E} = \partial \boldsymbol{B} / \partial t = 0$

- The second-term in R.H.S. becomes

$$\oint_c (\boldsymbol{v} \times \boldsymbol{B})_\phi \cdot ds = \oint_c (v_z B_r - v_r B_z) ds$$

Cowling's theorem (cont.)

- Since $B_r = B_z = 0$ on the path C, second term also vanishes.
- Thus,

$$\oint_c \frac{j_\phi}{\sigma} ds = 0$$

- So, $j_\phi = 0$ on C.
- Since j_ϕ should not vanish on C to maintain the poloidal field, this means that

the steady axisymmetric magnetic field cannot be maintained.

- But, Cowling's argument holds only for exact axisymmetry
- Slight departures from axisymmetry may allow a dynamo to work.

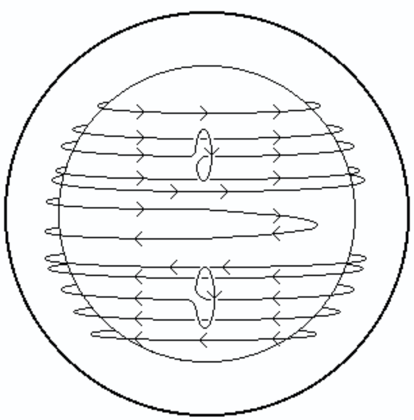
Beyond anti-dynamo theorem

- Parker (1955) pointed out that as rising blobs of plasma expand they also tend to rotate because of the Coriolis force in the Sun
- Such anticylonic (helical turbulent) motions are clockwise in the northern hemisphere and anti-clockwise in the southern hemisphere
- If they carry flux tubes up with them, the twist converts toroidal field into poloidal ones.
- He consider the rate of generation of B_p is proportional to B_ϕ
- So Parker modeled the net effect of many convection cells (turbulence) by adding an electric field (α-effect)

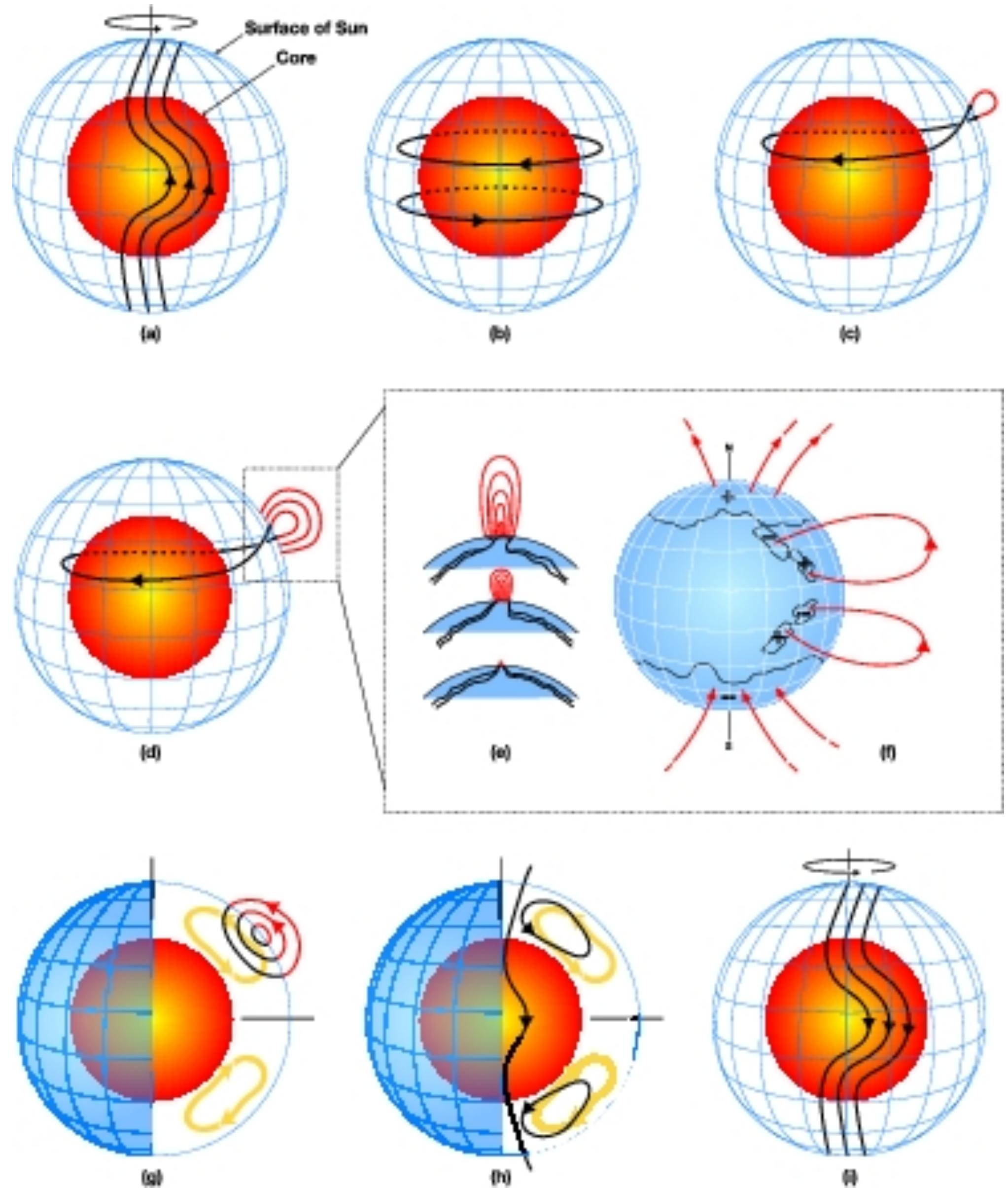
$$E_\phi = \alpha B_\phi$$

- B_ϕ => E_ϕ => j_ϕ => B_p

Beyond anti-dynamo theorem (cont.)



The α-effect



Mean field dynamo theory

- α-effect removes certain symmetry to work dynamo (overcome anti-dynamo theorem).
- The α-effect can work only rotating systems, where Coriolis force exists (differential rotation) and therefore results in non-zero helicity.
- Fortunately, most astrophysical systems rotate and also host sustain convective turbulence

velocity: $\boldsymbol{v}$

volticity: $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$

helicity: $H = \boldsymbol{v} \cdot \boldsymbol{\omega}$

- In a turbulent magnetized medium, the flow and field components can be expressed as a sum of fluctuating and mean components.
- This kind of dynamo theory is called mean-field dynamo

Averaging dynamo equations

- The basic idea is to split of the magnetic field and flow into mean and fluctuating parts,

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{B}', \quad \boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}'$$

- Apply the Reynolds averaging rules: assume linear averaging process

$$\overline{\boldsymbol{B}_1 + \boldsymbol{B}_2} = \overline{\boldsymbol{B}_1} + \overline{\boldsymbol{B}_2}, \quad \overline{\boldsymbol{v}_1 + \boldsymbol{v}_2} = \overline{\boldsymbol{v}_1} + \overline{\boldsymbol{v}_2}$$

- And

$$\overline{\overline{\boldsymbol{B}}} = \overline{\boldsymbol{B}}, \; \overline{\overline{\boldsymbol{v}}} = \overline{\boldsymbol{v}} \qquad \overline{\boldsymbol{B}'} = \overline{\boldsymbol{v}'} = 0$$

- Assume averaging commutes with differentiating, so

$$\overline{\frac{\partial \boldsymbol{B}}{\partial t}} = \frac{\partial}{\partial t}\overline{\boldsymbol{B}}, \quad \overline{\nabla \cdot \boldsymbol{B}} = \nabla \cdot \overline{\boldsymbol{B}}$$

Mean field induction equation

- we average the induction equation

$$\overline{\frac{\partial \boldsymbol{B}}{\partial t}} = \overline{\nabla \times (\boldsymbol{v} \times \boldsymbol{B})} + \eta \nabla^2 \overline{\boldsymbol{B}}$$

- Using the Reynolds averaging rules,

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times \overline{(\boldsymbol{v} \times \boldsymbol{B})} + \eta \nabla^2 \overline{\boldsymbol{B}} \qquad (9.4)$$

- The interesting term is $\overline{(\boldsymbol{v} \times \boldsymbol{B})}$

$$\overline{(\boldsymbol{v} \times \boldsymbol{B})} = \overline{\overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{v}} \times \boldsymbol{B}' + \boldsymbol{v}' \times \overline{\boldsymbol{B}} + \boldsymbol{v}' \times \boldsymbol{B}'}$$

- Therefore, we can write induction equation ($\overline{\boldsymbol{B}'} = \overline{\boldsymbol{v}'} = 0$)

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times (\overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}}) + \underline{\nabla \times \mathcal{E}} + \eta \nabla^2 \overline{\boldsymbol{B}}, \quad \mathcal{E} = \overline{\boldsymbol{v}' \times \boldsymbol{B}'} \qquad (9.5)$$

- ε is called mean e.m.f (electro-motive force) and it is new term of induction equation.

Mean field induction equation (cont.)

- We usually think of the primed quantities as being small scale turbulent fluctuations.
- The average mean e.m.f. can be nonzero if the turbulence has suitable averaged properties.

- No longer does Cowling's theorem apply (break axisymmetry by the small turbulent fluctuation).

Mean field induction equation (cont.)

- If we subtract the mean field equation (9.3) from the full equation (9.1),

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \nabla \times (\overline{\boldsymbol{v}} \times \boldsymbol{B}') + \nabla \times (\boldsymbol{v}' \times \overline{\boldsymbol{B}}) + \nabla \times \mathcal{G} + \eta \nabla^2 \boldsymbol{B}'$$

$$\mathcal{G} = \boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$$

- This is a linear equation for ***B***', with a forcing term $\nabla \times (\boldsymbol{v}' \times \overline{\boldsymbol{B}})$
- Therefore, *B*' can be thought of as the turbulent field generated by the turbulent ***v***' acting on the mean field $\overline{\boldsymbol{B}}$
- Therefore we can plausibly write (average mean e.m.f can be related to mean B-field, tensor approach)

$$\mathcal{E}_i = a_{ij} \overline{B}_j + b_{ijk} \frac{\partial \overline{B}_j}{\partial x_k} + \ldots$$

- Where the tensor a_{ij} and b_{ijk} depend on $\boldsymbol{v}'$ & $\overline{\boldsymbol{v}}$

Mean field induction equation (cont.)

- We don't know ***v***' and its unobservable, so we assume a_{ij} and b_{ijk} are simple isotropic tensors (realistically it may have directional dependence)

$$a_{ij} = \alpha(\boldsymbol{x})\delta_{ij}, \quad b_{ijk} = -\beta(\boldsymbol{x})\epsilon_{ijk} \tag{9.6}$$

- We now have the mean field dynamo theory (MFDT) equations in their usual form,

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times (\overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}}) + \nabla \times \alpha \overline{\boldsymbol{B}} - \nabla \times (\beta \nabla \times \overline{\boldsymbol{B}}) + \eta \nabla^2 \overline{\boldsymbol{B}} \tag{9.7}$$

- If β is constant, $\nabla \times (\beta \nabla \times \overline{\boldsymbol{B}}) = -\beta \nabla^2 \overline{\boldsymbol{B}}$ so the β-term acts like an enhanced diffusivity (turbulent diffusivity).
- We can now justify taking a large diffusion, choosing it to give agreement with observation.

First order smoothing

- The tensor approach is very general, but it gives lots of unknowns.
- Can we solve for ***B***’ in terms of ***v***’ directly?
- With a short correlation length, *l*, the mean velocity term (which is constant over the short length scale) can be removed by working in moving frame.
- Then, we have and order of magnitude is

$$\frac{\partial \boldsymbol{B}'}{\partial t} = (\overline{\boldsymbol{B}} \cdot \nabla)\boldsymbol{v}' + \nabla \times (\boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}) + \eta \nabla^2 \boldsymbol{B}'$$

O(B’/τ) O($B^{bar}v$’/l) O(B’ v’ /l) O(B’/l^2)

- If the small-scale magnetic Reynolds number v’l/η is small, curl term (2nd term in R.H.S.) is negligible.

First order smoothing (cont.)

- First-order smoothing assumption gives

$$\frac{\partial \boldsymbol{B}'}{\partial t} = (\overline{\boldsymbol{B}} \cdot \nabla)\boldsymbol{v}' + \eta \nabla^2 \boldsymbol{B}' \qquad (9.8)$$

- This implies that $\boldsymbol{B}' \ll \overline{\boldsymbol{B}}$ (but it is probably not true in Sun).
- Now suppose the turbulence to be a random superposition of waves,

$$\boldsymbol{v}' = Re\{\boldsymbol{v} \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)\}$$

- Then using eq (9.8)

$$\boldsymbol{B}' = Re\{\frac{i(\boldsymbol{k} \cdot \overline{\boldsymbol{B}})\boldsymbol{v}}{\eta k^2 - i\omega} \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)\}$$

First order smoothing (cont.)

- Now evaluate $\mathcal{E}$

$$\mathcal{E} = \overline{\boldsymbol{v}' \times \boldsymbol{B}'} = \frac{1}{2}\frac{i\eta k^2(\boldsymbol{k}\cdot\overline{\boldsymbol{B}})}{\eta^2 k^4 + \omega^2}(\boldsymbol{v}^* \times \boldsymbol{v})$$

- Where * denotes complex conjugate, equivalent to

$$\mathcal{E}_i = a_{ij}\overline{B}_j$$

$$a_{ij} = \frac{1}{2}\frac{i\eta k^2}{\eta^2 k^4 + \omega^2} k_j \epsilon_{imn} v_m^* v_n$$

- If the turbulence has no preferred direction, i.e. it is isotropic,

$$a_{ij} = \frac{1}{2}\frac{i\eta k^2}{\eta^2 k^4 + \omega^2} \delta_{ij} k_j \epsilon_{imn} v_m^* v_n \qquad (9.9)$$

First order smoothing (cont.)

- Now consider the helicity

$$H = \overline{\boldsymbol{v}' \cdot \nabla \times \boldsymbol{v}'} = \frac{1}{2} i\boldsymbol{k} \cdot (\boldsymbol{v}^* \times \boldsymbol{v})$$

velocity: $\boldsymbol{v}$

volticity: $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$

helicity: $H = \boldsymbol{v} \cdot \boldsymbol{\omega}$

- Taking the trace of eq (9.9) gives

$$\alpha = -\frac{1}{3} \frac{\eta k^2 H}{\eta^2 k^4 + \omega^2}$$

- This means that under first order smoothing, the mean e.m.f. is proportional to the helicity of the turbulence.
- Mirror-symmetric turbulence has zero helicity.
- Rotating convection has non-zero helicity in general.

Parker loop mechanism

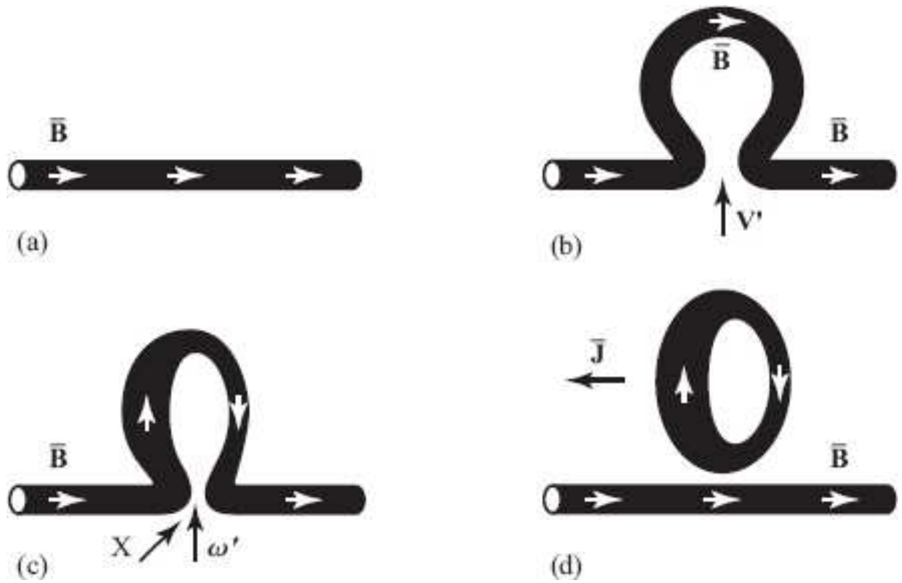
- Mean field theory predicts an e.m.f. parallel to the mean magnetic field,

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times (\overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}}) + \nabla \times \alpha \overline{\boldsymbol{B}} + \eta_T \nabla^2 \overline{\boldsymbol{B}}$$

Where η_T: turbulent diffusivity (resistivity) ($\beta >> \eta$, $\eta_{eff} = \beta = \eta_T$)

- This contrasts with $\overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}}$ which is perpendicular to the mean field.
- With constant α, the α-effect predicts growth of field parallel to the current $\mu_0 \nabla \times \boldsymbol{B}$
- Recalling that the α-effect depends on helicity, we can picture this process (next page)

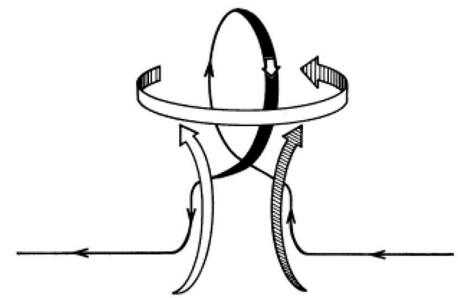
Parker loop mechanism (cont.)



velocity: $\boldsymbol{v}$

volticity: $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$

helicity: $H = \boldsymbol{v} \cdot \boldsymbol{\omega}$



- A rising twisting element of fluid brings up magnetic field.
- A loop of flux is created, which then twists due to helicity (volticity).
- The loop current is parallel to the original mean field.
- Poloidal field has been created out of toroidal field.

- Note that if there is too much twist, the current is in the opposite direction. First order smoothing assumes small twist.

Axisymmetric mean field dynamos

- The mean field dynamo equations with isotropic are derived from eq (9.3)

poloidal; α-effect; turbulent diffusion

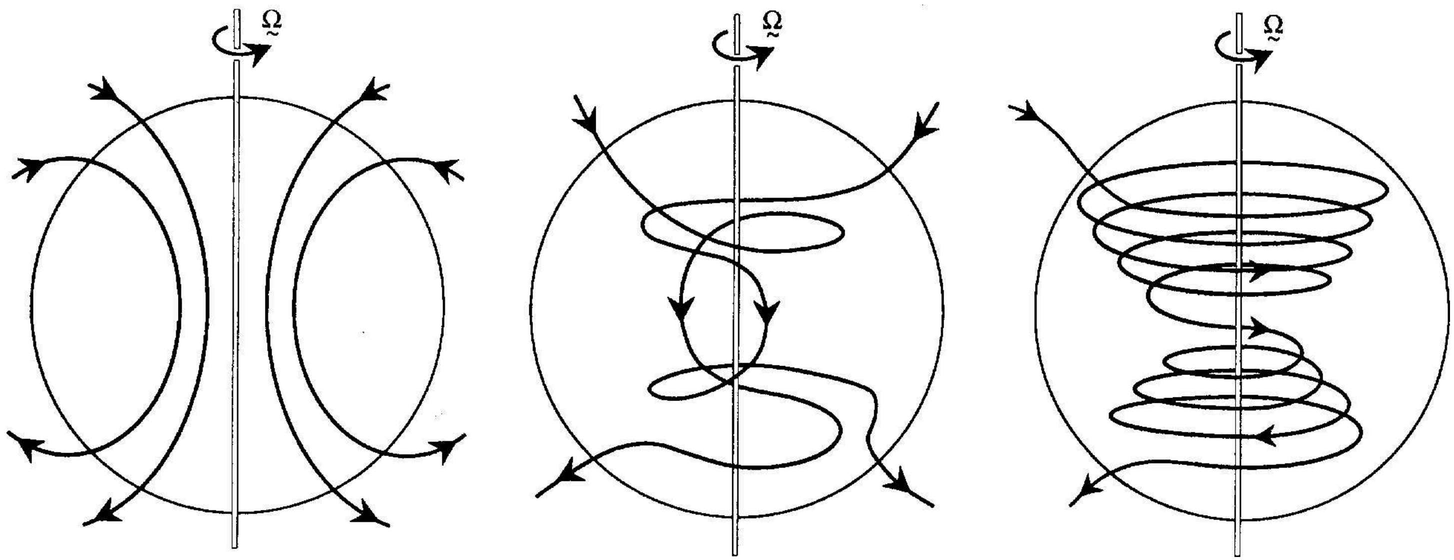
$$\frac{\partial A}{\partial t} + \frac{1}{R}(\boldsymbol{v}_p \cdot \nabla)(RA) = \alpha B + \eta \left(\nabla^2 - \frac{1}{R^2} \right) A, \qquad (9.10)$$

toroidal; Stretching (ω-effect)

$$\frac{\partial B}{\partial t} + R(\boldsymbol{v}_p \cdot \nabla)\frac{B}{R} = \nabla \times \alpha \boldsymbol{B}_p + \eta \left(\nabla^2 - \frac{1}{R^2} \right) B + R\boldsymbol{B}_p \cdot \nabla\Omega \qquad (9.11)$$

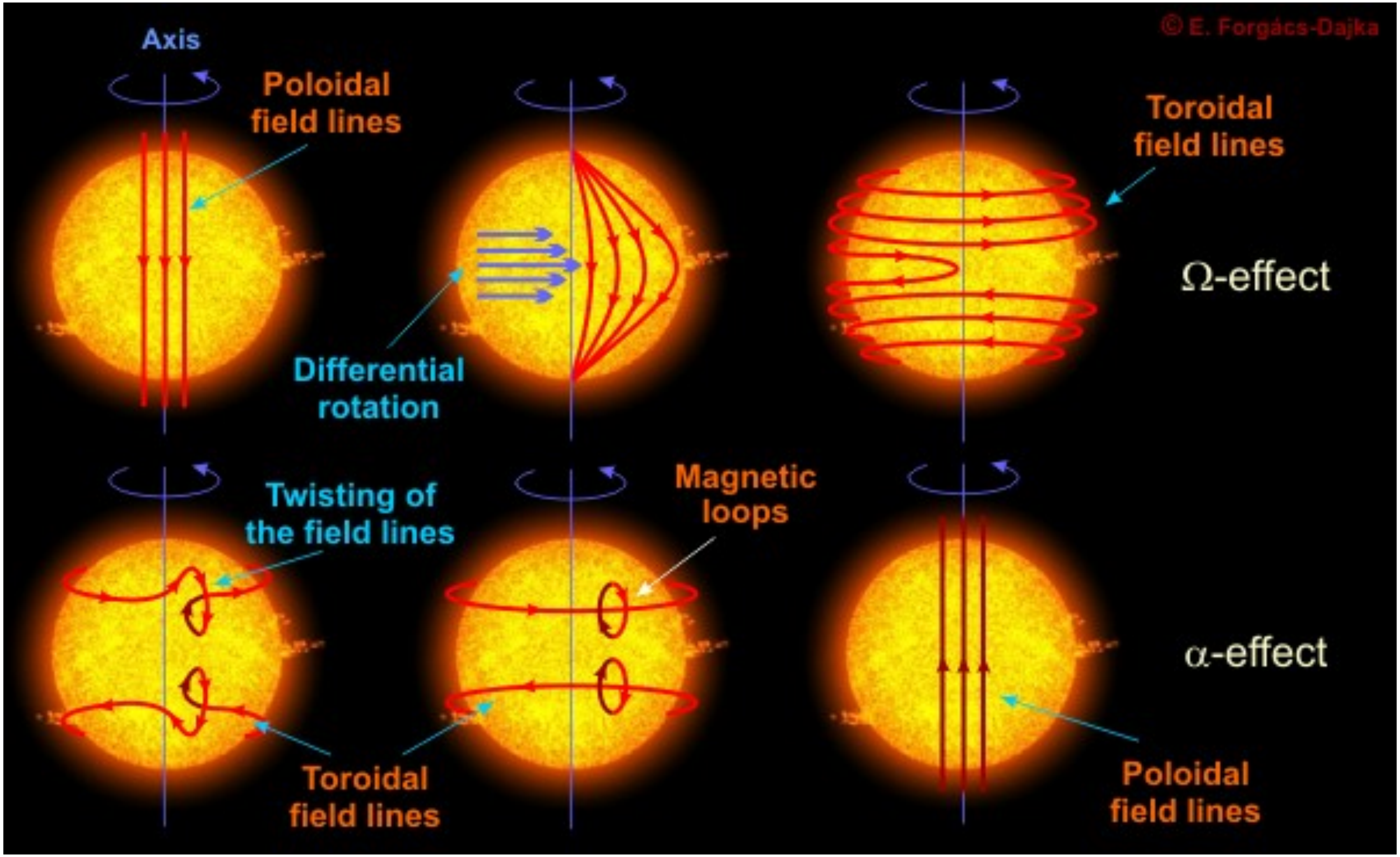
- The α-effect term is the source for generating poloidal field (A) from toroidal field (B) (no dynamo if $\alpha = 0$,).
- There are two ways of generating toroidal field $\boldsymbol{B}$ from poloidal field $\boldsymbol{B}_P$: the α-effect or the ω-effect (stretch).
- If the α-effect dominates, the dynamo is called an α^2-dynamo.
- If the ω-effect dominates its an $\alpha\omega$ dynamo
- We can also have $\alpha^2\omega$ dynamo where both mechanisms operate

The Omega effect



- In figure, an initial loop of meridional field threads through the sphere.
- The inside of the sphere is rotating faster than the outside: so we have differential rotation.
- The induction term $R\boldsymbol{B}_p \cdot \nabla\Omega$ generates toroidal field by stretching
- As seen in figure, opposite sign B_ϕ is generated on either side of the equator, as in the Sun.

αω dynamo



© E. Forgács-Dajka
Axis
Poloidal field lines
Differential rotation
Toroidal field lines
Ω-effect
Twisting of the field lines
Magnetic loops
Toroidal field lines
α-effect
Poloidal field lines

Dynamo wave

- One of the most remarkable property of the (linear) $\alpha\Omega$ dynamo equations is that they support travelling wave solutions.
- This was first demonstrated in Cartesian geometry by Parker (1955), who proposed that a latitudinally-travelling dynamo wave was at the origin of the observed equatorward drift of sunspot emergences in the course of the cycle.
- This finding was subsequently shown to hold in spherical geometry, as well as for non-linear models (Yoshimura 1975, Stix 1976)
- A result now known as the Parker-Yoshimura sign rule

Dynamo wave (cont.)

- The simplest analysis of dynamo waves uses Cartesian geometry, and we assume the waves are independent of y (toroidal).

$$\boldsymbol{B} = (-\partial A/\partial z, B, \partial A/\partial x), \quad \boldsymbol{v} = (-\partial \psi/\partial z, v_y, \partial \psi/\partial x)$$

$$\frac{\partial A}{\partial t} + \frac{\partial(\psi, A)}{\partial(x,z)} = \alpha B + \eta \nabla^2 A,$$

$$\frac{\partial B}{\partial t} + \frac{\partial(\psi, B)}{\partial(x,z)} = \frac{\partial(A, v_y)}{\partial(x,z)} - \nabla \cdot (\alpha \nabla A) + \eta \nabla^2 B$$

- Set ψ=0, α =const., $v_y = \Omega z$, a constant shear, ignore the α term in the B equation ($\alpha\omega$ model) and set $A = \exp[i(\omega t + \boldsymbol{k} \cdot \boldsymbol{x})]$
- The dispersion relation is

$$(i\omega + \eta k^2)^2 = i k_x \alpha \Omega$$

- And

$$\omega = (1+i)\sqrt{\alpha \Omega k_x / 2} - \eta k^2$$

Dynamo wave (cont.)

- Assume αΩ < 0, i.e., α > 0, Ω < 0 and take *k* < 0

$$\omega = i\eta k^2 - (1+i)\sqrt{|\alpha\Omega k_x/2|}$$

- Growth rate:

$$-Im(\omega) = -\eta k^2 + \sqrt{|\alpha\Omega k_x/2|} \geq 0 \text{ for } \sqrt{|\alpha\Omega k_x/2|} \geq \eta k^2$$

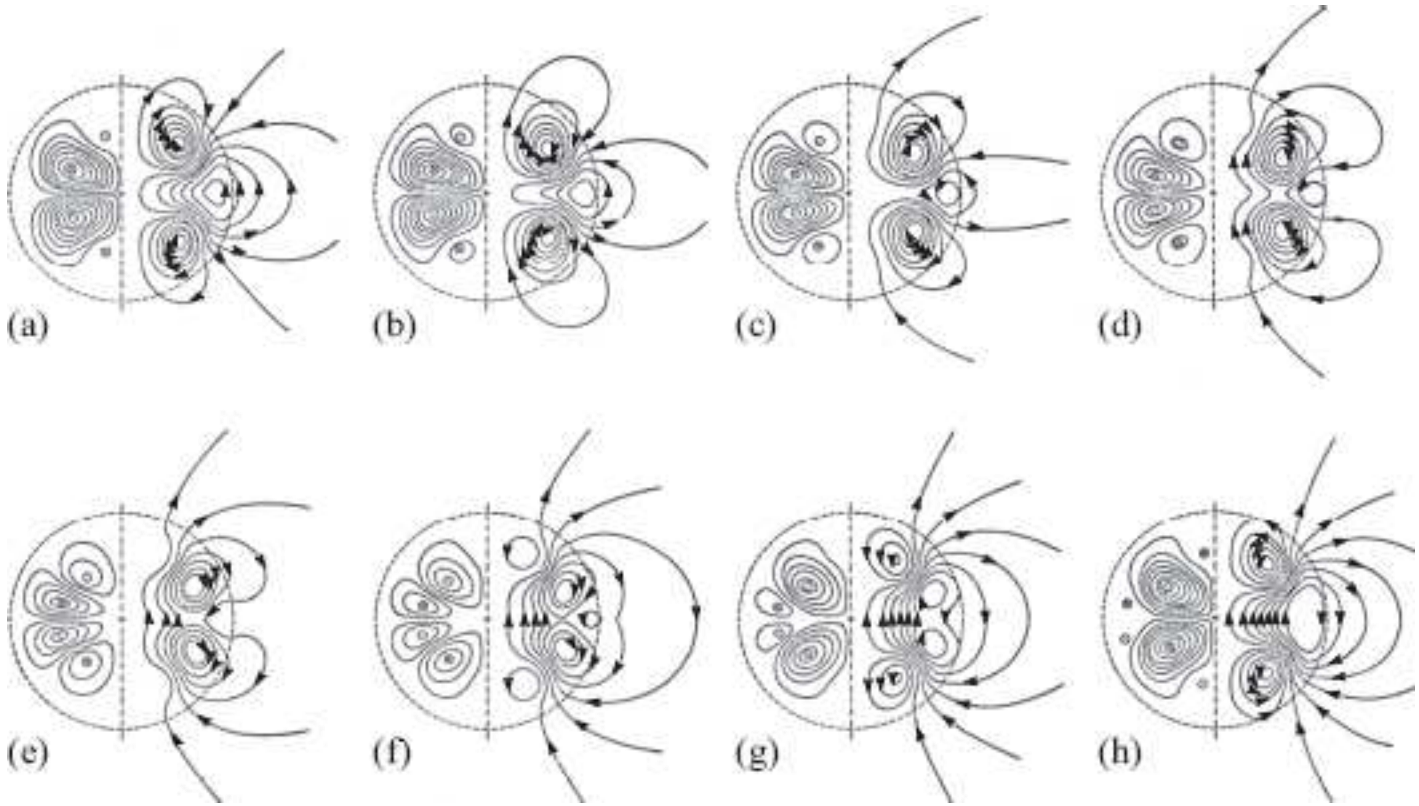
- Growing dynamo waves if αΩ term overcomes diffusion
- Frequency: $Re(\omega) = -\sqrt{|\alpha\Omega k_x/2|} < 0$
  - Wave propagation in positive x-direction
  - This is identical result for k < 0
  - if αΩ > 0, wave propagation in negative x-direction

- In general, wave propagates along surfaces of constant rotation (Yoshimura 1975)
- Direction of propagation depends on sign(αΩ)

Spherical αω dynamo

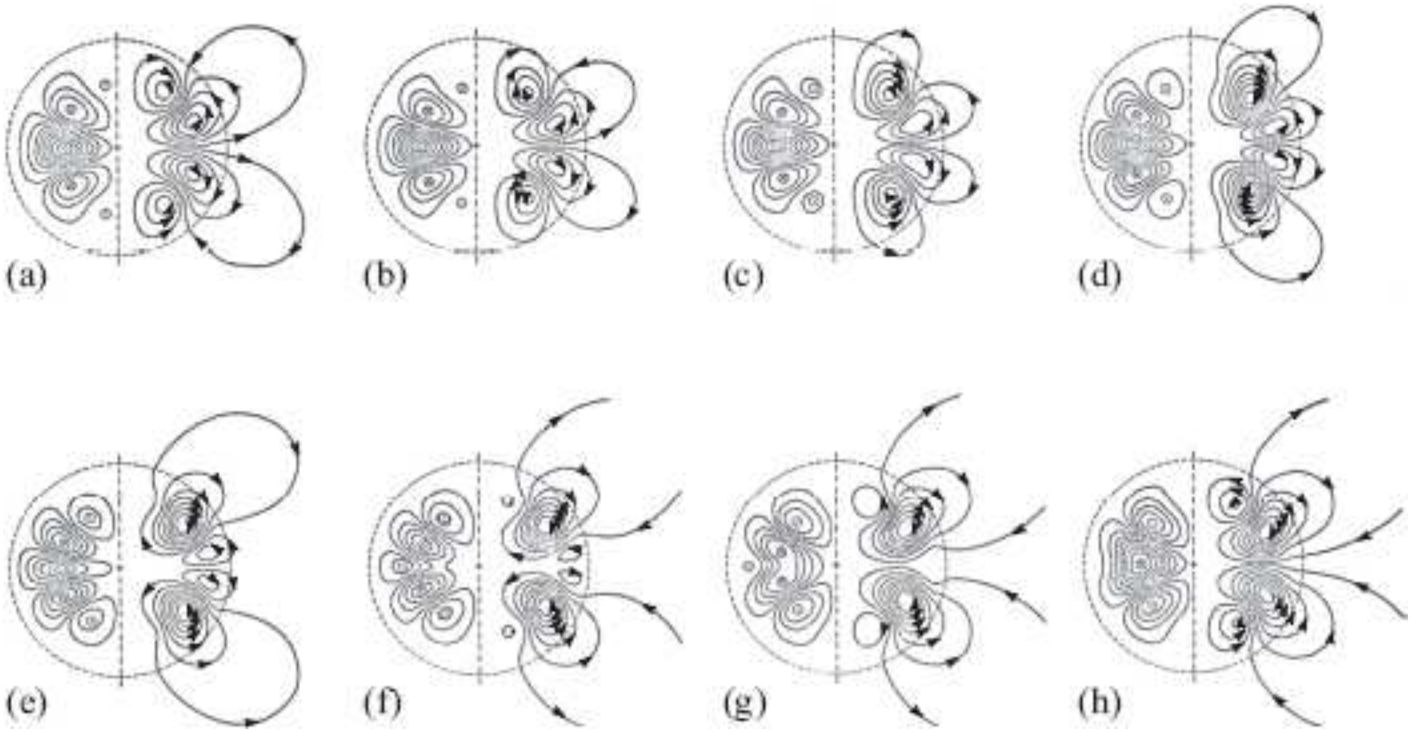
- Integrating the spherical geometry mean field dynamo equations (9.10) & (9.11) with $\alpha = f(r)\cos\theta$ and $\boldsymbol{v} = R\omega(r)\hat{\phi}$
- Various choices of the function $f(r)$ and $\omega(r)$ is considered in
- As expected from simple analysis of plane dynamo waves, αω dynamos give oscillatory solutions
- The results show that both dipolar and quadrupolar dynamos can occur
- Dipolar dynamos generally onset before quadrupolar dynamos if $\alpha\omega' < 0$

Spherical αω dynamo (cont.)



- Dipolar oscillatory solution of axisymmetric αω-dynamo in a sphere.
- (a)-(h) goes through one period.
- Right: meridional (poloidal) field, left: toroidal field
- ***B*** antisymmetric about equator, ***A*** symmetric

Spherical αω dynamo (cont.)



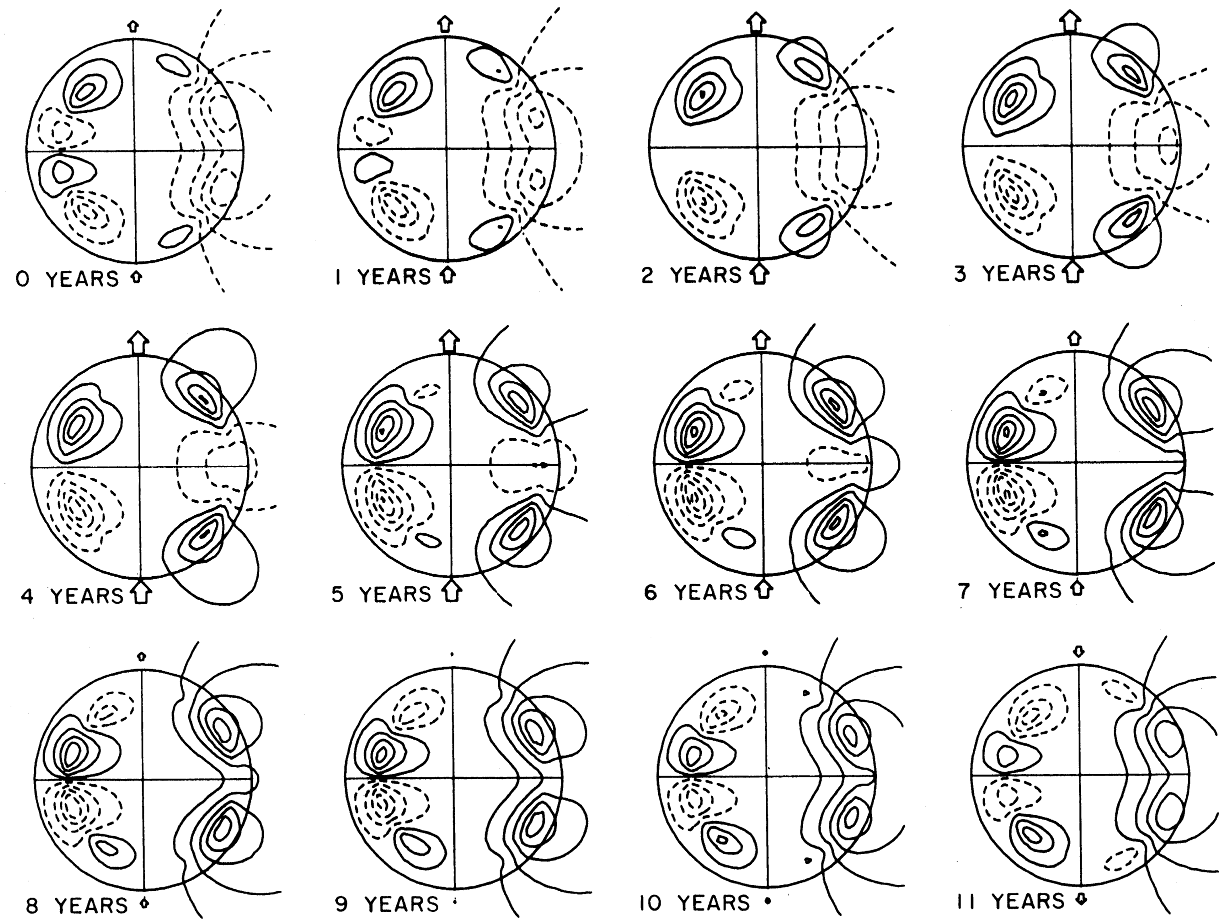
- quadrupolar oscillatory solution of axisymmetric αω-dynamo in a sphere.
- (a)-(h) goes through one period.
- Right: meridional (poloidal) field, left: toroidal field
- ***B*** antisymmetric about equator, ***A*** symmetric

Spherical mean field dynamo theory

- Brief summary of numerical findings about spherical MFDT model
  - Generally, α^2 models give steady dynamos (no dynamo wave), $\alpha\omega$ dynamos give oscillatory solutions. But, some α distributions, particularly if there are positive and negative values in the same hemisphere, can give steady $\alpha\omega$ dynamos.
  - Meridional circulation, non-zero ψ, can also help to steady $\alpha\omega$ dynamos
  - Unfortunately, it seems that a different dynamo behavior can be found depending on the spatial α distribution, even if α is restricted to the isotropic case. This is a major problem for modeling, as there is little prospect of determining α by observation

Solar dynamo

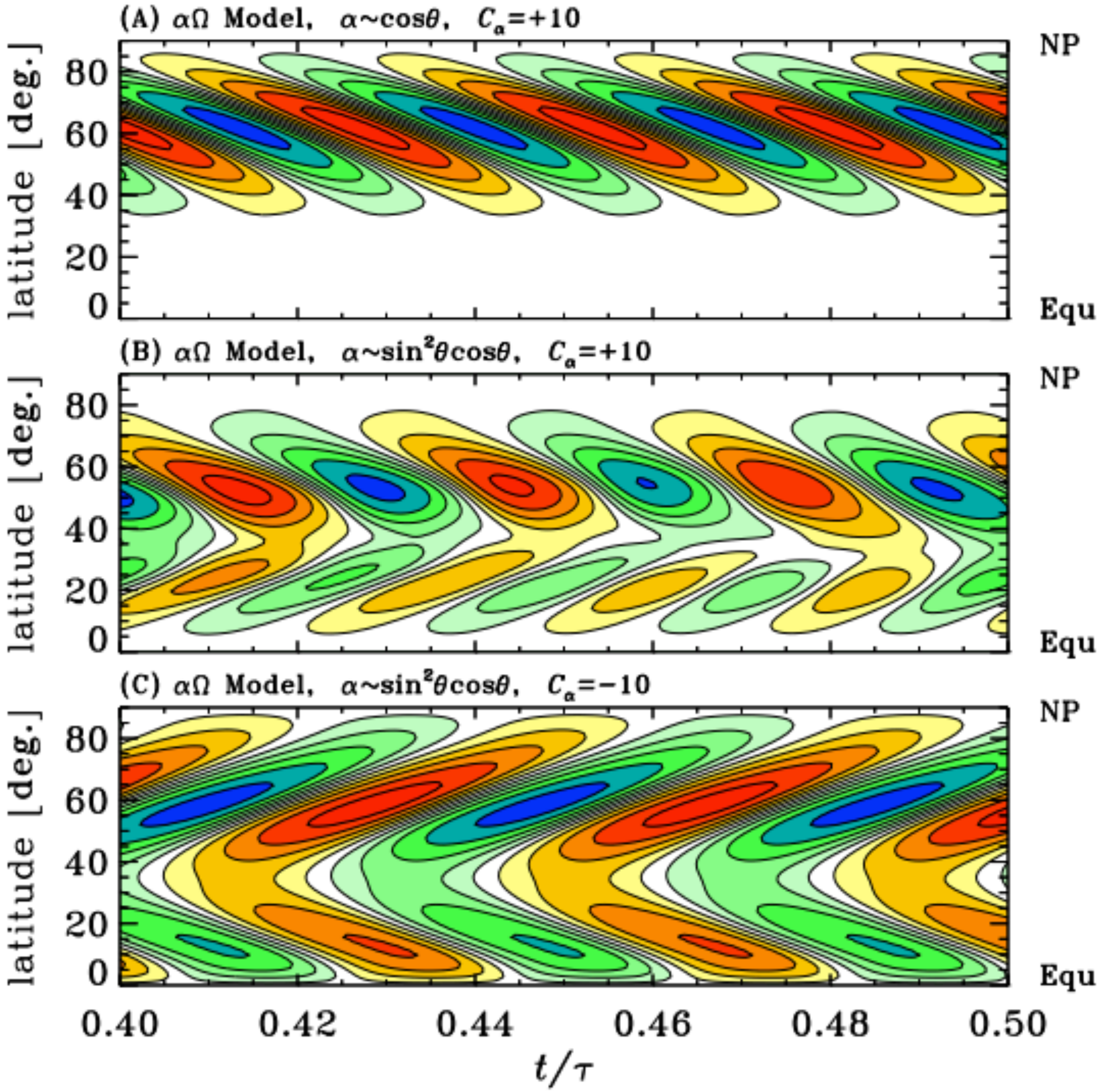
- αω dynamo in convection zone (in solar interiors), $\Omega(r)$ with $\partial\Omega/\partial r < 0, \alpha \sim \cos\theta, \eta_T = 10^{10}\mathrm{cm}^2/\mathrm{s}$



Stix 1976

Solar dynamo (cont.)

- Theoretical butterfly diagram

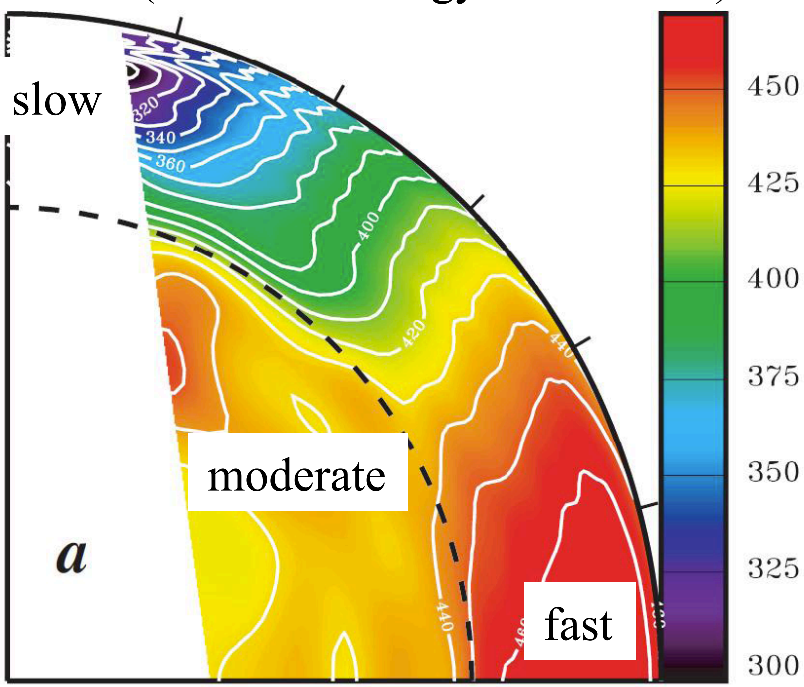


Charbnneau (2010)

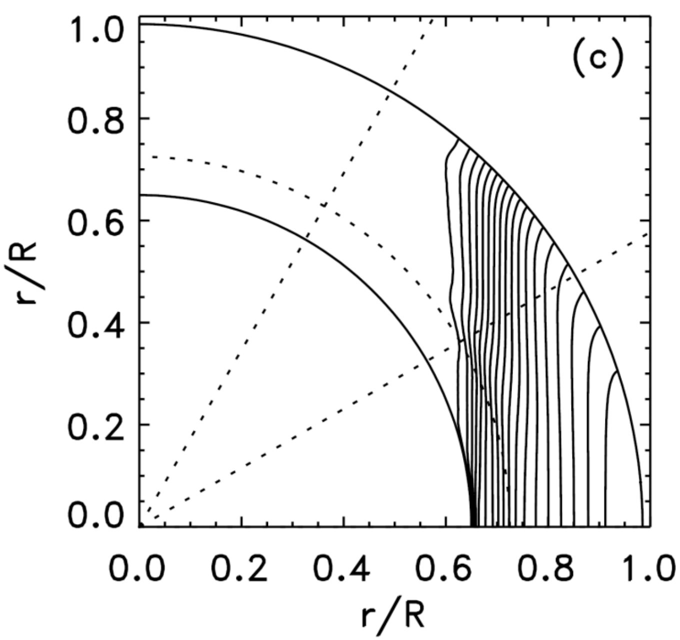
Solar dynamo (cont.)

- Difficulties of convection zone αω dynamos
  - Intermittency: B’ >> <B>
  - Polarity rules: strictly obeyed => B ~ 10^5 G (too strong to make dynamo by nonlinear saturation)
  - Magnetic buoyancy and storage problem: rise time << cycle length
  - Rotation law (does not fit theoretical model & observation)

Angular velocity distribution (solar seismology observation)



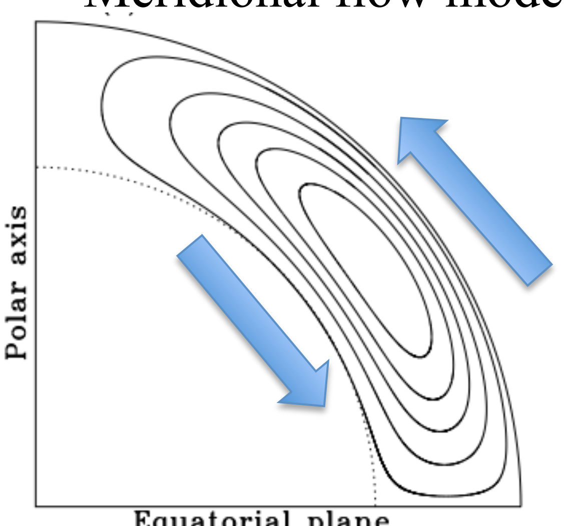
Taylor-Proudman state (theory)



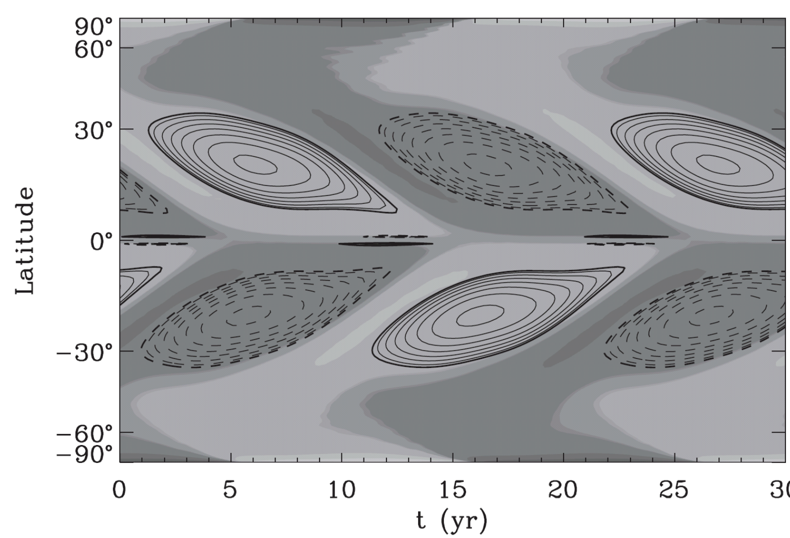
Flux-transport dynamo

- Flux transport dynamos (Choudhuri et al. 1995, Dikpati et al. 1999)
  - Regeneration of poloidal field through tilt of bipolar active regions close to surface (Babcock 1961, Leighton 1969)
  - rotational shear in tachocline (inside the Sun)
  - transport of magnetic flux by meridional circulation => determines migration direction and cycle period

Meridional flow model



Calculated butterfly diagram



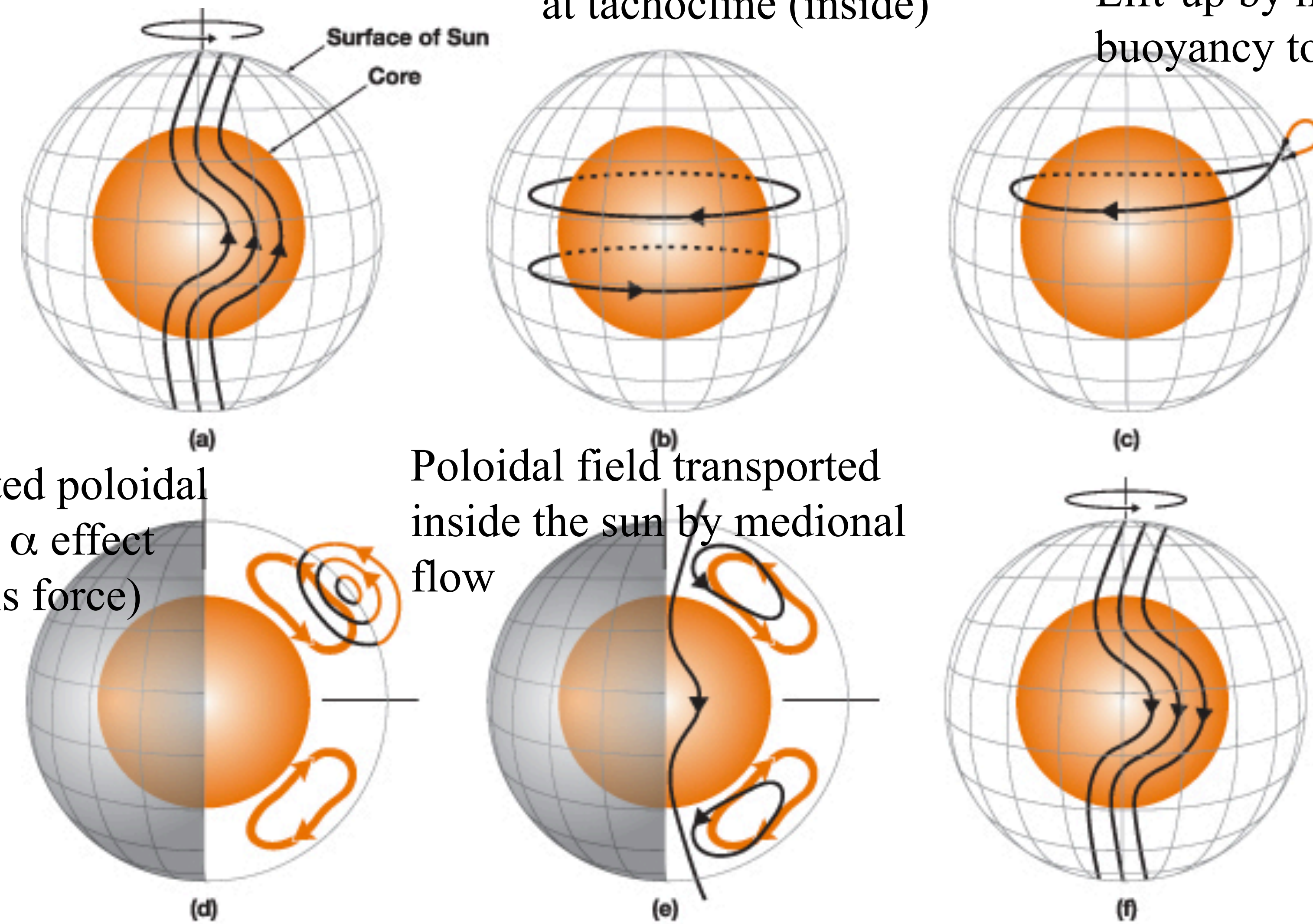
Flux-transport dynamo (cont.)

Toroidal filed generated at tachocline (inside)

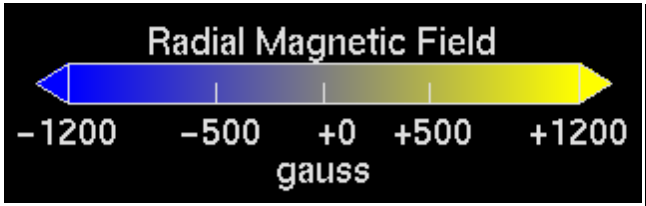
Lift-up by magnetic buoyancy to surface

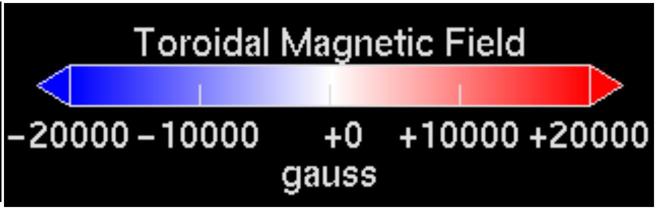
Generated poloidal filed by α effect (Coriolis force)

Poloidal field transported inside the sun by medional flow



Flux-transport dynamo (cont.)





Animation by
NASA-GSFC (2008)

Movie here

Fast and Slow dynamos

- If the magnetic diffusion time is much longer than the turn-over time of the flow, the induction equation (9.1) is

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{v} + \epsilon \nabla^2 \boldsymbol{B}$$

- Where $\epsilon = Rm^{-1}$ is small. Time is scaled on the turnover time L/V, L being the length scale of object and V is a typical velocity.

- For steady flow, a dynamo driven magnetic field grows exponentially $\boldsymbol{B} \sim e^{\sigma t}$, and if $\gamma = Re(\sigma)$
The flow is a fast dynamo if

$$\lim_{\epsilon \longrightarrow 0} \gamma(\epsilon) = \gamma_0 > 0$$

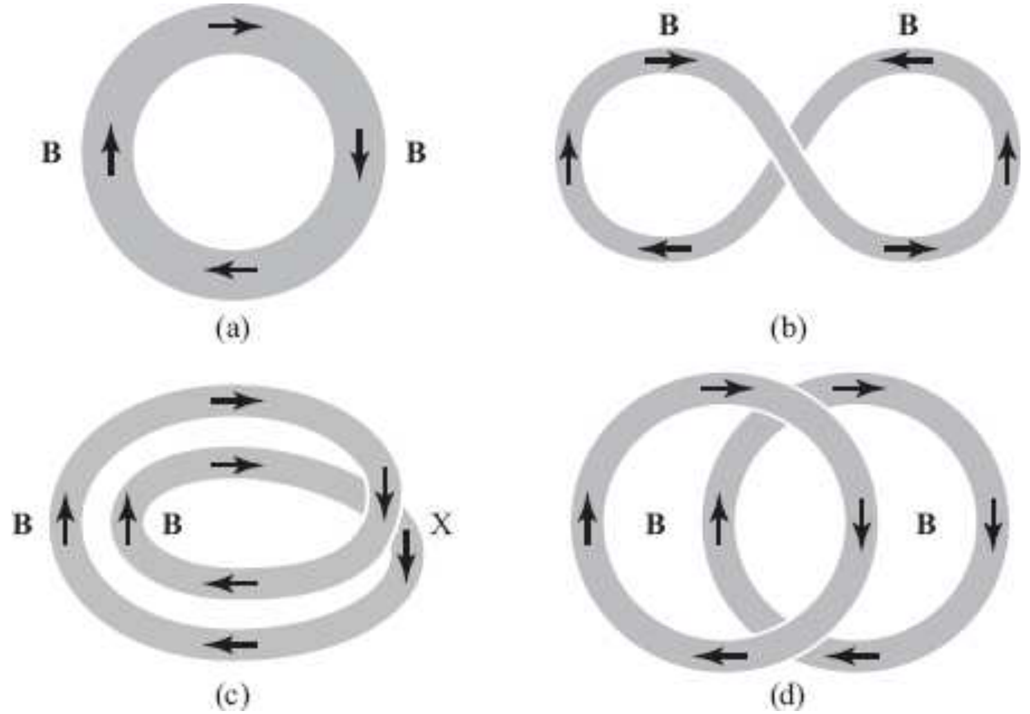
The flow is a slow dynamo if

$$\lim_{\epsilon \longrightarrow 0} \gamma(\epsilon) = \gamma_0 \leq 0$$

Fast and Slow dynamos (cont.)

- Fast dynamos grow on the turnover time (months in the Sun) not the magnetic diffusion time (millions of years in the Sun).
- The solar magnetic cycle operates on a twenty-two year cycle, much shorter time than the diffusion time, so it must be a fast dynamo.

The stretch-twist-fold dynamo



- A loop of flux is first stretched to twice its length, reducing the cross-section by half.
- Alfven’s theorem tells that the integrated flux through the loop cannot change if diffusion is small, so since the area is halved, the field strength must double.

The stretch-twist-fold dynamo (cont.)

- Now twist the loop to get to (b), and then fold to get to (c).
- Apply small diffusion at X to reconnection.
- Since the two loops in (d) both have the same flux as in (a), because each has half the area and double the field strength, we have doubled the total flux.
- Repeating the process doubles the flux again, so we have exponential growth in this process.

- This kind of dynamo process is called stretch-twist-fold dynamo or turbulent dynamo or small-scale dynamo

The stretch-twist-fold dynamo (cont.)

- The stretching phase of the process did work against the hoop stresses
- The Lorentz force can be written as

$$\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla \boldsymbol{B}) - \frac{1}{2\mu_0}\nabla B^2$$

- If $\boldsymbol{B} = B\hat{\phi}$

$$\boldsymbol{j} \times \boldsymbol{B} = -\frac{B^2}{\mu_0 R}\hat{R} - \hat{R}\frac{\partial}{\partial R}\frac{B^2}{2\mu_0} - \hat{z}\frac{\partial}{\partial z}\frac{B^2}{2\mu_0}$$

- The term $-B^2\hat{R}/\mu_0 R$ is hoop stress.
- Energy conservation means that, because energy is needed to generate magnetic field, the fluid flows must be doing work against the magnetic forces.
- This work must come from an energy source, such as thermal convection or mechanical driving (turbulence).

The stretch-twist-fold dynamo (cont.)

- This is a fast process, because it happens on the fluid velocity turn over time, L/V.
- It does however, appeal to 'small diffusion' to reconnect in step (c) to (d).
- The hope is that this reconnection occurs over a very short length scale over which diffusion can act quickly, so the small diffusion does not slow the process down significantly.

Basic ideas in nonlinear dynamos

- The induction equation is linear in $\boldsymbol{B}$, so it predicts dynamos that either decay or grow for ever.
- The field strength at which the dynamo stops growing is determined by terms nonlinear in $\boldsymbol{B}$.
- The Lorentz force $\boldsymbol{j} \times \boldsymbol{B} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}/\mu_0$ is the key nonlinear term
- Therefore, nonlinear dynamos require analysis of the equation of motion
- The dynamo stops growing when the Lorentz force changes the flow so that dynamo action is reduced.
- This process is called dynamo saturation

Dynamical regimes

- Saturation is poorly understood, and is probably different in different dynamical regimes.
- Here we focus on two problems:
  (i) dynamo saturation in moderately rotating systems, e.g. the Sun
  (ii) dynamo saturation in rapidly rotating systems, dominated by Coriolis force, e.g. the Earth's core

- The essential difference between (i) and (ii) is whether the rotation rate is fast or slow compared to the flow turnover time.
- The Rossby number is defined by Ro = $U/L\Omega$
  - In the interior of the Sun, Ro ~ 1 (it is larger near the surface)
  - In the Earth's core it is 10^{-7}, corresponding to very rapid rotation

Stellar dynamo saturation mechanisms

- Three different mechanisms of saturation have been proposed for stellar dynamos:

(1) omega-quenching:

- in most solar dynamo models, differential rotation generates toroidal field from poloidal field.
- The Lorentz force acts to stop the differential rotation, because the tension in the field lines opposes the shear.

(2) Magnetic buoyancy:

- a magnetic flux tube is lighter than its surroundings.
- Magnetic pressure in the tube means the gas pressure is reduced.
- Flux tubes therefore float upwards, removing themselves from the active dynamo region.

Stellar dynamo saturation mechanisms (cont.)

(3) Alpha-quenching:

- the magnetic field will stop the helical small-scale motions (turbulence) that create the mean field.
- Therefore we expect the helicity to drop when the field strength is large, and thus dynamo action to cease.
- This may be primarily mechanism for a mean field dynamo.
- Many models suggest that the alpha-effect should be quenched at relatively low field strengths, but nevertheless the Sun appears to achieve strong fields.

Stellar dynamo saturation mechanisms (cont.)

- In convection driven dynamos, the field can affect the stretching properties of the flow.
- Unfortunately, subtle changes in the flow pattern can radically alter stretching properties.
- The rate of creation of magnetic energy is through $\boldsymbol{v} \cdot \boldsymbol{j} \times \boldsymbol{B}$
- At large *Rm*, ***v*** and ***B*** are often nearly parallel.
- This means that small changes in the angle between ***v*** and ***j*** x ***B*** can strongly affect field generation.

Summary

- Astrophysical dynamo is a important mechanism to make magnetic field via electric current (field) and kinetic motion.
- From anti-dynamo theorem, a steady axisymmetric magnetic field cannot be maintained = need to remove axisymmetry for dynamo.
- In mean-field dynamo theory, small scale turbulent fluctuation contributes to produce poloidal magnetic field via electromotive force (α-effect).
- Omega-effect (stretching) via differential rotation produces toroidal magnetic field
- $\alpha\omega$ dynamo predicts dynamo-wave which can explain observed equatorward drift of sunspot emergences in the solar cycle.
- Turbulent motion is also derived stretch-twist-fold process to make magnetic field amplification.
- This kind of amplification mechanism is so-called turbulent dynamo or small-scale dynamo.