Conformal viscous hydrodynamics

and

finite temperature gauge/gravity duality

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• motivation: viscous effects in RHIC data

viscosity $\eta$ - relaxation time $\tau_{\pi}$, ...

• conformal invariance,

relativistic hydrodynamics - up to second-order gradients

• gauge/gravity duality -

(AdS/CFT) correspondence
mainly based on more recent work - and references therein - by:

R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov

“Relativistic viscous hydrodynamics, conformal invariance, and holography”

R. Loganayagam

“Entropy current in conformal hydrodynamics”

M. Luzum and P. Romatschke

“Conformal relativistic viscous hydrodynamics: Applications to RHIC at $\sqrt{s_{NN}} = 200$ GeV”

and
P. Romatschke

"New developments in relativistic viscous hydrodynamics"

arXiv: 0902.3663 [hep/ph]

"Relativistic viscous hydrodynamics - status update"

Plenary Talk at QM09, Knoxville, USA

and the seminal papers by the Frankfurt group:

D. H. Rischke and coworkers, A. Muronga, A. Dumitru, ..
A fluid which has no shear stresses, viscosity or heat conduction is called a

**PERFECT FLUID**

Top physics story of 2005 is the RHIC discovery of the strongly interacting quark-gluon plasma (called sQGP), which behaves almost like a perfect fluid, with very low viscosity

[T. D. Lee 06 ]

Is it correct? It requires to calculate corrections in a systematic way!

remarkable approach: **black hole theory**

is now used to explain properties of colliding nuclei

[L. Susskind 08]
dissipation: consequence of black hole absorption

(damped) quasinormal modes:
gravitational perturbation to a black hole
and to a hydrodynamic system

[from M. Natsuume]
energy momentum tensor: $\epsilon$ energy density and $p$ pressure

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} + \Pi^{\mu\nu}$$

$u^\mu$ fluid velocity, $u_\mu u^\mu = -1$, and $\Pi^{\mu\nu}$ shear stress tensor

with

$$u_\mu \Pi^{\mu\nu} = 0 \quad (u_\mu T^{\mu\nu} = -\epsilon u^\nu),$$

momentum density is due to energy flow, and

$$\Pi_\mu^\mu = 0$$

required from conformal invariance

local conservation law (covariant derivative $\nabla_\mu$): $\nabla_\mu T^{\mu\nu} = 0$

(assume: no net charge in the system)
Conformal fluid in $d = 4$ (curved) dimensions - line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Metric $g_{\mu\nu}$ with signature $(- + + +)$, $\mu, \nu$. Space-time indices $0, 1, 2, 3$

(geometric) covariant derivative: $\nabla_\mu$

$$\nabla_\mu \Phi(x) = \partial_\mu \Phi(x),$$
$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}^{\nu} V^\rho$$

etc., e.g. $\nabla_\mu g^{\nu\rho} = 0$

Christoffel symbols:

$$\Gamma_{\mu\rho}^{\nu} = \frac{1}{2} g^{\nu\sigma} (\partial_\mu g_{\rho\sigma} + \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho})$$

Riemann tensor: $[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu\sigma}^{\rho} V^\sigma$
approximation to $\Pi^{\mu\nu}$

only retaining shear viscosity terms - NO bulk viscosity

- **Navier - Stokes** = first-order theory in gradients with constitutive relation:

$$\Pi^{\mu\nu} = -2\eta < \nabla^\mu u^\nu > \equiv -2\eta \sigma^{\mu\nu}$$

with the traceless and symmetric shear strain tensor

$$\sigma^{\mu\nu} = \sigma^{\nu\mu} \equiv \frac{1}{2} (\Delta^{\mu\rho} \nabla^\rho u^\nu + \Delta^{\nu\rho} \nabla^\rho u^\mu) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} \nabla^\rho u^\sigma$$

and transverse projection

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu , \quad \Delta^{\mu\nu} u_\nu = 0$$
shear viscosity $\eta$

interacting field theories:

**non-vanishing shear viscosity** $\eta$

e.g. shear flow in $x-$ direction $\rightarrow$
force per unit area/momentum transfer

$$\frac{F_x}{A} = -\Pi_{xz} = \eta \frac{\partial u_x}{\partial z}, \quad \eta \propto \frac{1}{\sigma}$$

measure how quickly a system, perturbed from thermal equilibrium, goes back: it takes longer for a weakly coupled system, because particles interact less than for a strongly coupled system: $\eta_{strong} << \eta_{weak}$
viscosity/entropy density ratio

\[ \eta/s \text{ from quenched QCD lattice simulations [Nakamura and Sakai 05], also } \eta/s = 0.134(33) \text{ at } T = 1.65T_c \text{ [Meyer 07]} \]

compared with the perturbative result \( \sim \frac{1}{g^4 \ln(\mu/gT)} \sim O(1) \)

for strong coupling \( \lambda = g_Y^2 N_c \), \( \lambda \to \infty \) [Kovtun, Son, Starinets 05]

\[ \frac{\eta}{s} \geq \frac{1}{4\pi} \sim 0.08 : \text{ AdS/CFT universal (?) lower bound} \]
Navier-Stokes

transverse projection of EoM \[ \Delta^\mu_\alpha \nabla_\beta T^{\alpha\beta} = 0 \rightarrow \]
relativistic Navier-Stokes equation in first-order theory

\[
(\epsilon + p)u^\alpha \nabla_\alpha u^\mu + \Delta^{\mu\alpha} \nabla_\alpha p + \Delta^\mu_\alpha \nabla_\beta \left[ -2\eta < \nabla^\alpha u^\beta > \right] = 0
\]
i.e. parabolic equation:

time derivative is of first-order \((u^\alpha \nabla_\alpha \rightarrow \partial/\partial t)\)

while

space derivative is of second-order \((\nabla_j \sigma^{ij} \rightarrow \partial_j \partial_i u^j)\)

"relativistic first-order dissipative theory is highly pathological, and therefore should be discarded in favor of the second-order one"

[Hiscock and Lindblom 1983-1985]
transverse mode

small linear perturbation and first-order →
diffusion equation in shear channel:

\[ \delta u_\perp = \sqrt{\frac{(\epsilon + p)}{4\pi \eta t}} \exp \left[- \frac{(\epsilon + p) x^2}{4\eta t} \right] \]

i.e. propagates outside the light-cone (Gaussian)
starting from \( \delta u_\perp (x, t = 0) = \delta(x) \) !!

minimal modification → second-order:
relaxation time \( \tau_\pi > 0 \) → hyperbolic “telegraph” equation

\[ [\tau_\pi \partial_t^2 + \partial_t - \frac{\eta}{(\epsilon + p)} \partial_x^2] \delta u_\perp = 0 \]
a new (systematic) result

All second-order terms classified by conformal symmetry, by space-time dependent Weyl transformation:

\[ g^{\mu\nu} = e^{-2\phi(x)} \tilde{g}^{\mu\nu}, \quad T^{\mu\nu} = e^{-6\phi} \tilde{T}^{\mu\nu}, \quad T^\mu_\mu = 0, \ldots \]

Constitutive relation (d= 4) with five new terms:

\[
\Pi^{\mu\nu} = -2\eta \sigma^{\mu\nu} + 2\eta \tau_\pi u^\lambda D_\lambda \sigma^{\mu\nu} \\
+ 4\lambda_1 \sigma^{<\mu_1 \sigma^{\nu_1>\lambda}} + 2\lambda_2 \sigma^{<\mu_2 \Omega^{\nu_2>\lambda}} + \lambda_3 \Omega^{<\mu_3 \Omega^{\nu_3>\lambda}} \\
+ 2\kappa u_\alpha C^{\alpha\mu\nu\beta} u_\beta
\]

\( C^{\alpha\beta\gamma\delta} \) … Weyl tensor, \( \Omega^{\alpha\beta} \) … antisymmetric vorticity tensor, \( D_\lambda \) … Weyl derivative

[Baier, Romatschke, Son, Starinets, Stephanov 07]

[Bhattacharyya, Hubeny, Minwalla, Rangamani 07, Loganayagam 08]
hydrodynamic transport coefficients by gauge/gravity duality:
compare with quite involved AdS/CFT-gravity calculations at Hawking temperature $T = T_H$ and for momentum $\omega, k << T_H$

e.g. from sound channel dispersion up to $O(k^3)$, etc:

\[
\frac{\eta}{s} = \frac{1}{4\pi}, \quad \tau_\pi = \frac{2 - \ln 2}{2\pi T}, \quad \kappa = \frac{\eta}{\pi T} = 2\lambda_1,
\]

\[
\lambda_2 = -\frac{\ln 2}{2\pi T} \eta, \quad \lambda_3 = 0
\]
transport coefficients

kinetic QCD theory (LO weak coupling: $\alpha_s < 0.7$)

$$\frac{\eta/s}{\tau_\pi T} \simeq 0.17 - 0.2, \quad \lambda_1 \simeq (5.2 - 4.1) \frac{\eta^2}{sT}, \quad \lambda_2 = -2\tau_\pi \eta, \quad \lambda_3 = \kappa = 0$$

[York and Moore 08]

finite ’t Hooft coupling $\lambda \equiv g_{YM}^2 N_c, \quad \lambda \gg 1$ corrections to coefficients by gauge/gravity duality, e.g.:

[Buchel and Paulos 08]

$$\frac{\eta/s}{\tau_\pi T} = 0.383 \left(1 - 3.52 \lambda^{-3/2} + \ldots\right), \quad \lambda_1 = \frac{2\eta^2}{sT},$$

$$\lambda_2 = -0.530\tau_\pi \eta, \quad \kappa = \frac{\eta}{\pi T} \left(1 - \frac{145\zeta(3)}{8} \lambda^{-3/2} + \ldots\right)$$
conformal hydrodynamics
Weyl transformation

scaling transformation:

\[ g_{\mu\nu} = e^{2\phi} \tilde{g}_{\mu\nu}; \quad g^{\mu\nu} = e^{-2\phi} \tilde{g}^{\mu\nu}, \quad \phi = \phi(x) \]

leads to:

\[ \Gamma_{\lambda\mu}^{\nu} = \tilde{\Gamma}_{\lambda\mu}^{\nu} + \delta_{\lambda}^{\nu} \partial_{\mu} \phi + \delta_{\mu}^{\nu} \partial_{\lambda} \phi - \tilde{g}_{\lambda\mu} \tilde{g}^{\nu\sigma} \partial_{\sigma} \phi \]

\[ u^{\mu} = e^{-\phi} \tilde{u}^{\mu}, \quad \Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu} = e^{-2\phi} \tilde{\Delta}^{\mu\nu} \]

usefull quantities (d dimensions):

'divergence': \( \vartheta \equiv \nabla_{\mu} u^{\mu} = e^{-\phi} \left[ \tilde{\vartheta} + (d - 1) \tilde{u}^{\sigma} \partial_{\sigma} \phi \right] \),

'acceleration': \( a^{\nu} \equiv u^{\mu} \nabla_{\mu} u^{\nu} = e^{-2\phi} \left[ \tilde{a}^{\nu} + \tilde{\Delta}^{\nu\sigma} \partial_{\sigma} \phi \right] \),

'gauge field': \( A_{\nu} \equiv a_{\nu} - \frac{\vartheta}{d - 1} u_{\nu} = \tilde{A}_{\nu} + \partial_{\nu} \phi \)
Weyl derivative

cf. (abelian) gauge transform and derivative:

\[ \tilde{\psi} = e^{i\Lambda(x)}\psi, \quad \tilde{A}^\mu = A^\mu - \partial^\mu \Lambda, \]

\[ \tilde{D}^\mu \tilde{\psi} = e^{i\Lambda}(\partial^\mu + iA^\mu)\psi = e^{i\Lambda}D^\mu \psi \]

elegant way: define covariant Weyl derivative \[\text{[Loganayagam 07]}\]

\[ Q_{\nu \cdots}^\mu = e^{-w\phi} \tilde{Q}_{\nu \cdots}^\mu, \quad \mathcal{D}_\lambda Q_{\nu \cdots}^\mu = e^{-w\phi} \tilde{\mathcal{D}}_\lambda \tilde{Q}_{\nu \cdots}^\mu, \quad w..integer \]

where Weyl derivative - linear in \( A_\lambda \) -

\[ \mathcal{D}_\lambda Q_{\nu \cdots}^\mu \equiv \nabla_\lambda Q_{\nu \cdots}^\mu + w A_\lambda Q_{\nu \cdots}^\mu \]

\[ + \left[ g_{\lambda\alpha} A^\mu - \delta_\lambda^\mu A_\alpha - \delta_\alpha^\mu A_\lambda \right] Q_{\nu \cdots}^{\alpha \cdots} + \cdots \]

\[ - \left[ g_{\lambda\nu} A^{\alpha} - \delta_\lambda^\alpha A_\nu - \delta_\nu^\alpha A_\lambda \right] Q_{\nu \cdots}^{\mu \cdots} - \cdots \]
\[ \mathcal{D}_\lambda g_{\mu\nu} = 0 \ , \ \mathcal{D}_\mu u^\mu = 0 \ , \ u^{\mu} \mathcal{D}_\mu u^\nu = 0 \]

and

\[ \mathcal{D}_\mu u^\nu = \nabla_\mu u^\nu + u_\mu a^\nu - \frac{\vartheta}{d-1} \Delta_\mu u^\nu = \sigma_\mu^\nu + \Omega_\mu^\nu = e^{-\phi} \tilde{\mathcal{D}}_\mu \tilde{u}^\nu \]

\[ \sigma^{\mu\nu} \equiv \frac{1}{2} \left( \Delta^{\mu\lambda} \nabla_\lambda u^\nu + \Delta^{\nu\lambda} \nabla_\lambda u^\mu \right) - \frac{1}{d-1} \vartheta \Delta^{\mu\nu} = \frac{1}{2} \left( \mathcal{D}^\mu u^\nu + \mathcal{D}^\nu u^\mu \right) \]

\[ = e^{-3\phi} \tilde{\sigma}^{\mu\nu} \]

antisymmetric vorticity tensor:

\[ \Omega^{\mu\nu} \equiv \frac{1}{2} \left( \Delta^{\mu\lambda} \nabla_\lambda u^\nu - \Delta^{\nu\lambda} \nabla_\lambda u^\mu \right) = \frac{1}{2} \left( \mathcal{D}^\mu u^\nu - \mathcal{D}^\nu u^\mu \right) = e^{-3\phi} \tilde{\Omega}^{\mu\nu} \]
partition function $Z = \tilde{Z}$ is conformal invariant ($d = 4$ dimensions)

$$T^{\mu\nu} \propto \frac{1}{\sqrt{-g}} \frac{\delta \ln Z}{\delta g_{\mu\nu}} \rightarrow$$

$$T^{\mu\nu} = e^{-6\phi(x)} \tilde{T}^{\mu\nu} \text{ with } g_{\mu\nu} = e^{2\phi} \tilde{g}_{\mu\nu}$$

conservation: $\nabla_\mu T^{\mu\nu} = 0 \rightarrow \tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0 \rightarrow$

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = e^{6\phi} [\nabla_\mu T^{\mu\nu} + T^{\mu}_{\mu} \partial^\nu \phi] \rightarrow T^{\mu}_{\mu} = 0$$

only scale is temperature $T$:

$$T^{\mu\nu}_{\text{perfect}} = (g^{\mu\nu} + 4u^\mu u^\nu)p, \quad \epsilon = 3p$$

$$\rightarrow p = e^{-4\phi} \tilde{p}, \quad p \propto T^4, \quad T = e^{-\phi} \tilde{T}, \quad D_\mu T = e^{-\phi} \tilde{D}_\mu \tilde{T}$$
invariant Weyl tensor

\[(d - 2)C_{\mu\alpha\nu\beta}u^\alpha u^\beta = \Delta^\mu_\lambda \Delta^\nu_\sigma R_{\lambda\sigma} + (d - 2)\Delta^\mu_\lambda \Delta^\nu_\sigma R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta \]

\[-\frac{\Delta^\mu_\nu}{d - 1}(\Delta^\lambda_\sigma R_{\lambda\sigma} + (d - 2)\Delta^\lambda_\sigma R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta)\]

\[R_{\mu\nu} \ldots \text{ Ricci tensor}\]

symmetry properties (same as for the Riemann tensor):

\[C_{\mu\nu\lambda\sigma} = -C_{\nu\mu\lambda\sigma} = -C_{\mu\nu\sigma\lambda} = C_{\lambda\sigma\mu\nu}\]

and \[C_{\mu\alpha\lambda}^\alpha = 0\]

\[C_{\alpha\mu\nu}^\beta = \tilde{C}_{\alpha\mu\nu}^\beta \implies u_\alpha C_{\alpha\mu\nu}^\beta u_\beta = e^{-4\phi(x)} \tilde{u}_\alpha \tilde{C}_{\alpha\mu\nu}^\beta \tilde{u}_\beta\]

\[u_\mu u_\alpha C_{\alpha\mu\nu}^\beta u_\beta = 0\]
\( \Pi^{\mu\nu} \) has in \( d = 4 \) dimensions 5 independent tensor structures

first-order

\[
\Pi^{\mu\nu} = -2\eta \, \sigma^{\mu\nu}
\]

\[
\Pi^{\mu\nu} = e^{-6\phi} \tilde{\Pi}^{\mu\nu}, \quad \sigma^{\mu\nu} = e^{-3\phi} \tilde{\sigma}^{\mu\nu} \quad \rightarrow \quad \eta = e^{-3\phi} \tilde{\eta}, \text{ i.e. } \eta \propto T^3
\]

e.g. \( \eta = (\pi T)^3 \), \( p = (\pi T)^4 \),

\[
s = \frac{\partial p}{\partial T} = 4\pi (\pi T)^3 \quad \rightarrow \quad \eta/s = 1/4\pi \quad \text{i.e. only a number!}
\]

second-order

\[
u^\lambda D_\lambda \sigma^{\mu\nu} = e^{-4\phi} \tilde{u}^\lambda \tilde{D}_\lambda \tilde{\sigma}^{\mu\nu}, \quad \text{etc.}
\]

\[
\eta \tau_\pi = e^{-2\phi} \tilde{\eta} \tilde{\tau}_\pi, \quad \tau_\pi \propto \frac{1}{T}, \quad \kappa \propto T^2, \quad \lambda_{1,2,3} \propto T^2
\]
Conformal $\Pi^{\mu\nu}$

All 5 second-order terms classified by conformal symmetry

Constitutive relation (d= 4):

$$
\Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} + 2\eta\tau_{\pi} u^\lambda D^\lambda \sigma^{\mu\nu} \\
+ 4\lambda_1 \sigma^{<\mu \lambda \sigma^{\nu}>^\lambda} + 2\lambda_2 \sigma^{<\mu \lambda \Omega^{\nu}>^\lambda} + \lambda_3 \Omega^{<\mu \lambda \Omega^{\nu}>^\lambda} \\
+ 2\kappa u_\alpha C^{\alpha\mu\nu\beta} u_\beta
$$

$$
\eta \propto T^3, \quad \tau_{\pi} \propto \frac{1}{T}, \quad \kappa \propto \lambda_{1,2,3} \propto T^2
$$

[Baier, Romatschke, Son, Starinets, Stephanov 07]

[Bhattacharyya, Hubeny, Minwalla, Rangamani 07, Loganayagam 08]
consistent evaluation of transport coefficients in strong coupling limit

via AdS/CFT correspondence
duality $\sim$ equality

is between

Quantum Field Theory (a special one) in $d = 4$

and

Classical Gravity in $d = 5$ (for $N_c \gg 1$, $g_{YM}^2 N_c \gg 1$)

(at finite temperature)
strongly coupled quantized conformal gauge theory
in $d = 4$ dimensions (\(\mathcal{N} = 4\) SYM with \(8N_c^2\) (1 gauge and 6 scalar) bosons and \((4N_c^2)\) Weyl fermions)
[[ NOT QCD !]]

\[\iff\]

weakly coupled classical supergravity (type IIB)
in $d = 10$ dimensions (on \(AdS_5 \otimes S^5\))

via holographic property: radial coordinate \(r_0 \leq r < \infty\) with
gauge theory on the boundary at \(\infty\)
crucial limit of interest:

\[ \lambda = g_{YM}^2 N_c \text{ is large, } N_c \to \infty, \quad g_{YM}^2 \ll 1 \]

i.e. string coupling \( g_s = \frac{g_{YM}^2}{4\pi} \ll 1 \)

- NO LOOPS

and

small curvature \( \frac{l_s^4}{L^4} = \frac{1}{\lambda} \ll 1 \) – RADIUS \( L \) of CURVATURE

is LARGE compared to the STRING SCALE \( l_s = \sqrt{\alpha'} \)

- CLASSICAL GRAVITY
D–branes

[dynamical walls on which strings can end:]

theory of open strings living on $N_c - D3$–branes ($\mathcal{N} = 4$ SYM, $d = 4$) $\iff$

gravity theory of fields living in the space curved by the branes ($AdS_5$, $d = 5$)
AdS$_5$ metric at finite $T$

generalization of the Einstein-Hilbert-Maxwell equation
- strong analogy to Reissner-Nordström black hole

near extremal black $D3$–brane metric with horizon $r = r_0$
for $r_0 < r << L$, i.e. factorized metric for $AdS_5 \otimes S^5$

line element:

$$ds^2(5) = \frac{r^2}{L^2}(-f(r)dt^2 + d\vec{x}^2) + \frac{L^2}{r^2f(r)}dr^2$$

with $f(r) = 1 - \frac{r_0^4}{r^4}$

from $AdS_5$ action - negative cosmological constant $\Lambda = -\frac{6}{L^2}$:

$$S = \frac{1}{16\pi G(5)} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} \right],$$

with Newton constant $G(5) = \frac{\pi L^3}{2N_c^2}$
**Symmetries** ($T = 0$)

duality maps the operators of QFT to the boundary values of the SUGRA fields

- **QFT** ($\mathcal{N} = 4, d = 4$ CFT)

  \[
  S = -\frac{1}{g_{YM}^2} \int d^4 x \left[ \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} D_\mu \Phi_i^a D^\mu \Phi_i^a + \ldots \right]
  \]

  .... massless Weyl fermions + interactions , $i = 1, \ldots, 6, a = 1, \ldots, N_c^2$

  has Poincaré and conformal symmetry $SO(2, 4)$:

  $x^\mu \rightarrow \lambda x^\mu$, $\Phi_i^a(x) \rightarrow \lambda \Phi_i^a(\lambda x)$, $F_{\mu\nu}(x) \rightarrow \lambda^2 F_{\mu\nu}(\lambda x)$ etc,

  $\beta(\mu) = 0$, and $SU(\mathcal{N} = 4)$ symmetry and $SU(N_c)$

- **$AdS_5 \otimes S^5$**:

  \[
  SO(2, 4) \otimes SO(6) = SO(2, 4) \otimes SU(4)
  \]

  besides $SU(N_c)$ due to $N_c$ parallel $D - 3$ branes
Bekenstein-Hawking entropy

thermodynamics: \[ dS = \frac{dE}{T} \]

- Schwarzschild BH \((G = G_4, T_H = \frac{1}{4\pi r_0})\)

\[ E = M = \frac{r_0}{2G}, \quad dE = \frac{dr_0}{2G}, \quad dS = \frac{4\pi}{2G} r_0 dr_0, \]

entropy

\[ S = \frac{\pi r_0^2}{G} = \frac{A}{4G} \]

a universal relation with \(A = 4\pi r_0^2\)… area of the horizon
Bekenstein-Hawking entropy, cont.

$\text{AdS}_5$: $T_H = \frac{r_0}{\pi L^2}$, $S = \frac{A(3)}{4G(5)}$

$$A(3)(r = r_0) = \left(\frac{r_0}{L}\right)^3 V_3 = \pi^3 T_H^3 L^3 V_3, \quad G(5) = \frac{\pi L^3}{2N_c^2}$$

gives for the entropy density

$$s_{BH} = \frac{S}{V_3} = \frac{\pi^2 N_c^2}{2} T_H^3$$

[Note: $s_{BH} = \frac{3}{4} s_{Boltzmann}$]

$$s_{Boltzmann} = \frac{4p}{T} = \frac{4\pi^2 T^3}{90} \left[\frac{7}{4} \cdot 4 + 2 + 6\right] N_c^2 = \frac{2\pi^2 N_c^2}{3} T^3$$
energy density of QCD and SYM - via BH entropy

[from Myers and Vazquez 08]
master formula for AdS/CFT

schematically in terms of coinciding partition functions:

\[ \int e^{iS_{4d}^{\text{gauge}}} + i\Phi_0 O = \int e^{iS_{5d}[\Phi]} \approx e^{iS_{\text{classical}}[\Phi_0]} \]

\( S_{5d} \) is computed with non-trivial boundary condition (holography)

\[ \Phi(t, \vec{x}, r) \big|_{r \rightarrow \infty} = \Phi_0(t, \vec{x}) \]

\[ \implies \quad \text{quantum correlation} = \text{classical two-point function} \]

\[ G(x, y) = -i < TO(x)O(y) > = -\frac{\delta^2 S_{\text{classical}}}{\delta \Phi_0(x) \delta \Phi_0(y)} \big|_{r=\infty} \]

gauge: \( O = T_{\mu\nu} \) .. energy-momentum tensor

gravity: \( \phi = g_{\mu\nu} \) .. graviton
correlators from gravity, \( d = 4 \)

response of the fluid to small metric perturbations
e.g.: \( g_{xy} = h_{xy}(t, z, r) = -h^{xy}(t, z, r) \neq 0, \ u^0 = 1, \ T = \text{const} \)
it leads from Christoffel symbols in the covariant derivatives to linear approximation:

\[
\sigma^{xy} = \sigma_{xy} \approx \frac{1}{2} (\Gamma^y_{x0} + \Gamma^x_{y0}) \approx \frac{1}{2} \partial_t h_{xy}
\]

and for the energy-momentum tensor

\[
\delta T^{xy} \approx -p h_{xy} - \eta \partial_t h_{xy} + \eta \tau_\pi \partial^2_t h_{xy} - \frac{\kappa}{2} [\partial^2_t h_{xy} + \partial^2_z h_{xy}]
\]

Fourier transform \( h(t, z) = \exp (-i\omega t + ikz) \ h(\omega, k), \ \text{etc.} \)

\[
\delta T^{xy}(\omega, k) = -G_{R}^{xy,xy}(\omega, k) h_{xy}(\omega, k)
\]

via linear response
retarded Green function (in momentum space):

\[ G^{xy,xy}_R(\omega, k) = p - i\eta \omega + \eta\tau\pi\omega^2 - \frac{k}{2}(\omega^2 + k^2) + ... \]

e.g. Kubo formula

\[ \eta = \lim_{\omega \to 0} \frac{1}{\omega} iG^{xy,xy}_R(\omega, \vec{0}) \big|_{\omega=0} \]

\[ = \lim_{\omega \to 0} \frac{1}{\omega} \int dt d^3x \exp(i\omega t) \Theta(t) < [T_{xy}(t, \vec{x}), T_{xy}(0, \vec{0})] > \big|_{\omega=0} \]
classical action $S^{\text{classical}}[\Phi_0]$ from the solution $\Phi_0$ for $h_{xy}(\omega, k, r)$ at the boundary $r \to \infty$ and keeping only the surface contribution:

$$
G_{R}^{xy,xy}(\omega, k) = \frac{\pi N_c^2 T^3}{8} \left[ \pi T - i\omega + \frac{1 - 2 \ln 2}{2\pi T} \omega^2 - \frac{1}{2\pi T} k^2 \pm \ldots \right],
$$

$p = \frac{\pi^2 N_c^2 T^4}{8}$, $\eta = \frac{\pi N_c^2 T^3}{8}$, $\tau_\pi = \frac{2 - \ln 2}{2\pi T}$, $\kappa = \frac{\eta}{\pi T}$.
sound mode

consistency requirement from sound channel:

\[ \omega_{1,2} = \pm c_s k - i\Gamma k^2 \pm \frac{\Gamma}{c_s}(c_s^2 \tau \pi - \frac{\Gamma}{2}) k^3 + \mathcal{O}(k^4) \]

with \( \Gamma = \frac{2}{3} \frac{\eta}{\epsilon + p} \) and

gravity perturbation \( g_{tz} = h_{tz} \ll 1 \) [i.e. poles in \( G^{tz,tz}_{R}(\omega, k) \)]

\[ \omega_{1,2} = \pm \frac{1}{\sqrt{3}} k - i \frac{1}{6\pi T} k^2 \pm \frac{3 - 2\ln 2}{24\sqrt{3}\pi^2 T^2} k^3 \]

gives sound velocity \( c_s = \frac{1}{\sqrt{3}} \) and consistent values for

\[ \frac{\eta}{s} = \frac{1}{4\pi}, \quad \tau \pi = \frac{2 - \ln 2}{2\pi T} \]
RHIC physics
hydro + RHIC

heavy-ion collisions require beyond hydrodynamics:

- hydrodynamics = differential equations
  initial conditions!
- initial = equilibration time
- distribution of energy density (Glauber? CGC?)
- QCD equation of state
- hadronisation prescription (Cooper-Frye?)
- .......

Conformal viscous hydrodynamics – p.41
**main results [Luzum and Romatschke 08]**

- Viscous hydrodynamics simulation give a good description of RHIC data for
  \[
  \frac{\eta}{s} = 0.1 \pm 0.1 (\text{theory}) \pm 0.08 (\text{experiment})
  \]

- Modest estimate:
  \[
  \frac{\eta}{s} < 0.5
  \]

- Weak dependence on the values of the second-order parameters \( \tau_\pi, \lambda_1, \ldots \)

- Early thermalisation time is questioned, but
  \[
  \tau_0 < 2 fm
  \]
behaviour of $\eta/s$ as a function of the 't Hooft coupling $\lambda = gYM^2N_c$

$\frac{\eta}{s} \approx 0.5$: border-value between sQGP and pQPG
elliptic flow: $\eta/s > 0$ reduces $v_2$

[Luzum and Romatschke 08]
excitement about gauge/gravity correspondence: mainly to gain consistent information about STRONG COUPLING by analytic methods

ASKING FOR MORE:

Is there an experiment whose outcome could cast strong doubts on the relevance of AdS/CFT to understand QCD?

[P. Jacobs 08]

JET PHYSICS? - NO jets in $\mathcal{N} = 4$ SYM!

[Hatta, Iancu and Mueller 07 - 08]

MOST CHALLENGING TASK of the theory is to find the proper microscopic mechanism for the rather RAPID EQUILIBRATION of matter in RHIC collisions

[Heller, Janik and Peschanski 08; Chesler and Yaffe 08; ...]
EXTRAS
some classical gravity

Einstein-Hilbert with nonzero cosmological constant
\(d = 5\) dimensions

\[ R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\Lambda_{(5)} g_{\mu\nu}, \quad \mu, \nu = 0, 1, \ldots, 4 \]

ansatz:

\[ ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + \ldots, \quad g_{tt} = -\frac{1}{g_{rr}} = -\frac{r^2}{L^2} f(r) \]

\[ R(1 - \frac{5}{2}) = -5\Lambda_{(5)}, \quad R_{\mu\nu} = \frac{2\Lambda_{(5)}}{3} g_{\mu\nu} = -\frac{4}{L^2} g_{\mu\nu} \]
solution

\[ R_t^t = -\frac{1}{2} \left[ r^5 f' + 2r^4 f \right]' = -\frac{4}{L^2} \]

\[ r f' + 2f = 2 + \frac{const}{r^4}, \quad const = 2r_0^4 \]

\[ f(r) = 1 - \left( \frac{r_0}{r} \right)^4 \]

i.e.

\[ ds^2 = -\frac{r^2}{L^2} f(r) dt^2 + \frac{L^2}{r^2 f(r)} dr^2 + \ldots \]
Hawking temperature

black hole metric:

\[ ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + \ldots, \quad g_{tt} = -\frac{1}{g_{rr}} = -\left[1 - \left(\frac{r_0}{r}\right)^{(d-3)}\right] \]

expand around the horizon \( r \approx r_0 \):

\[ g_{tt} \approx -\frac{d - 3}{r_0} (r - r_0) = -\gamma_t (r - r_0), \quad g_{rr} \approx \frac{\gamma_r}{r - r_0}, \quad \gamma_r = \frac{r_0}{d - 3} \]

claim:

\[ T_H = \frac{1}{4\pi} \sqrt{\frac{\gamma_t}{\gamma_r}} = \frac{d - 3}{4\pi r_0} \]

and \( \text{AdS}_5 \):

\[ \gamma_t = 1/\gamma_r = 4r_0/R^2 \]

\[ T_H = \frac{r_0}{\pi R^2} \]
proof: with \( \rho^2 = \frac{4r_0(r-r_0)}{d-3} \)

\[
d\rho^2 = \rho^2 \left(\frac{d-3}{2r_0}\right)^2 d\tau^2 + d\rho^2 + ..., \quad \text{euclidean } d\tau^2 = -d\tau^2
\]

\[
\rightarrow \quad ds^2 = \rho^2 d\phi^2 + d\rho^2, \quad \phi = \frac{d-3}{2r_0} \tau
\]

**requirement:** periodicity \( \phi \rightarrow \phi + 2\pi \), i.e. NO conical singularity

\[
2\pi = \frac{(d-3)}{2r_0} \beta, \quad \beta = \frac{1}{T_H}
\]
quasinormal mode: some details

$G_R$ from gravity action with $AdS_5$ metric:

$$S_{(5d)} = \frac{N_c^2}{8\pi^2 L^3} \int d^5x \left[ \sqrt{-g} R_{(5d)} + \ldots \right] ,$$

weak field limit: scalar mode and EoM (of Heun’s type)

$$\sqrt{-g} R_{(5d)} \to -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi , \quad \Phi \equiv h_{xy} :$$

$$\Phi(t, z, u = \frac{r_0^2}{r^2}) = \exp(-i\omega t + ikz)\Phi_k(u) , \quad \Phi'_k = \frac{d}{du} \Phi_k ,$$

$$\Phi''_k - \frac{1 + u^2}{u(1 - u^2)} \Phi'_k + \frac{1}{4\pi^2 T^2} \frac{\omega^2 - k^2(1 - u^2)}{u(1 - u^2)^2} \Phi_k(u) = 0$$
solution with ansatz \( \Phi_k(u) = f_k(u)\Phi_0(k) \), \( f_k(0) = 1 \) and boundary condition: only the incoming wave, the one which moves toward the horizon at \( r \to r_0 \equiv u = 1 \), i.e. nothing comes out the horizon

\[
f_k(u) = (1 - u^2)^{-i\omega/(4\pi T)} F_k(u) \approx 1 + i \frac{\omega}{4\pi T} u^2 + \ldots
\]

note, near \( u \sim 1 \):

\[
\exp (-i\omega t)f_k(u) \to \exp \left[-i\omega(t + r^*)\right], \quad r^* = \frac{\ln(1-u)}{4\pi T}
\]

moves from \( u = 0 \) at \( t = 0 \) to \( u = 1 \) at \( t \to \infty \)
classical gravity action

inserting the solution into $S_{(5d)}$

$$S_{(5d)} \approx \frac{N_c^2}{8\pi^2 L^3} \int d^4x \int_0^1 du \sqrt{-g(-\frac{1}{2})} g^{uu} \partial_u \Phi \partial_u \Phi$$

and keeping only the surface contribution at $u = 0$  \Rightarrow

$$S^{\text{classical}} = -\frac{\pi^2 N_c^2 T^4}{8} \int \frac{d^4k}{(2\pi)^4} \Phi_0(-k) \left[ \frac{1 - u^2}{u} f'_k(u) f_k(u) \right] \bigg|_{u=0} \Phi_0(k)$$

and finally (including a counter term)  \Rightarrow

$$G_{R}^{xy,xy}(\omega, k)$$
Bjorken flow

boost-invariant (irrotational) $1 + 1$ flow [Bjorken 83]
second-order equations (proper time $\tau$, $\Phi$... viscous flow):

$$\partial_\tau \epsilon = -\frac{4\epsilon}{3\tau} + \frac{\Phi}{\tau}$$

$$\tau_\pi \partial_\tau \Phi = \frac{4\eta}{3\tau} - \Phi - \frac{4\tau_\pi}{3\tau} \Phi - \frac{\lambda_1}{2\eta^2} \Phi^2$$

non-linear term NOT in MIS theory! [BRSSS 07]

compare with AdS/CFT calculation:

$$\frac{\lambda_1 T}{\eta} = \frac{1}{2\pi} \left[ 1 + \frac{215}{8} \zeta(3) \lambda^{-3/2} + ... \right]$$

[Janik, Peschanski, Heller 06; Buchel, Paulos 08]
**COMPARISON**

<table>
<thead>
<tr>
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<th>QCD</th>
<th>$\mathcal{N}=4$ SYM</th>
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<tbody>
<tr>
<td>$T=0$</td>
<td>$N_c=3=N_f$, confinement, discrete spectrum, scattering, . . .</td>
<td>$N_c$ large, $N_f/N_c$ small, deconfined, conformal, supersymmetric, . . .</td>
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<td></td>
<td><strong>very different !!</strong></td>
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<tr>
<td>$T&gt;T_c$</td>
<td>strongly-coupled plasma of gluons &amp; <strong>fundamental</strong> matter</td>
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<td></td>
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<tr>
<td>$T&gt;&gt;T_c$</td>
<td>runs to weak coupling</td>
<td>remains strongly-coupled</td>
</tr>
<tr>
<td></td>
<td><strong>very different !!</strong></td>
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</tbody>
</table>

QCD and $\mathcal{N} = 4$ SYM as a function of temperature

[from Myers and Vazquez 08]
keeping essentially one term in the derivative expansion up to second order

\[ \Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} + 2\eta \tau_\pi \langle D\sigma^{\mu\nu} \rangle, \quad D = u \cdot \nabla \]

remark: does not match with AdS/CFT \( \mathcal{N} = 4 \) SYM

\[
\text{sound: } \tau_\pi = \frac{2 - \ln 2}{2\pi T} \\
\text{Bjorken flow: } \tau_\pi = \frac{1 - \ln 2}{2\pi T}
\]

⇒ all second-order terms consistent with conformal symmetry have to be included!
entropy current

Israel - Stewart 79: \[ s^\mu = (s - \frac{\tau \Pi_{\alpha\beta} \Pi^{\alpha\beta}}{4\eta T}) u^\mu \]

\[ \nabla_\mu s^\mu = \frac{\Pi_{\alpha\beta} \Pi^{\alpha\beta}}{2\eta T} \geq 0 \]

⇒ second law:

instead in causal viscous and conformal hydrodynamics

a more general current in terms of \( u^\mu \) and its derivatives:

\[ s^\mu = su^\mu + (\# \sigma^2 + \# \Omega^2) u^\mu + O(u \nabla^2 u) \]

with

\[ \nabla_\mu s^\mu = \frac{\eta}{2T} \sigma^{\mu\nu} \sigma_{\mu\nu} + \frac{1}{4T} (\kappa - 2\lambda_1) \sigma^{\mu}_{\nu} \sigma^{\nu}_{\lambda} \sigma^\lambda_{\mu} \]

in \( N = 4 \) SYM: \( \kappa = 2\lambda_1 \)

[Loganayagam 08]

Conformal viscous hydrodynamics – p.57
as important example

\[ \frac{dN}{dy dp_\perp d\phi} = \langle \frac{dN}{dy dp_\perp d\phi} \rangle_\phi (1 + 2v_2(p_\perp) \cos(2\phi) + \ldots) \]

- elliptic flow: \( v_2(p_\perp) \)
Glauber

$\eta/s=10^{-4}$
$\eta/s=0.08$
$\eta/s=0.16$

elliptic flow

[LUZUM AND ROMATSCHKE 08]
equilibration time

REMINDER:

claim of short \( \tau_0 \equiv \tau_{eq} \leq 0.5 \text{ fm at RHIC} \)

OPEN QUESTION:

\[ CGC \ (\alpha_s << 1) \rightarrow sQGP \ (\alpha_s > O(1)) \]

within a very short time \(< 0.5 \text{ fm} \)?
parametric pQCD estimate

for thermalisation in an expanding gluonic medium near equilibrium at $T(\tau)$: Knudsen number $Kn$

$$\frac{1}{Kn} = \frac{\text{longitudinal expansion time}}{\text{mean free path}} = \frac{\tau}{\lambda_f} \left( \sim \frac{1}{\eta s} \right) \gg 1$$

from many gluon interactions (including saturation):

Arnold et al.: $\tau_{eq} Q_s \geq \alpha_s^{-7/3}$

‘bottom-up’ [Baier, Son, Mueller and Schiff 01]

$\tau_{eq} Q_s \geq \alpha_s^{-13/5}$

**RHIC:** $\tau_{eq} \geq 2 - 3 \text{ fm}$