Exercise sheet X
July 3 [correction: July 10]

Problem 1 [Lattice discretisation of the harmonic oscillator]
Consider the Euclidean harmonic oscillator:

\[ Z[j] = \frac{1}{Z} \int [\mathcal{D}q] \exp \left\{ \frac{1}{2} \int d\tau q(\tau) \left( m \frac{d^2q}{d\tau^2} - m\omega^2 \right) q(\tau) + \int d\tau j(\tau)q(\tau) \right\} = \exp \left\{ \frac{1}{2} \int d\tau_1 \int d\tau_2 j(\tau_1) \left[ \frac{e^{-\omega|\tau_1-\tau_2|}}{2m\omega} \right] j(\tau_2) \right\} ; \]

assume, in particular, a periodic and finite time extent \( T \).

(i) Discretise now the temporal direction, by dividing \( T \) into \( N \) intervals of length \( \Delta t \). Write the corresponding action \( S_{ij} \) and generating functional \( Z(j) \), where now \( S_{ij} \) is a matrix acting on \( N \)-component vectors \( q = (q_1, \ldots, q_n) \), the discretised form of the trajectories \( q(\tau) \). The source term \( j \) will also be a \( N \)-component vector.

(ii) Evaluate the inverse of \( S_{ij} \) with a discrete Fourier transform:

\[ (S^{-1}_{k\ell}) = G_{k-\ell} = \sum_{n=1}^{N} \exp \left( -\frac{2\pi in(k-\ell)}{N} \right) \tilde{G}_n . \]

(Hint: you should get a closed form only in Fourier space, while in coordinate space an explicit summation will remain, over a finite number of terms).

(iii) Express the lattice two-point function

\[ \langle 0| T \left( q(j \Delta t)q(k \Delta t) \right) |0 \rangle = C \left( \frac{(k-j)\Delta t}{t} \right) \]

in terms of \( S^{-1}_{k\ell} \). Compute numerically the resulting sum with your favourite method (Python, C++, Fortran, ...): to do this, choose \( mT = 8 \), \( \omega/m = 1 \) and take \( N = 8, 16, 32 \). Compare, by means of a plot, the discretised results with the \( T \to \infty \) continuum equation you have seen in the Lecture. Are there differences?

(iv) Determine the slope of \( -\log \left( C(t) \right) \) for “intermediate” \( t \). Interpret what you find.

(v) Look at your results for \( t \ll T \): can you imagine a technique to make the comparison to the continuum more accurate?