The spectrum of and forces between $B$ mesons from lattice QCD

Seminar, University of Nicosia, Cyprus

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I will discuss two lattice QCD projects (which are related):

(1) Computation of the spectrum of radially/orbitally excited $B$ mesons.

(2) Computation of spin/isospin/parity dependent forces between $B$ mesons.
Part 1

Computation of the spectrum of radially/orbitally excited $B$ mesons

[K. Jansen, C. Michael, A. Shindler and M.W. [ETM Collaboration], JHEP 0812, 058 (2008)]

[C. Michael, A. Shindler and M.W. [ETM Collaboration], JHEP 1008, 009 (2010)]
• Quantum field theory of quarks (six flavors $u$, $d$, $s$, $c$, $t$, $b$, which differ in mass) and gluons.

• Part of the standard model explaining the formation of hadrons (mesons = $q\bar{q}$, baryons = $qqq/\bar{q}\bar{q}\bar{q}$) and their masses; essential for decays involving hadrons.

• Definition of QCD by means of an action simple:

$$S = \int d^4x \left( \sum_{f \in \{u,d,s,c,t,b\}} \bar{\psi}(f) \left( \gamma_\mu \left( \partial_\mu - i A_\mu \right) + m(f) \right) \psi(f) + \frac{1}{2g^2} \text{Tr} \left( F_{\mu\nu} F^{\mu\nu} \right) \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu].$$

• However, no analytical solutions for low energy QCD observables, e.g. hadron masses, known, because of the absence of any small parameter (i.e. perturbation theory not applicable).
**B mesons, static-light mesons**

- **B meson**: a meson made from a heavy $b$ quark ($m_b \approx 4200$ MeV) and a light $u$, $d$ or $s$ quark ($m_l \lesssim 100$ MeV), e.g. $B = \{\bar{b}u, \bar{b}d\}$, $B_s = \bar{b}s$.

- **Static limit**, i.e. $m_b \to \infty$:
  - No interactions involving the static quark spin.
  - Classify states according to parity $\mathcal{P}$ and half-integer total angular momentum of the light cloud $j$.

- $m_b$ finite, but heavy:
  - Classify states according to parity $\mathcal{P}$ and total angular momentum $J$.

<table>
<thead>
<tr>
<th>$j^P$ (static-light)</th>
<th>$J^P$ (finite $m_b$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1/2)^- \equiv S$</td>
<td>$0^- \equiv B^*_{(s)}$</td>
</tr>
<tr>
<td></td>
<td>$1^- \equiv B^*_{(s)}$</td>
</tr>
<tr>
<td>$(1/2)^+ \equiv P_-$</td>
<td>$0^+ \equiv B^*_{(s)0}$ (not in PDG)</td>
</tr>
<tr>
<td></td>
<td>$1^+ \equiv B^*_{(s)1}$ (not in PDG)</td>
</tr>
<tr>
<td>$(3/2)^+ \equiv P_+$</td>
<td>$1^+ \equiv B_{(s)1}$</td>
</tr>
<tr>
<td></td>
<td>$2^+ \equiv B^*_{(s)2}$</td>
</tr>
<tr>
<td>$(3/2)^- \equiv D_-$</td>
<td>$1^-$ (no experiment)</td>
</tr>
<tr>
<td>$(5/2)^- \equiv D_+$</td>
<td>$2^-$ (no experiment)</td>
</tr>
<tr>
<td></td>
<td>$2^-$ (no experiment)</td>
</tr>
<tr>
<td></td>
<td>$3^-$ (no experiment)</td>
</tr>
<tr>
<td>$(5/2)^+ \equiv F_-$</td>
<td>$2^+$ (no experiment)</td>
</tr>
<tr>
<td></td>
<td>$3^+$ (no experiment)</td>
</tr>
</tbody>
</table>

...
**How to compute** $M(\text{static-light meson})$? (1)

- Let $\mathcal{O}(x)$ be a suitable “static-light meson creation operator”, i.e. an operator such that $\mathcal{O}(x)\ket{\Omega}$ is a state containing a static-light meson at position $x$ ($\ket{\Omega}$: vacuum).

- More precisely: ... an operator such that $\mathcal{O}(x)\ket{\Omega}$ has the same quantum numbers ($j^P$, flavor) as the static-light meson of interest.

- Determine the mass of the ground state of the corresponding static-light meson from the exponential behavior of the corresponding correlation function $C$ at large Euclidean times $T$:

$$
C(T) = \bra{\Omega} \left( \mathcal{O}(x, T) \right) \dagger \mathcal{O}(x, 0) \ket{\Omega} = \bra{\Omega} e^{+HT} \left( \mathcal{O}(x, 0) \right) \dagger e^{-HT} \mathcal{O}(x, 0) \ket{\Omega} = \\
= \sum_n \left| \bra{n} \mathcal{O}(x, 0) \ket{\Omega} \right|^2 \exp \left( - (E_n - E_\Omega) T \right) \approx \quad \text{(for } T \gg 1) \\
\approx \left| \bra{0} \mathcal{O}(x, 0) \ket{\Omega} \right|^2 \exp \left( - \frac{(E_0 - E_\Omega)}{M(\text{static-light meson})} T \right).
$$
How to compute $\mathcal{M}$ (static-light ...)

- General form of a static-light meson creation operator:

$$O(x) = \bar{Q}(x) \int d\hat{n} \Gamma(\hat{n}) U(x; x + d\hat{n}) q(x + d\hat{n}).$$

- $\bar{Q}(x)$ creates an infinitely heavy i.e. static antiquark at position $x$.
- $q(x + d\hat{n})$ creates a light quark at position $x + d\hat{n}$ separated by a distance $d$ from the static antiquark.
- The spatial parallel transporter

$$U(x; x + d\hat{n}) = P \left\{ \exp \left( +i \int_{x}^{x+d\hat{n}} dz_j A_j(z) \right) \right\}$$

connects the antiquark and the quark in a gauge invariant way via gluons.
- The integration over the unit sphere $\int d\hat{n}$ combined with a suitable weight factor $\Gamma(\hat{n})$ yields well defined total angular momentum $J$ and parity $P$ ($\Gamma(\hat{n})$ is a combination of spherical harmonics [→ angular momentum] and $\gamma$-matrices [→ spin]; Wigner-Eckart theorem).

Marc Wagner, "The spectrum of and forces between $B$ mesons from lattice QCD", March 7, 2012
How to compute $M$ (static-light ...)

- General form of a static-light meson creation operator:

$$O(x) = \bar{Q}(x) \int d\hat{n} \Gamma(\hat{n}) U(x; x + d\hat{n}) q(x + d\hat{n}).$$

- List of operators ($J$: total angular momentum; $j$: total angular momentum of the light cloud; $P$: parity):

<table>
<thead>
<tr>
<th>$\Gamma(\hat{n})$</th>
<th>$J^P$</th>
<th>$j^P$</th>
<th>$O_h$</th>
<th>lattice $j^P$</th>
<th>notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_5, \gamma_5 \gamma_j \hat{n}_j$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1, \gamma_j \hat{n}_j$</td>
<td>0⁻ [1⁻]</td>
<td>(1/2)⁻</td>
<td>$A_1$</td>
<td>(1/2)⁻ , (7/2)⁻ , ...</td>
<td>$S$</td>
</tr>
<tr>
<td></td>
<td>0⁺ [1⁺]</td>
<td>(1/2)⁺</td>
<td></td>
<td>(1/2)⁺ , (7/2)⁺ , ...</td>
<td>$P_-$</td>
</tr>
<tr>
<td>$\gamma_1 \hat{n}_1 - \gamma_2 \hat{n}_2$ (and cyclic)</td>
<td>2⁺ [1⁺]</td>
<td>(3/2)⁺</td>
<td></td>
<td>(3/2)⁺ , (5/2)⁺ , ...</td>
<td>$P_+$</td>
</tr>
<tr>
<td>$\gamma_5 (\gamma_1 \hat{n}_1 - \gamma_2 \hat{n}_2)$ (and cyclic)</td>
<td>2⁻ [1⁻]</td>
<td>(3/2)⁻</td>
<td></td>
<td>(3/2)⁻ , (5/2)⁻ , ...</td>
<td>$D_\pm$</td>
</tr>
<tr>
<td>$\gamma_1 \hat{n}_2 \hat{n}_3 + \gamma_2 \hat{n}_3 \hat{n}_1 + \gamma_3 \hat{n}_1 \hat{n}_2$</td>
<td>3⁻ [2⁻]</td>
<td>(5/2)⁻</td>
<td>$A_2$</td>
<td>(5/2)⁻ , (7/2)⁻ , ...</td>
<td>$D_+$</td>
</tr>
<tr>
<td>$\gamma_5 (\gamma_1 \hat{n}_2 \hat{n}_3 + \gamma_2 \hat{n}_3 \hat{n}_1 + \gamma_3 \hat{n}_1 \hat{n}_2)$</td>
<td>3⁺ [2⁺]</td>
<td>(5/2)⁺</td>
<td></td>
<td>(5/2)⁺ , (7/2)⁺ , ...</td>
<td>$F_\pm$</td>
</tr>
</tbody>
</table>

Marc Wagner, “The spectrum of and forces between $B$ mesons from lattice QCD”, March 7, 2012
• Goal: compute correlation functions $C(T)$ of the previously discussed static-light meson creation operators (the corresponding meson masses can directly be read off from their exponential decays).

• Use the path integral formulation of QCD,

\[
C(T) = \langle \Omega | \left( \mathcal{O}(x, T) \right)^\dagger \mathcal{O}(x, 0) | \Omega \rangle = \frac{1}{Z} \int \left( \prod_f D\psi^{(f)} D\bar{\psi}^{(f)} \right) DA_\mu \left( \mathcal{O}(x, T) \right)^\dagger \mathcal{O}(x, 0) e^{-S[\psi^{(f)}, \bar{\psi}^{(f)}, A_\mu]}.
\]

– $|\Omega\rangle$: ground state/vacuum.

– $(\mathcal{O}(x, T))^\dagger \mathcal{O}(x, 0)$: function of the quark and gluon fields (cf. previous slides).

– $\int \left( \prod_f D\psi^{(f)} D\bar{\psi}^{(f)} \right) DA_\mu$: integral over all possible quark and gluon field configurations $\psi^{(f)}(x, t)$ and $A_\mu(x, t)$.

– $e^{-S[x]}$: weight factor containing the QCD action.
Lattice QCD (2)

- Numerical implementation of the path integral formalism in QCD:
  - Discretize spacetime with sufficiently small lattice spacing
    \[ a \approx 0.05 \text{ fm} \ldots 0.10 \text{ fm} \]
    \[ \rightarrow \text{“continuum physics”}. \]
  - “Make spacetime periodic” with sufficiently large extension
    \[ L \approx 2.0 \text{ fm} \ldots 4.0 \text{ fm} \] (4-dimensional torus)
    \[ \rightarrow \text{“no finite size effects”}. \]

\[ x_\mu = (n_0, n_1, n_2, n_3) \in \mathbb{Z}^4 \]
Lattice QCD (3)

- Numerical implementation of the path integral formalism in QCD:
  - After discretization the path integral becomes an ordinary multidimensional integral:

\[ \int D\psi D\bar{\psi} DA \ldots \rightarrow \prod_{x_\mu} \left( \int d\psi(x_\mu) d\bar{\psi}(x_\mu) dU(x_\mu) \right) \ldots \]

- Typical present-day dimensionality of a discretized QCD path integral:
  * \( x_\mu: 32^4 \approx 10^6 \) lattice sites.
  * \( \psi = \psi^a_i(f) \): 24 quark degrees of freedom for every flavor (\( \times 2 \) particle/antiparticle, \( \times 3 \) color, \( \times 4 \) spin), 2 flavors.
  * \( U = U^{ab}_{\mu} \): 32 gluon degrees of freedom (\( \times 8 \) color, \( \times 4 \) spin).
  * In total: \( 32^4 \times (2 \times 24 + 32) \approx 83 \times 10^6 \) dimensional integral.

→ standard approaches for numerical integration not applicable
→ sophisticated algorithms mandatory (stochastic integration techniques, so-called Monte-Carlo algorithms).
Lattice setup

- Various lattice spacings: $a \approx 0.051 \text{ fm}, 0.064 \text{ fm}, 0.080 \text{ fm}$ (corresponding to $48^3 \times 96, 32^3 \times 64, 24^3 \times 48$ lattice sites).

- Lattice extensions: $L \approx 2.45 \text{ fm}, 2.05 \text{ fm}, 1.92 \text{ fm}$ (periodic boundary conditions).

- Many different pion mass: $m_{PS} \approx 284 \ldots 637 \text{ MeV}$. 

Marc Wagner, “The spectrum of and forces between $B$ mesons from lattice QCD”, March 7, 2012
Masses of $B$ and $B_s$ mesons (1)

- Compute static-light meson masses ($B/B_s$ mesons with $m_b \to \infty$) for different light $u/d$ quark masses and different lattice spacings:
  - Different $u/d$ quark masses to extrapolate to the physical $u/d$ quark mass (due to technical reasons $m_{PS}^{(lattice)} \gtrsim 284$ MeV, $m_{PS}^{physical} \approx 135$ MeV).
  - Different lattice spacings to extrapolate to the continuum.
  - Horizontal axis: pion mass ($m_{PS}^{2}$).
  - Vertical axis: $M(j^P) - M((1/2)^-)$ mass difference between radially and orbitally excited “$B$ mesons” ($B_0^*, B_1^*, B_1, B_2^*$, ...) and the “ground state $B$ meson” ($B/B^* \equiv j^P = (1/2)^-$) ... analogous for “$B_s$ mesons”.

---

$M((1/2)^+) - M((1/2)^-)$

\[
\frac{(M(P_+ - M(S))_{physical}}{\chi^2/\text{d.o.f.} = 0.95} = (406.03 \pm 18.74) \text{ MeV}
\]

\[
\frac{(M(D_- - M(S))_{physical}}{\chi^2/\text{d.o.f.} = 0.95} = (870.35 \pm 26.68) \text{ MeV}
\]

\[
\frac{(M(D_+ - M(S))_{physical}}{\chi^2/\text{d.o.f.} = 0.95} = (930.13 \pm 29.78) \text{ MeV}
\]

\[
\frac{(M(F_- - M(S))_{physical}}{\chi^2/\text{d.o.f.} = 0.95} = (1196.41 \pm 29.78) \text{ MeV}
\]

\[
\frac{(M(S^*) - M(S))_{physical}}{\chi^2/\text{d.o.f.} = 0.95} = (755.05 \pm 16.13) \text{ MeV}
\]
**Masses of $B$ and $B_s$ mesons (2)**

- Summary of the computed static-light meson spectrum:

<table>
<thead>
<tr>
<th>$j^P$</th>
<th>alternative notation</th>
<th>$B$ mesons ($\bar{b}u$ or $\bar{b}d$): $M(j^P) - M((1/2)^-)$ in MeV</th>
<th>$B_s$ mesons ($\bar{b}s$): $M(j^P) - M((1/2)^-)$ in MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1/2)^+$</td>
<td>$P_-$</td>
<td>406(19)</td>
<td>413(12)</td>
</tr>
<tr>
<td>$(3/2)^+$</td>
<td>$P_+$</td>
<td>516(18)</td>
<td>504(12)</td>
</tr>
<tr>
<td>$(3/2)^-,(5/2)^-$</td>
<td>$D_{\pm}$</td>
<td>870(27)</td>
<td>770(26)</td>
</tr>
<tr>
<td>$(5/2)^-$</td>
<td>$D_+$</td>
<td>930(28)</td>
<td>960(24)</td>
</tr>
<tr>
<td>$(5/2)^+, (7/2)^+$</td>
<td>$F_{\pm}$</td>
<td>1196(30)</td>
<td>1179(37)</td>
</tr>
<tr>
<td>$(1/2)^-$</td>
<td>$S^*$</td>
<td>755(16)</td>
<td>751(26)</td>
</tr>
</tbody>
</table>

- Motivation/achievements:
  
  - Continuum limit (among the first).
  
  - Dependence on the light $u/d$ sea quark mass (for the first time).
  
  - Valuable input for model builders (e.g. no reversal of $M(P_-)$ and $M(P_+)$, ...).
Masses of $B$ and $B_s$ mesons (3)

• Comparison to experimental results:
  
  – Extrapolation to the physical (finite) $b$ quark mass $m_B \approx 4200$ MeV:
    
    * Use rather precise experimental results for $c$ quarks, i.e. $D$ mesons.
    * Assume that Heavy Quark Effective Theory (HQET) up to $O(1/m_Q)$ is “valid” down to the physical charm quark mass.
    * Amounts to “reincluding” hyperfine splitting.

<table>
<thead>
<tr>
<th>name</th>
<th>$M - M(B)$ in MeV</th>
<th>name</th>
<th>$M - M(B_s)$ in MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B^*_0$</td>
<td>443(21)</td>
<td>$B^*_0$</td>
<td>391(8)</td>
</tr>
<tr>
<td>$B^*_1$</td>
<td>460(22)</td>
<td>$B^*_s0$</td>
<td>440(8)</td>
</tr>
<tr>
<td>$B_1$</td>
<td>530(12)</td>
<td>$B^*_s1$</td>
<td>464(5)</td>
</tr>
<tr>
<td>$B^*_1$</td>
<td>464(5)</td>
<td>$B^*_s1$</td>
<td>526(8)</td>
</tr>
<tr>
<td>$B_2^*$</td>
<td>543(12)</td>
<td>$B^*_s2$</td>
<td>539(8)</td>
</tr>
<tr>
<td>$B_J^*$</td>
<td>418(8)</td>
<td>$B^*_sJ$</td>
<td>487(15)</td>
</tr>
</tbody>
</table>

• Difference between lattice and experimental results: scale setting problem?
Part 2

Computation of spin/isospin/parity dependent forces between $B$ mesons.

[M. W. [ETM Collaboration], PoS LATTICE2010, 162 (2010)]
• Goal: compute the potential of (or equivalently the force between) two $B$ mesons from first principles by means of lattice QCD:

– Treat the $b$ quark in the static approximation.

– Consider only pseudoscalar/vector mesons ($j^P = (1/2)^-$, denoted by $S$, PDG: $B, B^*$) and scalar/pseudovector mesons ($j^P = (1/2)^+$, denoted by $P_-$, PDG: $B_0^*, B_1^*$), which are among the lightest static-light mesons.

– Study the dependence of the mesonic potential $V(R)$ on

  * the light quark flavor $u$ and/or $d$ (isospin),
  * the light quark spin (the static quark spin is irrelevant),
  * the type of the meson $S$ and/or $P_-$. 

\[ \bar{Q} \uparrow \quad \bar{Q} \downarrow \]

\[ V(R) = ? \]

\[ P = - \quad P = + \]
Motivation:

- First principles computation of a hadronic force.
- Possible application: determine, whether two $B$ mesons may form bound states (tetraquarks).
- Until now
  - it has mainly been studied in the quenched approximation,
  - only pseudoscalar ($S$), but no scalar ($P_{-}$) $B$ mesons have been considered.

[G. Bali and M. Hetzenegger, PoS LATTICE2010, 142 (2010)]
(Pseudo)scalar $B$ mesons

- Symmetries and quantum numbers of static-light mesons:
  - Isospin: $I = 1/2$, $I_z = \pm 1/2$, i.e. $B \equiv \bar{Q}u$ or $B \equiv \bar{Q}d$.
  - Parity: $\mathcal{P} = \pm$,
    * $\mathcal{P} = - \equiv S$ (wave),
    * $\mathcal{P} = + \equiv P_-$ (wave).
  - Rotations:
    * Light cloud angular momentum $j = 1/2$ (for $S$ and $P_-$), $j_z = \pm 1/2$.
    * Static quark spin: irrelevant (static quarks can also be treated as spinless color charges).

- Examples of static-light meson creation operators:
  - $\bar{Q}\gamma_5 q$ (pseudoscalar, i.e. $S$), $q \in \{u, d\}$,
  - $\bar{Q}q$ (scalar, i.e. $P_-$)

($j_z$ is not well-defined, when using these operators).
**BB systems (1)**

- Symmetries and quantum numbers of a pair of static-light mesons (separated along the $z$-axis):
  - Isospin: $I = 0, 1$, $I_z = -1, 0, +1$.
  - Rotations around the $z$-axis:
    * Angular momentum of the light degrees of freedom $j_z = -1, 0, +1$.
    * Static quark spin: irrelevant (static quarks can also be treated as spinless color charges).
  - Parity: $\mathcal{P} = \pm$.
  - If $j_z = 0$, reflection along the $x$-axis: $\mathcal{P}_x = \pm$.
  - Instead of using $j_z = \pm 1$ one can also label states by $|j_z| = 1$, $\mathcal{P}_x = \pm$.

→ Label $BB$ states by $(I, I_z, |j_z|, \mathcal{P}, \mathcal{P}_x)$. 

$\bar{Q}$
To extract the potential(s) of a given sector (characterized by $(I, I_z, |j_z|, \mathcal{P}, \mathcal{P}_x)$), compute the temporal correlation function of the trial state

$$(C\Gamma)_{AB}\left(\bar{Q}_C(-R/2)q_A^{(1)}(-R/2)\right)\left(\bar{Q}_C(+R/2)q_B^{(2)}(+R/2)\right)|\Omega\rangle,$$

where

- $C = \gamma_0\gamma_2$ (charge conjugation matrix),
- $q^{(1)}q^{(2)} \in \{ud - du , uu, dd, ud + du\}$ (isospin $I, I_z$),
- $\Gamma$ is an arbitrary combination of $\gamma$ matrices (spin $|j_z|$, parity $\mathcal{P}, \mathcal{P}_x$).
**BB systems (3)**

- **BB** creation operators for $I_z = +1$: 16 operators of type

\[
(C\Gamma)_{AB}\left(\bar{Q}_C(-R/2)q_A^{(u)}(-R/2)\right)\left(\bar{Q}_C(+R/2)q_B^{(u)}(+R/2)\right).
\]

| $\Gamma$    | $|j_z|, \mathcal{P}, \mathcal{P}_x$ |
|-------------|-----------------------------------|
| 1           | 0, $-,-$                          |
| $\gamma_0\gamma_5$ | 0, $+,$ $+$          |
| $\gamma_5$   | 0, $+,-$                          |
| $\gamma_0$   | 0, $+,-$                          |
| $\gamma_3$   | 0, $-,-$                          |
| $\gamma_0\gamma_3\gamma_5$ | 0, $+,$ $+$        |
| $\gamma_3\gamma_5$ | 0, $-,-$                       |
| $\gamma_0\gamma_3$ | 0, $-,-$                        |
| $\gamma_1$   | 1, $-,-$                          |
| $\gamma_0\gamma_1\gamma_5$ | 1, $+,$ $+$          |
| $\gamma_1\gamma_5$ | 1, $-,-$                       |
| $\gamma_0\gamma_1$ | 1, $-,-$                     |
| ...          | ...                              |
$BB$ systems (4)

- $BB$ creation operators for $I_z = 0$: 32 operators of type

$$
(C\Gamma)_{AB}\left(\bar{Q}_C(-R/2)q_A^{(u)}(-R/2)\right)\left(\bar{Q}_C(+R/2)q_B^{(d)}(+R/2)\right) \pm (u \leftrightarrow d).
$$

| $\Gamma$, $\pm$ | $|j_z|$, $I$, $\bar{P}$, $\bar{P}$ |
|----------------------|------------------|
| $\gamma_5$, $-$     | 0, 0, $-$, $+$   |
| $\gamma_0$, $-$     | 0, 0, $-$, $-$   |
| 1, $-$               | 0, 0, $+$, $-$   |
| $\gamma_0\gamma_5$, $-$ | 0, 0, $-$, $+$ |
| $\gamma_3\gamma_5$, $-$ | 0, 0, $+$, $+$ |
| $\gamma_0\gamma_3\gamma_5$, $-$ | 0, 0, $+$, $-$ |
| $\gamma_3$, $-$     | 0, 0, $+$, $-$   |
| $\gamma_0\gamma_3\gamma_5$, $-$ | 0, 0, $-$, $+$ |
| $\gamma_5$, $+$     | 0, 1, $+$, $+$   |
| $\gamma_0$, $+$     | 0, 1, $+$, $-$   |
| 1, $+$               | 0, 1, $-$, $-$   |
| $\gamma_0\gamma_5$, $+$ | 0, 1, $+$, $+$ |

...
Lattice setup

- Lattice spacing: \( a \approx 0.079 \) fm.
- Lattice extension: \( L \approx 1.90 \) fm (periodic boundary conditions).
- Pion mass: \( m_{PS} \approx 340 \) MeV.
Discussion of results (1)

- Four “types of potentials”:
  - Two attractive, two repulsive.
  - Two have asymptotic values, which are larger by \( \approx 400 \text{ MeV} \).

- There are cases, where two potentials with identical quantum numbers are completely different (i.e. of different type)
  \( \rightarrow \) at least one of the corresponding trial states must have very small ground state overlap
  \( \rightarrow \) physical understanding, i.e. interpretation of trial states needed.
• Expectation at large meson separation $R$: $V(R) \approx 2 \times \text{meson mass}$.
  
  − Potentials with smaller asymptotic value at $\approx 2 \times m(S)$.
  − $m(P_-) - m(S) \approx 400 \text{ MeV}$: approximately the observed difference between different types of potentials.

→ Two types correspond to $S \leftrightarrow S$ potentials.
→ Two types correspond to $S \leftrightarrow P_-$ potentials.

• Can this be understood in detail on the level of the used $BB$ creation operators?
Discussion of results (3)

- Express the $BB$ creation operators in terms of static-light meson creation operators (use suitable spin and parity projectors for the light quarks).
  
  - Examples:
    
    $\ast$ $uu, \Gamma = 1 \rightarrow \mathcal{P} = -, \mathcal{P}_x = -$:
    
    \begin{equation*}
    (C1)_{AB} \left( \bar{Q}_C (-R/2)q_A^u (-R/2) \right) \left( \bar{Q}_C (+R/2)q_B^u (+R/2) \right) \propto S_\uparrow P_\downarrow - S_\downarrow P_\uparrow + P_\uparrow S_\downarrow - P_\downarrow S_\uparrow.
    \end{equation*}

    $\ast$ $uu, \Gamma = \gamma_3 \rightarrow \mathcal{P} = -, \mathcal{P}_x = -$:
    
    \begin{equation*}
    (C\gamma_3)_{AB} \left( \bar{Q}_C (-R/2)q_A^u (-R/2) \right) \left( \bar{Q}_C (+R/2)q_B^u (+R/2) \right) \propto S_\uparrow S_\downarrow + S_\downarrow S_\uparrow - P_\uparrow P_\downarrow - P_\downarrow P_\uparrow.
    \end{equation*}

  - $\text{SS/SP}_-$ content and asymptotic values in agreement for all 64 correlation functions/potentials

  $\rightarrow$ asymptotic differences understood.

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Marc Wagner, “The spectrum of and forces between $B$ mesons from lattice QCD”, March 7, 2012
• Is there a general rule, about when a potential is repulsive and when attractive?

- $S \leftrightarrow S$ potentials:
  * $(I = 0, \ s = 0)$ or $(I = 1, \ s = 1)$, i.e. $I = s$ $\rightarrow$ attractive
  $(I = 0, \ s = 1)$ or $(I = 1, \ s = 0)$, i.e. $I \neq s$ $\rightarrow$ repulsive
  ($s$: combined angular momentum of the two mesons).

* Example: $uu, \ \Gamma = \gamma_3$ $\rightarrow$ $\mathcal{P} = -, \ \mathcal{P}_x = -$:

$$
(C\gamma_3)_{AB}\left(\bar{Q}_C(-R/2)q_A^{(u)}(-R/2)\right)\left(\bar{Q}_C(+R/2)q_B^{(u)}(+R/2)\right) \propto S^\uparrow S^\downarrow + S^\downarrow S^\uparrow - P^\uparrow P^\downarrow - P^\downarrow P^\uparrow.
$$

i.e. $I = 1, \ s = 1$; the numerically obtained potential is attractive, i.e. in agreement with the above stated rule.

* All 32 $S \leftrightarrow S$ correlation functions/potentials fulfill the rule.

* Agreement with similar quenched lattice studies.


Discussion of results (5)

- $S \leftrightarrow P_-$ potentials:
  * Do not obey the above stated rule.
  * It can, however, easily be generalized by including parity, i.e. symmetry or antisymmetry under exchange of $S$ and $P_-$:
    trial state symmetric under meson exchange $\rightarrow$ attractive
    trial state antisymmetric under meson exchange $\rightarrow$ repulsive
    (meson exchange $\equiv$ exchange of flavor, spin and parity).
  * Example: $uu$, $\Gamma = \gamma_0$ $\rightarrow$ $P = +$, $P_x = -$:
    $\left( C\gamma_0 \right)_{AB} \left( \bar{Q}_C(-R/2)q^{(u)}_A(-R/2) \right) \left( \bar{Q}_C(+R/2)q^{(u)}_B(+R/2) \right) \propto S_\uparrow P_\downarrow - S_\downarrow P_\uparrow - P_\uparrow S_\downarrow + P_\downarrow S_\uparrow$,
    i.e. $I = 1$ (symmetric), $s = 0$ (antisymmetric), antisymmetric with respect to $S \leftrightarrow P_-$; the numerically obtained potential is attractive, i.e. in agreement with the above stated general rule.
  * All 32 $S \leftrightarrow P_-$ correlation functions/potentials (and all 32 $S \leftrightarrow S$ correlation functions/potentials) fulfill the generalized rule.
Discussion of results (6)

- Improvements after having understood the extraction and interpretation of $BB$ potentials from single correlation functions:
  - Linearly combine $BB$ operators to either eliminate $P_- \leftrightarrow P_-$ or $S \leftrightarrow S$ combinations.
  - Example:
    \[
    ud - du, \quad \Gamma = \gamma_5 \rightarrow -S_\uparrow S_\downarrow + S_\downarrow S_\uparrow - P_\uparrow P_\downarrow + P_\downarrow P_\uparrow
    \]
    \[
    ud - du, \quad \Gamma = \gamma_0 \gamma_5 \rightarrow -S_\uparrow S_\downarrow + S_\downarrow S_\uparrow + P_\uparrow P_\downarrow - P_\downarrow P_\uparrow
    \]
    → use $\gamma_5 + \gamma_0 \gamma_5$ to obtain a better signal for the $S \leftrightarrow S$ potential
    → use $\gamma_5 - \gamma_0 \gamma_5$ to extract the $P_- \leftrightarrow P_-$ potential.

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![Graph showing potential $V$ vs meson separation $r$ in fm](image)
Discussion of results (7)

- Improvements after having understood the extraction and interpretation of $BB$ potentials from single correlation functions:
  - Use correlation matrices instead of single correlation functions to avoid mixing with $BB$ states of lower energy, which is present, because
    * although the product of two specific $B$ meson creation operators closely resembles the corresponding $BB$ state, it will still have a non-vanishing overlap to $BB$ states corresponding to $B$ mesons with different isospin, spin and/or parity,
    * twisted mass lattice QCD explicitly breaks isospin and parity (the breaking is proportional to the lattice spacing $a$; isospin and parity will be restored in the continuum limit).

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Summary of $BB$ states and degeneracies

- Two $B$ mesons, each can have $I_z = \pm 1/2$, $j_z = \pm 1/2$, $\mathcal{P} = \pm$ → $8 \times 8 = 64$ states.

- $S \leftrightarrow S$ potentials:
  - Attractive: $1 \oplus 3 \oplus 6$  
    $I=0, |j_z|=0 \quad I=1, |j_z|=0 \quad I=1, |j_z|=1$  
    (10 states).
  - Repulsive: $1 \oplus 3 \oplus 2$  
    $I=0, |j_z|=0 \quad I=1, |j_z|=0 \quad I=0, |j_z|=1$  
    (6 states).

- $S \leftrightarrow P_-$ potentials:
  - Attractive: $1 \oplus 1 \oplus 3 \oplus 3 \oplus 2 \oplus 6$  
    $|j_z|=0 \quad |j_z|=1$  
    (16 states).
  - Repulsive: $1 \oplus 1 \oplus 3 \oplus 3 \oplus 2 \oplus 6$  
    $|j_z|=0 \quad |j_z|=1$  
    (16 states).

- $P_- \leftrightarrow P_-$ potentials: identical to $S \leftrightarrow S$ potentials.

- In total 24 different potentials.

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Attractive $S \leftrightarrow S$ potentials

- Attractive $S \leftrightarrow S$ potentials are relevant, when trying to determine, whether $BB$ may form a bound state.

- Three different attractive $S \leftrightarrow S$ potentials: $I=0, |j_z|=0 \oplus 3 \oplus 6$.

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Summary, conclusions, future plans (1)

• Computation of $BB$ potentials (arbitrary flavor, spin and parity) with “light” dynamical quarks ($m_{PS} \approx 340 \text{ MeV}$).
  
  – Qualitative agreement with existing quenched results for $S \leftrightarrow S$ potentials.
  
  – First lattice computation of $S \leftrightarrow P_-$ and $P_- \leftrightarrow P_-$ potentials.
  
  – Clear statements about whether a potential of a given channel is attractive or repulsive.

• Statistical accuracy problematic (exponentially decaying correlation functions are quickly lost in statistical noise):
  
  – Reasonable accuracy for attractive $S \leftrightarrow S$ potentials (interesting, when trying to determine, whether $BB$ may form a bound state).
  
  – Other (higher) potentials:
    
    → $BB$ potentials are extracted at rather small temporal separations
    → slight contamination from excited states cannot be excluded.
Further plans and possibilities:

- Other values of the lattice spacing, the spacetime volume and/or the $u/d$ quark mass.
- Partially quenched computations, to obtain $B_s B_s$ and/or $B_s B$ potentials.
- Improve lattice meson potentials at small separations (where the suppression of UV fluctuations due to the lattice cutoff yields wrong results) with corresponding perturbative potentials.
- Use lattice meson potentials to study, whether $BB$ may form a bound state.