udbb tetraquark resonances with lattice QCD potentials
and the Born-Oppenheimer approximation

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We study tetraquark resonances with lattice QCD potentials computed for a static $b\bar{b}$ pair in the presence of two lighter quarks $ud$, the Born-Oppenheimer approximation and the emergent wave method. As a proof of concept we focus on the system with isospin $I = 0$, but consider different relative angular momenta $l$ of the heavy quarks $\bar{b}b$. For $l = 0$ a bound state has already been predicted with quantum numbers $I(J^P) = 0(1^-)$. Exploring various angular momenta we now compute the phase shifts and search for $S$ and $T$ matrix poles in the second Riemann sheet. We predict a tetraquark resonance for $l = 1$, decaying into two $B$ mesons, with quantum numbers $I(J^P) = 0(1^-)$, mass $m = 10.576^{+4}_{-4}$ MeV and decay width $\Gamma = 112^{+90}_{-103}$ MeV.


I. INTRODUCTION

A long standing problem in QCD is to understand exotic hadrons, i.e. hadrons which have a structure more complicated than a quark-antiquark pair or a triplet of quarks [1]. The problem of identifying exotic hadrons, say tetraquarks, pentaquarks, hexaquarks, hybrids or glueballs, turned out to be much harder than initially expected [2]. The observed candidates are resonances high in the spectrum, not only difficult to observe, but also technical to address in hadronic models and very difficult to study theoretically from first principles e.g. with lattice QCD.

Our main motivation is to investigate tetraquarks by combining lattice QCD and quantum mechanics techniques. We specialize in systems with two heavy antiquarks, which are expected to form bound states, when sufficiently heavy [3–13]. The starting point are potentials of two static antiquarks in the presence of two light quarks, which can be computed with state of the art lattice QCD techniques (cf. e.g. [14–19]). If the masses of the two heavy quarks are much larger than the scale of QCD, which is the case for two $b$ quarks, their dynamics can then be described by a quantum mechanical Hamiltonian with the aforementioned lattice QCD potentials. This two-step approach is the Born-Oppenheimer approximation [20]. Using this approach, a $udb\bar{b}$ tetraquark bound state with quantum numbers $I(J^P) = 0(1^-)$ has recently been predicted [15][19][21][23] and confirmed by a lattice QCD computation with four quarks of finite mass [24]. So far, however, resonances have not been studied in this framework.

In this work we extend the previous Born-Oppenheimer studies with lattice QCD potentials, reviewed in Section [1] We utilize the emergent wave method, a technique from scattering theory detailed in Section [III] to compute phase shifts, $S$ and $T$ matrix poles in the second Riemann sheet and the corresponding resonance masses and decay widths. Our results are presented in Section [IV] We conclude in Section [V].

II. LATTICE QCD POTENTIALS OF TWO STATIC ANTIQUARKS IN THE PRESENCE OF TWO LIGHT QUARKS

Potentials $V(r)$ of two static antiquarks $\bar{Q}Q$ in the presence of two light quarks $qq$ have been computed using lattice QCD for many different quantum numbers including light flavor combinations $qq$ with $q \in \{u, d, s, c\}$, parity $P$ and light total angular momentum $j$ (cf. e.g. [19][22]). There are both attractive and repulsive channels. Most promising with respect to the existence of tetraquark bound states or resonances are light quarks $q \in \{u, d\}$ together with $I = 0, j = 0$ or $I = 1, j = 1$, where $I$ denotes isospin, since the corresponding potentials $V(r)$ are not only attractive, but also rather wide and deep.

The lattice QCD results for these two potentials can be parameterized by a screened Coulomb potential,

$$V(r) = -\frac{\alpha}{r} e^{-r^2/\alpha^2} .$$

This ansatz is inspired by one-gluon exchange at small $\bar{Q}Q$ separations $r$ and a screening of the Coulomb potential due to the formation of two $B$ mesons at large $r$, as illustrated in Fig. [1] The values of the two parameters $\alpha$ and $d$ as determined in [19] are listed in Table [I]. Clearly, the $(I = 0, j = 0)$ potential is more attractive than the $(I = 1, j = 1)$ potential.
Applying the Born-Oppenheimer approximation, Eq. (7) is used as a potential for two heavy antiquarks, i.e. \( \bar{b}b \), in the presence of two light quarks \( uu \) or for two heavy-light mesons, i.e. \( B^{(*)} \bar{B}^{(*)} \). Solving the Schrödinger equation for the \((I = 0, j = 0)\) potential and angular momentum \( l = 0 \) of the two \( \bar{b} \) quarks a bound state has been predicted with binding energy \( 90^{+35}_{-36} \text{ MeV} \) \([19]\).

### III. THE EMERGENT WAVE METHOD

We now summarize the emergent wave method, explained in detail for instance in Ref. \([2]\), which is suited to study phase shifts and resonances. Let us consider the same Schrödinger equation utilized in the bound state study,

\[
\left( H_0 + V(r) \right) \Psi = E \Psi . \tag{2}
\]

The first step is to split the wave function into two parts,

\[
\Psi = \Psi_0 + X , \tag{3}
\]

where \( \Psi_0 \) is the incident wave, a solution of the free Schrödinger equation,

\[
H_0 \Psi_0 = E \Psi_0 , \tag{4}
\]

and \( X \) is the emergent wave. Substituting Eq. (3) into Eq. (2) and using Eq. (4) we obtain

\[
\left( H_0 + V(r) - E \right) X = -V(r) \Psi_0 . \tag{5}
\]

For any energy \( E \) we can use this equation to calculate the emergent wave \( X \) by providing the corresponding \( \Psi_0 \) and fixing the appropriate boundary conditions. From the asymptotic behaviour of \( X \) we then determine the phase shifts, the \( S \) matrix and the \( T \) matrix.

The problem can be continued to complex energies in a straightforward way and we can, therefore, find the poles of the \( S \) matrix and the \( T \) matrix in the complex plane. We identify a resonance with a pole, when located in the second Riemann sheet at \( m - i\Gamma / 2 \), where \( m \) is the mass and \( \Gamma \) is the decay width of the resonance.

### Table I. Parameters \( \alpha \) and \( d \) of the potential of Eq. (1) for two static antiquarks \( \bar{Q}Q \), in the presence of two light quarks \( qq \) with quantum numbers \( I \) and \( j \), as determined in \([19]\).

<table>
<thead>
<tr>
<th>( I )</th>
<th>( j )</th>
<th>( \alpha )</th>
<th>( d ) in fm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0</td>
<td></td>
<td>0.34±0.03</td>
<td>0.45±0.12</td>
</tr>
<tr>
<td>1 1</td>
<td></td>
<td>0.25±0.03</td>
<td>0.16±0.02</td>
</tr>
</tbody>
</table>

### A. Partial wave decomposition

The Hamiltonian describing the two heavy antiquarks \( \bar{b}b \) at vanishing total momentum, i.e. in the rest frame of the system, is

\[
H = H_0 + V(r) = -\frac{\hbar^2}{2\mu} \Delta + V(r) \tag{6}
\]

with reduced mass \( \mu = M / 2 \), where \( M = 5280 \text{ MeV} \) is the mass of the \( B \) meson from the PDG \([25]\). For simplicity we omit the additive constant \( 2M \) in (6), i.e. all resulting energy eigenvalues are energy differences with respect to \( 2M \). We consider an incident plane wave \( \Psi_0 = e^{ikr} \), which can be expressed as a sum of spherical waves,

\[
\Psi_0 = e^{ikr} = \sum_i (2l + 1) i^l j_l(kr) P_l(\vec{k} \cdot \vec{r}) , \tag{7}
\]

where \( j_l \) are spherical Bessel functions, \( P_l \) are Legendre polynomials and the relation between energy and momentum is \( \hbar k = \sqrt{2\mu E} \). For a spherically symmetric potential \( V(r) \) as in Eq. (2) and an incident wave \( \Psi_0 = e^{ikr} \) the emergent wave \( X \) can also be expanded in terms of Legendre polynomials \( P_l \),

\[
X = \sum_i (2l + 1) i^l \frac{\chi_l(r)}{kr} P_l(\vec{k} \cdot \vec{r}) . \tag{8}
\]

Inserting Eq. (7) and Eq. (8) into Eq. (6) leads to a set of ordinary differential equations for \( \chi_l \),

\[
\left( -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + l(l+1) - \frac{\hbar^2}{2\mu r^2} + V(r) - E \right) \chi_l(r) = -V(r)kr j_l(kr) . \tag{9}
\]

### B. Solving the differential equations for the emergent wave

The potentials \( V(r) \), Eq. (7), are exponentially screened, i.e. \( V(r) \approx 0 \) for \( r \geq R \), where \( R \gg d \). For large separations \( r \geq R \) the emergent wave is, hence, a superposition of outgoing spherical waves, i.e.

\[
\frac{\chi_l(r)}{kr} = i t_l h_l^{(1)}(kr) , \tag{10}
\]
where $h^{(1)}_l$ are the spherical Hankel functions of first kind.

Our aim is now to compute the complex prefactors $t_l$, which will eventually lead to the phase shifts. To this end we solve the ordinary differential equation \( \chi_l \) for different approaches: (1) a fine uniform discretization of the interval \( [0, R] \), which reduces the differential equation to a large set of linear equations, which can be solved rather efficiently, since the corresponding matrix is tridiagonal; (2) a standard 4-th order Runge-Kutta shooting method. The numerical solution of the differential equation \( \chi_l \) for real energy \( E \) and angular momenta \( l = 0,1,2,3,4 \) is rather straightforward. To check our results and to exclude any numerical artefacts we implemented two different approaches: (1) a fine uniform discretization of the interval \( [0, R] \), which reduces the differential equation to a large set of linear equations, which can be solved rather efficiently, since the corresponding matrix is tridiagonal; (2) a standard 4-th order Runge-Kutta shooting method.

### C. Phase shifts and S and T matrix poles

The quantity $t_l$ is a T matrix eigenvalue (cf. standard textbooks on quantum mechanics and scattering, e.g. [20]). From $t_l$ we can calculate the phase shift $\delta_l$ and also read off the corresponding S matrix eigenvalue $s_l$ [21],

\[
s_l \equiv 1 + 2it_l = e^{2i\delta_l}.
\]

Moreover, note that both the S matrix and the T matrix are analytical in the complex plane. They are well-defined for complex energies $E$. Thus, our numerical method can as well be applied to solve the differential equation \( \chi_l \) for complex $E$. We find the S and T matrix poles by scanning the complex plane (Re($E$), Im($E$)) and applying Newton’s method to find the roots of $1/t_l(E)$. The poles of the S and the T matrix correspond to complex energies of resonances. Note the resonance poles must be in the second Riemann sheet with a negative imaginary part both for the energy $E$ and the momentum $k$.

IV. RESULTS FOR PHASE SHIFTS, S MATRIX AND T MATRIX POLES AND RESONANCES

We first consider the more attractive $ud\bar{b}\bar{b}$ potential corresponding to isospin $I = 0$ and light spin $j = 0$ (cf. Sec. II). We compute $t_l$ and via Eq. (9) the phase shift $\delta_l$ for real energy $E$ and angular momenta $l = 0, 1, 2, \ldots$. A very clear signal for a resonance would be a fast increase of the phase shift $\delta_l$ as a function of $E$ from 0 to $\approx \pi$, almost like a step function. However, we do not find such a pronounced increase (cf. Fig. 2). Thus, we must search more thoroughly for possibly existing resonances.

Starting with angular momentum $l = 1$ we first search for clear resonance signals by making the potential more and more attractive. We increase the parameter $\alpha$, while keeping the parameter $d = 0.45 \text{ fm}$ fixed, to preserve the scale of the potential. The corresponding results for the phase shift $\delta_l$ are shown in Fig. 3. Indeed, for $\alpha \geq 0.65$ we find clear resonances with $\delta_l$ increasing from 0 to $\approx \pi$. Then, for $\alpha = 0.72$, we find a bound state, since the phase shift $\delta_l$ starts at $\pi$ and decreases monotonically to 0, when increasing the energy $E$. However, from these phase shifts it is not clear, for which values of $\alpha$ a resonance exists or not, i.e. it is not possible to say, whether there is a resonance for e.g. $\alpha \approx 0.50$ or even for the physical $\alpha = 0.34$.

Thus, we search directly for poles of the T matrix eigenvalues $t_l$. With this technique we clearly find a pole for angular momentum $l = 1$ and physical values of the
parameters, $\alpha = 0.34$ and $d = 0.45$ fm. We show this pole in Fig. 4 by plotting the $t_1$ as a function of the complex energy $E$. The pole is clearly visible as a sharp peak.

To understand the dependence of the resonance pole on the shape of the potential, we again scan different values of the parameter $\alpha$ and determine each time the norm $|t_1|$, while the phase $\arg(t_1)$ corresponds to different colours.

Finally we perform a detailed statistical and systematic error analysis of the pole of $t_1$ for the physical values of the parameters, $\alpha = 0.34$ and $d = 0.45$ fm. We use the same analysis method as for our previous study of the bound state for $l = 0$, which is rather evolved and explained in detail in Ref. [22]. Our systematic errors are depicted in Fig. 5. Indeed, starting with $\alpha = 0.21$ we find a pole. This confirms our prediction of a resonance for angular momentum $l = 1$ and physical values of the parameters, $\alpha = 0.34$ and $d = 0.45$ fm.

In what concerns angular momenta $l \neq 1$, we find no clear signal for a resonance pole (except for the bound state pole for $l = 0$). We also find no poles for any $l$ in the less attractive case of $(I = 1, j = 1)$.

V. CONCLUSIONS AND OUTLOOK

As a case study for the investigation of resonances above the $BB$ meson pair threshold, we have explored the $udbb$ four-quark system. We have utilized lattice QCD potentials computed for two static antiquarks in the presence of two light quarks, the Born-Oppenheimer approximation and the emergent wave method for the $BB$ system. First we have computed scattering phase shifts. Then we have performed the analytic continuation of the S matrix and the T matrix to the second Riemann sheet and have searched for poles as signals of resonances.

From these results we have predicted a new resonance, with quantum numbers $I(J^P) = 0(1^-)$. Performing a careful statistical and systematic error analysis has led to a resonance mass $m = 10576^{+4}_{-3} \text{MeV}$ and a decay width $\Gamma = 112^{+90}_{-103} \text{MeV}$.

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Calculations on the LOEWE-CSC and on the on the

![Figure 4](image4.png)

Figure 4. (Colour online.) $T$ matrix eigenvalue $t_1$ as a function of the complex energy $E$ for the $(I = 0, j = 0)$ potential ($\alpha = 0.34, d = 0.45$ fm). Along the vertical axis we show the norm $|t_1|$, while the phase $\arg(t_1)$ corresponds to different colours.

![Figure 5](image5.png)

Figure 5. (Colour online) Trajectory of the pole of the eigenvalue $t_1$ of the $T$ matrix in the complex plane $(\text{Re}(E), \text{Im}(E))$, corresponding to a variation of $\alpha$ for the $(I = 0, j = 0)$ potential ($d = 0.45$ fm). We also illustrate with a cloud of diamond points the computation of the systematic error, utilizing the technique of Ref. [22].
FUCHS-CSC high-performance computer of the Frankfurt University were conducted for this research. We would like to thank HPC-Hessen, funded by the State Ministry of Higher Education, Research and the Arts, for programming advice.

[27] At large distances \( r \geq R \), the radial wavefunction is
\[
kr_1^2 \left( kr \right) + it_1^2 \left( kr \right) = \left( kr / 2 \right) h_1^{(2)} \left( kr \right) + e^{2i\theta} h_1^{(1)} \left( kr \right).
\]