Problem 5

Quite often it is convenient to consider Euclidean versions of relativistic field theories, which mainly arise by replacing the Minkowski metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ by the Euclidean metric $g_{\mu\nu} = \text{diag}(+1, +1, +1, +1)$. Since there is no difference between covariant and contravariant components anymore, one usually writes lower indices only.

(a) Consider the Euclidean version of Maxwell theory, i.e.

$$S_E = \int d^Dx \frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  

What is the minimum value of $S_E$ and which field configurations yield that value? For which number of spacetime dimensions $D$ is it possible to exclude the existence of further solutions of the equations of motion with finite Euclidean action?

(b) Consider the Euclidean version of SU(2) Yang-Mills theory, i.e.

$$S_E = \int d^Dx \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$$

(the additional upper index $a$ of the gauge field $A^a_\mu$ is a color index, where $a = 1, 2, 3$, i.e. the gauge field has three times as many components as in Maxwell theory). What is the minimum value of $S_E$ and which field configurations yield that value? For which number of spacetime dimensions $D$ is it possible to exclude the existence of further solutions of the equations of motion with finite Euclidean action?
Problem 6

The Abelian Higgs model,

\[ \mathcal{L} = (D^\mu \phi)^* D_\mu \phi - V(|\phi|) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad V(|\phi|) = \frac{\lambda}{4} (|\phi|^2 - a^2)^2, \]

is a model closely related to superconductivity.

One of the major phenomena of superconductivity is the ejection of any sufficiently weak magnetic field from the interior of a superconductor (Meissner effect).

(a) Derive the equations of motion for \( \phi \) and \( A_\mu \).

(b) Show that for \( \phi = a \) and \( E = 0 \) the spatial components of the equation of motion for \( A_\mu \)
reduce to the London equation\(^1\)

\[ \left( \text{rot} \, B \right)_j = 2e^2a^2 A^j. \]

(c) Solve the London equation. Interpret the obtained solution with respect to the Meissner effect. Are the remaining equations of motion of the Abelian Higgs model also fulfilled?

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\(^1\)Developed in 1935 by F. and H. London by means of phenomenological considerations.