On the definition and interpretation of a static quark anti-quark potential in the colour-adjoint channel

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Original motivation (1)

- Determination of $\Lambda_{\overline{\text{MS}}}$ from the (singlet) static potential for $n_f = 0$ (Yang-Mills theory) and $n_f = 2, 3$ (QCD) and gauge group $SU(3)$:
  
  - Fit the perturbative result, which depends on the perturbative scale $\Lambda_{\overline{\text{MS}}}$, to the corresponding lattice result, where the scale has been set e.g. by the typical non-perturbative scale $r_0 \approx 0.45 \text{ fm} \ldots 0.50 \text{ fm}$ (or by other hadronic quantities, e.g. $m_\pi$ and $f_\pi$).
  
  - Similar problem: relate the perturbative scale $\Lambda_{\overline{\text{MS}}}$ and the non-perturbative scale $r_0$ by determining the dimensionless quantity $\Lambda_{\overline{\text{MS}}}r_0$.
  
  - Instead of $\Lambda_{\overline{\text{MS}}}$ one can also determine $\alpha_s$ at some fixed scale.


• Perturbative calculation of the color adjoint static potential up to 2 loops, recently also up to 3 loops (the “octet static potential” for gauge group $SU(3)$).


• Plan:
  – Compute the octet static potential using lattice QCD.
  – Determine $\Lambda_{\overline{MS}}$ using perturbative and lattice results for the octet static potential.

• We encountered some conceptual problems, lattice results and perturbative results show strong qualitative differences ...
This work is concerned with the interpretation of the colour adjoint static potential from Wilson loops with generator insertions (using different gauges).

We discuss both non-perturbative (lattice) and perturbative calculations; the focus, however, will be on the non-perturbative side.
Lattice Yang Mills theory/QCD (1)

- Lattice gauge theory is based on the path integral formulation of Yang Mills theory/QCD,

\[
\langle \Omega | \mathcal{O}[\psi^{(f)}, \bar{\psi}^{(f)}, A_\mu] | \Omega \rangle = \frac{1}{Z} \int \left( \prod_f D\psi^{(f)} D\bar{\psi}^{(f)} \right) DA_\mu \mathcal{O}[\psi^{(f)}, \bar{\psi}^{(f)}, A_\mu] e^{-S[\psi^{(f)}, \bar{\psi}^{(f)}, A_\mu]}.
\]

- $| \Omega \rangle$: ground state/vacuum.
- $\mathcal{O}[\psi^{(f)}, \bar{\psi}^{(f)}, A_\mu]$: functional of the quark and gluon fields.
- $\int (\prod_f D\psi^{(f)} D\bar{\psi}^{(f)}) DA_\mu$: integral over all possible quark and gluon field configurations $\psi^{(f)}(x, t)$ and $A_\mu(x, t)$.
- $e^{-S[x]}$: weight factor containing the Yang-Mills/QCD action.
• Numerical implementation of the path integral formalism in Yang Mills theory/QCD:

  – Discretise spacetime with sufficiently small lattice spacing
    \( a \approx 0.05 \text{ fm} \ldots 0.10 \text{ fm} \)
    \( \rightarrow \) “continuum physics”.
  
  – “Make spacetime periodic” with sufficiently large extension
    \( L \approx 2.0 \text{ fm} \ldots 4.0 \text{ fm} \) (4-dimensional torus)
    \( \rightarrow \) “no finite size effects”.

\[ x_\mu = (n_0, n_1, n_2, n_3) \in \mathbb{Z}^4 \]
• After discretization the path integral becomes an ordinary multidimensional integral:

\[ \int D\psi \, D\bar{\psi} \, DA \ldots \to \prod_{x_\mu} \left( \int d\psi(x_\mu) \, d\bar{\psi}(x_\mu) \, dU(x_\mu) \right) \ldots , \]

where

\[ U_\nu(x_\mu) = P\left( \exp \left( ig \int_{x_\mu}^{x_\mu+ae_\mu^{(\nu)}} dz_\rho \, A_\rho(z) \right) \right) , \]

i.e. the lattice gauge field is stored in parallel transporters connecting neighbouring lattice sites (so-called links).

• Advantages/disadvantages of lattice Yang Mills theory/QCD:

(+) Exact Yang Mills/QCD results (no approximations, no model assumptions, etc.).

(−) Only numerical results, i.e. numbers, no analytical functions, etc.
Introduction: singlet static potential (1)

- The (singlet) static potential $V^1$ is a very common and important observable in lattice gauge theory.

- It is the energy of a static antiquark $\bar{Q}(x)$ and a static quark $Q(y)$ in a colour singlet (i.e. a gauge invariant) orientation as a function of the separation $r \equiv |x - y|$.

- The spin of a static quark is irrelevant, i.e. in the following
  - no spin indices or $\gamma$ matrices,
  - only spinless colour charges,
  $\bar{Q}^a_A(x) = (Q^{a,\dagger}(x)\gamma_0)_A \rightarrow Q^{a,\dagger}(x)$,
  $Q^a_A(y) \rightarrow Q^a(y)$,
  where $a$ denotes a colour index and $A$ a spin index.
The singlet static potential for gauge group $SU(N)$ can be obtained as follows:

1. Define a trial state
   \[ |\Phi^1\rangle \equiv \bar{Q}(x)U(x,y)Q(y)|0\rangle. \]

2. The temporal correlation function of this trial state simplifies to the well-known Wilson loop,
   \[ \langle \Phi^1(t_2)|\Phi^1(t_1)\rangle = e^{-2M\Delta t}N\langle W_1(r, \Delta t) \rangle, \quad \Delta t \equiv t_2 - t_1 > 0, \]
   where
   \[ W_1(r, \Delta t) = \frac{1}{N}\text{Tr}\left(P\left(\exp\left(ig\oint dz_\mu A_\mu(z)\right)\right)\right). \]
• The singlet static potential for gauge group $SU(N)$ can be obtained as follows:

(3) The singlet static potential $V^1 \equiv V_0^1$ can be obtained from the asymptotic exponential behaviour of the Wilson loop,

$$\langle W_1(r, \Delta t) \rangle \propto \langle \Phi^1(t_2)|\Phi^1(t_1) \rangle = e^{+E_0\Delta t}\langle \Phi^1(t_1)|e^{-H\Delta t}|\Phi^1(t_1) \rangle =$$

$$= \sum_{n=0}^{\infty} \langle \Phi^1|n \rangle e^{-V_n^1(r)\Delta t} \langle n|\Phi^1 \rangle =$$

$$= \sum_{n=0}^{\infty} \left| \langle \Phi^1|n \rangle \right|^2 e^{-V_n^1(r)\Delta t} \Delta t \rightarrow \infty \exp \left( -V^1(r)\Delta t \right)$$

$$V^1(r) = -\lim_{\Delta t \rightarrow \infty} \frac{\langle \dot{W}_1(r, \Delta t) \rangle}{\langle W_1(r, \Delta t) \rangle}$$

($\sum_n$ is the sum over eigenstates of the Hamiltonian, which have the quantum numbers of $|\Phi^1 \rangle$, in particular a static $Q\bar{Q}$ pair at $x$ and $y$).
Goal of this work: compute and interpret the potential of a static antiquark $\bar{Q}(x)$ and a static quark $Q(y)$ in a colour adjoint (i.e. a gauge variant) orientation in various gauges as a function of the separation $r \equiv |x - y|$.

A colour adjoint orientation of a static antiquark and a static quark can be obtained by inserting the generators of the colour group $T^a$ (e.g. for $SU(3)$, $T^a = \lambda^a/2$), i.e. $\bar{Q}T^aQ|0\rangle$.

If the static antiquark and the static quark are separated in space, a straightforward generalisation is

$$|\Phi^{T^a}\rangle \equiv \bar{Q}(x)U(x, x_0)T^aU(x_0, y)Q(y)|0\rangle.$$

A corresponding definition of the colour adjoint static potential has been proposed and used in pNRQCD (a framework based on perturbation theory).

We discuss non-perturbative calculations analogous as for the singlet static potential in various gauges,

$$
\langle \Phi^T_a(t_2)|\Phi^T_a(t_1)\rangle = e^{-2M\Delta t}N\langle W_T^a(r, \Delta t) \rangle,
$$

$$
W_T^a(r, \Delta t) \equiv \frac{1}{N}\text{Tr}\left(T^aU_R T^a, U_L^\dagger\right)
$$

$$
\langle W_T^a(r, \Delta t) \rangle = \sum_{n=0}^{\infty} c_n \exp\left(-V_n^T(a)(r)\Delta t\right) \xrightarrow{\Delta t \to \infty} \exp\left(-V^T_T(a)(r)\Delta t\right).
$$

In particular we are interested,

- whether the colour adjoint static potential $V^T_T \equiv V_0^T$ is gauge invariant (i.e. whether the obvious gauge dependence of the correlation function $\langle W_T^a(r, \Delta t) \rangle$ only appears in the matrix elements $c_n$),

- whether $V^T_T$ indeed corresponds to the potential of a static antiquark and a static quark in a colour adjoint orientation, or whether it has to be interpreted differently.
Without gauge fixing

\[ \langle W_{T^a}(r, \Delta t) \rangle = 0, \]

because this correlation function is gauge variant (and does not contain any
gauge invariant contribution).

→ Without gauge fixing the calculation of a colour adjoint static potential fails.
\[ V^{T^a} \text{ in Coulomb gauge} \]

- Coulomb gauge: \( \nabla A_g^g(x) = 0 \), which amounts to an independent condition on every time slice \( t \).
- The remaining residual gauge symmetry corresponds to global independent colour rotations \( h^{\text{res}}(t) \in SU(N) \) on every time slice \( t \); with respect to this residual gauge symmetry the colour adjoint Wilson loop transforms as

\[
\langle W_{T^a}(r, \Delta t) \rangle = \frac{1}{N} \text{Tr} \left( T^a U_R T^a, \dagger U_L \right) \rightarrow h^{\text{res}}
\]

\[
\rightarrow h^{\text{res}} \frac{1}{N} \text{Tr} \left( h^{\text{res}, \dagger}(t_1) T^a h^{\text{res}}(t_1) U_R h^{\text{res}}(t_2) T^a, \dagger h^{\text{res}, \dagger}(t_2) U_L \right).
\]

- Since \( h^{\text{res}}(t_1) \) and \( h^{\text{res}}(t_2) \) are independent, the situation is analogous to that without gauge fixing, i.e.

\[
\langle W_{T^a}(r, \Delta t) \rangle_{\text{Coulomb gauge}} = 0.
\]

\( \rightarrow \) In Coulomb gauge the calculation of a colour adjoint static potential fails.
\( V^{T^a} \) in Lorenz gauge

- Lorenz gauge: \( \partial_\mu A_\mu^g(x) = 0 \).

- In Lorenz gauge a Hamiltonian or a transfer matrix does not exist.

- Only gauge invariant correlation functions like the ordinary Wilson loop \( \langle W_1(r, \Delta t) \rangle \) exhibit an asymptotic exponential behaviour and, therefore, allow the determination of energy eigenvalues.

- The colour adjoint Wilson loop \( \langle W_{T^a}(r, \Delta t) \rangle_{\text{Lorenz gauge}} \) does not decay exponentially in the limit of large \( \Delta t \).

\[ \rightarrow \] The physical meaning of a colour adjoint static potential determined from \( \langle W_{T^a}(r, \Delta t) \rangle_{\text{Lorenz gauge}} \) is unclear.
\( V^{T^a} \) in temporal gauge (1)

- Temporal gauge: \( \partial_\mu A_g^0(x) = 0 \) or equivalently \( U_0^g(x) = 1 \).

- Temporal links gauge transform as

\[
U_0^g(t, x) = g(t, x) U_0(t, x) g^\dagger(t + a, x) , \quad g(t, x) \in SU(N).
\]

- A possible choice to implement temporal gauge is

\[
\begin{align*}
g(t = 2a, x) &= U_0(t = a, x), \\
g(t = 3a, x) &= g(t = 2a, x) U_0(t = 2a, x) = U_0(t = a, x) U_0(t = 2a, x), \\
g(t = 4a, x) &= g(t = 3a, x) U_0(t = 3a, x) = \ldots, \\
\ldots &= \ldots
\end{align*}
\]
\( VT^a \) in temporal gauge (2)

- By inserting the transformation to temporal gauge \( g(t, x) \), the gauge variant colour adjoint Wilson loop turns into a gauge invariant observable:

\[
\langle W_{T^a}(r, \Delta t) \rangle_{\text{temporal gauge}} = \frac{1}{N} \langle \text{Tr} \left( U^{T^a,g(t_1; x, y)} U^{T^a,g(t_2; y, x)} \right) \rangle_{\text{temporal gauge}} = \ldots = \frac{2}{N(N^2 - 1)} \sum_a \sum_b \left( \text{Tr} \left( T^a U_R T^b U_L \right) \text{Tr} \left( T^a U(t_1, t_2; x_0) T^b U(t_2, t_1; x_0) \right) \right)
\]

\((U^{T^a}(x, y) = U(x, x_0) T^a U(x_0, y))\).

- \( \text{Tr}(T^a U_R T^b U_L) \): Wilson loop with generator insertions.

- \( \text{Tr}(T^a U(t_1, t_2; x_0) T^b U(t_2, t_1; x_0)) \): propagator of a static adjoint quark.

\(\rightarrow\) The colour adjoint Wilson loop in temporal gauge is a correlation function of a gauge invariant three-quark state, one fundamental static quark, one fundamental static anti-quark, one adjoint static quark.

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\[ V^{Ta} \text{ in temporal gauge (3)} \]

- Equivalently, after defining
  \[ |\Phi Q\bar{Q}Q^{ad}\rangle \equiv Q^{ad,a}(x_0)(\bar{Q}(x)U^{Ta}(x, y)Q(y))|0\rangle, \]

one can verify

\[ \langle \Phi Q\bar{Q}Q^{ad}(t_2)|\Phi Q\bar{Q}Q^{ad}(t_1)\rangle \propto \left\langle W_{Ta}(r, \Delta t) \right\rangle_{\text{temporal gauge}}. \]

\[ \rightarrow V^{Ta} \text{ in temporal gauge should not be interpreted as the potential of a static quark and a static anti-quark, which form a colour-adjoint state.} \]

\[ \rightarrow V^{Ta} \text{ in temporal gauge is the potential of a colour-singlet three-quark state.} \]

\[ \rightarrow V^{Ta} \text{ in temporal gauge does not only depend on the } Q\bar{Q} \text{ separation } r = |x - y|, \text{ but also on the position } s = |x - x_0|/2 - |y - x_0|/2 \text{ of the static adjoint quark } Q^{ad}, \text{ i.e. } V^{Ta}(r, s) \text{ (in the following we work with the symmetric alignment } x_0 = (x + y)/2). \]
\( V^T_a \) in temporal gauge (4)

- A different approach, leading to the same result, is the transfer matrix formalism.
  
  \[ [O. \text{ Jahn and O. Philipsen, Phys. Rev. D 70, 074504 (2004) [hep-lat/0407042]}] \]
  \[ [O. \text{ Philipsen, Nucl. Phys. B 628, 167 (2002) [hep-lat/0112047]}] \]

- One can perform a spectral analysis of the colour adjoint Wilson loop:

\[
\left\langle W_{Ta}(r, \Delta t) \right\rangle_{\text{temporal gauge}} = \frac{1}{N} \sum_k e^{-(V^T_a(r)-E_0)\Delta t} \sum_{\alpha,\beta} \left| \left\langle k^a_{\alpha\beta} \right| U^{Ta}_{\alpha\beta}(x, y) \right| 0 \rangle \right|^2,
\]

where \( |k^a_{\alpha\beta} \rangle \) denotes states containing three static quarks (one fundamental static quark, one fundamental static anti-quark, one adjoint static quark).

→ Again the conclusion is that \( V^T_a \) in temporal gauge is the potential of a colour-singlet three-quark state.
A gauge invariant definition via $B$ fields?

- In the literature one can also find a proposal of a gauge invariant quantity to determine a colour adjoint static potential,

$$W_B(r, \Delta t) \equiv \frac{1}{N} \text{Tr} \left( T^a U_R T^{b,\dagger} U_L \right) B^a(x_0, t_1) B^b(x_0, t_2),$$

i.e. open colour indices are saturated by colour magnetic fields.


- Using the transfer matrix formalism one can again perform a spectral analysis and show that only states with a fundamental quark and a fundamental antiquark $|k_{\alpha\beta}\rangle$ (i.e. singlet static potentials) contribute:

$$\langle W_B(r, \Delta t) \rangle = \sum_k e^{-(V^1_{k,\alpha} - E_0) \Delta t} \sum_{\alpha,\beta} \left| \langle k_{\alpha\beta} | U_{\alpha\beta}^{T^a B^a}(x, y) | 0 \rangle \right|^2.$$

$\langle W_B(r, \Delta t) \rangle$ is suited to extract colour singlet static potentials only (quantum numbers “parity” $[PC, P_x]$ and angular momentum may differ from the ordinary singlet static potential $\rightarrow$ hybrid potentials).

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Numerical lattice results for $SU(2)$

- $SU(2)$ colour group, four different lattice spacings $a = 0.038\text{ fm} \ldots 0.102\text{ fm}$.
- In temporal gauge the colour adjoint (or rather $Q\bar{Q}Q^\text{ad}$) static potential $V^{T^a}$ is attractive,
  - for small separations stronger than the singlet static potential $V^1$,
  - for large separations the slope is the same as for the singlet static potential $V^1$ (indicates flux tube formation between $QQ^\text{ad}$ and $\bar{Q}Q^\text{ad}$).

![SU(2) singlet and colour-adjoint static potential in temporal gauge](image)
LO perturbative calculations (1)

- Perturbation theory for static potentials is a good approximation for small quark separations and should agree in that region with corresponding non-perturbative results.

- **Singlet static potential** (gauge invariant, i.e. the gauge is not important):

  \[
  V^1(r) = -\frac{(N^2 - 1)g^2}{8N\pi r} + \text{const} + O(g^4). 
  \]

- **Colour adjoint static potential** (in Lorenz gauge):

  \[
  V^{T\alpha}(r) = +\frac{g^2}{8N\pi r} + \text{const} + O(g^4). 
  \]

  - In Lorenz gauge a Hamiltonian or a transfer matrix does not exist, i.e. the physical meaning is unclear; appears frequently in the literature.
  - The repulsive behaviour is not reproduced by any of the presented non-perturbative considerations or computations.
• Colour adjoint static potential (“in temporal gauge”; more precisely: perturbative calculation in Lorenz gauge of the gauge invariant observable, which is equivalent to the colour adjoint Wilson loop in temporal gauge):

\[ V_{T}^{a}(r, s = 0) = V_{Q\bar{Q}^{ad}}(r, s = 0) = -\frac{(4N^2 - 1)g^2}{8N\pi r} + \text{const} + \mathcal{O}(g^4). \]

– Attractive and stronger by a factor 4...5 than the singlet static potential (depending on \( N \)).
– Qualitative agreement with numerical lattice results for \( SU(2) \).
Lattice results for the static potential exhibit large discretisation errors for \( r < 2a \) (for our ensembles \( 2a \approx 0.08 \text{ fm} \ldots 0.20 \text{ fm} \)).

Perturbative results for the static potential are only trustworthy for separations \( \lesssim 0.2 \text{ fm} \).

\[ \rightarrow \text{Small region of overlap between lattice and perturbative results.} \]

The leading order of perturbation theory, which we will use in the following,

\[ V^{1, \text{LO}}(r) = -\frac{3g^2}{16\pi r} + \text{const} , \quad V^{T^a, \text{LO}}(r, s = 0) = -\frac{15g^2}{16\pi r} + \text{const} \]

(here specialized to gauge group \( SU(2) \)) is known to be a rather poor approximation.

\[ \rightarrow \text{Only qualitative agreement expected, when comparing to lattice results.} \]
Matching lattice/perturbative results (2)

• We determine $\alpha_s \equiv g^2/4\pi$ from the corresponding static forces $F^X(r) = dV^X(r)/dr$, $X \in \{1, T^a\}$; on the lattice the derivative is defined by a finite difference,

$$\begin{align*}
\frac{V^{1,\text{lattice}}(3a) - V^{1,\text{lattice}}(2a)}{a} &= \frac{3\alpha_s^1}{4(2.5 \times a)^2} \\
\frac{V^{T^a,\text{lattice}}(6a) - V^{T^a,\text{lattice}}(4a)}{2a} &= \frac{15\alpha_s^{T^a}}{4(5 \times a)^2}
\end{align*}$$

(static colour charges are separated by at least $2a$, while at the same time their separation is still quite small).

• $\Delta\alpha_s^{\text{rel}}$ is quite small.
  → A clear sign of agreement between lattice and perturbative results.

• $\alpha_s < 0.5$ for $\beta = 2.60, 2.70$.
  → Perturbation theory “valid”.

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Conclusions

• We have discussed the non-perturbative definition of a static potential $V^T_a$ for a quark antiquark pair in a colour adjoint orientation, based on Wilson loops with generator insertions $\langle W^T_a(r, \Delta t) \rangle$ in various gauges:

  – **Without gauge fixing/Coulomb gauge:** $\langle W^T_a(r, \Delta t) \rangle = 0$, i.e. the calculation of a potential $V^T_a$ fails.

  – **Lorenz gauge:** a Hamiltonian or a transfer matrix does not exist, the physical meaning of a corresponding potential $V^T_a$ is unclear.

  – **Temporal gauge:** a strongly attractive potential $V^T_a$, which should be interpreted as the potential of three quarks, i.e. $V^T_a = V_{QQQ}^{ad}$.

• Saturating open colour indices with $B^a$, yields a singlet static potential (a hybrid potential).

• LO perturbation theory in Lorenz gauge has long predicted $V^T_a$ to be repulsive; it appears impossible, to reproduce this repulsive behaviour by a non-perturbative computation based on Wilson loops.