Problem 1 [Zassenhaus formula]

Using the Zassenhaus formula $e^{A+B} = e^A e^B e^{Z_2} e^{Z_3} e^{Z_4} \ldots$ with $Z_2 = \frac{1}{2} [B, A]$, $Z_3 = -\frac{1}{6} (2[B, [B, A]] + [A, [B, A]])$ and exponents $Z_n, n > 3$ containing higher powers of $A$ and $B$, show the relation

$$e^{-i\epsilon H} = e^{-i\epsilon (H_0 + V)} = e^{-i\epsilon V/2} e^{-i\epsilon H_0} e^{-i\epsilon V/2} + O(\epsilon^3).$$

(1)

Problem 2 [Harmonic oscillator – partition function]

You are to obtain the partition function of the (quantum mechanical and classical) harmonic oscillator in three different ways:

(i) Starting from the formula of the harmonic oscillator seen in the Lectures

$$\langle x, t | x, 0 \rangle = \sqrt{\frac{m\omega}{2\pi i \sin(\omega t)}} \exp \left( i \frac{x^2 m\omega}{2 \sin(\omega t)} (\cos(\omega t) - 1) \right)$$

(2)

go to Euclidean time ($t = -i\tau$) and calculate the partition function $Z = \int dx \, \langle x | e^{-\tau H} | x \rangle$.

(ii) The oscillator spectrum is discrete. Make use of this fact and calculate $Z = \sum_n e^{-\tau E_n}$. Compare to (i).

(iii) Now calculate the partition function of the classical oscillator using $Z = \int dx dp \, e^{-\tau (T + V)}$.

Compute the free energy $F = -1/\tau \log Z$ for the three cases and show that they agree for $\tau \ll 1$. Why?

Problem 3 [Functional derivative, chain rule]

Calculate the functional derivative, with respect to $f(x)$, of the functional $F[g(f)] = \int dx \, g(f(x))$ with $g(f) = (\frac{dx}{df})^n$ using:

(i) the definition of the (functional) chain rule $\frac{\delta F[g(\phi)]}{\delta \phi(y)} = \int ds \frac{\delta F[g(\phi)]}{\delta g(\phi(s))} \frac{\delta g(\phi(s))}{\delta \phi(y)}$;

(ii) the chain rule of ordinary differential calculus and the equivalent definition

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \frac{d}{ds} F[\phi(x) + s\delta(x - y)] \bigg|_{s=0}.$$