Problem 1 [Natural units] How much is 1 kg in GeV and 1 s in GeV$^{-1}$? Use the results to express Newton’s gravitational constant $G_N = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$ and the value of the Planck mass $M_{Pl} = G_N^{-1/2}$ in natural units.

Problem 2 [Hamiltonian mechanics] Starting from the definition of the Hamiltonian,

$$H(x, p) \equiv p \dot{x} - L(x, \dot{x}),$$

and using the Euler-Lagrange equation, derive Hamilton’s equations,

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}$$

[Hint: Be careful about what the independent variables of a function are.]

Problem 3 [Continuity equation] Using the Schrödinger equation for the wavefunction $\Psi(x, t)$,

$$\left[ -\frac{\hbar^2 \nabla^2}{2m} + V(x) \right] \Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t),$$

show that the probability density, $\rho = \Psi^* \Psi$, satisfies the continuity equation

$$\frac{\partial}{\partial t} \rho + \nabla j = 0$$

where

$$j = \frac{\hbar}{2mi} [\Psi^* \nabla \Psi - (\nabla \Psi^*) \Psi].$$

Problem 4 [Heisenberg equation of motion] Let $\hat{O}$ be a time-independent operator in the Schrödinger picture, and $\hat{H}$ the time-independent Hamiltonian of the system. Starting from the definition of a Heisenberg operator,

$$\hat{O}_H(t) = e^{i\hat{H}(t-t_0)/\hbar} \hat{O} e^{-i\hat{H}(t-t_0)/\hbar},$$

derive the Heisenberg equation of motion,

$$i\hbar \frac{d}{dt} \hat{O}_H = [\hat{O}_H, \hat{H}_H].$$