Problem 1 [Functional differentiation]: The formal definition of the functional derivative is as follows: with $F[\phi]$ a functional of $\phi(x)$,

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \left. \frac{d}{ds} F[\phi(x) + s\delta(x - y)] \right|_{s=0} .$$

(i) In calculating the transition probability $\langle x_2, t_2 | x_1, t_1 \rangle$ for the harmonic oscillator,

$$S[x] = \int_{t_1}^{t_2} dt \left( \frac{m}{2} \dot{x}^2(u) - \frac{m\omega^2}{2} x^2(u) \right) ; \quad x = x(t) ,$$

one can write the trajectory as $x(t) = x_{cl}(t) + y(t)$, with boundary conditions $x_{cl}(t_1) = x_1$, $x_{cl}(t_2) = x_2$ and $y(t_1) = y(t_2) = 0$.

Expanding around the classical solution to the equations of motion, $x_{cl}$, the following exact result holds:

$$S[x] = S[x_{cl}] + \int_{t_1}^{t_2} dt \left( \frac{m}{2} y^2(t) - \frac{m\omega^2}{2} y^2(t) \right) .$$

(a) Show, using the formal definition above, that indeed

$$\frac{\delta S[x]}{\delta x(t)} = -m\dddot{x}(t) - m\omega^2 x(t) .$$

(b) Show that

$$\frac{\delta^2 S[x]}{\delta x(t) \delta x(t')} = \delta(t - t') \left( -m \frac{d^2}{dt'^2} - m\omega^2 \right) .$$

(c) Show that, for any $n > 2$,

$$\frac{\delta^n S[x]}{\delta x(t) \delta x(t') \cdots \delta x(t^{(n-1)})} = 0 ;$$

for which class of potentials do you expect this result to hold?

(ii) Starting from the real Klein-Gordon action in four dimensions, with metric $g^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$,

$$S[\phi] = \int d^4x \left( \partial_\mu \phi(x) \partial^\mu \phi(x) + m^2 \phi^2(x) \right) ,$$

evaluate the derivative

$$\frac{\delta S[\phi]}{\delta \phi(x)}$$

and verify that setting it to zero yields the Klein-Gordon equation $(\Box - m^2)\phi(x) = 0$.

[Hint: in integration by parts, the finite contribution can be neglected.]
Problem 2 [Contour integration, residues]:

(i) Prove that
\[ I_1 = \int_0^{2\pi} \frac{d\theta}{\frac{5}{2} + 3\cos \theta} = \frac{\pi}{2}. \]

(ii) Prove that
\[ I_2 = \int_0^\infty \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx = \frac{\pi}{6}. \]

(iii) Prove that
\[ I_3 = \int_{-\infty}^{+\infty} \frac{\cos x}{1 + x^2} dx = \frac{\pi}{e}. \]
\text{Hint: different parts of the calculation may require different choices of half-planes...}

(iv) Prove that
\[ I_4 = \int_0^\pi \frac{\sin x}{x} dx = \frac{\pi}{2}. \]
\text{Hint: remember the freedom granted by Cauchy's theorem...}

(v) An integral representation for the Heaviside step function \( \theta(x) \) is the following:
\[ \theta(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ixz}}{z - i\epsilon} dz; \]
verify, using the residue theorem, that it gives indeed the usual \( \theta(x) \). What happens for \( x = 0 \)? [Hint: how is \( \log(z) \) defined in \( \mathbb{C} \)?]