Exercise sheet XI
July 6 [correction: July 13]

Problem 1 [Fermions on the lattice]
Consider, in the following, a $(1 + 1)$-dimensional Minkowski spacetime. Here the metric tensor is $g_{\mu\nu} = \text{diag}(+1, -1)$, the two $2 \times 2$ gamma-matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, and spinors have two components.

(i) Solve the Dirac equation on the continuum
\[ i\gamma_0 \partial_0 \psi(x, t) + i\gamma_1 \partial_x \psi(x, t) - m\psi(x, t) = 0 \]
up to identifying the dispersion relation $E = E(p)$. To do so:
- get rid of derivatives with the plane-wave Ansatz
  \[ \psi(x, t) = e^{i(Et - px)}u(p, E) ; \]
- get rid of the gamma-matrix structure with the Ansatz
  \[ u(p, E) = (-\gamma_0 E + \gamma_1 p + m)v , \]
  with $v$ some two-component spinor.

How many solutions $p$ do you find for a given energy $E > m$?

(ii) Now discretise only the spatial direction $x$ with a spacing $a$, choosing a symmetric derivative for rewriting the Dirac equation:
\[ \partial_x \psi(x) \mapsto \frac{\psi(x + a) - \psi(x - a)}{2a} . \]
Proceed as in the previous case, with an adequate $u \rightarrow v$ Ansatz ($\text{Hint:}$ is it still true that $-\infty < p < +\infty$ ?), and write down the lattice version of the dispersion $E = E_{\text{lat}}(p)$. Compare it with the continuum result: in particular, how many solutions (for a given energy $E$) do you find on the lattice?

(iii) Repeat the above exercise, but now with the following discretised derivative:
\[ \partial_x \psi(x) \mapsto \frac{\psi(x + a) - \psi(x)}{a} . \]
Which essential property of the continuum action is lost in this case?