The pseudoparticle approach

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Outline

Part I: the pseudoparticle approach, a model for SU(2) Yang-Mills theory
  • Basic principle.
  • Numerical results: static quark antiquark potential, topological susceptibility, critical temperature, gluelump masses.

Part II: properties of confining gauge field configurations
  • Pseudoparticles of different profile.
  • Instantons, antiinstantons and akyrons.

Part III: fermions in the pseudoparticle approach
  • Problems with fermionic fields in the pseudoparticle approach.
  • The Gross-Neveu model as testing ground.

Summary and outlook

Part I: the pseudoparticle approach, a model for SU(2) Yang-Mills theory
Basic principle of the PP approach (1)

- A numerical technique to approximate Euclidean path integrals (in this talk: mainly SU(2) Yang-Mills theory):

\[ \langle \mathcal{O} \rangle = \frac{1}{Z} \int DA \mathcal{O}[A] e^{-S[A]} \]

\[ S[A] = \frac{1}{4g^2} \int d^4x \ F^a_{\mu\nu} F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu. \]

- A tool to analyze the importance of certain classes of gauge field configurations with respect to confinement.

- Related work:
  - Ensembles of calorons with non-trivial holonomy (P. Gerhold, E.-M. Ilgenfritz, M. Müller-Preussker, 2006).
  - Ensembles of dyons (D. Diakonov, V. Petrov, 2007).
• PP: any gauge field configuration \( a_{\mu}^{a} \), which is localized in space and in time.

• Consider only those gauge field configurations, which can be written as a sum of a fixed number (\( \approx 400 \)) of PPs:

\[
A_{\mu}^{a}(x) = \sum_{j} \rho^{ab}(j) a_{\mu}^{b}(x - z(j))
\]

\((j): \text{PP index; } \rho^{ab}(j): \text{degrees of freedom of the } j\text{-th PP, i.e. amplitude and color orientation; } z(j): \text{position of the } j\text{-th PP).}\)

• Define the functional integration as an integration over the PP degrees of freedom:

\[
\int DA \ldots \rightarrow \int \left( \prod_{j} d\rho^{ab}(j) \right) \ldots
\]
Building blocks of PP ensembles


$$a^{a}_{\mu, \text{instanton}}(x) = \eta^{a}_{\mu\nu} \frac{x_{\nu}}{x^2 + \lambda^2}$$

$$a^{a}_{\mu, \text{antiinstanton}}(x) = \bar{\eta}^{a}_{\mu\nu} \frac{x_{\nu}}{x^2 + \lambda^2}$$

$$a^{a}_{\mu, \text{akyron}}(x) = \delta^{a1}_{\mu} \frac{x_{\mu}}{x^2 + \lambda^2}.$$

- Instantons, antiinstantons and akyrons form a basis of all gauge field configurations in the “continuum limit”.

- Degrees of freedom: amplitudes $A(j) \in \mathbb{R}$, color orientations $C^{ab}(j) \in \text{SO}(3)$, positions $z(j) \in \mathbb{R}^4$.

$$A^{a}_{\mu}(x) = \sum_j A(j) C^{ab}(j) a^{a}_{\mu, \ldots}(x - z(j)).$$
PP ensembles (1)

- PP ensemble: a fixed number of PPs inside a spacetime hypersphere.
- Gauge field:

\[
A^a_\mu(x) = \sum_j A(j)C^{ab}(j)a^b_{\mu,\text{instanton}}(x - z(j)) + \sum_k A(k)C^{ab}(k)a^b_{\mu,\text{antiinstanton}}(x - z(k)) + \sum_l A(l)C^{ab}(l)a^b_{\mu,\text{akyron}}(x - z(l)).
\]

- Choose color orientations \(C^{ab}(j)\) and positions \(z(j)\) randomly.

- \(A^a_\mu\) is no classical solution (not even close to a classical solution)!

- Long range interactions between PPs.
• Approximation of the path integral:

\[
\langle \mathcal{O} \rangle = \frac{1}{Z} \int \left( \prod_j dA(j) \right) \mathcal{O}(A(j)) e^{-S(A(j))}
\]

(integration over PP amplitudes).

• Solve this multidimensional integral via Monte-Carlo simulations.

• Exclude boundary effects: observables have to be “measured” sufficiently far away from the boundary.
Numerical results (1)

- Static quark antiquark potential:
  - Linear for large separations, i.e. confinement.
  - Fit with $V(R) = V_0 - \alpha/R + \sigma R$:
    * String tension $\sigma > 0$ (in the following $\sigma$ is used to set the scale).
    * Attractive $1/R$-correction, which is of the right order of magnitude compared to lattice results and the bosonic string picture.
  - Potentials for different quark representations exhibit Casimir scaling.

![Graphs of quark antiquark potential vs separation](image-url)
Numerical results (2)

- Further dimensionful quantities:
  - Topological susceptibility \( \chi = \langle Q_V^2 \rangle / V \).
  - Critical temperature of the confinement deconfinement phase transition \( T_{\text{critical}} \).
  - Mass of magnetic and electric gluelumps (adjoint representation) \( m_B \) and \( m_E \) (work done by Ch. Szasz).

- Dimensionless quantities (physically meaningful):
  \( \chi^{1/4}/\sigma^{1/2} \), \( T_{\text{critical}}/\sigma^{1/2} \), \( R^{B}_{SB}\sigma^{1/2} \), \( R^{E}_{SB}\sigma^{1/2} \).

- Considering different \( g \) amounts to considering different spacetime volumes.

- Qualitative agreement with lattice results.

- Consistent scaling behavior.
Part II: properties of confining gauge field configurations
Properties of confining gauge field ...

• What are essential properties of confining gauge field configurations?
• Which gauge field configurations are responsible for confinement?
• Apply the PP approach with different types of PPs to study the effect of different classes of gauge field configurations on confinement:
  – PPs with a limited range of interaction (PPs with Gaussian profile).
  – PPs without topological charge (akyrons).
  – ...
PPs with Gaussian profile (1)

- Consider ensembles with Gaussian localized PPs of different size $\lambda$:

\[
a^a_{\mu,\text{instanton}}(x) = \eta^a_{\mu\nu} x_{\nu} e^{-x^2/2\lambda^2}, \quad a^a_{\mu,\text{antiinstanton}}(x) = \bar{\eta}^a_{\mu\nu} x_{\nu} e^{-x^2/2\lambda^2},
\]

\[
a^a_{\mu,\text{akyron}}(x) = \delta^a_1 x_{\mu} e^{-x^2/2\lambda^2}.
\]

- Gaussian localized PPs have a limited range of interaction, which is proportional to their size $\lambda$.

- Typical PP profiles:

\[
(x_0/\lambda)\exp(-x_0^2/2\lambda^2) \text{ plotted against } x_0
\]

\[
(x_1/\lambda)\exp(-(x_0^2 + x_1^2)/2\lambda^2)|_{x_1=\lambda} \text{ plotted against } x_0
\]
PPs with Gaussian profile (2)

- Short range PPs ($\lambda \leq 0.50$)
  → little overlap between neighboring PPs.
  → no confinement.

- Long range PPs ($\lambda \geq 1.00$)
  → significant overlap between neighboring PPs.
  → confinement.

- PP percolation $\leftrightarrow$ confinement.

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Typical gauge field configurations ($\lambda = 0.25 \leftrightarrow \lambda = 1.00$)

- $\lambda = 0.25$: local UV fluctuations $\rightarrow$ no confinement.
- $\lambda = 1.00$: global excitations $\rightarrow$ confinement.

Gauge field configurations responsible for confinement contain extended structures and large area excitations.
Instantons, antiinstantons and akyrons

- Consider the following ensembles:
  - Akyron ensemble: 400 akyrons (topological charge density $q = 0$).
  - Standard ensemble: 150 instantons, 150 antiinstantons, 100 akyrons.
  - Instanton ensemble: 200 instantons, 200 antiinstantons.

- No confinement in the akyron ensemble
  $\rightarrow$ akyrons alone are not suited to reproduce Yang-Mills physics.
  $\rightarrow$ supports the expectation that confinement and topological charge are closely related.

- Standard ensemble $\leftrightarrow$ instanton ensemble
  $\rightarrow (\chi^{1/4}/\sigma^{1/2})_{\text{standard}} = 0.35$
  $\quad (\chi^{1/4}/\sigma^{1/2})_{\text{instanton}} = 0.26$
  $\quad (\chi^{1/4}/\sigma^{1/2})_{\text{lattice}} = 0.49$.
  $\rightarrow$ using akyrons is beneficial with respect to quantitative results.
Part III: fermionic fields in the pseudoparticle approach
Basic principle (fermionic fields) (1)

- How can fermions be included in the PP approach?

- Action and partition function of any theory with quadratic fermion interaction (no restriction to SU(2) Yang-Mills theory anymore):

\[
S[\psi, \bar{\psi}, \phi] = \int dx \left( \bar{\psi} Q(\phi) \psi + \mathcal{L}(\phi) \right)
\]

\[
Z = \int D\psi D\bar{\psi} D\phi e^{-S[\psi, \bar{\psi}, \phi]}
\]

\(Q: \) Dirac operator; \(\phi: \) any type and number of bosonic fields, e.g. the non-Abelian gauge field in QCD.

Basic principle (fermionic fields) (2)

- Consider only those fermionic field configurations, which can be represented by a linear superposition a fixed number of localized building blocks:

\[ \psi(x) = \sum_j \eta_j G_j(x) \]

\( \eta_j \): Grassmann valued spinors; \( G_j \): functions, which are localized in space as well as in time, i.e. PPs).

- Define the functional integration over all fermionic field configurations as an integration over the Grassmann valued spinors:

\[ \int D\psi D\bar{\psi} \ldots = \int \left( \prod_j d\eta_j d\bar{\eta}_j \right) \ldots \]
Basic principle (fermionic fields) (3)

- Integrate out the fermions:

\[ S_{\text{effective}}[\phi] = \int d^{d+1}x \, \mathcal{L}(\phi) - \ln \left( \det \left( \langle G_j | Q | G_{j'} \rangle \right) \right) \]

\[ Z \propto \int D\phi e^{-S_{\text{effective}}[\phi]} \]

(\[ \langle G_j | Q | G_{j'} \rangle \] is a finite matrix; “Q-regularization”).

- If \( \det(Q) \) is real and positive, \( \det(Q) = \sqrt{\det(Q^\dagger Q)} \). This suggests another PP regularization:

\[ S_{\text{effective}}[\phi] = \int d^{d+1}x \, \mathcal{L}(\phi) - \frac{1}{2} \ln \left( \det \left( \langle G_j | Q^\dagger Q | G_{j'} \rangle \right) \right) \]

(“\( Q^\dagger Q \)-regularization”).

- The “\( Q^\dagger Q \)-regularization” has significant advantages over the naive “Q-regularization”.

\( \mathcal{Q} \) versus \( \mathcal{Q}^\dagger \mathcal{Q} \) (1)

- For the sake of simplicity: consider all PPs \( G_j \) to be orthonormal, i.e.
  \( \langle G_j | G_{j'} \rangle = \delta_{jj'} \) (this is not a restriction!).

- The problem of the \( \mathcal{Q} \)-regularization:
  
  - **Applying the Dirac operator** \( \mathcal{Q} \) **to one of the PPs** \( G_{j'} \) **in general yields a function, which is (partially) outside the PP function space** \( \text{span}\{G_n\} \), i.e.
    \[
    QG_{j'}(x) = \sum_k a_{j'k} G_k(x) + h_{j'} H_{j'}(x)
    \]
    \( (H_{j'} \text{ normalized, } H_{j'} \perp \text{span}\{G_n\}) \).
  
  - If \( |\sum_k a_{j'k} G_k| \gg |h_{j'}| \) → no problem.
  
  - If \( |\sum_k a_{j'k} G_k| \lesssim |h_{j'}| \) → when computing the fermionic matrix
    \( \langle G_j | \mathcal{Q} | G_{j'} \rangle \), a significant part of \( \mathcal{Q} G_{j'} \) is simply ignored, just because
    \( H_{j'} \) is perpendicular to the PP function space \( \text{span}\{G_n\} \).
The advantage of the $Q^\dagger Q$-regularization:

- Both the left hand sides $\langle G_j | Q^\dagger$ and the right hand sides $Q | G_{j'} \rangle$ of the matrix elements $\langle G_j | Q^\dagger Q | G_{j'} \rangle$ might be outside the PP function space $\text{span}\{G_n\}$, but they form the same function space, $\text{span}\{QG_n\}$, in which their overlap is computed.

Testing ground: Gross-Neveu model (1)

- Action and partition function of the 1+1-dimensional Gross-Neveu model:

\[
S = \int d^2 x \left( \sum_{n=1}^{N} \overline{\psi}^{(n)} \left( \gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 \right) \psi^{(n)} \right) - \frac{g^2}{2} \left( \sum_{n=1}^{N} \overline{\psi}^{(n)} \psi^{(n)} \right)^2
\]

\[
Z = \int \left( \prod_{n=1}^{N} D\psi^{(n)} D\overline{\psi}^{(n)} \right) e^{-S}
\]

\((N: \text{number of flavors; } \mu: \text{chemical potential; } g: \text{coupling constant})\).
Testing ground: Gross-Neveu model (2)

- Introduce a real scalar field $\sigma$ and integrate out the fermions:

$$S_{\text{effective}} = N \left( \frac{1}{2\lambda} \int d^2 x \, \sigma^2 - \ln \left( \det \left( \gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 + \sigma \right) \right) \right)$$

$$Z \propto \int D\sigma e^{-S_{\text{effective}}}$$

$(\lambda = N g^2)$.

- Large-$N$ limit:
  - $N \to \infty$, $\lambda = N g^2 = \text{constant}$.
  - There is no need to compute the $\sigma$-path integral anymore.
  - It is sufficient to minimize $S_{\text{effective}}$ with respect to $\sigma$.
  - $\sigma = -g^2 \sum_{n=1}^{N} \bar{\psi}^{(n)} \psi^{(n)}$ (chiral condensate).
Fermionic PPs

- Fermionic PPs (in this talk): a large number of uniformly distributed “hat functions” (B-spline basis functions of degree 2).
  - “Sensible set of field configurations” (any not too heavily oscillating field configuration can be approximated)
    → we can expect to reproduce correct Gross-Neveu results.
  - Piecewise polynomial functions
    → certain integrals can be calculated analytically.

Phase diagram

- $Q$-regularization: completely wrong and useless results.
  - No improvement, when using a larger number of PPs.
  - No improvement, when using a different type of PPs.

- $Q^\dagger Q$-regularization: excellent agreement with analytical results.
Summary and Outlook
Summary and conclusions (1)

• The PP approach with \( \approx 400 \) instantons, antiinstantons and akyrons is able to reproduce many essential features of SU(2) Yang-Mills theory:
  
  – **Linear quark antiquark potential, i.e. confinement.**
  – Casimir scaling for different quark representations.
  – **Consistent scaling behavior of** \( \sigma, \chi, T_{\text{critical}}, m_B \) **and** \( m_E \).
  – Dimensionless quantities are in qualitative agreement with lattice results.

• Essential properties of confining gauge field configurations:
  
  – Long range PPs necessary for confinement (PP percolation)
    \( \rightarrow \) **confinement** \( \leftrightarrow \) **extended structures and large area excitations.**
  – Instantons and antiinstantons (PPs with non-vanishing topological charge density) necessary for confinement
    \( \rightarrow \) **confinement** \( \leftrightarrow \) **topological charge.**
Summary and conclusions (2)

- Inclusion of fermions in the PP approach:
  - **Always apply the** $Q^\dagger Q$-**regularization and not the naive** $Q$-**regularization.**
  - The application of the PP approach to compute the phase diagram of the $1+1$-dimensional Gross-Neveu model in the large-$N$-limit has been a successful test.
  - Next steps:
    - Apply the PP approach to QCD.
    - Try to identify a small number of physically relevant fermionic PPs (PPs, which are able to approximate the low lying eigenmodes of the Dirac operator?).
Outlook

• Current research:
  – Improve the static quark antiquark potential:
    * Use improved operators (smeared Wilson loops) to extract the static quark antiquark potential (o.k.).
    * String breaking for the adjoint representation (???).
  – Consider SU(2) Yang-Mills theory with dynamical fermions:
    * Chiral symmetry breaking by computing the low lying eigenmodes of the Dirac operator in the quenched approximation (Banks-Casher relation) (???).

• Goal: obtain a model with a small number of degrees of freedom, which exhibits chiral symmetry breaking and a confinement deconfinement phase transition at the same time.

• Compute further observables: pion masses, ...