Problem 1 [Electrons in LEP] In the LEP storage ring at CERN, head-on collisions between (equally accelerated) electrons and positrons were produced, such that the total energy in the center of mass frame was equal to the mass of the $Z$ boson ($m_Z = 91 \text{ GeV}$). What is the velocity of each particle before the collision? If an electron is accelerated toward a positron at rest, what velocity does it need in order to reach the same center-of-mass total energy?

Problem 2 [Continuity equation revisited] The Klein-Gordon equation for a free particle reads
\[ -\frac{\partial^2}{\partial t^2} \phi = -\nabla^2 \phi + m^2 \phi. \]
Derive the corresponding continuity equation using a similar procedure as in Problem 3 of exercise sheet I.

Verify that the continuity equation can be written as $\partial_\mu J^\mu = 0$, where
\[ J^\mu = i(\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi). \]
What happens to $J^0 = \rho$ for a negative energy solution $\phi \sim \exp(ipx)$, and what does this imply for the interpretation of $\rho$?

Problem 3 [Invariant measure] Given the relativistic invariance of the measure $d^4k$, show that the integration measure
\[ \frac{d^3k}{(2\pi)^3 2E(k)} \]
is Lorentz invariant, provided that $E(k) = \sqrt{k^2 + m^2}$. Use this result to argue that $2E(k)\delta(k-k')$ is a Lorentz invariant distribution. [Hint: Start from the Lorentz invariant expression $\frac{d^4k}{(2\pi)^4}\delta(k^2 - m^2)\delta(k_0)$ and use $\delta(x^2 - x_0^2) = \frac{1}{2|x|}(\delta(x-x_0) + \delta(x+x_0))$. What is the significance of the $\delta$ and the $\theta$ functions above?]

Problem 4 [Lorentz transformations] An infinitesimal Lorentz transformation and its inverse can be written as
\[ x'^\alpha = (g^{\alpha\beta} + \epsilon^{\alpha\beta}) x_\beta, \quad x^\alpha = (g^{\alpha\beta} + \epsilon^{\alpha\beta}) x'_\beta, \]
where $\epsilon^{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are infinitesimal. Show from the definition of the inverse that $\epsilon^{\alpha\beta} = -\epsilon^{\alpha\beta}$.

Show from the preservation of the norm that
\[ \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}. \]