Fermionic fields in the pseudoparticle approach

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Introduction (1)

• Models for SU(2) Yang-Mills theory with a small number of degrees of freedom:
  – Calorons with non-trivial holonomy (P. Gerhold, E.-M. Ilgenfritz, M. Müller-Preussker, 2006).

• Basic principle: restrict the path integral to those gauge field configurations, which can be represented by a linear superposition of a small number of localized building blocks (instantons, merons, akyrons, calorons, dyons, ...).
• Successes of these models:
  – Linear potential between two static charges at large separations (confinement).
  – Confinement-deconfinement phase transition.
  – String tension, topological susceptibility, critical temperature, ... in qualitative agreement with lattice results.

• Intention: get a better understanding of confining gauge field configurations and the mechanism of confinement.

• **How can fermions be included in such models?**

• In this talk: first steps in this direction.
  – Basic principle of the PP approach in fermionic theories.
  – Testing ground: the 1+1-dimensional Gross-Neveu model in the large-$N$-limit (phase diagram, chiral condensate).
Basic principle (1)

- Starting point: action and partition function of any **theory with quadratic fermion interaction**:

\[
S[\psi, \bar{\psi}, \phi] = \int d^{d+1}x \left( \bar{\psi} Q(\phi) \psi + L(\phi) \right)
\]

\[
Z = \int D\psi \ D\bar{\psi} \ D\phi \ e^{-S[\psi, \bar{\psi}, \phi]}
\]

(Q: Dirac operator; \(\phi\): any type and number of bosonic fields, e.g. the non-Abelian gauge field in QCD).
Basic principle (2)

- Consider only those fermionic field configurations, which can be represented by a linear superposition a fixed number of localized building blocks:

\[ \psi(x) = \sum_j \eta_j G_j(x) \]

(\( \eta_j \): Grassmann valued spinors; \( G_j \): functions, which are localized in space as well as in time, i.e. PPs).

- Define the functional integration over all fermionic field configurations as an integration over the Grassmann valued spinors:

\[ \int D\psi \, D\bar{\psi} \ldots = \int \left( \prod_j d\eta_j \, d\bar{\eta}_j \right) \ldots \]
Basic principle (3)

- Integrate out the fermions:

\[ S_{\text{effective}}[\phi] = \int d^{d+1}x \, \mathcal{L}(\phi) - \ln \left( \det \left( \langle G_j | Q | G_{j'} \rangle \right) \right) \]

\[ Z \propto \int D\phi \, e^{-S_{\text{effective}}[\phi]} \]

\( \langle G_j | Q | G_{j'} \rangle \) is a finite matrix; \textit{“Q-regularization”}.

- If \( \det(Q) \) is real and positive, \( \det(Q) = \sqrt{\det(Q^{\dagger}Q)} \). This suggests another PP regularization:

\[ S_{\text{effective}}[\phi] = \int d^{d+1}x \, \mathcal{L}(\phi) - \frac{1}{2} \ln \left( \det \left( \langle G_j | Q^{\dagger}Q | G_{j'} \rangle \right) \right) \]

\textit{(“Q^{\dagger}Q-regularization”)}.

- The \textit{“Q^{\dagger}Q-regularization”} has significant advantages over the naive \textit{“Q-regularization”}.

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\( Q \) versus \( Q^\dagger Q \) (1)

- For the sake of simplicity: consider all PPs \( G_j \) to be orthonormal, i.e. \( \langle G_j | G_{j'} \rangle = \delta_{jj'} \) (this is not a restriction!).

- The problem of the \( Q \)-regularization:

  - Applying the Dirac operator \( Q \) to one of the PPs \( G_{j'} \) in general yields a function, which is (partially) outside the PP function space \( \text{span}\{G_n\} \), i.e.

\[
QG_{j'}(x) = \sum_k a_{j'k} G_k(x) + h_{j'} H_{j'}(x)
\]

(\( H_{j'} \) normalized, \( H_{j'} \perp \text{span}\{G_n\} \)).

  - If \( |\sum_k a_{j'k} G_k| \gg |h_{j'}| \rightarrow \) no problem.

  - If \( |\sum_k a_{j'k} G_k| \lesssim |h_{j'}| \rightarrow \) when computing the fermionic matrix \( \langle G_j | Q | G_{j'} \rangle \), a significant part of \( QG_{j'} \) is simply ignored, just because \( H_{j'} \) is perpendicular to the PP function space \( \text{span}\{G_n\} \).
The advantage of the $Q^\dagger Q$-regularization:

- Both the left hand sides $\langle G_j | Q^\dagger$ and the right hand sides $Q | G_{j'} \rangle$ of the matrix elements $\langle G_j | Q^\dagger Q | G_{j'} \rangle$ might be outside the PP function space $\text{span}\{G_n\}$, but they form the same function space, $\text{span}\{QG_n\}$, in which their overlap is computed.

Testing ground: Gross-Neveu model (1)

- Action and partition function of the 1+1-dimensional Gross-Neveu model:

\[
S = \int d^2 x \left( \sum_{n=1}^{N} \bar{\psi}^{(n)} \left( \gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 \right) \psi^{(n)} - \frac{g^2}{2} \left( \sum_{n=1}^{N} \bar{\psi}^{(n)} \psi^{(n)} \right)^2 \right)
\]

\[
Z = \int \left( \prod_{n=1}^{N} D\psi^{(n)} D\bar{\psi}^{(n)} \right) e^{-S}
\]

\(N: \text{number of flavors; } \mu: \text{chemical potential; } g: \text{coupling constant}\).
• Introduce a real scalar field $\sigma$ and integrate out the fermions:

$$S_{\text{effective}} = N \left( \frac{1}{2\lambda} \int d^2 x \sigma^2 - \ln \left( \det \left( \gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 + \sigma \right) \right) \right)$$

$$Z \propto \int D\sigma e^{-S_{\text{effective}}}$$

($\lambda = Ng^2$).

• Large-$N$ limit:
  
  – $N \to \infty$, $\lambda = Ng^2 = \text{constant}$.
  
  – There is no need to compute the $\sigma$-path integral anymore.
  
  – It is sufficient to minimize $S_{\text{effective}}$ with respect to $\sigma$.
  
  – $\sigma = -g^2 \sum_{n=1}^{N} \bar{\psi}^{(n)} \psi^{(n)}$ (chiral condensate).
Fermionic PPs

- Fermionic PPs in this talk: a large number of uniformly distributed “hat functions”.
  - B-spline basis functions of degree 2.
  - 8 ... 56 PPs in time direction, 144 PPs in space direction.
  - Antiperiodic boundary conditions in time direction, periodic boundary conditions in space direction.

- “Sensible set of field configurations” (any not too heavily oscillating field configuration can be approximated)
  → we can expect to reproduce correct Gross-Neveu results.

- Piecewise polynomial functions
  → certain integrals can be calculated analytically.
Phase diagram

- $Q$-regularization: completely wrong and useless results.
  - No improvement, when using a larger number of PPs.
  - No improvement, when using a different type of PPs.

- $Q^\dagger Q$-regularization: excellent agreement with analytical results.
Chiral condensate

- $Q^\dagger Q$-regularization: excellent agreement with analytical results.
  - The $\sin$-like behavior “inside the crystal phase” changes to a kink-antikink structure, when approaching the left phase boundary.

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Summary

- Always apply the $Q^\dagger Q$-regularization and not the naive $Q$-regularization, when including fermionic fields in the pseudoparticle approach.

- The application of the PP approach to compute the phase diagram of the 1+1-dimensional Gross-Neveu model in the large-$N$-limit has been a successful test.
Outlook

• Next steps:
  – Apply the PP approach to QCD.
  – Try to identify a small number of physically relevant fermionic PPs (PPs, which are able to approximate typical low lying eigenmodes of the Dirac operator?).

• Current research:
  – Chiral symmetry breaking by computing the low lying eigenmodes of the Dirac operator in the quenched approximation (Banks-Casher relation).

• Goals:
  – Obtain a model with a small number of degrees of freedom, which exhibits chiral symmetry breaking and a confinement deconfinement phase transition at the same time.
  – Compute further fermionic observables: pion masses, ...