Exercise sheet IV
November 9 [correction: November 16]

Problem 1 [3-vector bilinear] Show that
\[ \bar{\psi}_1 \gamma^0 \gamma^i \gamma^5 \psi_2 \]
transforms as a 3-vector (e.g. \( x^i \)) under rotations.

Problem 2 [Chiral projectors] The projectors on the left- and right-components of Dirac spinors are given by \( P_L = \frac{1 - \gamma^5}{2} \) and \( P_R = \frac{1 + \gamma^5}{2} \), respectively. Use the properties of \( \gamma_5 \) to prove the following projector identities:
\[ P_L P_R = P_R P_L = 0, \quad P_R P_R = P_R, \quad P_L P_L = P_L. \]
Verify that, given a spinorial solution to the massless Dirac equation, the spinors \( P_{L,R} u_s(p) \) are eigenstates of the helicity operator
\[ \mathfrak{h} = \frac{1}{2|p|} \begin{pmatrix} \sigma \cdot p & 0 \\ 0 & \sigma \cdot p \end{pmatrix} \]
with eigenvalues \( \pm \frac{1}{2} \). Does this hold also when \( m \neq 0 \)?

Problem 3 [spin and spinors] Let us consider the following two spinors
\[ u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_1+ip_2}{E+m} \end{pmatrix}, \quad \text{and} \quad u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_1-ip_2}{E+m} \\ -\frac{p_3}{E+m} \end{pmatrix}. \]
It is easy to see that they are eigenstates of \( S_3 \) with eigenvalues \( \pm \frac{1}{2} \), if \( p = (0, 0, p_3) \).

a) Construct the eigenstates of \( S_1 \) and \( S_2 \) as linear combinations of \( u_1 \) and \( u_2 \), and the appropriate choice of \( (p_1, p_2, p_3) \).

b) Alternatively, to obtain those eigenstates, you can perform a rotation of \( \pi/2 \) about the \( y \) and \( x \) axis of the spinors \( u_1 \) and \( u_2 \). Show that the result is the same as in a), up to an irrelevant phase factor.
Note: Be careful with the sign of the rotation.