The colour adjoint static potential from Wilson loops with generator insertions and its physical interpretation

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Introduction: singlet static potential (1)

- The (singlet) static potential $V^1$ is a very common and important observable in lattice gauge theory.

- It is the energy of a static antiquark $\bar{Q}(x)$ and a static quark $Q(y)$ in a colour singlet (i.e. a gauge invariant) orientation as a function of the separation $r \equiv |x - y|$.

- The spin of a static quark is irrelevant, i.e. in the following
  - no spin indices or $\gamma$ matrices,
  - only spinless colour charges,

$$\bar{Q}^a_A(x) = (Q^{a,\dagger}(x)\gamma_0)_A \rightarrow Q^{a,\dagger}(x),$$

$$Q^a_A(y) \rightarrow Q^a(y),$$

where $a$ denotes a colour index and $A$ a spin index.
The singlet static potential for gauge group $SU(N)$ can be obtained as follows:

1. Define a trial state
   \[ |\Phi^1\rangle \equiv \overline{Q}(x)U(x,y)Q(y)|0\rangle. \]

2. The temporal correlation function of this trial state simplifies to the well known Wilson loop,
   \[ \langle \Phi^1(t_2)|\Phi^1(t_1)\rangle = e^{-2M\Delta t}N\langle W_1(r,\Delta t)\rangle, \quad \Delta t \equiv t_2 - t_1 > 0. \]

3. The singlet static potential $V^1 \equiv V^1_0$ can be obtained from the asymptotic exponential behaviour,
   \[ \langle W_1(r,\Delta t)\rangle = \sum_{n=0}^{\infty} c_n \exp\left(-V^1_n(r)\Delta t\right)^{\Delta t \to \infty} \exp\left(-V^1(r)\Delta t\right) \]
   \[ V^1(r) = -\lim_{\Delta t \to \infty} \frac{\langle \dot{W}_1(r,\Delta t)\rangle}{\langle W_1(r,\Delta t)\rangle}. \]
**Goal of this work:** compute and interpret the potential of a static antiquark \( \bar{Q}(x) \) and a static quark \( Q(y) \) in a colour adjoint (i.e. a gauge variant) orientation in various gauges as a function of the separation \( r \equiv |x - y| \).

A colour adjoint orientation of a static antiquark and a static quark can be obtained by inserting the generators of the colour group \( T^a \) (e.g. for \( SU(3) \), \( T^a = \lambda^a / 2 \)), i.e. \( \bar{Q} T^a Q |0\rangle \).

If the static antiquark and the static quark are separated in space, a straightforward generalisation is

\[
|\Phi^{T^a}\rangle \equiv \bar{Q}(x) U(x, x_0) T^a U(x_0, y) Q(y) |0\rangle.
\]

A corresponding definition of the colour adjoint static potential has been proposed and used in pNRQCD (a framework based on perturbation theory).

We discuss non-perturbative calculations analogous as for the singlet static potential in various gauges,

\[
\langle \Phi^T_a(t_2) | \Phi^T_a(t_1) \rangle = e^{-2M \Delta t} N \langle W_T^a(r, \Delta t) \rangle ,
\]

\[
W_T^a(r, \Delta t) \equiv \frac{1}{N} \text{Tr} \left( T^a U_R T^a, \dagger U_L \right)
\]

\[
\langle W_T^a(r, \Delta t) \rangle = \sum_{n=0}^{\infty} c_n \exp \left( -V_n^T(r) \Delta t \right) \xrightarrow{\Delta t \to \infty} \exp \left( -V_T^a(r) \Delta t \right).
\]

In particular we are interested,

– whether the colour adjoint static potential \( V_T^a \equiv V_0^T \) is gauge invariant (i.e. whether the obvious gauge dependence of the correlation function \( \langle W_T^a(r, \Delta t) \rangle \) only appears in the matrix elements \( c_n \)),

– whether \( V_T^a \) indeed corresponds to the potential of a static antiquark and a static quark in a colour adjoint orientation, or whether it has to be interpreted differently.
Without gauge fixing

$\langle W_{T^a}(r, \Delta t) \rangle = 0,$

because this correlation function is gauge variant (and does not contain any gauge invariant contribution).

→ Without gauge fixing the calculation of a colour adjoint static potential fails.
\( V^T_a \) in Coulomb gauge

- Coulomb gauge: \( \nabla A^g(x) = 0 \), which amounts to an independent condition on every time slice \( t \).

- The remaining residual gauge symmetry corresponds to global independent colour rotations \( h^\text{res}(t) \in SU(N) \) on every time slice \( t \); with respect to this residual gauge symmetry the colour adjoint Wilson loop transforms as

\[
\langle W_{T^a}(r, \Delta t) \rangle = \frac{1}{N} \text{Tr} \left( T^a U_R T^a, U_L \right) \rightarrow h^\text{res}
\]

\[
\rightarrow h^\text{res} \frac{1}{N} \text{Tr} \left( h^\text{res},(t_1) T^a h^\text{res}(t_1) U_R h^\text{res}(t_2) T^a, h^\text{res},(t_2) U_L \right).
\]

- Since \( h^\text{res}(t_1) \) and \( h^\text{res}(t_2) \) are independent, the situation is analogous to that without gauge fixing, i.e.

\[
\langle W_{T^a}(r, \Delta t) \rangle_{\text{Coulomb gauge}} = 0.
\]

\( \rightarrow \) In Coulomb gauge the calculation of a colour adjoint static potential fails.

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\( V_{T^a} \) in Lorenz gauge

- Lorenz gauge: \( \partial_\mu A^g_\mu(x) = 0 \).

- In Lorenz gauge a Hamiltonian or a transfer matrix does not exist.

- Only gauge invariant correlation functions like the ordinary Wilson loop \( \langle W_1(r, \Delta t) \rangle \) exhibit an asymptotic exponential behaviour and, therefore, allow the determination of energy eigenvalues.

- The colour adjoint Wilson loop \( \langle W_{T^a}(r, \Delta t) \rangle_{\text{Lorenz gauge}} \) does not decay exponentially in the limit of large \( \Delta t \).

→ The physical meaning of a colour adjoint static potential determined from \( \langle W_{T^a}(r, \Delta t) \rangle_{\text{Lorenz gauge}} \) (as frequently done in perturbation theory) is unclear.
$V^T_a$ in temporal gauge (1)

- Temporal gauge: $\partial_\mu A^g_0(x) = 0$ or equivalently $U^g_0(x) = 1$.

- Temporal links gauge transform as

  $$U^g_0(t, x) = g(t, x)U_0(t, x)g^\dagger(t + a, x) \quad , \quad g(t, x) \in SU(N).$$

- A possible choice to implement temporal gauge is

  $$g(t = 2a, x) = U_0(t = a, x),$$
  $$g(t = 3a, x) = g(t = 2a, x)U_0(t = 2a, x) = U_0(t = a, x)U_0(t = 2a, x),$$
  $$g(t = 4a, x) = g(t = 3a, x)U_0(t = 3a, x) = \ldots,$$
  $$\ldots = \ldots$$
$V^T_T^a$ in temporal gauge (2)

- By inserting the transformation to temporal gauge $g(t, x)$, the gauge variant colour adjoint Wilson loop turns into a gauge invariant observable:

$$
\langle W_{T^a}(r, \Delta t) \rangle \text{temporal gauge} =
\frac{1}{N} \langle \text{Tr} \left( U^{T^a,g(t_1; x, y)} U^{T^a,g(t_2; y, x)} \right) \rangle \text{temporal gauge} = \ldots =
\frac{2}{N(N^2 - 1)} \sum_a \sum_b \langle \text{Tr} \left( T^a U^R T^b U^L \right) \text{Tr} \left( T^a U(t_1, t_2; x_0) T^b U(t_2, t_1; x_0) \right) \rangle
$$

$$
(U^{T^a}(x, y) = U(x, x_0) T^a U(x_0, y)).
$$

- $\text{Tr}(T^a U^R T^b U_L)$: Wilson loop with generator insertions.

- $\text{Tr}(T^a U(t_1, t_2; x_0) T^b U(t_2, t_1; x_0))$: propagator of a static adjoint quark.

$\rightarrow$ The colour adjoint Wilson loop in temporal gauge is a correlation function of a gauge invariant three-quark state, one fundamental static quark, one fundamental static anti-quark, one adjoint static quark.
$V_{T^a}$ in temporal gauge (3)

- Equivalently, after defining
  \[ |\Phi^{Q\bar{Q}\text{ad}}\rangle \equiv Q_{\text{ad},a}(x_0)(\bar{Q}(x)U^{T^a}(x, y)Q(y))|0\rangle, \]
  one can verify
  \[ \langle \Phi^{Q\bar{Q}\text{ad}}(t_2)|\Phi^{Q\bar{Q}\text{ad}}(t_1)\rangle \propto \left\langle W_{T^a}(r, \Delta t) \right\rangle_{\text{temporal gauge}}. \]

$V_{T^a}$ in temporal gauge should not be interpreted as the potential of a static quark and a static anti-quark, which form a colour-adjoint state.

$V_{T^a}$ in temporal gauge is the potential of a colour-singlet three-quark state.

$V_{T^a}$ in temporal gauge does not only depend on the $Q\bar{Q}$ separation $r = |x - y|$, but also on the position $s = |x - x_0|/2 - |y - x_0|/2$ of the static adjoint quark $Q_{\text{ad}}$, i.e. $V_{T^a}(r, s)$ (in the following we work with the symmetric alignment $x_0 = (x + y)/2$).
\( V^{T^a} \) in temporal gauge (4)

- A different approach, leading to the same result, is the transfer matrix formalism.


- One can perform a spectral analysis of the colour adjoint Wilson loop:

\[
\langle W_{T^a}(r, \Delta t) \rangle_{\text{temporal gauge}} = \frac{1}{N} \sum_k e^{-\left( V^{T^a}_k(r) - \varepsilon_0 \right) \Delta t} \sum_{\alpha, \beta} \left| \langle k^a_{\alpha\beta} | U^{T^a}_{\alpha\beta}(x, y) | 0 \rangle \right|^2 ,
\]

where \( |k^a_{\alpha\beta} \rangle \) denotes states containing three static quarks (one fundamental static quark, one fundamental static anti-quark, one adjoint static quark).

→ Again the conclusion is that \( V^{T^a} \) in temporal gauge is the potential of a colour-singlet three-quark state.
A gauge invariant definition via B fields?

- In the literature one can also find a proposal of a gauge invariant quantity, from which the colour adjoint static potential can possibly be determined,

\[ W_B(r, \Delta t) \equiv \frac{1}{N} \text{Tr} \left( T^a U_R T^{b,\dagger} U_L \right) B^a(x_0, t_1) B^b(x_0, t_2), \]

i.e. open colour indices are saturated by colour magnetic fields.


- However, using the transfer matrix formalism one can again perform a spectral analysis and show that only states with a fundamental quark and a fundamental antiquark \( |k_{\alpha\beta}\rangle \) (i.e. singlet static potentials) contribute to the correlation function:

\[
\langle W_B(r, \Delta t) \rangle = \sum_k e^{-(V_k^{1,-}(r) - \mathcal{E}_0)\Delta t} \sum_{\alpha,\beta} \left| \langle k_{\alpha\beta} | U^{T^a_B}(x, y) | 0 \rangle \right|^2.
\]

\[ \rightarrow \langle W_B(r, \Delta t) \rangle \] is not suited to extract a colour adjoint static potential.
Numerical lattice results for $SU(2)$

- $SU(2)$ colour group, four different lattice spacings $a = 0.038$ fm $\ldots 0.102$ fm.
- In temporal gauge the colour adjoint (or rather $Q\bar{Q}Q^{\text{ad}}$) static potential $V^{T^a}$ is attractive,
  - for small separations stronger than the singlet static potential $V^1$,
  - for large separations the slope is the same as for the singlet static potential $V^1$ (indicates flux tube formation between $QQ^{\text{ad}}$ and $\bar{Q}Q^{\text{ad}}$).
LO perturbative calculations (1)

- Perturbation theory for static potentials is a good approximation for small quark separations and should agree in that region with corresponding non-perturbative results.

- Singlet static potential (gauge invariant, i.e. the gauge is not important):

\[ V^1(r) = -\frac{(N^2 - 1)g^2}{8N\pi r} + \text{const} + \mathcal{O}(g^4). \]

- Colour adjoint static potential (in Lorenz gauge):

\[ V^{T^a}(r) = +\frac{g^2}{8N\pi r} + \text{const} + \mathcal{O}(g^4). \]

  - In Lorenz gauge a Hamiltonian or a transfer matrix does not exist, i.e. the physical meaning is unclear; appears frequently in the literature.
  - The repulsive behaviour is not reproduced by any of the presented non-perturbative considerations or computations.
LO perturbative calculations (2)

- Colour adjoint static potential ("in temporal gauge"; more precisely: perturbative calculation in Lorenz gauge of the gauge invariant observable, which is equivalent to the colour adjoint Wilson loop in temporal gauge):

\[ V^{Q\bar{Q}Q^{\text{ad}}}(r, s = 0) = -\frac{(4N^2 - 1)g^2}{8N\pi r} + \text{const} + \mathcal{O}(g^4). \]

- Attractive and stronger by a factor 4...5 than the singlet static potential (depending on \(N\)).
- Qualitative agreement with numerical lattice results for \(SU(2)\).
Conclusions

- We have discussed the non-perturbative definition of a static potential $V^{Ta}$ for a quark antiquark pair in a colour adjoint orientation, based on Wilson loops with generator insertions $\langle W^{Ta}(r, \Delta t) \rangle$ in various gauges:
  - Without gauge fixing/Coulomb gauge: $\langle W^{Ta}(r, \Delta t) \rangle = 0$, i.e. the calculation of a potential $V^{Ta}$ fails.
  - Lorenz gauge: a Hamiltonian or a transfer matrix does not exist, the physical meaning of a corresponding potential $V^{Ta}$ is unclear.
  - Temporal gauge: a strongly attractive potential $V^{Ta}$, which should be interpreted as the potential of three quarks, i.e. $V^{Ta} = VQ\bar{Q}Q^{ad}$.

Clearly the resulting potential $V^{Ta}$ is gauge dependent.

- Saturating open colour indices with $B^{a}$, yields a singlet static potential.

- LO perturbation theory in Lorenz gauge has long predicted $V^{Ta}$ to be repulsive; it appears impossible, to reproduce this repulsive behaviour by a non-perturbative computation based on Wilson loops.