Shear viscosity in strongly coupled quantum systems

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A. Jakovac, arXiv:0901.2802
1. Introduction
   - Gas vs. fluid
   - Calculation of $\eta/s$
   - Status of the lower bound

2. $\eta/s$ at low temperatures
   - Generic formulae
   - Quasiparticle and non-quasiparticle systems
   - Realistic strongly coupled systems
   - Behavior of $\eta/s$ in strongly coupled systems

3. Strongly coupled Yang-Mills theories
   - Euclidean formulation
   - Strong coupling expansion
   - Viscosity and entropy at finite temperature

4. Conclusions
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Gas and fluid

What is the difference:

- Usually gases are rarer: but what if the (energy) density is the same?
- Mean free path ($\ell$) is much smaller in liquid phase!

\[ \ell \rightarrow \infty : \text{ideal gas} \quad \ell \rightarrow 0 : \text{perfect fluid} \]

- in ideal gas the particles, in perfect fluid the streamlines are independent
- Cross sections are much larger in fluids: $\sigma n \ell \sim 1$ ($\sigma$ cross section, $n$ density) \[ \Rightarrow \ell \rightarrow 0 : \sigma \rightarrow \infty, \ell \rightarrow \infty : \sigma \rightarrow 0. \]

Fluids: strongly interacting matter, gas: weakly interacting matter.
**Transport**

**Dynamical appearance:** Information propagation (transport) is much slower.

Consider for example layers with different flow velocity. Assume diffusion equation for momentum equilibration

\[
\rho \frac{\delta v}{\tau} \sim \eta \frac{\delta v}{\ell^2} \quad \Rightarrow \quad \eta \sim \rho u \ell \sim \epsilon \tau \sim \frac{mu}{\sigma},
\]

**Estimate \( \eta \):** insert typical lifetime \((\tau)\), path length \((\ell)\), velocity \((u = \ell/\tau)\); \(\sigma\) cross section, \(\epsilon\) energy density, \(m\) particle mass

Ideal fluid: \(\eta = 0\), ideal gas \(\eta = \infty\).
Relaxation in fluids

Damping rate of small perturbations: $\Gamma = \frac{4k^2 \eta}{3T} s$

Typical values of $\eta/s$:
- water at room temperature $\sim 30$
- superfluid $^4$He at lambda-point $\sim 0.8$
- smallest at the phase transition point

Non-central heavy ion collisions have initial anisotropy. Time evolution of anisotropy: the larger the viscosity, the more extent the initial anisotropy is washed out

⇒ flow seen by RHIC is very close to ideal hydro

\[ \frac{\eta}{s} \lesssim 0.16 \]
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Kubo formula

Linear response theory for conserved currents $\Rightarrow$ Kubo formula

Define

$$C(x) = \frac{1}{10} \left< \left[ \pi_{ij}(x), \pi_{ij}(0) \right] \right>,$$

$$\pi_{ij} = T_{ij} - \frac{1}{3} T_{i}^{i},$$

with it

$$\eta = \lim_{\omega \to 0} \frac{C(\omega, k = 0)}{\omega}.$$
Perturbation theory

- Perturbation theory is expansion around the ideal gas $\eta = \infty$ 
  \[ \Rightarrow \] small correction still yield large viscosity
- we expect $\sigma \sim g^4$  \[ \Rightarrow \] $\eta/s \sim 1/g^4$
- more precise calculation needs resummation of ladder diagrams (cf. P.Arnold, G.D.Moore, L.G.Yaffe, JHEP 0305, 051 (2003))

\[ \frac{\eta}{s} = \frac{\kappa}{g^4 \ln 1/g} \]

- taking into account higher order processes may lower the $\eta/s$ ratio (Z. Xu, C. Greiner, H. Stoecker, Phys.Rev.Lett.101:082302,2008.)

importance of bremsstrahlung, LPM effect

Weakly interacting quasiparticles with QCD symmetries describe gas

To explain experiments we have give up some of these features.
Calculation of $\eta/s$

MC studies

... give up only weak interactions

- measure $\langle T_{12}(x) T_{12}(0) \rangle$ correlator on lattice $\Rightarrow$ Euclidean discrete time

- we need the spectral function, which is related to the correlator as
  \[
  \int d^3 x \langle T_{12}(\tau, x) T_{12}(0) \rangle = \int_0^\infty \frac{d\omega}{\pi} \frac{C(\omega, k = 0)}{\sinh \beta \tau/2} \frac{\cosh(\omega(\beta/2 - \tau))}{\cosh(\omega(\beta/2 - \tau))}
  \]

- invert this relation with the prior knowledge $C(\omega > 0) > 0$
  Maximal Entropy Method, or ad hoc solutions

- too little sensitivity to small $\omega$ regime $\Rightarrow$ large systematical uncertainties; additional assumptions are needed

- best estimates $\eta/s = 0.102 (56)$ at $T = 1.24 T_c$


$\Rightarrow$ needs analytic control!
AdS/CFT methods

...strong interaction & non-QCD symmetries

- for $\mathcal{N} = 4$ SYM theory at $N_c \gg 1$, $\lambda = g^2 N_c \gg 1$ from graviton absorption in the dual theory:

$$\frac{\eta}{s} = \frac{1}{4\pi}$$


- for weaker coupling we expect to increase the ratio:

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + \frac{15\zeta(3)}{\lambda^{3/2}} + \frac{5\lambda^{1/2}}{16N^2} + \ldots \right] > \frac{1}{4\pi}$$

(R.C. Myers, M.F. Paulos, A. Sinha, arXiv:0806.2156)


- QCD is not $\mathcal{N} = 4$ SYM theory! For other theories supplementary argumentations are needed: lower bound.
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Status of the lower bound

Generic arguments


$$\eta \sim \epsilon \tau, \quad s \sim n \quad \Rightarrow \quad \frac{\eta}{s} \sim E_{\text{part}} \tau$$

$\epsilon$: energy density, $\tau$: lifetime, $n$: particle density, $E_{\text{part}}$: particle energy

From uncertainty relation:

$$\frac{\eta}{s} \geq \hbar$$

Coefficient can be found from gravity dual models

$$\frac{\eta}{s} \geq \frac{1}{4\pi}, \quad \text{in } \frac{\hbar}{k_B} \text{ units.}$$

For known matters the bound is respected (RHIC quark matter near the bound; water: $380 \times$, liquid He: $9 \times$ above bound)

At the bound tight packing: mean free path $\sim$ de Broglie wavelength!
Caveats: gravity side

- in higher derivative AdS gravity:

\[ S = \int d^D x \sqrt{-g} \left[ \frac{1}{2\kappa} R - \Lambda + cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right] \Rightarrow 4\pi \frac{\eta}{s} = 1 - \#c. \]


\( \eta/s < 1/(4\pi) \) is possible, even \( \eta/s = 0 \) can be achieved.


- since then also in other models violation of the bound is found


caveats: particle physics

- counterexample (T.D. Cohen, Phys.Rev.Lett.99:021602,2007.): nonrelativistic gas with lot of species with the same interaction: dynamics is unchanged, but entropy is enhanced by Gibbs mixing entropy $\eta/s$ can approach zero!

- Controversal: metastability – are $\eta$ and $s$ both sensible quantities on the same physical size? (D T. Son, Phys.Rev.Lett.100:029101,2008)

- same effect can be obtained if the excitations are in the same quantum channel (A. Cherman, T.D. Cohen, P.M. Hohler, JHEP0802:026,2008)

Question is not settled, further investigations are going on
New directions

- **new phenomena in QFT**: KSS heuristic argumentation is based on quasiparticle picture, but spectral functions (density of states) always have a continuous part.

  continuum $\Leftrightarrow$ “infinitely many” species in Cohen’s construction, without metastability.

- **strong coupling results** were known only for the conformal SUSY case – how well can it describe real QCD or Yang-Mills theories?

  $\Rightarrow$ strong coupling expansion
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## Conclusions
Study the effect of continuous spectrum: consider a generic theory where the **lowest lying states** are represented by a spectral function (density of states, DoS):

\[
\sum_n \langle n | \ldots | n \rangle = \sum_Q \int \frac{d^4 p}{(2\pi)^4} \varrho_Q(p) \langle p, Q | \ldots | p, Q \rangle,
\]

where \( Q \) denotes the quantum channel, \( p = (p_0, \mathbf{p}) \) is the total energy-momentum of the state.

- QM-based description
  - use volume normalization to calculate densities
- effects of interaction
  - continuous DoS
  - temperature dependent DoS
Transport coefficients

Consider \( C_J(x) = \langle [J_i(x), J^i(0)] \rangle \) correlator with some current \( J \).

Inserting a complete basis of energy-momentum eigenstates

\[
C_J(x) = \frac{1}{Z} \sum_{n,m} \left[ \text{Tr} \langle n | e^{-\beta H} J(x) | m \rangle \langle m | J(0) | n \rangle - \{x \leftrightarrow 0\} \right]
\]

- \( J(x) = e^{iP_x} A(0) e^{-iP_x} \)
- Fourier transform

\[
|\langle q, Q| J| q, P \rangle|^2 = q^2 A_{QP}, \text{ since it is a current.}
\]

In case of viscosity: gradient of the local momentum

- \( \eta_J = \lim_{\omega \to 0} C(\omega, k = 0)/\omega \)

results in:

\[
\eta_J = \frac{\beta}{Z} \sum_{PQ} \int \frac{d^4 q}{(2\pi)^4} \varrho_Q(q) \varrho_P(q) q^2 e^{-\beta q_0} A_{QP}.
\]
Small temperature limit

assume: only one quantum channel contributes, and $A_{QP} \approx A$ constant.

\[
Z - 1 = \sum_{|n\rangle \neq |0\rangle} \langle n | e^{-\beta H} | n \rangle = \sum_{Q \neq 0} \int \frac{d^4 q}{(2\pi)^4} \varrho_Q(q) e^{-\beta q_0}.
\]

$\Rightarrow$ $Z \approx 1$ can be used for $\eta_J$, and the free energy density

\[
f = -T \int \frac{d^4 q}{(2\pi)^4} \varrho(q) e^{-\beta q_0} \quad \Rightarrow \quad s = -\frac{\partial f}{\partial T}
\]

Thus the eta/s ratio reads

\[
\frac{\eta_J}{s} = \frac{A_{\beta} \int \frac{d^4 q}{(2\pi)^4} \varrho^2(q) q^2 e^{-\beta q_0}}{\frac{\partial}{\partial T} T \int \frac{d^4 q}{(2\pi)^4} \varrho(q) e^{-\beta q_0}}.
\]
Structure of the ratio

The rough structure of $\eta/s$ is

$$\frac{\eta J}{s} \sim \frac{\int_V \varrho^2}{\int_V \varrho}, \quad V = \text{relevant region}$$

- large peak in $\varrho$ $\Rightarrow$ $\varrho^2$ even larger $\Rightarrow$ $\eta/s$ large
- shallow $\varrho$ $\Rightarrow$ $\varrho^2$ even shallower $\Rightarrow$ $\eta/s$ small

in non-quasiparticle (large width) systems

$\eta/s$ is naturally small!

... irrespective on the strength of interactions.
Assume that the lowest lying states can be approximated with Breit-Wigner form:

\[ \varrho(q) = \frac{2\Gamma}{(q_0 - \varepsilon_q)^2 + \Gamma^2}. \]

Assume relativistic dispersion relation \( \varepsilon^2 = m^2 + q^2 \).

In the small width limit \( \varrho(q)^2 \approx \frac{2}{\pi} \frac{2\pi\delta(q_0 - \varepsilon_q)}{2\pi\delta(q_0 - \varepsilon_q)}. \)

Integrations can be exactly performed:

\[ f = -\frac{T^4}{2\pi^2}(\beta m)^2 K_2(\beta m), \quad \eta_J = \frac{3Am^3 T}{\Gamma\pi^2} K_3(\beta m). \]
Quasiparticle case

Both in $m = 0$ and in $T \ll m$ limit the ratio is the same:

$$\frac{\eta J}{s} = 6A \frac{T}{\Gamma}.$$ 

- $T \sim$ particle energy, $\Gamma^{-1} \sim$ lifetime $\Rightarrow E_T$
- Boltzmann-eq. result with 2-2 scattering
- value of $\Gamma$
  - in conformal case $\Gamma \sim T \Rightarrow \frac{\eta J}{s} \sim$ constant
  - lower limit may come from infinite coupling, $1/4\pi$.
  - massive case: at $T = 0 \Gamma = 0$, at finite $T$ the width is the consequence of scattering on thermal particles
  $\Rightarrow$ abundance is $e^{-M/T}$ ($M$: energy of scattering state)
  $\Rightarrow \frac{\eta J}{s} \sim Te^{M/T} \rightarrow \infty.$
\( \rho^\# e^{-\beta q_0} \) enhances the lowest lying states \( \Rightarrow \) in this case it is not a peak, but a continuum above a threshold \( M \). Keeping only the first terms in \( q - M \) expansion:

\[
\rho(q) = C \Theta(q - M)(q - M)^w, \quad q = \sqrt{q^2}.
\]

\( C \) is dimensionfull: \([C] = [E]^{-(1+w)}\)

\( \eta_J \sim C^2 \) and \( f \sim C \) \( \Rightarrow \) \( C \) remains in the ratio.

After evaluation \( M \) turns out to be canceled from the ratio:

\[
\frac{\eta_J}{s} \sim C T^{1+w} \quad T \to 0 \quad 0
\]

for an integrable threshold \((w > -1)\)
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Simplest interacting field theories at zero temperature has a spectral function (DoS) like

- the Dirac-delta at $m$ represents a stable particle
- the continuum at $m + m'$ represents a multiparticle state.
- infinite lifetime $\Rightarrow$ gas

What could spoil this simple picture?
Finite temperature effects

- at finite temperature $\varrho \neq 0$ for all energies $\Rightarrow$ a (would-be) quasiparticle acquires thermal width
- high temperature strong coupling system: thermal width $\sim g^2 T$, thermal mass $\sim gT$ $\Rightarrow$ for $g \geq 1$ non-quasiparticle excitations
If there are in the system zero mass particles, then \( m' = 0 \)
\[ \Rightarrow \text{ cut and the delta-peak melt together} \]

1-loop threshold behavior linear
\[ \Rightarrow \rho \sim 1/(E - E_{\text{thr}}) \]
If there are in the system zero mass particles, then \( m' = 0 \)
\[ \Rightarrow \] cut and the delta-peak melt together

1-loop threshold behavior linear \( \Rightarrow \) \( \varrho \sim \frac{1}{(E - E_{\text{thr}})} \)

This is not normalizable! \( \Rightarrow \) IR divergences near the threshold, which smear out the \( 1/x \) singularity
Zero mass excitations

- Interpretation: no single charged particle (electron, quark), it is always surrounded by soft gauge bosons
Zero mass excitations

- Interpretation: no single charged particle (electron, quark), it is always surrounded by soft gauge bosons
- In QCD: from fitting to MC pressure data one obtains similar distribution of quasiparticle masses

Strongly coupled systems

$g \to \infty$, and assume that the system converges to somewhere.

- The coupling constant “suppressed” structures will be more and more enhanced.
- The cuts starting at $n \times m$ is enhanced by $g^{2n} \Rightarrow$ in case of convergence $n \times m$ is finite $\Rightarrow m$ is renormalized to 0.
- Mass gap is a quantity of $\infty \times 0$, can be finite or zero.

As a result the spectral function will be similar to the zero mass systems although there may remain no zero mass asymptotic states any more.
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Relevant quantity: spectral function multiplied by $e^{-E/T}$

Notation: $\Delta m = E_{\text{peak}} - E_{\text{thr}}$
at $T \gg \Delta m$: small suppression
at $T \gg \Delta m$: small suppression

$\downarrow$

quasiparticle peak dominates

$\downarrow$

practically a quasiparticle system, KSS bound is respected
at $T \ll \Delta m$: large suppression
at $T \ll \Delta m$: large suppression

\[ \Downarrow \]

threshold dominates, not a quasiparticle system!!

small \eta/s, at $T = 0$ zero
Behavior of \( \eta / s \) in strongly coupled systems

Small temperature behavior

- at small temperatures non-quasiparticle behavior, \( \eta / s \to 0 \) as \( T \to 0 \).
- at larger temperatures quasiparticle behavior, \( \eta / s \sim T / \Gamma \) decresing with increasing \( T \).
- at even larger temperatures \( T / \Gamma \sim 1 / g^2 \), grows in asymptotically free theories.
- The turning point depends on the system!
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We want to calculate Euclidean time the quantity

$$\mathcal{M}_{\mu\nu\varrho\sigma}(x) = \langle T\left(T_{\mu\nu}(x)T_{\varrho\sigma}(0)\right)\rangle,$$

which has the properties $\mathcal{M}_{\mu\nu\varrho\sigma} = \mathcal{M}_{\nu\mu\varrho\sigma} = \mathcal{M}_{\mu\nu\sigma\varrho} = \mathcal{M}_{\varrho\sigma\mu\nu}$.

We are interested in the index combination

$$\mathcal{M}(x) = \frac{1}{10} \sum_{ij} \langle T\left(\pi_{ij}(x)\pi_{ij}(0)\right)\rangle = \frac{1}{10} \sum_{ij} \left[M_{ij\,ij} - \frac{1}{3}M_{ii\,jj}\right].$$

We need $\pi_{ij}$ spectral function:

$$\mathcal{M}(-ik_0 + \varepsilon, k) = i\mathcal{M}^{\text{ret}}(k_0, k), \quad C(k) = -2 \text{Im} i\mathcal{M}^{\text{ret}}(k_0 + i\varepsilon, k).$$
Lattice discretization

To determine $M(x)$ we use lattice!

Discretize the spacetime to a lattice with lattice spacing $a$. Yang-Mills theories in Wilson representation

$$Z = \int \mathcal{D}U e^{-S}, \quad S = \sum_p S_p, \quad S_p = \beta \left( 1 - \frac{1}{N} \text{Re} \text{ Tr} \ U_p \right),$$

where $\beta = 2N/g_0^2$, $p$ means plaquettes,

$U_p(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\nu^\dagger(x + a\hat{\nu}) U_\mu^\dagger(x)$ plaquette variables, $U_\mu(x) \in \text{SU}(N)$ link gauge variables $U_\mu(x) = \mathcal{P} e^{-i \int_0^a ds A_\mu(x+s\hat{\mu})}$.

Diagonal energy-momentum tensor elements can be expressed through plaquettes:

$$T_{\mu\mu} = \frac{\beta}{N} \left[ - \sum_{\nu \neq \mu} P_{\mu\nu} + \sum_{\sigma,\nu \neq \mu; \sigma > \nu} P_{\sigma\nu} \right],$$

$\Rightarrow$ we have to calculate plaquette-plaquette correlators (F. Karsch, et al., Phys. Rev. D 35, 2518 (1987)).
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For small $\beta$ values (large $g_0^2$), we expand $e^{-S_P}$ in powers of $\beta$. For computational purposes it is better to write

$$e^{-S_P} = c(\beta) \left( 1 + \sum_{R \neq 0} d_R a_R(\beta) \chi_R(U_P) \right),$$

where $\chi_R$ is the characters of the irreducible representation $R$ of $SU(N)$, $d_R$ is the dimension of the representation and $c(\beta)$ and $a_R(\beta)$ are some coefficients. At leading order in $\beta$ expansion

$$c(\beta) = 1, \quad a_f(\beta) = \begin{cases} \frac{\beta}{N^2} & N = 2 \\ \frac{\beta}{2N^2} & N > 2 \end{cases}.$$ 

$f$ is the fundamental representation.
Known positive properties of the strong coupling expansion:

- describes confinement, chiral symmetry breaking; bound state spectrum and masses can be computed fairly well
  (a recent calculation: J. Langelage, G. Munster, O. Philipsen, JHEP 0807:036, 2008.)
- perturbation series converge

Known negative properties of the strong coupling expansion:

- difficulty in reaching continuum limit: asymptotic freedom requires $g_0 \to 0$, strong coupling
- towards continuum limit: roughening phase transition; infinite order phase transition for Wilson loops between area and perimeter law.
- renormalization? results not $a$-independent.

Our strategy: identify robust results ($a$-independent); for $a$-dependent parts try to connect to physical observables.
Strong coupling expansion leading order

To leading orders:
- draw closed surfaces
- assign a representation to each surface: to leading order the fundamental representation \( \sim \beta \)
- on-axis plaquette-plaquette correlator, leading order

\[
G_{\text{straight}}^E (\ell) \sim a_f^{4n} = e^{-\sigma \ell}, \quad \sigma = -\frac{4}{a} \ln a_f (\beta).
\]

\( \ell = na \) physical length

April 23, 2009. Frankfurt
**Open tube contribution**

In Fourier space it reads

\[ G^E(k) = \frac{\Delta}{(k^2/\sigma^2 + 1)^{5/2}}, \quad \Delta = \frac{12\pi^2}{(a\sigma)^4} \]

like “anomalous dimension” \( \eta = 3 \) (but it is not valid in UV)

**Corrections**

- not straight (zigg-zagging) lines: important, see later
- local corrections to the tube: self-energy, glueball mass correction
- roughening of the tube surface \( \Rightarrow \) this is relevant for the restauration of the free propagator at small coupling, large \( \beta \).
Almost straight lines are just a little longer than straight lines, but there is a large number of such surfaces \(\Rightarrow\) can compensate the small suppression. Contribution of zig-zagging lines with \(m\) breaking point is an \(m\)-fold convolution; in Fourier space \((G^E(k))^n\). All zig-zagging lines:

\[
G^E_{\text{zig-zag}}(k) = \frac{1}{G^E_{\text{straight}}(k) - 1}
\]

\(\Rightarrow\) like self-energy expression: junction contribution is 1.
\( G^E \rightarrow G^\text{ret} \rightarrow \varrho \) with analytic methods:

\[
\varrho(k) = \text{sgn}(k_0) \Theta(k^2 - \sigma^2) \frac{2 \Delta (k^2/\sigma^2 - 1)^{5/2}}{\Delta^2 + (k^2/\sigma^2 - 1)^5}
\]
mass gap until $\sigma$, then threshold behavior and a peak
zig-zagging influences the vicinity of the threshold
because mass gap $C(k_0 < \sigma) = 0 \implies$ zero temperature viscosity is zero $\implies$ superfluidity!
Thermodynamics requires $s \to 0$ as $T \to 0 \implies$ for not-superfluids $\eta/s \to \infty$ at $T \to 0$
Now we have $0/0$ contribution $\implies$ deeper investigations are needed!
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Viscosity at finite temperature

Filling the gap in spectral function for small momenta $\Rightarrow$ like Landau damping; leading contribution comes from a loop diagram, where one tube wraps around the imaginary axis, the other goes straight:

This results in (assuming $T \ll \sigma$)

$$\eta = \frac{\Delta^2 N^2 a^4}{2\pi^3 T} \int_0^\infty dk k^2 \int_0^{\infty} d\omega_k e^{-k_0/T} \varphi^2(k),$$

where $\omega_k^2 = k^2 + \sigma^2$. 

April 23, 2009. Frankfurt
The entropy

The entropy density $s = -\frac{\partial f}{\partial T}$, $f$ is the free energy density.

In strong coupling theory: closed surfaces.

The smallest surfaces (cube, double cube, etc.) do not feel the temperature $\Rightarrow$ yield zero entropy

First nontrivial surface which wraps around the temperature direction: we can express this contribution with help of the opened tube propagator

$$f_T = -6(aT) \left( G^E_T(0) - G^E_{T=0}(0) \right).$$

With algebraic manipulations and assuming $T \ll \sigma$ from here

$$s = \frac{3a^5}{\pi^3 T} \int_0^\infty dk k^2 \int_{\omega_k}^\infty dk_0 k_0 \, e^{-k_0/T} \bar{\varrho}(k).$$
The $\eta/s$ ratio

After simplifications

$$\frac{\eta}{s} = \frac{N^2}{2} \Delta^{9/4} f\left(\frac{2T}{\Delta^{2/5} \sigma}\right),$$

$$f(w) = \frac{1}{3(12\pi^2)^{1/4}} \int_0^\infty \frac{dz}{dz} e^{-z/w} \left(\frac{2z^{5/2}}{1 + z^5}\right)^2.$$

Asymptotic values can be computed analytically

$$f(w \ll 1) \sim T^{5/2}$$ and

$$f(w \gg 1) = 0.056.$$
Conclusions

- gas and fluid ⇔ weak and strong coupling
  ⇔ transport coefficients large and small

why do we observe small transport coefficients in QGP?

- one (standard) explanation: quasiparticles in conformal limit
  ⇒ dimensionless ratios are numbers, restricted by QM
  ⇒ there is (a small) lower bound for $\eta/s$

- alternative scenario: non-quasiparticles: for broad spectral functions the transport coefficients are naturally small
  ⇒ no lower bound for $\eta/s$

⇒ off-shell effects may be very important
Hydrodynamics

System with local collective flow: QM description?
Construction of the system: at \( t = -\infty \) equilibrium in rest 
(\( \bar{u} = (1, 0, 0, 0) \)), then a modified time evolution corresponding to 
the flow \( u - \) denote \( \Delta u = u - \bar{u} \):

\[
H^{(0)} = \bar{u}_\mu P^{(0)\mu} = u_\mu P^\mu = H + \Delta u_\mu T^{0\mu} \quad \Rightarrow \quad \delta H = - \int d^3 x \Delta u_\mu T^{0\mu}
\]

\[
\delta H = \int_{-\infty}^{t} dt' \partial_0 \delta H = - \int_{-\infty}^{t} dt' d^3 x \left[ \partial_0 u_\mu T^{0\mu} + \Delta u_\mu \partial_0 T^{0\mu} \right]
\]

With energy-momentum conservation \( \partial_0 T^{0\mu} = - \partial_i T^{i\mu} \) and 
partial integration

\[
\delta H = - \int_{-\infty}^{t} dt' d^3 x \partial_\mu u_\nu T^{\mu\nu}
\]
Linear response theory (→ the flowing and the original systems are not too far ⇒ nonrelativistic):
\[ \delta \langle X(t) \rangle = i \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' d^3x'' \langle [X(t), T^{\mu\nu}(x'')] \rangle \partial_{\mu} u_{\nu}. \]

Hydrodynamical approximation: \( \partial u \approx \text{const.} \) ⇒ \( \delta \langle X \rangle \) time independent.

Spatial rotational symmetry of the ground state ⇒
\[ \delta \langle \pi_{ij} \rangle \equiv \delta \langle T_{ij} - \frac{1}{3} \delta_{ij} T^k_k \rangle = \frac{\eta}{2} \left[ \partial_k v_\ell + \partial_\ell v_k - \frac{2}{3} \delta_{k\ell} \partial \nu \right], \]
the coefficient from above (denote \( C(x) = \langle [T_{12}(0), T_{12}(x)] \rangle \))
\[ \eta = i \int_{-\infty}^{0} dt \int_{-\infty}^{t} dt' d^3x' C(x') = \lim_{\omega \to 0} \frac{C(\omega, k = 0)}{\omega} \]

Kubo formula
Zero temperature limit of viscosity

The viscosity \( \eta \) and the entropy have a common form

\[
F_{n,m} = \frac{3a^5}{2\pi^3 T} \int \frac{d^4k}{(2\pi)^4} \Theta(k_0)\Theta(k^2 - \sigma^2)e^{-k_0/T}(ak_0)^n \varrho^m(k),
\]

since

\[
\eta = N^2 \Delta^2 F_{0,2}, \quad s = 2F_{1,1}.
\]

After reducing the integrals

\[
a^3 F_{n,m} = C(a\sigma)^n (a^2 \sigma T)^{3/2} e^{-\sigma/T} \int_0^\infty dz \, e^{-z} \left( \frac{2(wz)^{5/2}}{1 + (wz)^5} \right)^m,
\]

where \( w = 2\Delta^{-2/5} T/\sigma \) rescaled temperature.

BOTH \( \eta \) and \( s \) goes to zero at zero temperature