UNRUH EFFECT

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Introduction

What is the Unruh effect?
- Vacuum state in Minkowski spacetime: state where no particles are present; lowest energy eigenstate.
- Unruh effect: an accelerated observer moving through the Minkowski vacuum detects particles (Minkowski vacuum and vacuum in the frame of the accelerated observer are different quantum states).
- In this talk: observer moves in 1+1 dimensions with constant acceleration.

Accelerated motion (1)

Three different frames
- Laboratory frame (inertial frame; coordinates $x^\mu = (t, x)$): the usual inertial frame.
- Accelerated frame or proper frame (non inertial frame; coordinates $(\tau, \xi)$): the frame where the accelerated observer is at rest.
- Comoving frames (inertial frames; coordinates $x'^\mu = (t', x')$: frames where the accelerated observer is momentarily at rest.

Literature
  www.theorie.physik.uni-muenchen.de/~serge/T6/
Accelerated motion (2)

Constant acceleration (i)

- Constant acceleration: constant four acceleration in the comoving frames.
- Any inertial frame:
  \[ u^\mu u_\mu = 1 \to 0 = \frac{d}{d\tau} (u^\mu u^\mu) = 2u_\mu \frac{d}{d\tau} u^\mu = 2u_\mu a^\mu. \] (1)
- Comoving frame (at that time when the accelerated observer is at rest):
  \[ u'_\mu = (1, 0) \to a'^\mu = (0, A). \] (2)

Accelerated motion (3)

Constant acceleration (ii)

- \( A \) is the ordinary three acceleration \( a' \) in the comoving frame (at the time when the accelerated observer is at rest):
  \[ u'^0 = \frac{dt'}{d\tau} = \frac{d}{d\tau} \left( \frac{a'^0}{d\tau} \right) = \frac{d}{d\tau} \left( u'^0 v' \right) = a'^0 v' + (u'^0)^2 \left( \frac{d}{d\tau} v' \right) = a'. \] (4)

Accelerated frame (coordinates) (1)

- To describe quantum fields in the accelerated frame and to compare them with quantum fields in the laboratory frame we need coordinates \((\tau, \xi)\) in the accelerated frame and transformation laws \( t = t(\tau, \xi) \) and \( x = x(\tau, \xi) \).
- \( \tau \) is proper time of the observer (or anybody moving along the trajectory \( \xi = 0 \)).
- \( \xi \) is spatial distance from the observer at \( \xi = 0 \).
- Consider a measuring stick of length \( \xi_0 \) in the accelerated frame. In the current comoving frame it is represented by the four vector \( s'^\mu = (0, \xi_0) \) (the measuring stick is momentarily at rest in the current comoving frame).
- Four vector of the measuring stick in the laboratory system:
  \[ s^\mu = \frac{1}{\sqrt{1 - v^2}} \begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \xi_0 \\ 0 & 0 & \xi_0 \end{pmatrix} = \begin{pmatrix} u^0 & u^1 & 0 \\ u^1 & u^0 & 0 \\ 0 & 0 & \xi_0 \\ u^0 & u^1 & \xi_0 \end{pmatrix}. \] (9)
Accelerated frame (coordinates) (2)

• The far end of the measuring stick has proper coordinates $(\tau, \xi_0)$. From that, (8) and (9), the transformation law between laboratory coordinates $(t, x)$ and proper coordinates $(\tau, \xi)$ can be derived:

\[
t = \frac{1}{a} \sinh(a'\tau) + u \xi = \frac{1 + a'\xi}{a'} \sinh(a'\tau) \quad (10)
\]
\[
x = \frac{1}{a} \cosh(a'\tau) + u \xi = \frac{1 + a'\xi}{a'} \cosh(a'\tau). \quad (11)
\]

• Inverse transformation law:

\[
\tau = \frac{1}{2a'} \ln \left( \frac{x + t}{x - t} \right), \quad \tau \in (-\infty, \infty) \quad (12)
\]
\[
\xi = \sqrt{x^2 - t^2} - \frac{1}{a'}, \quad \xi \in [-1/a', \infty). \quad (13)
\]

Rindler spacetime (1)

• (10) and (11):

\[
dt = \frac{dt}{d\tau} d\tau + \frac{dt}{d\xi} d\xi = (1 + a'\xi) \cosh(a'\tau) d\tau + \sinh(a'\tau) d\xi \quad (14)
\]
\[
dx = \frac{dx}{d\tau} d\tau + \frac{dx}{d\xi} d\xi = (1 + a'\xi) \sinh(a'\tau) d\tau + \cosh(a'\tau) d\xi. \quad (15)
\]

• Rindler spacetime:

\[
ds^2 = dt^2 - dx^2 = (1 + a'\xi)^2 d\tau^2 - d\xi^2. \quad (16)
\]

Rindler spacetime (2)

Conformally flat Rindler spacetime (i)

• Quantising fields in conformally flat spacetime in 1+1 dimensions is as easy as quantising fields in Minkowski spacetime.

• To get a conformally flat metric we need a coordinate transformation $\xi = \xi(\tilde{\xi})$ with

\[
d\xi = (1 + a'\xi) d\tilde{\xi}. \quad (17)
\]

• Separation of variables yields

\[
\xi = \int d\xi \frac{1}{1 + a'\xi} = \frac{1}{a'} \ln(1 + a'\xi), \quad \xi \in (-\infty, \infty) \quad (18)
\]

• This is a rescaling of the spatial coordinate $\xi$, $\tilde{\xi}$ is not the spatial distance but parameterises the spatial distance $\xi$.

• Conformally flat Rindler spacetime:

\[
ds^2 = e^{2\xi} \left( d\tau^2 - d\tilde{\xi}^2 \right). \quad (19)
\]

Rindler spacetime (3)

Conformally flat Rindler spacetime (ii)

• Transformation law between laboratory coordinates $(t, x)$ and conformally flat Rindler coordinates $(\tau, \tilde{\xi})$:

\[
t = e^{\frac{\xi}{a'}} \sinh(a'\tau) \quad (20)
\]
\[
x = e^{\frac{\xi}{a'}} \cosh(a'\tau). \quad (21)
\]
Massless scalar field (1)

- Action of a minimally coupled massless scalar field:
  \[ S[\phi] = \int d^2 x \sqrt{-g} \frac{1}{2} \left( \partial_\mu \phi \right) \left( \partial_\nu \phi \right). \]  
  \[ (22) \]

- Laboratory frame (Minkowski spacetime):
  \[ S[\phi] = \int dt dx \left( \frac{1}{2} \left( \partial_t \phi \right)^2 - \frac{1}{2} \left( \partial_x \phi \right)^2 \right). \]  
  \[ (23) \]

- Accelerated frame (conformally flat Rindler spacetime; \( \sqrt{-g} = e^{2a'\eta} \), \( g_{\mu\nu} = \text{diag}(e^{-2a'\eta}, e^{-2a'\eta}) \)):
  \[ S[\phi] = \int d\tau d\tilde{\xi} \left( \frac{1}{2} \left( \partial_\tau \phi \right)^2 - \left( \partial_{\tilde{\xi}} \phi \right)^2 \right). \]  
  \[ (24) \]

- In 1+1 dimensions minimal coupling is equivalent to conformal coupling. Therefore the action in conformally flat Rindler spacetime is identical to the action in Minkowski spacetime. Quantising the field \( \phi \) in conformally flat Rindler spacetime is therefore as easy as in Minkowski spacetime.

Massless scalar field (2)

Quantisation in Minkowski spacetime

- Field operator in laboratory coordinates \((a(k) = \text{annihilation operators}, a^\dagger(k) = \text{creation operators})\):
  \[ \phi(t, x) = \frac{1}{\sqrt{2\pi}} \int dk \frac{1}{\sqrt{2|k|}} \left( e^{-i|k|t + ilx} a(k) + e^{i|k|t - ilx} a^\dagger(k) \right). \]  
  \[ (25) \]

- Minkowski vacuum:
  \[ a(k)|0_M\rangle = 0. \]  
  \[ (26) \]

- Expectation values of certain operators, e.g. \( H \) (energy), \( P \) (momentum), \( T_{\mu\nu} \) (energy momentum tensor, i.e. energy and momentum density), allow a physical interpretation of \( a \)-particle states.

- Example: \( a^\dagger(k)|0_M\rangle \) represents a particle with definite momentum \( k \).

Massless scalar field (3)

Quantisation in Rindler spacetime

- Field operator in conformally flat Rindler coordinates \((b(k) = \text{annihilation operators}, b^\dagger(k) = \text{creation operators})\):
  \[ \phi(\tau, \tilde{\xi}) = \frac{1}{\sqrt{2\pi}} \int dk \frac{1}{\sqrt{2|k|}} \left( e^{-i|k|\tau + il\tilde{\xi}} b(k) + e^{i|k|\tau - il\tilde{\xi}} b^\dagger(k) \right). \]  
  \[ (27) \]

- Rindler vacuum:
  \[ b(k)|0_R\rangle = 0. \]  
  \[ (28) \]

- Physical interpretation of \( b \)-particle states: analogous to physical interpretation of \( a \)-particle states (the action which is identical in both cases determines the physical meaning of all quantum states).

Massless scalar field (4)

- The field operators represent the same quantum field, i.e. \( \phi(t, x) = \phi(\tau, \tilde{\xi}) \).

- The creation and annihilation operators \( a(k) \), \( a^\dagger(k) \) and \( b(k) \), \( b^\dagger(k) \) are different, i.e. they create or annihilate different field excitations.

- Therefore the Minkowski vacuum \( |0_M\rangle \) and the Rindler vacuum \( |0_R\rangle \) are different quantum states.
**Lightcone coordinates**

- Get the relation between \(a(k), a^\dagger(k)\) and \(b(k), b^\dagger(k)\) by comparing the left and right hand side of
  \[
  \phi(t, x) = \phi(\tau, \xi). \tag{29}
  \]

- Lightcone coordinates will simplify this procedure:
  \[
  u = t - x, \quad v = t + x \tag{30}
  \]

- Relation between \((u, v)\) and \((\tilde{u}, \tilde{v})\):
  \[
  u = t - x = \frac{e^{\ell \xi}}{a^\dagger} \sinh(a^\dagger \tau) - \frac{e^{\ell \xi}}{a^\dagger} \cosh(a^\dagger \tau) = - \frac{1}{a^\dagger} e^{\ell \xi} (-\tau) = - \frac{1}{a^\dagger} e^{-\xi \tilde{u}} \tag{32}
  \]

- \[
  v = t + x = \frac{e^{\ell \xi}}{a^\dagger} \sinh(a^\dagger \tau) + \frac{e^{\ell \xi}}{a^\dagger} \cosh(a^\dagger \tau) = \frac{1}{a^\dagger} e^{\ell \xi} (\tau - \xi) = \frac{1}{a^\dagger} e^{\xi \tilde{v}} \tag{33}
  \]

- Lightcone coordinates do not mix: \(u = u(\tilde{u}), v = v(\tilde{v})\).

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**Bogolyubov transformation (1)**

- Field operator in \((u, v)\)-coordinates:
  \[
  \phi(u, v) = \frac{1}{\sqrt{2\pi}} \int dk \frac{1}{2|k|} \left( e^{ik\tilde{u} + i\xi} a(k) + e^{ik\tilde{u} - i\xi} a^\dagger(k) \right) = \frac{1}{\sqrt{2\pi}} \int d\omega \int \frac{1}{2\omega} \left( e^{-i\omega(t-x)} a(\omega) + e^{i\omega(t-x)} a^\dagger(\omega) \right) + \frac{1}{\sqrt{2\pi}} \int d\omega \int \frac{1}{2\omega} \left( e^{i\omega(t+x)} a(\omega) + e^{-i\omega(t+x)} a^\dagger(\omega) \right) = A(u) + B(v) \tag{34}
  \]

- Advantage: \(\phi(u, v)\) now is a sum of a \(u\)-dependent part and a \(v\)-dependent part.

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**Bogolyubov transformation (2)**

- Field operator in \((\tilde{u}, \tilde{v})\)-coordinates:
  \[
  \phi(\tilde{u}, \tilde{v}) = \frac{1}{\sqrt{2\pi}} \int d\Omega \frac{1}{\sqrt{2\Omega}} \left( e^{-i\Theta \tilde{u}} b(\Omega) + e^{i\Theta \tilde{u}} b^\dagger(\Omega) + e^{-i\Theta \tilde{v}} a(\Omega) + e^{i\Theta \tilde{v}} a^\dagger(\Omega) \right) = P(\tilde{u}) + Q(\tilde{v}). \tag{35}
  \]

- The \(u\)-dependent parts of (34) and (35) must be equal:
  \[
  \frac{1}{\sqrt{2\pi}} \int d\omega \frac{1}{\sqrt{2\omega}} \left( e^{-i\omega \tilde{u}} a(\omega) + e^{i\omega \tilde{u}} a^\dagger(\omega) \right) = \frac{1}{\sqrt{2\pi}} \int d\omega \frac{1}{\sqrt{2\omega}} \left( e^{-i\omega \tilde{v}} b(\omega) + e^{i\omega \tilde{v}} b^\dagger(\omega) \right). \tag{36}
  \]

- Can be solved for \(b(\Omega)\) and \(b^\dagger(\Omega)\) by performing a Fourier transformation on both sides:
  \[
  \frac{1}{\sqrt{2\pi}} \int d\tilde{u} e^{i\tilde{u} \Theta} \ldots. \tag{37}
  \]

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**Bogolyubov transformation (3)**

- Right hand side:
  \[
  \frac{1}{\sqrt{2\pi}} \int d\tilde{u} e^{i\tilde{u} \Theta} \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tilde{\Omega} \frac{1}{\sqrt{2\tilde{\Omega}}} \left( e^{-i\tilde{u} \Theta} b(\tilde{\Omega}) + e^{i\tilde{u} \Theta} b^\dagger(\tilde{\Omega}) \right) = \int_0^\infty d\tilde{\Omega} \frac{1}{\sqrt{2\tilde{\Omega}}} \frac{1}{\sqrt{2\tilde{\Omega}}} \int d\tilde{u} e^{i\tilde{u} \Theta} \left( e^{-i\tilde{u} \Theta} b(\tilde{\Omega}) + e^{i\tilde{u} \Theta} b^\dagger(\tilde{\Omega}) \right) = \int_0^\infty d\tilde{\Omega} \frac{1}{\sqrt{-2\tilde{\Omega}}} \left( b(\Omega - \tilde{\Omega}) b(\tilde{\Omega}) + \delta(\Omega + \tilde{\Omega}) b^\dagger(\tilde{\Omega}) \right) = \left\{ b^\dagger(-\Omega) / \sqrt{-2\Omega} \right\} \text{ for } \Omega > 0 \tag{38}
  \]
Expectation value of the number of $b$-particles with "momentum" $\Omega$ in the Minkowski vacuum ($\Omega > 0$):

$$
\langle 0|b^\dagger(\Omega)b(\Omega)|0\rangle_M = \langle 0| \int d\omega \left( a^{\dagger}_\omega a(\omega) + \beta^{\dagger}_\omega a(\omega) \right) = \int d\omega \left( a^{\dagger}_\omega a(\omega) + \beta^{\dagger}_\omega a(\omega) \right) |0\rangle_M = \int d\omega \int d\omega' \beta^{\dagger}_\omega \beta^{\dagger}_\omega |0\rangle_M |0\rangle_M = \int d\omega |[\beta^{\dagger}_\omega]^2|.
$$

The integral on the right hand side of (44) can be solved (Mukhanov et al., page 112 and 113):

$$
\langle 0|b^\dagger(\Omega)b(\Omega)|0\rangle_M = \frac{1}{e^{\Omega T} - 1}.
$$

An analogous calculation for $\Omega < 0$ can be carried out. The result for arbitrary $\Omega$ is

$$
\langle 0|b^\dagger(\Omega)b(\Omega)|0\rangle_M = \frac{1}{e^{\Omega T} - 1}.
$$

Result $(\Omega > 0)$:

$$
b(\Omega) = \int d\omega \left( \alpha^{\dagger}_\omega a(\omega) + \beta^{\dagger}_\omega a(\omega) \right) = \sqrt{\frac{\Omega}{\omega}} \int d\omega \frac{1}{2\pi} \exp \left( i\Omega \alpha^\dagger - i\alpha \Omega \left( \frac{1}{\omega} - \frac{1}{\omega'} \right) \right) \exp \left( i\beta^\dagger \beta \right). \tag{40}
$$

$\alpha^{\dagger}_\omega = \sqrt{\frac{\Omega}{\omega}} \int d\omega' \frac{1}{2\pi} \exp \left( i\Omega \alpha^\dagger + i\alpha \Omega \left( \frac{1}{\omega} - \frac{1}{\omega'} \right) \right) \exp \left( i\beta^\dagger \beta \right). \tag{41}$

$\beta^{\dagger}_\omega = \sqrt{\frac{\Omega}{\omega}} \int d\omega' \frac{1}{2\pi} \exp \left( i\Omega \alpha^\dagger - i\alpha \Omega \left( \frac{1}{\omega} - \frac{1}{\omega'} \right) \right). \tag{42}$

Transformations like (40) and (43) which relate different sets of creation and annihilation operators are called Bogolyubov transformations. The coefficients (41) and (42) are called Bogolyubov coefficients.

Comparing (46) with the Bose distribution

$$
n(\Omega) = \frac{1}{e^{\Omega T} - 1} \tag{47}
$$

yields the Unruh temperature

$$
T = \frac{a}{2\pi}. \tag{48}
$$

Conclusion: An accelerated observer moving through the Minkowski vacuum has the impression of moving through a thermal bath of $b$-particles with temperature $T$. 