Exercise sheet V
November 12 [solution: November 21]

Problem 1 [Classical real scalar field]

(i) Using the Euler-Lagrange equation, find the equation of motion of the real scalar (Klein-Gordon) field whose Lagrangian density is:

\[ L = \frac{1}{2} \left( \partial_\mu \phi \right)^2 - \frac{1}{2} m^2 \phi^2. \]  

(ii) Check that the solution of the Klein-Gordon equation is a plane wave:

\[ \phi(x) = e^{\pm ip^\mu x_\mu} \equiv e^{\pm ipx}. \]

Justify that it implies that a general solution of this equation is:

\[ \phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E(p)}} \left( a(p)e^{-ipx} + a^*(p)e^{ipx} \right), \]

where \( a(p) \) is an arbitrary function and \( E(p) = \sqrt{m^2 + p^2} \).

(iii) Calculate the energy-momentum tensor for the real scalar field:

\[ T^{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu \phi)} \partial^\nu \phi - L g^{\mu\nu}, \]

where \( g^{\mu\nu} \) is the metric tensor.

(iv) Using result from (iii) and eq. (3), calculate the total energy and momentum of the Klein-Gordon field:

\[ H = \int T^{00} d^3x, \quad P^i = \int T^{0i} d^3x. \]

Hint: use the complex representation of the 3D Dirac delta function:

\[ \delta(p \pm p') = \frac{1}{(2\pi)^3} \int d^3xe^{i(p \pm p') \cdot x}. \]  

Problem 2 [Classical complex scalar field]

(i) Show that the complex Klein-Gordon field Lagrangian:

\[ L = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \]

is invariant with respect to the global gauge transformation:

\[ \phi \rightarrow \phi' = e^{-ia} \phi, \quad \phi^* \rightarrow (\phi^*)' = e^{ia} \phi^*. \]

(ii) Calculate the conserved Noether current \( j^\mu \):

\[ j^\mu = \left. \frac{\partial L}{\partial (\partial_\mu \phi)} \partial^\phi \right|_{a=0} + \left. \frac{\partial L}{\partial (\partial_\mu \phi^*)} \partial^\phi^* \right|_{a=0}, \]

related to the global gauge transformation (8). Check that \( \partial_\mu j^\mu = 0 \).