Problem 1 [Fermions on the lattice]
Consider, in the following, a $(1 + 1)$-dimensional Minkowski spacetime. Here the metric tensor is $g_{\mu \nu} = \text{diag}(+1, -1)$, the two $2 \times 2$ gamma-matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu \nu}$, and spinors have two components.

(i) Solve the Dirac equation on the continuum

$$i\gamma^0 \partial_0 \psi(x, t) + i\gamma^1 \partial_x \psi(x, t) - m\psi(x, t) = 0$$

up to identifying the dispersion relation $E = E(p)$. To do so:

• get rid of derivatives with the plane-wave Ansatz

$$\psi(x, t) = e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{E} t)} u(p, E) ;$$

• get rid of the gamma-matrix structure with the Ansatz

$$u(p, E) = (-\gamma_0 E - \gamma_1 p + m)v ,$$

with $v$ some two-component spinor.

How many solutions $p$ do you find for a given energy $E > m$?

(ii) Now discretise only the spatial direction $x$ with a spacing $a$, choosing a symmetric derivative for rewriting the Dirac equation:

$$\partial_x \psi(x) \mapsto \frac{\psi(x + a) - \psi(x - a)}{2a} .$$

Proceed as in the previous case, with an adequate $u \rightarrow v$ Ansatz (Hint: is it still true that $-\infty < p < +\infty$ ?), and write down the lattice version of the dispersion $E = E_{\text{lat}}(p)$. Compare it with the continuum result: in particular, how many solutions (for a given energy $E$) do you find on the lattice?

(iii) Repeat the above exercise, but now with the following discretised derivative:

$$\partial_x \psi(x) \mapsto \frac{\psi(x + a) - \psi(x)}{a} .$$

Which essential property of the continuum action is lost in this case?