Lattice calculation of the Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$ with dynamical quarks

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Introduction

- Consider semileptonic decays of $B$ mesons ($B$, $B^*$) into orbitally excited $P$ wave $D$ mesons ($D^{**}$):

  $$B^{(*)} \rightarrow D^{**} l \nu.$$

- Precise knowledge of the corresponding branching fractions important, e.g. to reduce the systematic uncertainty in the measurements of the CKM matrix element $|V_{cb}|$.

- There is a persistent conflict (“1/2 versus 3/2 puzzle”) between theory and experiment:
  - Experiment favors the decay into “1/2 $P$ wave $D^{**}$’s”.
  - Theory favors the decay into “3/2 $P$ wave $D^{**}$’s”.
  - Lattice calculations can help to resolve this conflict.

Marc Wagner, “Lattice calculation of the Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$ with dynamical quarks”, April 27, 2009
Outline

• Heavy-light mesons.
• The $1/2$ versus $3/2$ puzzle:
  – Experimental side.
  – Theory side.
  – Possible explanations to resolve the puzzle.
• Lattice computation of the Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$:
  – Simulation setup, static and light quark propagators.
  – Static-light meson creation operators.
  – Static-light meson masses.
  – 2-point functions, ground state norms.
  – 3-point functions, Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$.
  – Extrapolation to the $u/d$ quark mass.
• Conclusions.
Heavy-light mesons

- Heavy-light meson: a meson made from a heavy quark \((b, c)\) and a light quark \((u, d)\), i.e. \(B = \{\bar{b}u, \bar{b}d\}, \ D = \{\bar{c}u, \bar{c}d\}\).

- Static limit, i.e. \(m_b, m_c \to \infty\):
  - No interactions involving the static quark spin.
  - Classify states according to parity \(P\) and total angular momentum of the light cloud (light quarks and gluons) \(j\).

- \(m_b, m_c\) finite, but heavy:
  - Classify states according to parity \(P\) and total angular momentum \(J\).
  - Although \(j\) is not a “true quantum number” anymore, it is still an approximate quantum number \(\to \) notation \(D_J^j\).
  - \(D^{**} = \{D_0^*, D_1', D_1, D_2^*\}\).

<table>
<thead>
<tr>
<th>(j^P)</th>
<th>(J^P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1/2)^-)</td>
<td>(S)</td>
</tr>
<tr>
<td>((1/2)^{+})</td>
<td>(P_-)</td>
</tr>
<tr>
<td>((3/2)^+)</td>
<td>(P_+)</td>
</tr>
</tbody>
</table>

- \(0^- \equiv B, D\)
- \(1^- \equiv B^*, D^*\)
- \(0^+ \equiv D_0^* \equiv D_0^{1/2}\)
- \(1^+ \equiv D_1' \equiv D_1^{1/2}\)
- \(1^+ \equiv D_1 \equiv D_1^{3/2}\)
- \(2^+ \equiv D_2^* \equiv D_2^{3/2}\)
Consider the semileptonic decay $B \rightarrow X_c l \nu$.

Experiments, which have studied this decay: ALEPH, BaBar, BELLE, CDF, DELPHI, DØ.

What is $X_c$?

- $\approx 75\%$ $D$ and $D^*$, i.e. $S$ wave states (agreement with theory).
- $\approx 10\%$ $D^{3/2}_1$ and $D^{3/2}_2$, i.e. $j = 3/2$ $P$ wave states (agreement with theory).

- For the remaining $\approx 15\%$ the situation is not clear:
  * A natural candidate would be $D^{1/2}_0$ and $D^{1/2}_1$, i.e. $j = 1/2$ $P$ wave states.
  * This would imply $\Gamma(B \rightarrow D^{1/2}_{0,1} l \nu) > \Gamma(B \rightarrow D^{3/2}_{1,2} l \nu)$, which is in conflict with theory.
  * This conflict between experiment and theory is called the “$1/2$ versus $3/2$ puzzle”.

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Marc Wagner, “Lattice calculation of the Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$ with dynamical quarks”, April 27, 2009
1/2 versus 3/2: experimental side (2)

- Example plot from BaBar/SLAC:
  - Horizontal axis:
    \( m(D^{(*)}\pi) - m(D^{(*)}) \) in GeV/c^2.
  - Vertical axis:
    events/(20 MeV/c^2).
  - Simultaneous fit of four probability distribution functions (\(D^*_0, D'_1, D_1, D^*_2\)) to \(m(D^{(*)}\pi) - m(D^{(*)})\) data:
    a) \(B^- \rightarrow D^{*+}\pi^-l^-\bar{\nu}_l\).
    b) \(B^- \rightarrow D^+\pi^-l^-\bar{\nu}_l\).
  - Two states (\(D_1\) and \(D^*_2\), i.e. the \(j = 3/2\) P wave states) have small widths and can “clearly” be identified.
  - Two states (\(D^*_0\) and \(D'_1\), i.e. the \(j = 1/2\) P wave states) have very large widths.

1/2 \textit{versus} 3/2: theory side (1)

- Static limit, i.e. $m_b, m_c \rightarrow \infty$.

- Parameterization of the matrix elements relevant for decays $B \rightarrow X_c l \nu$ by a small set of form factors (Isgur-Wise functions) due to heavy quark symmetry. [N. Isgur and M. B. Wise, Phys. Rev. D 43, 819 (1991)]

- In particular for $B \rightarrow D^{**} l \nu$,

$$
\langle D_{0}^{1/2}(v') | \bar{c} \gamma_5 \gamma_\mu b | B(v) \rangle \propto \tau_{1/2}(w)(v - v')_\mu
$$

$$
\langle D_{2}^{3/2}(v', \epsilon) | \bar{c} \gamma_5 \gamma_\mu b | B(v) \rangle \propto \tau_{3/2}(w) \left((w + 1)\epsilon^*_\mu \nu^\alpha - \epsilon^*_\alpha \nu^\beta v^\beta v'_\nu\right).
$$

where $w = v'v \geq 1$. 
1/2 versus 3/2: theory side (2)

- Relation to decay rates:

\[
\frac{d\Gamma(B \to D_{J}^{1/2} l\nu)}{dw} \propto G_F^2 |V_{cb}|^2 K_{J}^{1/2}(w) \left| \tau_{1/2}(w) \right|^2, \quad J = 0, 1
\]

\[
\frac{d\Gamma(B \to D_{J}^{3/2} l\nu)}{dw} \propto G_F^2 |V_{cb}|^2 K_{J}^{3/2}(w) \left| \tau_{3/2}(w) \right|^2, \quad J = 1, 2,
\]

where \( K_{J}^{j} \) are analytically known kinematical factors, e.g.

\[
K_{0}^{1/2}(w) = 4r^3(w^2 - 1)^{3/2}(1 - r)^2
\]

\[
K_{1}^{1/2}(w) = 4r^3(w - 1)(w^2 - 1)^{1/2}\left((w - 1)(1 + r)^2 + 4w(1 + r^2 - 2rw)\right)
\]

\[
\ldots
\]

with \( r = m(D)/m(B) \).
1/2 versus 3/2: theory side (3)

- By means of OPE a couple of sum rules have been derived in the static limit:
  - Most prominent sum rule in this context: Uraltsev sum rule,
    \[ \sum_n \left| \tau_{3/2}^{(n)}(1) \right|^2 - \left| \tau_{1/2}^{(n)}(1) \right|^2 = \frac{1}{4} \]
    \((\tau_{1/2} \equiv \tau_{1/2}^{(0)} \text{ and } \tau_{3/2} \equiv \tau_{3/2}^{(0)}; \text{ the sum is over all } 1/2 \text{ and } 3/2 \text{ } P \text{ wave meson states respectively).} \]
  
  
- From experience with sum rules one expects approximate saturation from the ground states, i.e.
  \[ \left| \tau_{3/2}^{(0)}(1) \right|^2 - \left| \tau_{1/2}^{(0)}(1) \right|^2 \approx \frac{1}{4}, \]
  which implies \( |\tau_{1/2}(1)| < |\tau_{3/2}(1)| \). This strongly suggests
  \( \Gamma(B \to D_{0,1}^{1/2} l \nu) < \Gamma(B \to D_{1,2}^{3/2} l \nu) \), which is in conflict with experiment.
Phenomenological models:

- $|\tau_{1/2}(1)| < |\tau_{3/2}(1)|$ and $\Gamma(B \to D_{0,1}^{1/2} l\nu) < \Gamma(B \to D_{1,2}^{3/2} l\nu)$, which is in “conflict” with experiment.


- Same qualitative picture also beyond the static limit, i.e. for finite $m_b$ and $m_c$.

1/2 versus 3/2: possible explanations (1)

- **Experiment:**
  
  (A) The signal for the remaining 15% of $X_c$ is rather vague; therefore, only a small part might be $D_{0,1}^{1/2}$.

- **OPE:**
  
  (B) Sum rules hold in the static limit and might change for finite quark masses.
  
  (C) Sum rules make statements about $\tau_{1/2}(w = 1)$ and $\tau_{3/2}(w = 1)$; to obtain decay rates, however, one has to integrate over $w$.

- **Phenomenological models:**
  
  (B) Models might give a wrong answer.

- **Most probable scenario:** a combination of (A), (B) and (C).

1/2 versus 3/2: possible explanations (2)

- A lattice calculation of $\tau_{1/2}$ and $\tau_{3/2}$ could shed some light on this puzzle.

- Exploratory quenched lattice study confirmed the theory side:
  \[
  \tau_{1/2}(1) = 0.38(4), \quad \tau_{3/2}(1) = 0.53(8).
  \]

- In the following I will report about the first unquenched lattice calculation of $\tau_{1/2}(1)$ and $\tau_{3/2}(1)$.
Lattice calculation of $\tau_{1/2}$ and $\tau_{3/2}$ (1)

- The “Isgur-Wise relations”

\[
\langle D_0^{1/2}(v')|\bar{c}\gamma_5\gamma_\mu b|B(v)\rangle \propto \tau_{1/2}(w)(v - v')_\mu
\]
\[
\langle D_2^{3/2}(v', \epsilon)|\bar{c}\gamma_5\gamma_\mu b|B(v)\rangle \propto \tau_{3/2}(w)\left((w + 1)\epsilon^*_\mu v^\alpha - \epsilon^*_{\alpha\beta}v^\alpha v^\beta v^\nu\right).
\]

are not directly useful to compute $\tau_{1/2}(1)$ and $\tau_{3/2}(1)$.

- They can be rewritten in the following form, which is directly accessible to a lattice calculation:

\[
\langle D_0^{1/2}(v)|\bar{c}\gamma_5\gamma_j D_k b|B(v)\rangle = -ig_{jk}\left(m(D_0^{1/2}) - m(B)\right)\tau_{1/2}(1)
\]
\[
\langle D_2^{3/2}(v, \epsilon)|\bar{c}\gamma_5\gamma_j D_k b|B(v)\rangle = +i\sqrt{3}\epsilon_{jk}\left(m(D_2^{3/2}) - m(B)\right)\tau_{3/2}(1).
\]

[arXiv:hep-ph/9705467]]
Lattice calculation of $\tau_{1/2}$ and $\tau_{3/2}$ (2)

- We compute

$$
\langle D_0^{1/2}(v)|\bar{c}\gamma_5\gamma_j D_k b|B(v)\rangle = -ig_{jk}\left(m(D_0^{1/2}) - m(B)\right)\tau_{1/2}(1)
$$

via

$$
\tau_{1/2}(1) = \lim_{t_0-t_1\to\infty, t_1-t_2\to\infty} \tau_{1/2,\text{effective}}(t_0-t_1, t_1-t_2)
$$

$$
\tau_{1/2,\text{effective}}(t_0 - t_1, t_1 - t_2) =
$$

$$
= \frac{1}{Z_D}\left| \frac{N(P_-) N(S)}{\left(m(P_-) - m(S)\right)} \langle \left(\mathcal{O}^{(P-)}(t_0)\right)^\dagger (\bar{Q}\gamma_5\gamma_3 D_3 Q)(t_1) \mathcal{O}^{(S)}(t_2) \rangle \right|
$$

- We need:

  - Static-light meson creation operators $\mathcal{O}^{(S)}$, $\mathcal{O}^{(P-)}$, $\mathcal{O}^{(P+)}$.
  - Static-light meson masses $m(S)$, $m(P_-)$ and $m(P_+)$.
  - 2-point and 3-point functions (and norms $N(S)$, $N(P_-)$, $N(P_+)$).
Simulation setup (1)

- Lattice volume: $L^3 \times T = 24^3 \times 48$.
- Gauge action: tree-level Symanzik improved,

$$S_G[U] = \frac{\beta}{6} \left( b_0 \sum_{x,\mu \neq \nu} \text{Tr} \left( 1 - P^{1\times1}(x; \mu, \nu) \right) + b_1 \sum_{x,\mu \neq \nu} \text{Tr} \left( 1 - P^{1\times2}(x; \mu, \nu) \right) \right),$$

$b_0 = 1 - 8b_1$, $b_1 = -1/12$.
- Gauge coupling $\beta = 3.9$ corresponds to $a = 0.0855 \text{ fm}$.
Simulation setup (2)

- Fermionic action: Wilson twisted mass, $N_f = 2$ degenerate flavors,

  \[ S_F[\chi, \bar{\chi}, U] = a^4 \sum_x \bar{\chi}(x) \left( D_W + i \mu_q \gamma_5 \tau_3 \right) \chi(x) \]

  \[ D_W = \frac{1}{2} \left( \gamma_\mu (\nabla_\mu + \nabla^*_\mu) - a \nabla^*_\mu \nabla_\mu \right) + m_0 \]

  ($m_0$: untwisted mass; $\mu_q$: twisted mass; $\tau_3$: third Pauli matrix acting in flavor space).

- Relation between the physical basis $\psi$ and the twisted basis $\chi$ (in the continuum):

  \[ \psi = \frac{1}{\sqrt{2}} \left( \cos(\omega/2) + i \sin(\omega/2) \gamma_5 \tau_3 \right) \chi \]

  \[ \bar{\psi} = \frac{1}{\sqrt{2}} \bar{\chi} \left( \cos(\omega/2) + i \sin(\omega/2) \gamma_5 \tau_3 \right) \]

  ($\omega$: twist angle; $\omega = \pi/2$: maximal twist).
Simulation setup (3)

- Untwisted mass $m_0$, tuned to maximal twist ($\kappa = 1/(8 + 2m_0) = 0.160856$) → “automatic $\mathcal{O}(a)$ improvement of physical quantities”.

<table>
<thead>
<tr>
<th>$\mu_q$</th>
<th>$m_{PS}$ in MeV</th>
<th>number of gauge configurations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0040</td>
<td>314(2)</td>
<td>1400</td>
</tr>
<tr>
<td>0.0064</td>
<td>391(1)</td>
<td>1450</td>
</tr>
<tr>
<td>0.0085</td>
<td>448(1)</td>
<td>1350</td>
</tr>
</tbody>
</table>
Static and light quark propagators

- Static quark propagators:
  \[
  \left\langle Q(x)\bar{Q}(y) \right\rangle_{Q,\bar{Q}} = \delta^3(x - y)U^{(\text{HYP2})}(x; y) \left( \Theta(y_0 - x_0) \frac{1 - \gamma_0}{2} + \Theta(x_0 - y_0) \frac{1 + \gamma_0}{2} \right).
  \]
  - Essentially Wilson lines in time direction.
  - HYP2 static action to improve the signal-to-noise ratio.

- Light quark propagators:
  - Stochastic timeslice propagators.
Static-light meson creation operators (1)

• In the continuum, physical basis:

\[ \mathcal{O}^{(\Gamma)}(x) = \bar{Q}(x) \int d\hat{n} \, \Gamma(\hat{n}) U(x; x + r\hat{n}) \psi^{(u)}(x + r\hat{n}). \]

- \( \bar{Q}(x) \) creates an infinitely heavy i.e. static antiquark at position \( x \).
- \( \psi^{(u)}(x + r\hat{n}) \) creates a light quark at position \( x + r\hat{n} \) separated by a distance \( d \) from the static antiquark.
- The spatial parallel transporter

\[ U(x; x + d\hat{n}) = P \left\{ \exp \left( +i \int_{x}^{x+d\hat{n}} dz_j A_j(z) \right) \right\} \]

connects the antiquark and the quark in a gauge invariant way via gluons.
- The integration over the unit sphere \( \int d\hat{n} \) combined with a suitable weight factor \( \Gamma(\hat{n}) \) yields well defined total angular momentum \( J \) and parity \( \mathcal{P} \) (\( \Gamma(\hat{n}) \) is a combination of spherical harmonics [\( \rightarrow \) angular momentum] and \( \gamma \)-matrices [\( \rightarrow \) spin]; Wigner-Eckart theorem).
In the continuum, physical basis:

\[ O^{(\Gamma)}(x) = \bar{Q}(x) \int d\hat{n} \Gamma(\hat{n}) U(x; x + r\hat{n}) \psi^{(u)}(x + r\hat{n}). \]

List of operators (\( J \): total angular momentum; \( j \): total angular momentum of the light cloud; \( P \): parity):

<table>
<thead>
<tr>
<th>( \Gamma(\hat{n}) )</th>
<th>( J^P )</th>
<th>( j^P )</th>
<th>( O_h )</th>
<th>lattice ( j^P )</th>
<th>notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_5 )</td>
<td>0(^-)</td>
<td>(1/2)(^-)</td>
<td>( A_1 )</td>
<td>(1/2)(^-), (7/2)(^-), ...</td>
<td>( S )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>0(^+)</td>
<td>(1/2)(^+)</td>
<td></td>
<td>(1/2)(^+), (7/2)(^+), ...</td>
<td>( P_- )</td>
</tr>
<tr>
<td>( \gamma_1\hat{n}_1 - \gamma_2\hat{n}_2 ) (cyclic)</td>
<td>2(^+)</td>
<td>(3/2)(^+)</td>
<td>( E )</td>
<td>(3/2)(^+), (5/2)(^+), ...</td>
<td>( P_+ )</td>
</tr>
<tr>
<td>( \gamma_5(\gamma_1\hat{n}_1 - \gamma_2\hat{n}_2) ) (cyclic)</td>
<td>2(^-)</td>
<td>(3/2)(^-)</td>
<td></td>
<td>(3/2)(^-), (5/2)(^-), ...</td>
<td>( D_\pm )</td>
</tr>
</tbody>
</table>

On the lattice, twisted basis:

\[ O^{(\Gamma)}(x) = \bar{Q}(x) \sum_{n=\pm\hat{e}_1, \pm\hat{e}_2, \pm\hat{e}_3} \Gamma(\hat{n}) U(x; x + rn) \chi^{(u)}(x + rn). \]
Static-light meson creation operators (3)

- On the lattice, twisted basis:

\[ O^{(\Gamma)}(x) = \bar{Q}(x) \sum_{n=\pm \hat{e}_1, \pm \hat{e}_2, \pm \hat{e}_3} \Gamma(\hat{n}) U(x; x + r n) \chi^{(u)}(x + r n). \]

- Due to the twisted basis each operator creates both a \( \mathcal{P} = + \) part and a \( \mathcal{P} = - \) part (e.g. \( \bar{Q}\gamma_5 \chi \approx (\bar{Q}\gamma_5 \psi - i\bar{Q}\psi)/\sqrt{2} \)).

- Smearing techniques to optimize the ground state overlaps:
  * APE smearing for spatial links \( U \).
  * Gaussian smearing for light quark fields \( \chi^{(u)} \).
Static-light meson masses (1)

- Consider $2 \times 2$ correlation matrices:

$$C_{JK}(t) = \left\langle \left( O^{(\Gamma_J)}(t) \right)^\dagger O^{(\Gamma_K)}(0) \right\rangle.$$

- For $S$ and $P_-$, $\Gamma_J \in \{\gamma_5, 1\}$.
- For $P_+$, $\Gamma_J \in \{\gamma_1 \hat{n}_1 - \gamma_2 \hat{n}_2, \gamma_5(\gamma_1 \hat{n}_1 - \gamma_2 \hat{n}_2)\}$.

- Solve a generalized eigenvalue problem:

$$C_{JK}(t)v_K^{(n)}(t) = C_{JK}(t_0)v_K^{(n)}(t)\lambda^{(n)}(t, t_0).$$

- Determine static-light meson masses from effective mass plateaus:

$$m_{\text{effective}}^{(n)}(t) = \ln \left( \frac{\lambda^{(n)}(t, t_0)}{\lambda^{(n)}(t + 1, t_0)} \right).$$
Static-light meson masses (2)

- The generalized eigenvalue problem,

\[ C_{JK}(t) v_K^{(n)}(t) = C_{JK}(t_0) v_K^{(n)}(t) \lambda^{(n)}(t, t_0), \]

also yields appropriate linear combinations of twisted basis meson creation operators with well defined parity:

\[
\begin{align*}
O^{(S)} &= v^{(S)}_{\gamma_5}(t) O^{(\gamma_5)} + v^{(S)}_1(t) O^{(1)} \\
O^{(P-)} &= v^{(P-)}_{\gamma_5}(t) O^{(\gamma_5)} + v^{(P-)}_1(t) O^{(1)} \\
O^{(P+)} &= v^{(P+)}_{\gamma_x \hat{n}_x - \gamma_y \hat{n}_y}(t) O^{(\gamma_x \hat{n}_x - \gamma_y \hat{n}_y)} + v^{(P+)}_{\gamma_5(\gamma_x \hat{n}_x - \gamma_y \hat{n}_y)}(t) O^{(\gamma_5(\gamma_x \hat{n}_x - \gamma_y \hat{n}_y))}. 
\end{align*}
\]
2-point functions, ground state norms

- 2-point functions are now straightforward to compute:

\[ \langle \left( \mathcal{O}^{(S)}(t) \right)^\dagger \mathcal{O}^{(S)}(0) \rangle, \quad \langle \left( \mathcal{O}^{(P-)}(t) \right)^\dagger \mathcal{O}^{(P-)}(0) \rangle, \quad \langle \left( \mathcal{O}^{(P+)}(t) \right)^\dagger \mathcal{O}^{(P+)}(0) \rangle. \]

- Ground state norms \( N(S) \), \( N(P_-) \) and \( N(P_+) \) by fitting exponentials at large temporal separations, e.g. \( N(S)^2 e^{-mt} \) to \( \langle (\mathcal{O}^{(S)}(t))^\dagger \mathcal{O}^{(S)}(0) \rangle \).
3-point functions, $\tau_{1/2}$ and $\tau_{3/2}$

- Compute the Isgur-Wise function

$$\tau_{1/2}(1) = \left| \frac{\langle P_- | \bar{Q} \gamma_5 \gamma_3 D_3 Q | S \rangle}{m(P_-) - m(S)} \right|$$

via “effective form factors”:

$$\tau_{1/2}(1) = \lim_{t_0-t_1 \to \infty, t_1-t_2 \to \infty} \tau_{1/2,\text{effective}}(t_0-t_1, t_1-t_2)$$

$$\tau_{1/2,\text{effective}}(t_0-t_1, t_1-t_2) =$$

$$= \frac{1}{Z_D} \frac{N(P_-) N(S)}{(m(P_-) - m(S))} \frac{\left| \langle \mathcal{O}^{(P-)}(t_0)^\dagger (\bar{Q} \gamma_5 \gamma_3 D_3 Q)(t_1) \mathcal{O}^{(S)}(t_2) \rangle \right|}{\left| \langle \mathcal{O}^{(P-)}(t_0)^\dagger \mathcal{O}^{(P-)}(t_1) \rangle \langle \mathcal{O}^{(S)}(t_1)^\dagger \mathcal{O}^{(S)}(t_2) \rangle \right|}.$$

- $\tau_{3/2}(1)$ analogously: replace

$$P_- \rightarrow P_+ , \quad \gamma_3 D_3 \rightarrow \frac{\gamma_5 (\gamma_1 D_1 - \gamma_2 D_2)}{\sqrt{6}}.$$
3-point functions, $\tau_{1/2}$ and $\tau_{3/2}$ (2)

- $Z_D$ in

$$
\tau_{1/2,\text{effective}}(t_0 - t_1, t_1 - t_2) = \frac{1}{Z_D} \left| \frac{N(P_-) N(S)}{m(P_-) - m(S)} \right| \left\langle \left( O^{(P_-)}(t_0) \right)^\dagger (\bar{Q} \gamma_5 \gamma_3 D_3 Q)(t_1) O^{(S)}(t_2) \right\rangle
$$

is the renormalization constant of the heavy-heavy current $\bar{Q} \gamma_5 \gamma_3 D_3 Q$, i.e.

$$
(\bar{Q} \gamma_5 \gamma_3 D_3 Q)^R = \frac{(\bar{Q} \gamma_5 \gamma_3 D_3 Q)^B}{Z_D},
$$

to first order in perturbation theory.

- Analytical formulae long and “complicated”.
- Tree-level Symanzik improved gauge action, HYP2 static action: $Z_D = 0.976$. 

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3-point functions, $\tau_{1/2}$ and $\tau_{3/2}$

- Results for various light quark masses:

<table>
<thead>
<tr>
<th>$\mu_q$</th>
<th>$\tau_{1/2}(1)$</th>
<th>$\tau_{3/2}(1)$</th>
<th>$(\tau_{3/2})^2 - (\tau_{1/2})^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0040</td>
<td>0.299(14)</td>
<td>0.519(13)</td>
<td>0.180(16)</td>
</tr>
<tr>
<td>0.0064</td>
<td>0.312(10)</td>
<td>0.538(13)</td>
<td>0.193(13)</td>
</tr>
<tr>
<td>0.0085</td>
<td>0.308(12)</td>
<td>0.522(8)</td>
<td>0.177(9)</td>
</tr>
</tbody>
</table>

- The Uraltsev sum rule,

$$\sum_n \left| \tau_{3/2}^{(n)}(1) \right|^2 - \left| \tau_{1/2}^{(n)}(1) \right|^2 = \frac{1}{4},$$

is almost fulfilled by the ground state contributions $\tau_{1/2}^{(0)}(1) \equiv \tau_{1/2}(1)$ and $\tau_{3/2}^{(0)}(1) \equiv \tau_{3/2}(1)$.
Extrapolation to the $u/d$ quark mass

- Linear extrapolation in $(m_{PS})^2$ to the $u/d$ quark mass $m_{PS} = 135$ MeV:
  - $\tau_{1/2} = 0.296(26)$.
  - $\tau_{3/2} = 0.526(23)$.

Marc Wagner, “Lattice calculation of the Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$ with dynamical quarks”, April 27, 2009
Conclusions

- First dynamical lattice computation of the Isgur-Wise functions $\tau_{1/2}(1)$ and $\tau_{3/2}(1)$:
  - $\tau_{1/2}(1) = 0.296(26)$, $\tau_{3/2}(1) = 0.526(23)$.
  - This indicates $\Gamma(B \to D_{0,1}^{1/2} l \nu) < \Gamma(B \to D_{1,2}^{3/2} l \nu)$ in the static limit.
  - Expectation from sum rules confirmed:
    * Ural'tsev sum rule is approximately fulfilled by the ground states.
    * $\tau_{1/2}(1) \ll \tau_{3/2}(1)$.
    * Numerical values in agreement with sum rule expectation.
- Phenomenological models qualitatively and quantitatively confirmed.
- Experiment:
  * Fair agreement with the experimentally measured $\tau_{3/2}(1) \approx 0.75$.
  * No agreement with the experimentally measured $\tau_{1/2}(1) \approx 1.28$.

[arXiv:0711.3252 [hep-ex]]]
Outlook

- “To do list” and “wish list”:
  - Perform the continuum limit.
  - Use $N_f = 2 + 1 + 1$ flavors of dynamical quarks.
  - Non-perturbative renormalization.
  - Compute $1/m_Q$ corrections.
  - Compute the slopes of the form factors $d\tau_{1/2}/dw \bigg|_{w=1}$, $d\tau_{3/2}/dw \bigg|_{w=1}$.
  - Study other decays of the type $B \rightarrow X_{c\ell\nu}$, e.g. decays into radially excited states.

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