An Introduction to Quantum Field Theory

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Contents

When I became a student of Pomeranchuk in 1950 I heard from him a kind of joke that the Book of Physics had two volumes: vol.1 is "Pumps and Manometers", vol.2 is "Quantum Field Theory"

Lev Okun

0 Prologue

The development of Quantum Field Theory is surely one of the most important achievements in modern physics. Presently, all observational evidence points to the fact that Quantum Field Theory (QFT) provides a good description of all known elementary particles, as well as for particle physics beyond the Standard Model for energies ranging up to the Planck scale $\sim 10^{19}$ GeV, where quantum gravity is expected to set in and presumably requires a new and different description. Historically, Quantum Electrodynamics (QED) emerged as the prototype of modern QFT's. It was developed in the late 1940s and early 1950s chiefly by Feynman, Schwinger and Tomonaga, and is perhaps the most successful theory in physics: the anomalous magnetic dipole moment of the electron predicted by QED agrees with experiment with a stunning accuracy of one part in 10^{10}!

The scope of these lectures is to provide an introduction to the *formalism* of Quantum Field Theory, and as such is somewhat complementary to the other lectures of this school. It is natural to wonder why QFT is necessary, compelling us to go through a number of formal rather than physical considerations, accompanied by the inevitable algebra. However, thinking for a moment about the high precision experiments, with which we hope to detect physics beyond the Standard Model, it is clear that comparison between theory and experiment is only conclusive if the numbers produced by either side are "water-tight". On the theory side this requires a formalism for calculations, in which every step is justified and reproducible, irrespective of subjective intuition about the physics involved. In other words, QFT aims to provide the bridge from the building blocks of a theory to the evaluation of its predictions for experiments.

This program is best explained by restricting the discussion to the quantum theory of scalar fields. Furthermore, I shall use the Lagrangian formalism and canonical quantisation, thus leaving aside the quantisation approach via path integrals. Since the main motivation for these lectures is the discussion of the underlying formalism leading to the derivation of *Feynman rules*, the canonical approach is totally adequate. The physically relevant theories of QED, QCD and the electroweak model are covered in the lectures by Nick Evans, Sacha Davidson and Stefano Moretti.

The outline of these lecture notes is as follows: to put things into perspective, we shall review the Lagrangian formalism in classical mechanics, followed by a brief reminder of the basic principles of quantum mechanics in Section 1. Section 2 discusses the step from classical mechanics of non-relativistic point particle to a classical, relativistic theory for non-interacting scalar fields. There we will also derive the wave equation for free scalar fields, i.e. the Klein-Gordon equation. The quantisation of this field theory is done is Section 3, where also the relation of particles to the quantised fields will be elucidated. The more interesting case of interacting scalar fields is presented in Section 4: we shall

introduce the S-matrix and examine its relation with the Green's functions of the theory. Finally, in Section 5 the general method of perturbation theory is presented, which serves to compute the Green functions in terms of a power series in the coupling constant. Here, Wick's Theorem is of central importance in order to understand the derivation of Feynman rules.

1 Introduction

Let us begin this little review by considering the simplest possible system in classical mechanics, a single point particle of mass m in one dimension, whose coordinate and velocity are functions of time, $x(t)$ and $\dot{x}(t) = dx(t)/dt$, respectively. Let the particle be exposed to a time-independent potential $V(x)$. It's motion is then governed by Newton's law

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} = F(x), \tag{1.1}$$

where $F(x)$ is the force exerted on the particle. Solving this equation of motion involves two integrations, and hence two arbitrary integration constants to be fixed by initial conditions. Specifying, e.g., the position $x(t_0)$ and velocity $\dot{x}(t_0)$ of the particle at some initial time t_0 completely determines its motion: knowing the initial conditions and the equations of motion, we also know the evolution of the particle at all times (provided we can solve the equations of motion).

1.1 Lagrangian formalism in classical mechanics

The equation of motion in the form of Newton's law was originally formulated as an equality of two forces, based on the physical principle *actio = reactio*, i.e. the external force is balanced by the particle's inertia. The Lagrangian formalism allows to derive the same physics through a formal algorithm. It is formal, rather than physical, but as will become apparent throughout the lectures, it is an immensely useful tool allowing to treat all kinds of physical systems by the same methods.

To this end, we introduce the Lagrange function

$$L(x, \dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - V(x), \tag{1.2}$$

which is a function of coordinates and velocities, and given by the difference between the kinetic and potential energies of the particle. Next, the action functional is defined as

$$S = \int_{t_0}^{t_1} dt\, L(x, \dot{x}). \tag{1.3}$$

From these expressions the equations of motion can be derived by the *Principle of least Action*: consider small variations of the particle's trajectory, cf. Fig. 1,

$$x'(t) = x(t) + \delta x(t), \qquad \delta x/x \ll 1, \tag{1.4}$$

with its initial and end points fixed,

$$\left.\begin{array}{l} x'(t_1) = x(t_1) \\ x'(t_2) = x(t_2) \end{array}\right\} \Rightarrow \delta x(t_1) = \delta x(t_2) = 0. \tag{1.5}$$

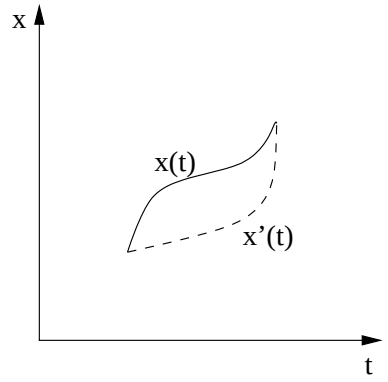


Figure 1: *Variation of particle trajectory with identified initial and end points.*

The true trajectory the particle will take is the one for which

$$\delta S = 0, \tag{1.6}$$

i.e. the action along $x(t)$ is stationary. In most systems of interest to us the stationary point is a minimum, hence the name of the principle, but there are exceptions as well (e.g. a pencil balanced on its tip). We can now work out the variation of the action by doing a Taylor expansion to leading order in the variation δx,

$$\begin{aligned} S + \delta S &= \int_{t_1}^{t_2} L(x + \delta x, \dot{x} + \delta\dot{x})\, dt, \qquad \delta\dot{x} = \frac{d}{dt}\delta x \\ &= \int_{t_1}^{t_2} \left\{ L(x, \dot{x}) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \ldots \right\} dt \\ &= S + \frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right\} \delta x\, dt, \end{aligned} \tag{1.7}$$

where we performed an integration by parts on the last term in the second line. The second and third term in the last line are the variation of the action, δS, under variations of the trajectory, δx. The second term vanishes because of the boundary conditions for the variation, and we are left with the third. Now the Principal of least Action demands $\delta S = 0$. For the remaining integral to vanish for arbitrary δx is only possible if the integrand vanishes, leaving us with the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0. \tag{1.8}$$

If we insert the Lagrangian of our point particle, Eq. (1.2), into the Euler-Lagrange equation we obtain

$$\begin{aligned} \frac{\partial L}{\partial x} &= -\frac{\partial V(x)}{\partial x} = F \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{d}{dt}\, m\dot{x} = m\ddot{x} \\ &\Rightarrow \quad m\ddot{x} = F = -\frac{\partial V}{\partial x} \quad \text{(Newton's law)}. \end{aligned} \tag{1.9}$$

Hence, we have derived the equation of motion by the Principal of least Action and found it to be equivalent to the Euler-Lagrange equation. The benefit is that the latter can be easily generalised to other systems in any number of dimensions, multi-particle systems, or systems with an infinite number of degrees of freedom, such as needed for field theory. For example, if we now consider our particle in the full three-dimensional Euclidean space, the Lagrangian depends on all coordinate components, $L(\mathbf{x}, \dot{\mathbf{x}})$, and all of them get varied independently in implementing Hamilton's principle. As a result one obtains Euler-Lagrange equations for the components,

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0. \tag{1.10}$$

In particular, the Lagrangian formalism makes symmetries and their physical consequences explicit and thus is a convenient tool when constructing different kinds of theories based on symmetries observed (or speculated to exist) in nature.

For later purposes in field theory we need yet another, equivalent, formal treatment, the Hamiltonian formalism. In our 1-d system, we define the 'conjugate momentum' p by

$$p \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{1.11}$$

and the Hamiltonian H via

$$\begin{aligned} H(x,p) &\equiv p\dot{x} - L(x,\dot{x}) \\ &= m\dot{x}^2 - \tfrac{1}{2}m\dot{x}^2 + V(x) \\ &= \tfrac{1}{2}m\dot{x}^2 + V(x) = T + V. \end{aligned} \tag{1.12}$$

The Hamiltonian $H(x,p)$ corresponds to the total energy of the system; it is a function of the position variable x and the conjugate momentum[1] p. It is now easy to derive Hamilton's equations

$$\frac{\partial H}{\partial x} = -\dot{p}, \quad \frac{\partial H}{\partial p} = \dot{x}. \tag{1.13}$$

These are two equations of first order, while the Euler-Lagrange equation was a single equation of second order. Taking another derivative in Hamilton's equations and substituting one into the other, it is easy to convince oneself that the Euler-Lagrange equations and Hamilton's equations provide an entirely equivalent description of the system. Again, this generalises obviously to three-dimensional space yielding equations for the components,

$$\frac{\partial H}{\partial x_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{x}_i. \tag{1.14}$$

1.2 Quantum mechanics

Having set up some basic formalism for classical mechanics, let us now move on to quantum mechanics. In doing so we shall use 'canonical quantisation', which is historically what was used first and what we shall later use to quantise fields as well. We remark, however, that one can also quantise a theory using path integrals.

[1] It should be noted that the conjugate momentum is in general not equal to $m\dot{x}$.

Canonical quantisation consists of two steps. Firstly, the dynamical variables of a system are replaced by operators, which we denote by a hat. For example, in our simplest one particle system,

$$\begin{aligned} \text{position: } x_i &\rightarrow \hat{x}_i \\ \text{momentum: } p_i &\rightarrow \hat{p}_i = -i\hbar\frac{\partial}{\partial x_i} \\ \text{Hamiltonian: } H &\rightarrow \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}) = -\frac{\hbar^2\nabla^2}{2m} + V(\hat{\mathbf{x}}). \end{aligned} \tag{1.15}$$

Secondly, one imposes commutation relations on these operators,

$$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij} \tag{1.16}$$

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0. \tag{1.17}$$

The physical state of a quantum mechanical system is encoded in state vectors $|\psi\rangle$, which are elements of a Hilbert space $\mathcal{H}$. The hermitian conjugate state is $\langle\psi| = (|\psi\rangle)^\dagger$, and the modulus squared of the scalar product between two states gives the probability for the system to go from state 1 to state 2,

$$|\langle\psi_1|\psi_2\rangle|^2 = \text{probability for} \quad |\psi_1\rangle \rightarrow |\psi_2\rangle. \tag{1.18}$$

On the other hand physical observables O, i.e. measurable quantities, are given by the expectation values of hermitian operators, $\hat{O} = \hat{O}^\dagger$,

$$O = \langle\psi|\hat{O}|\psi\rangle, \quad O_{12} = \langle\psi_2|\hat{O}|\psi_1\rangle. \tag{1.19}$$

Hermiticity ensures that expectation values are real, as required for measurable quantities. Due to the probabilistic nature of quantum mechanics, expectation values correspond to statistical averages, or mean values, with a variance

$$(\Delta O)^2 = \langle\psi|(\hat{O} - O)^2|\psi\rangle = \langle\psi|\hat{O}^2|\psi\rangle - \langle\psi|\hat{O}|\psi\rangle^2. \tag{1.20}$$

An important concept in quantum mechanis is that of eigenstates of an operator, defined by

$$\hat{O}|\psi\rangle = O|\psi\rangle. \tag{1.21}$$

Evidently, between eigenstates we have $\Delta O = 0$. Examples are coordinate eigenstates, $\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$, and momentum eigenstates, $\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$, describing a particle at position $\mathbf{x}$ or with momentum $\mathbf{p}$, respectively. However, a state vector cannot be simultaneous eigenstate of non-commuting operators. This leads to the Heisenberg uncertainty relation for any two non-commuting operators $\hat{A}, \hat{B}$,

$$\Delta A \Delta B \geq \frac{1}{2}|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle|. \tag{1.22}$$

Finally, sets of eigenstates can be orthonormalized and we assume completeness, i.e. they span the entire Hilbert space,

$$\langle\mathbf{p}'|\mathbf{p}\rangle = \delta(\mathbf{p} - \mathbf{p}'), \quad 1 = \int d^3p\; |\mathbf{p}\rangle\langle\mathbf{p}|. \tag{1.23}$$

As a consequence, an arbitrary state vector can always be expanded in terms of a set of eigenstates. In particular, the Schrödinger wave function of a particle in coordinate representation is given by $\psi(\mathbf{x}) = \langle \mathbf{x}|\psi\rangle$.

Having quantised our system, we now want to describe its time evolution. This can be done in different quantum pictures.

1.3 The Schrödinger picture

In this approach state vectors are functions of time, $|\psi(t)\rangle$, while operators are time independent, $\partial_t \hat{O} = 0$. The time evolution of a system is described by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{x},t) = \hat{H}\psi(\mathbf{x},t). \tag{1.24}$$

If at some initial time t_0 our system is in the state $\Psi(\mathbf{x},t_0)$, then the time dependent state vector

$$\Psi(\mathbf{x},t) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi(\mathbf{x},t_0) \tag{1.25}$$

solves the Schrödinger equation for all later times t.

The expectation value of some hermitian operator $\hat{O}$ at a given time t is then defined as

$$\langle \hat{O}\rangle_t = \int d^3x\, \Psi^*(\mathbf{x},t)\hat{O}\Psi(\mathbf{x},t), \tag{1.26}$$

and the normalisation of the wavefunction is given by

$$\int d^3x\, \Psi^*(\mathbf{x},t)\Psi(\mathbf{x},t) = \langle 1\rangle_t. \tag{1.27}$$

Since $\Psi^*\Psi$ is positive, it is natural to interpret it as the probability density for finding a particle at position $\mathbf{x}$. Furthermore one can derive a conserved current $\mathbf{j}$, as well as a continuity equation by considering

$$\Psi^* \times (\text{Schr.Eq.}) - \Psi \times (\text{Schr.Eq.})^*. \tag{1.28}$$

The continuity equation reads

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\mathbf{j} \tag{1.29}$$

where the density ρ and the current $\mathbf{j}$ are given by

$$\rho = \Psi^*\Psi \qquad \text{(positive)}, \tag{1.30}$$

$$\mathbf{j} = \frac{\hbar}{2im}\left(\Psi^*\nabla\Psi - (\nabla\Psi^*)\Psi\right) \quad \text{(real)}. \tag{1.31}$$

Now that we have derived the continuity equation let us discuss the probability interpretation of Quantum Mechanics in more detail. Consider a finite volume V with boundary S. The integrated continuity equation is

$$\begin{aligned}\int_V \frac{\partial\rho}{\partial t}\, d^3x &= -\int_V \nabla\cdot\mathbf{j}\, d^3x \\ &= -\int_S \mathbf{j}\cdot \underline{dS}\end{aligned} \tag{1.32}$$

where in the last line we have used Gauss's theorem. Using Eq. (1.27) the lhs. can be rewritten and we obtain

$$\frac{\partial}{\partial t}\langle 1\rangle_t = -\int_S \mathbf{j}\cdot \underline{dS} = 0. \tag{1.33}$$

In other words, provided that $\mathbf{j} = 0$ everywhere at the boundary S, we find that the time derivative of $\langle 1\rangle_t$ vanishes. Since $\langle 1\rangle_t$ represents the total probability for finding the particle anywhere inside the volume V, we conclude that this probability must be conserved: particles cannot be created or destroyed in our theory. Non-relativistic Quantum Mechanics thus provides a consistent formalism to describe a single particle. The quantity $\Psi(\mathbf{x}, t)$ is interpreted as a one-particle wave function.

1.4 The Heisenberg picture

Here the situation is the opposite to that in the Schrödinger picture, with the state vectors regarded as constant, $\partial_t|\Psi_H\rangle = 0$, and operators which carry the time dependence, $\hat{O}_H(t)$. This is the concept which later generalises most readily to field theory. We make use of the solution Eq. (1.25) to the Schrödinger equation in order to *define* a Heisenberg state vector through

$$\Psi(x,t) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi(x,t_0) \equiv e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi_H(x), \tag{1.34}$$

i.e. $\Psi_H(\mathbf{x}) = \Psi(\mathbf{x}, t_0)$. In other words, the Schrödinger vector at some time t_0 is defined to be equivalent to the Heisenberg vector, and the solution to the Schrödinger equation provides the transformation law between the two for all times. This transformation of course leaves the physics, i.e. expectation values, invariant,

$$\langle\Psi(t)|\hat{O}|\Psi(t)\rangle = \langle\Psi(t_0)|e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\Psi(t_0)\rangle = \langle\Psi_H|\hat{O}_H(t)|\Psi_H\rangle, \tag{1.35}$$

with

$$\hat{O}_H(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}. \tag{1.36}$$

From this last equation it is now easy to derive the equivalent of the Schrödinger equation for the Heisenberg picture, the Heisenberg equation of motion for operators:

$$i\hbar\frac{d\hat{O}_H(t)}{dt} = [\hat{O}_H, \hat{H}]. \tag{1.37}$$

Note that all commutation relations, like Eq. (1.16), with time dependent operators are now intended to be valid for all times. Substituting $\hat{x}, \hat{p}$ for $\hat{O}$ into the Heisenberg equation readily leads to

$$\begin{aligned}\frac{d\hat{x}_i}{dt} &= \frac{\partial\hat{H}}{\partial\hat{p}_i},\\ \frac{d\hat{p}_i}{dt} &= -\frac{\partial\hat{H}}{\partial\hat{x}_i},\end{aligned} \tag{1.38}$$

the quantum mechanical equivalent to the Hamilton equations of classical mechanics.

1.5 The quantum mechanical harmonic oscillator

Because of similar structures later in quantum field theory, it is instructive to also briefly recall the harmonic oscillator in one dimension. Its Hamiltonian is given by

$$\hat{H}(\hat{x},\hat{p}) = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right). \tag{1.39}$$

Employing the canonical formalism we have just set up, we easily identify the momentum operator to be $\hat{p}(t) = m\partial_t\hat{x}(t)$, and from the Hamilton equations we find the equation of motion to be $\partial_t^2\hat{x} = -\omega^2\hat{x}$, which has the well known plane wave solution $\hat{x} \sim \exp i\omega t$.

An alternative path useful for later field theory applications is to introduce new operators, expressed by the old ones,

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\sqrt{\frac{\hbar}{m\omega}}\,\hat{p}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\sqrt{\frac{\hbar}{m\omega}}\,\hat{p}\right). \tag{1.40}$$

Using the commutation relation for $\hat{x},\hat{p}$, one readily derives

$$[\hat{a},\hat{a}^\dagger] = 1, \quad [\hat{H},\hat{a}] = -\hbar\omega\hat{a}, \quad [\hat{H},\hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger. \tag{1.41}$$

With the help of these the Hamiltonian can be rewritten in terms of the new operators,

$$\hat{H} = \frac{1}{2}\hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega. \tag{1.42}$$

With this form of the Hamiltonian it is easy to construct a complete basis of energy eigenstates $|n\rangle$,

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{1.43}$$

Using the above commutation relations, one finds

$$\hat{a}^\dagger\hat{H}|n\rangle = (\hat{H}\hat{a}^\dagger - \hbar\omega\hat{a}^\dagger)|n\rangle = E_n\hat{a}^\dagger|n\rangle, \tag{1.44}$$

and from the last equation

$$\hat{H}\hat{a}^\dagger|n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger|n\rangle. \tag{1.45}$$

Thus, the state $\hat{a}^\dagger|n\rangle$ has energy $E_n + \hbar\omega$, and therefore $\hat{a}^\dagger$ may be regarded as a "creation operator" for a quantum with energy $\hbar\omega$. Along the same lines one finds that $\hat{a}|n\rangle$ has energy $E_n - \hbar\omega$, and $\hat{a}$ is an "annihilation operator".

Let us introduce a vacuum state $|0\rangle$ with no quanta excited, for which $\hat{a}|n\rangle = 0$, because there cannot be any negative energy states. Acting with the Hamiltonian on that state we find

$$\hat{H}|0\rangle = \hbar\omega/2, \tag{1.46}$$

i.e. the quantum mechanical vacuum has a non-zero energy, known as vacuum oscillation or zero point energy. Acting with a creation operator onto the vacuum state one easily finds the state with one quantum excited, and this can be repeated n times to get

$$\begin{aligned} |1\rangle = \hat{a}^\dagger|0\rangle \quad &, \quad E_1 = (1+\frac{1}{2})\hbar\omega, \quad \dots \\ |n\rangle = \frac{\hat{a}^\dagger}{\sqrt{n}}|n-1\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \quad &, \quad E_n = (n+\frac{1}{2})\hbar\omega. \end{aligned} \tag{1.47}$$

The root of the factorial is there to normalise all eigenstates to one. Finally, the "number operator" $\hat{N} = \hat{a}^\dagger \hat{a}$ returns the number of quanta in a given energy eigenstate,

$$\hat{N}|n\rangle = n|n\rangle. \tag{1.48}$$

Problems

1.1 Starting from the definition of the Hamiltonian,

$$H(x,p) \equiv p\dot{x} - L(x,\dot{x}),$$

derive Hamilton's equations

$$\frac{\partial H}{\partial x} = -\dot{p}, \qquad \frac{\partial H}{\partial p} = \dot{x}.$$

[**Hint:** the key is to keep track of what are the independent variables]

1.2 Using the Schrödinger equation for the wavefunction $\Psi(\mathbf{x},t)$,

$$\left\{-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{x})\right\}\Psi(\mathbf{x},t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t),$$

show that the probability density $\rho = \Psi^*\Psi$ satisfies the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{j} = 0,$$

where

$$\mathbf{j} = \frac{\hbar}{2im}\left\{\Psi^*\nabla\Psi - (\nabla\Psi^*)\,\Psi\right\}.$$

[**Hint:** Consider $\Psi^*\times$ (Schr.Eq.) – $\Psi\times$ (Schr.Eq.)*]

1.3 Let $|\psi\rangle$ be a simultaneous eigenstate of two operators $\hat{A}, \hat{B}$. Prove that this implies a vanishing commutator $[\hat{A}, \hat{B}]$.

1.4 Let $\hat{O}$ be an operator in the Schrödinger picture. Starting from the definition of a Heisenberg operator,

$$\hat{O}_H(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)},$$

derive the Heisenberg equation of motion

$$i\hbar\frac{d\hat{O}_H}{dt} = [\hat{O}_H, \hat{H}].$$

1.5 Consider the Heisenberg equation of motion for the momentum operator $\hat{p}$ of the harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right),$$

and show that it is equivalent to Newton's law for the position operator $\hat{x}$.

2 Classical Field Theory

2.1 From N-point mechanics to field theory

In the previous sections we have reviewed the Lagrangian formalism for a single point particle in classical mechanics. A benefit of that formalism is that it easily generalises to any number of particles or dimensions. Let us return to one dimension for the moment but consider an N-particle system, i.e. we have N coordinates and N momenta, $x_i(t), p_i(t), i = 1, \ldots N$. For such a system we get $2N$ Heisenberg equations,

$$-\frac{\partial H}{\partial x_i} = \frac{dp_i}{dt}, \quad \frac{\partial H}{\partial p_i} = \frac{dx_i}{dt} \ . \tag{2.1}$$

To make things more specific, consider a piece of a guitar string, approximated by N coupled oscillators, as in Fig. 2. Each point mass of the string can only move in the

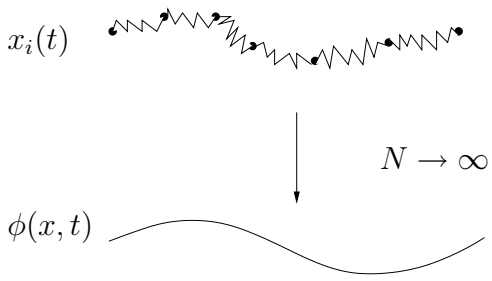


Figure 2: *From N coupled point masses to a continuous string, i.e. infinitely many degrees of freedom.*

direction perpendicular to the string, i.e. is a particle moving in one dimension. This approximation of a string gets better and better the more points we fill in between the springs, and a continuous string obtains in the limit $N \to \infty$. The displacement of the string at some particular point x along its length is now given by a field coordinate $\phi(x,t)$. Going back to the N-point system and comparing what measures the location of a point and its displacement, we find the following "dictionary" between point mechanics and field theory:

Classical Mechanics:		Classical Field Theory:
$x(t)$	$\longrightarrow$	$\phi(x,t)$
$\dot{x}(t)$	$\longrightarrow$	$\dot{\phi}(x,t)$
i	$\longrightarrow$	x
$L(x,\dot{x})$	$\longrightarrow$	$\mathcal{L}[\phi,\dot{\phi}]$

(2.2)

In the last line we have introduced a new notation: the square brackets indicate that $L[\phi,\dot{\phi}]$ depends on the functions $\phi(x,t), \dot{\phi}(x,t)$ at every space-time point, but not on the coordinates directly. Such an object is called a "functional", as opposed to a function which depends on the coordinate variables only.

Formally the above limit of infinite degrees of freedom can also be taken if we are dealing with particles in a three-dimensional Euclidean space, for which there are N three-vectors $\mathbf{x}_i$ specifying the positions. We then obtain a field $\phi(\mathbf{x}, t)$, defined at every point in space and time.

2.2 Relativistic field theory

Before continuing to set up the formalism of field theory, we want to make it relativistic as well. Coordinates are combined into four-vectors, $x^\mu = (t, x_i)$ or $x = (t, \mathbf{x})$, whose length $x^2 = t^2 - \mathbf{x}^2$ is invariant under Lorentz transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu. \tag{2.3}$$

A general function transforms as $f(x) \to f'(x')$, i.e. both the function and its argument transform. A Lorentz scalar is a function which is the same in all inertial frames,

$$\phi'(x') = \phi(x) \quad \text{for all } \Lambda. \tag{2.4}$$

On the other hand a vector function transforms as

$$V'^\mu(x') = \Lambda^\mu_\nu V^\nu(x). \tag{2.5}$$

An example is the covariant derivative of a scalar field,

$$\partial^\mu \phi(x) = \frac{\partial \phi(x)}{\partial x_\mu}, \quad \partial_\mu \phi(x) = \frac{\partial \phi(x)}{\partial x^\mu}, \tag{2.6}$$

whose square evaluates to

$$(\partial^\mu \phi)(\partial_\mu \phi) = (\partial^0 \phi)^2 - (\nabla \phi)^2. \tag{2.7}$$

2.3 Action for a scalar field

We are now ready to write down the action for a relativistic scalar field. According to our dictionary, the action from point mechanics, Eq. (1.3), should go into

$$S = \int dt\ L[\phi, \dot{\phi}]. \tag{2.8}$$

However, for a relativistic theory we require Lorentz invariance of the action, and this is not obvious in the current form. The integration is over time only, rather than over the Lorentz-invariant four-volume element $d^4x = dt\, d^3x$, and so the non-invariance of the integration measure has to cancel against that of the Lagrange function in order to have an invariant action. Similar reasoning applies to the arguments of the Lagrangian. In order to have the symmetries manifest, we instead rewrite

$$S = \int d^4x\ \mathcal{L}[\phi, \partial^\mu], \quad L[\phi, \dot{\phi}] = \int d^3x\ \mathcal{L}[\phi, \partial^\mu \phi]. \tag{2.9}$$

Now everything is expressed in covariant quantities, and the action is Lorentz-invariant as soon as the newly defined Lagrangian density $\mathcal{L}$ is.

We now follow the same procedure as in point mechanics and apply the Hamiltonian principle by demanding $\delta S = 0$. For the variation of the field and its derivative we have

$$\phi \to \phi + \delta\phi, \quad \partial_\mu\phi \to \partial_\mu\phi + \delta\partial_\mu\phi, \qquad \delta\partial_\mu\phi = \partial_\mu\delta\phi. \tag{2.10}$$

Using the rule for functional differentiation, $\delta\phi(x)/\delta\phi(y) = \delta^4(x-y)$, the variation of the action then is (to first order in a Taylor expansion)

$$\begin{aligned} \delta S &= \int d^4x \left\{ \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right\} \\ &= \underbrace{\left. \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi \right|}_{=0 \text{ at boundaries}} + \int d^4x \left\{ \frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \right\} \delta\phi. \end{aligned} \tag{2.11}$$

Again the integrand itself must vanish if $\delta S = 0$ for arbitrary variations of the field, $\delta\phi$. This yields the Euler-Lagrange equations for a classical field theory:

$$\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = 0, \tag{2.12}$$

where in the second term a summation over the Lorentz index μ is implied.

Let us now consider the specific Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2. \tag{2.13}$$

The functional derivatives yield

$$\frac{\delta\mathcal{L}}{\delta\phi} = -m^2\phi, \quad \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = \partial^\mu\phi, \tag{2.14}$$

so that

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = \partial_\mu\partial^\mu\phi = \Box\phi. \tag{2.15}$$

The Euler-Lagrange equation then implies

$$(\Box + m^2)\phi(x) = 0. \tag{2.16}$$

This is the Klein-Gordon equation for a scalar field. It is the simplest relativistic wave equation and can be deduced from relativistic energy considerations. Here we have derived it from the Lagrange density following our canonical formalism, in complete analogy to point mechanics. Relativistic invariance of the equations of motion is ensured because we started from an invariant Lagrange density. This is the power of the formalism.

In keeping the analogy with point mechanics, we can define a conjugate momentum π through

$$\pi(x) \equiv \frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial\dot{\phi}(x)} = \frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial(\partial_0\phi(x))} = \partial_0\phi(x). \tag{2.17}$$

Note that the momentum variables p_μ and the conjugate momentum π are *not* the same. The word "momentum" is used only as a semantic analogy to classical mechanics. Further, we define the Hamilton function and a corresponding Hamilton density,

$$H(t) = \int d^3x\, \mathcal{H}[\phi,\pi], \quad \mathcal{H}[\phi,\pi] = \pi\dot{\phi} - \mathcal{L}. \tag{2.18}$$

For the Lagrangian density we considered, this gives

$$\mathcal{H} = \frac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right]. \tag{2.19}$$

2.4 Plane wave solution to the Klein-Gordon equation

Let us consider real solutions to Eq. (2.16), characterised by $\phi^*(x) = \phi(x)$. To find them we try an ansatz of plane waves

$$\phi(x) \propto \mathrm{e}^{i(k^0 t - \mathbf{k}\cdot\mathbf{x})}. \tag{2.20}$$

The Klein-Gordon equation is satisfied if $(k^0)^2 - \mathbf{k}^2 = m^2$ so that

$$k^0 = \pm\sqrt{\mathbf{k}^2 + m^2}. \tag{2.21}$$

If we choose the positive branch of the square root then we can define the energy as

$$E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2} > 0, \tag{2.22}$$

and obtain two types of solutions which read

$$\phi_+(x) \propto \mathrm{e}^{i(E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x})}, \quad \phi_-(x) \propto \mathrm{e}^{-i(E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x})}. \tag{2.23}$$

The general solution is a superposition of ϕ_+ and ϕ_-. Using

$$E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x} = k^\mu k_\mu = k_\mu k^\mu = k\cdot x \tag{2.24}$$

this solution reads

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})}\, \left(\mathrm{e}^{ik\cdot x}\alpha^*(\mathbf{k}) + \mathrm{e}^{-ik\cdot x}\alpha(\mathbf{k})\right), \tag{2.25}$$

where $\alpha(\mathbf{k})$ is an arbitrary complex coefficient. From the general solution one easily reads off that ϕ is real, i.e. $\phi = \phi^*$.

2.5 Symmetries and conservation laws

Symmetries play such a fundamental role in physics because they are related to conservation laws. This is stated in Noether's theorem. In a nutshell, Noether's theorem says that invariance of the action under a symmetry transformation implies the existence of a conserved quantity. For instance, the conservation of 3-momentum $\mathbf{p}$ is associated with translational invariance of the Lagrangian, i.e. the transformation

$$\mathbf{x} \to \mathbf{x} + \mathbf{a}, \quad \mathbf{a}: \text{ constant 3-vector}, \tag{2.26}$$

while the conservation of energy comes from the invariance under time translations

$$t \to t + \tau, \quad \tau: \text{ constant time interval}. \tag{2.27}$$

Let us apply this to our relativistic field theory and consider four-translations, $x^\mu \to x^\mu + \epsilon^\mu$. The variation of the Lagrangian is

$$\begin{aligned} \delta\mathcal{L} &= = \frac{\delta\mathcal{L}}{\delta\phi}\frac{\partial\phi}{\partial x^\nu}\epsilon^\nu + \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\frac{\partial(\partial^\mu\phi)}{\partial x^\nu}\epsilon^\nu \\ &= \frac{\partial}{\partial x_\mu}\left[\frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\frac{\partial\phi}{\partial x^\nu}\epsilon^\nu\right], \end{aligned} \tag{2.28}$$

where we have made use of the Euler-Lagrange Eqs. (2.12), to get to the last expression. If the action is to be invariant under such translations, its variation has to vanish for arbitrary ϵ^ν, which leads to

$$\frac{\partial}{\partial x_\mu}\left[\frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\right] = 0. \tag{2.29}$$

The quantity in square brackets is called the energy-momentum tensor $\Theta_{\mu\nu}$, and thus we have

$$\partial^\mu\Theta_{\mu\nu} \equiv \partial^0\Theta_{0\nu} - \partial^j\Theta_{j\nu} = 0, \tag{2.30}$$

i.e. four conservation laws (one for every value of ν). Let us look in more detail at the components of the energy-momentum tensor,

$$\begin{aligned} \Theta_{00} &= \frac{\partial\mathcal{L}}{\partial(\partial^0\phi)}\partial_0\phi - g_{00}\mathcal{L} = \pi(x)(\partial_0\phi(x)) - \mathcal{L}, \\ \Theta_{0j} &= \frac{\partial\mathcal{L}}{\partial(\partial^0\phi)}\partial_j\phi - g_{0j}\mathcal{L} = \pi(x)\partial_j\phi. \end{aligned} \tag{2.31}$$

The first line is nothing but the Hamiltonian density, and integrating it over space will thus be the Hamiltonian, or the energy. Its conservation can then be shown by considering

$$\begin{aligned} \frac{\partial}{\partial t}\int_V d^3x\,\Theta_{00} &= \int_V d^3x\,\partial^0\Theta_{00} \\ &= \int_V d^3x\,\partial^j\Theta_{j0} = \int_S dS_j\cdot\Theta_{0j} = 0, \end{aligned} \tag{2.32}$$

where we have used Eq. (2.30) in the second line. The Hamiltonian density is a conserved quantity, provided that there is no energy flow through the surface S which encloses the volume V. In a similar manner one can show that the 3-momentum p_j, which is related to Θ_{0j}, is conserved as well. It is then useful to define a conserved energy-momentum four-vector

$$P_\mu = \int d^3x\;\Theta_{0\mu}. \tag{2.33}$$

In analogy to point mechanics, we thus see that invariances of the Lagrangian density correspond to conservation laws. An entirely analogous procedure leads to conserved quantities like anguluar mometum and spin. Furthermore one can study so-called internal symmetries, i.e. ones which are not related to coordinate but other transformations. Examples are conservation of all kinds of charges, isospin, etc.

We have thus established the Lagrange-Hamilton formalism for classical field theory: we derived the equation of motion (Euler-Lagrange equation) from the Lagrangian and introduced the conjugate momentum. We then defined the Hamiltonian (density) and considered conservation laws by studying the energy-momentum tensor $\Theta_{\mu\nu}$.

Problems

2.1 Given the relativistic invariance of the measure d^4k, show that the integration measure

$$\frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}$$

is Lorentz-invariant, provided that $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$.

[**Hint:** Start from the Lorentz-invariant expression

$$\frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m^2)\,\theta(k_0)$$

and use

$$\delta(x^2 - x_0^2) = \frac{1}{2|x|}(\delta(x - x_0) + \delta(x + x_0)).$$

What is the significance of the δ and θ functions above? If you're really keen, you may prove the relation for $\delta(x^2 - x_0^2)$.]

2.2 Verify that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\,\left\{e^{ik\cdot x}a(\mathbf{k}) + e^{-ik\cdot x}b(\mathbf{k})\right\}$$

is a solution of the Klein-Gordon equation. Show that a real scalar field $\phi^*(x) = \phi(x)$ requires the condition $b(\mathbf{k}) = a^*(\mathbf{k})$.

2.3 Show that the Hamiltonian density $\mathcal{H}$ for a free scalar field is given by

$$\mathcal{H} = \frac{1}{2}\left\{(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right\}.$$

Derive the components $\hat{P}_0$, $\hat{\mathbf{P}}$ of the energy-momentum four-vector $\hat{P}^\mu$ in terms of the field operators $\hat{\phi}, \hat{\pi}$.

3 Quantum Field Theory

After many preparations, we have finally arrived at the proper subject of the lecture. In this section we shall apply the canonical quantisation formalism to field theory.

3.1 Canonical field quantisation

To lighten notation, let us follow common practice in quantum field theory and set $\hbar = c = 1$. Our starting point is the Lagrangian density for the free scalar field,

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2, \tag{3.1}$$

which led to the Klein-Gordon equation in the previous section. We have seen that in field theory the field $\phi(x)$ plays the role of the coordinates in ordinary point mechanics, and

we defined a canonically conjugate momentum, $\pi(x) = \delta\mathcal{L}/\delta\dot{\phi} = \dot{\phi}(x)$. We then continue the analogy to point mechanics through the quantisation procedure, i.e. we now take our canonical variables to be operators,

$$\phi(x) \to \hat{\phi}(x), \quad \pi(x) \to \hat{\pi}(x). \tag{3.2}$$

Next we impose equal-time commutation relations on them,

$$\begin{aligned} \left[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)\right] &= i\delta^3(\mathbf{x}-\mathbf{y}), \\ \left[\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{y},t)\right] &= [\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)] = 0. \end{aligned} \tag{3.3}$$

As in the case of quantum mechanis, the canonical variables commute among themselves, but not the canonical coordinate and momentum with each other. Note that the commutation relation is entirely analogous to the quantum mechanical case. There would be an $\hbar$, if it hadn't been set to one earlier, and the delta-function accounts for the fact that we are dealing with fields. It is one if the fields are evaluated at the same space-time point, and zero otherwise.

After quantisation, our fields have turned into field operators. Note that within the relativistic formulation they depend on time, and hence they are Heisenberg operators.

3.2 Causality and commutation relations

In the previous paragraph we have formulated commutation relations for fields evaluated at equal time, which is clearly a special case when considering fields at general x, y. The reason has to do with maintaining causality in a relativistic theory. Let us recall the light cone about an event at y, as in Fig. 3. One important postulate of special relativity states that no signal and no interaction can travel faster than the speed of light. This has important consequences about the way in which different events can affect each other. For instance, two events which are characterised by space-time points x^μ and y^μ are said to be causal if the distance $(x-y)^2$ is time-like, i.e. $(x-y)^2 > 0$. By contrast, two events characterised by a space-like separation, i.e. $(x-y)^2 < 0$, cannot affect each other, since the point x is not contained inside the light cone about y.

In non-relativistic Quantum Mechanics the commutation relations among operators indicate whether precise and independent measurements of the corresponding observables can be made. If the commutator does not vanish, then a measurement of one observable affects that of the other. From the above it is then clear that the issue of causality must be incorporated into the commutation relations of the relativistic version of our quantum theory: whether or not independent and precise measurements of two observables can be made depends also on the separation of the 4-vectors characterising the points at which these measurements occur. Clearly, events with space-like separations cannot affect each other, and hence all fields must commute,

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = [\hat{\pi}(x), \hat{\pi}(y)] = \left[\hat{\phi}(x), \hat{\pi}(y)\right] = 0 \quad \text{for} \quad (x-y)^2 < 0. \tag{3.4}$$

This condition is sometimes called micro-causality. Writing out the four-components of the time interval, we see that as long as $|t'-t| < |\mathbf{x}-\mathbf{y}|$, the commutator vanishes in

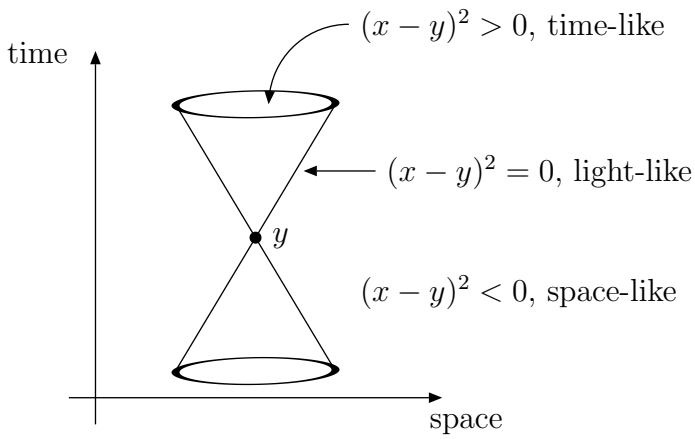


Figure 3: The light cone about y. Events occurring at points x and y are said to be time-like (space-like) if x is inside (outside) the light cone about y.

a finite interval $|t'-t|$. It also vanishes for $t'=t$, as long as $\mathbf{x} \neq \mathbf{y}$. Only if the fields are evaluated at an equal space-time point can they affect each other, which leads to the equal-time commutation relations above. They can also affect each other everywhere within the light cone, i.e. for time-like intervals. It is not hard to show that in this case

$$\begin{aligned} \left[\hat{\phi}(x), \hat{\phi}(y)\right] = [\hat{\pi}(x), \hat{\pi}(y)] &= 0, \quad \text{for} \quad (x-y)^2 > 0 \\ \left[\hat{\phi}(x), \hat{\pi}(y)\right] &= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left(e^{ip\cdot(x-y)} + e^{-ip\cdot(x-y)}\right). \end{aligned} \tag{3.5}$$

3.3 Creation and annihilation operators

After quantisation, the Klein-Gordon equation we derived earlier turns into an equation for operators. For its solution we simply promote the classical plane wave solution, Eq. (2.25), to operator status,

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} \left(\mathrm{e}^{ik\cdot x}\hat{a}^\dagger(\mathbf{k}) + \mathrm{e}^{-ik\cdot x}\hat{a}(\mathbf{k})\right). \tag{3.6}$$

Note that the complex conjugation of the Fourier coefficient turned into hermitian conjugation for an operator.

Let us now solve for the operator coefficients of the positive and negative energy solutions. In order to do so, we invert the Fourier integrals for the field and its time derivative,

$$\int d^3x\ \hat{\phi}(\mathbf{x},t)e^{ikx} = \frac{1}{2E}\left[\hat{a}(\mathbf{k}) + \hat{a}^\dagger(\mathbf{k})e^{2ik_0x_0}\right], \tag{3.7}$$

$$\int d^3x\ \dot{\hat{\phi}}(\mathbf{x},t)e^{ikx} = -\frac{i}{2}\left[\hat{a}(\mathbf{k}) - \hat{a}^\dagger(\mathbf{k})e^{2ik_0x_0}\right], \tag{3.8}$$

and then build the linear combination $iE(k)(3.7)-(3.8)$ to find

$$\int d^3x \left[iE(k)\hat{\phi}(\mathbf{x},t) - \dot{\hat{\phi}}(\mathbf{x},t)\right] e^{ikx} = i\hat{a}(\mathbf{k}), \tag{3.9}$$

Following a similar procedure for $\hat{a}^\dagger(k)$, and using $\hat{\pi}(x) = \dot{\hat{\phi}}(x)$ we find

$$\begin{aligned}\hat{a}(\mathbf{k}) &= \int d^3x \left[E(k)\hat{\phi}(\mathbf{x},t) + i\hat{\pi}(\mathbf{x},t)\right] e^{ikx}, \\ \hat{a}^\dagger(\mathbf{k}) &= \int d^3x \left[E(k)\hat{\phi}(\mathbf{x},t) - i\hat{\pi}(\mathbf{x},t)\right] e^{-ikx}. \end{aligned} \tag{3.10}$$

Note that, as Fourier coefficients, these operators do not depend on time, even though the right hand side does contain time variables. Having expressions in terms of the canonical field variables $\hat{\phi}(x), \hat{\pi}(x)$, we can now evaluate the commutators for the Fourier coefficients. Expanding everything out and using the commutation relations Eq. (3.3), we find

$$\left[\hat{a}^\dagger(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k}_2)\right] = 0 \tag{3.11}$$

$$\left[\hat{a}(\mathbf{k}_1), \hat{a}(\mathbf{k}_2)\right] = 0 \tag{3.12}$$

$$\left[\hat{a}(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k}_2)\right] = (2\pi)^3\, 2E(\mathbf{k}_1)\delta^3(\mathbf{k}_1 - \mathbf{k}_2) \tag{3.13}$$

We easily recognise these for every $\mathbf{k}$ to correspond to the commutation relations for the harmonic oscillator, Eq. (1.41). This motivates us to also express the Hamiltonian and the energy momentum four-vector of our quantum field theory in terms of these operators. This yields

$$\begin{aligned}\hat{H} &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\, E(\mathbf{k}) \left(\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})\right), \\ \hat{\mathbf{P}} &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\, \mathbf{k} \left(\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})\right). \end{aligned} \tag{3.14}$$

We thus find that the Hamiltonian and the momentum operator are nothing but a continuous sum of excitation energies/momenta of one-dimensional harmonic oscillators! After a minute of thought this is not so surprising. We expanded the solution of the Klein-Gordon equation into a superposition of plane waves with momenta $\mathbf{k}$. But of course a plane wave solution with energy $E(\mathbf{k})$ is also the solution to a one-dimensional harmonic oscillator with the same energy. Hence, our free scalar field is simply a collection of infinitely many harmonic oscillators distributed over the whole energy/momentum range. These energies sum up to that of the entire system. We have thus reduced the problem of handling our field theory to oscillator algebra. From the harmonic oscillator we know already how to construct a complete basis of energy eigenstates, and thanks to the analogy of the previous section we can take this over to our quantum field theory.

3.4 Energy of the vacuum state and renormalisation

In complete analogy we begin again with the postulate of a vacuum state $|0\rangle$ with norm one, which is annihilated by the action of the operator a,

$$\langle 0|0\rangle = 1, \quad \hat{a}(\mathbf{k})|0\rangle = 0 \quad \text{for all} \quad \mathbf{k}. \tag{3.15}$$

Let us next evaluate the energy of this vacuum state, by taking the expectation value of the Hamiltonian,

$$E_0 = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})}\, E(\mathbf{k}) \left\{\langle 0|\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})|0\rangle + \langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle\right\}. \tag{3.16}$$

The first term in curly brackets vanishes, since a annihilates the vacuum. The second can be rewritten as

$$\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle = \left\{\left[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k})\right] + \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\right\}|0\rangle. \tag{3.17}$$

It is now the second term which vanishes, whereas the first can be replaced by the value of the commutator. Thus we obtain

$$E_0 = \langle 0|\hat{H}|0\rangle = \delta^3(0)\frac{1}{2}\int d^3k\, E(\mathbf{k}) = \delta^3(0)\frac{1}{2}\int d^3k\,\sqrt{\mathbf{k}^2+m^2} = \infty, \tag{3.18}$$

which means that the energy of the ground state is infinite! This result seems rather paradoxical, but it can be understood again in terms of the harmonic oscillator. Recall that the simple quantum mechanical oscillator has a finite zero-point energy. As we have seen above, our field theory corresponds to an infinite collection of harmonic oscillators, i.e. the vacuum receives an infinite number of zero point contributions, and its energy thus diverges.

This is the first of frequent occurrences of infinities in quantum field theory. Fortunately, it is not too hard to work around this particular one. Firstly, we note that nowhere in nature can we observe absolute values of energy, all we can measure are energy differences relative to some reference scale, at best the one of the vacuum state, $|0\rangle$. In this case it does not really matter what the energy of the vacuum is. This then allows us to redefine the energy scale, by always subtracting the (infinite) vacuum energy from any energy we compute. This process is called "renormalisation".

We then *define* the renormalised vacuum energy to be zero, and take it to be the expectation value of a renormalised Hamiltonian,

$$E_0^R \equiv \langle 0|\hat{H}^R|0\rangle = 0. \tag{3.19}$$

According to this recipe, the renormalised Hamiltonian is our original one, minus the (unrenormalised) vacuum energy,

$$\begin{aligned}
\hat{H}^R &= \hat{H} - E_0 \\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\,E(\mathbf{k})\left\{\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}) - \langle 0|\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle\right\} \\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\,E(\mathbf{k})\left\{2\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \left[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k})\right] - \langle 0|\left[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k})\right]|0\rangle\right\}
\end{aligned} \tag{3.20}$$

Here the subtraction of the vacuum energy is shown explicitly, and we can rewrite is as

$$\begin{aligned}
\hat{H}^R &= \int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \\
&\quad + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\left\{\left[\hat{a}(\mathbf{p}),\hat{a}^\dagger(\mathbf{p})\right] - \langle 0|\left[\hat{a}(\mathbf{p}),\hat{a}^\dagger(\mathbf{p})\right]|0\rangle\right\}. \\
&= \int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\,\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{H}^{\rm vac}
\end{aligned} \tag{3.21}$$

The operator $\hat{H}^{\rm vac}$ ensures that the vacuum energy is properly subtracted: if $|\psi\rangle$ and $|\psi'\rangle$ denote arbitrary N-particle states, then one can convince oneself that $\langle\psi'|\hat{H}^{\rm vac}|\psi\rangle = 0$. In particular we now find that

$$\langle 0|\hat{H}^R|0\rangle = 0, \tag{3.22}$$

as we wanted. A simple way to automatise the removal of the vacuum contribution is to introduce *normal ordering*. Normal ordering means that all annihilation operators appear to the right of any creation operator. The notation is

$$: \hat{a}\hat{a}^\dagger : = \hat{a}^\dagger \hat{a}, \tag{3.23}$$

i.e. the normal-ordered operators are enclosed within colons. For instance

$$: \tfrac{1}{2}\left(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\right) : \; = \; \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}). \tag{3.24}$$

It is important to keep in mind that $\hat{a}$ and $\hat{a}^\dagger$ *always* commute inside $: \cdots :$. This is true for an arbitrary string of $\hat{a}$ and $\hat{a}^\dagger$. With this definition we can write the normal-ordered Hamiltonian as

$$\begin{aligned} : \hat{H} : \; &= \; : \frac{1}{2} \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\, E(\mathbf{p}) \left(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\right) : \\ &= \; \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\, E(\mathbf{p})\, \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}), \end{aligned} \tag{3.25}$$

and thus have the relation

$$\hat{H}^R =: \hat{H} : + \hat{H}^{\text{vac}}. \tag{3.26}$$

Hence, we find that

$$\langle \psi' | : \hat{H} : |\psi\rangle = \langle \psi' | \hat{H}^R | \psi \rangle, \tag{3.27}$$

and, in particular, $\langle 0| : \hat{H} : |0\rangle = 0$. The normal ordered Hamiltonian thus produces a renormalised, sensible result for the vacuum energy.

3.5 Fock space and particle number representation

After this lengthy grappling with the vacuum state, we can continue to construct our basis of states in analogy to the harmonic oscillator, making use of the commutation relations for the operators $\hat{a}, \hat{a}^\dagger$. In particular, we define the state $|\mathbf{k}\rangle$ to be the one obtained by acting with the operator $a^\dagger(\mathbf{k})$ on the vacuum,

$$|\mathbf{k}\rangle = \hat{a}^\dagger(\mathbf{k})|0\rangle. \tag{3.28}$$

Using the commutator, its norm is found to be

$$\begin{aligned} \langle \mathbf{k}|\mathbf{k}'\rangle \; &= \; \langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}')|0\rangle = \langle 0|[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')]|0\rangle + \langle 0|\hat{a}^\dagger(\mathbf{k}')a(\mathbf{k})|0\rangle \\ &= \; (2\pi)^3 2E(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}'), \end{aligned} \tag{3.29}$$

since the last term in the first line vanishes ($\hat{a}(\mathbf{k})$ acting on the vacuum). Next we compute the energy of this state, making use of the normal ordered Hamiltonian,

$$\begin{aligned} : \hat{H} : |\mathbf{k}\rangle \; &= \; \int \frac{d^3k'}{(2\pi)^3\, 2E(\mathbf{k}')} E(\mathbf{k}')\hat{a}^\dagger(\mathbf{k}')\hat{a}(\mathbf{k}')\hat{a}^\dagger(\mathbf{k})|0\rangle \\ &= \; \int \frac{d^3k'}{(2\pi)^3\, 2E(\mathbf{k}')} E(\mathbf{k}')(2\pi)^3 2E(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\hat{a}^\dagger(\mathbf{k})|0\rangle \\ &= \; E(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle = E(\mathbf{k})|\mathbf{k}\rangle, \end{aligned} \tag{3.30}$$

and similarly one finds

$$: \hat{\mathbf{P}} : |\mathbf{k}\rangle = \mathbf{k}|\mathbf{k}\rangle. \tag{3.31}$$

Observing that the normal ordering did its job and we obtain renormalised, finite results, we may now interpret the state $|\mathbf{k}\rangle$. It is a one-particle state for a relativistic particle of mass m and momentum $\mathbf{k}$, since acting on it with the energy-momentum operator returns the relativistic one particle energy-mometum dispersion relation, $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$. The $a^\dagger(\mathbf{k}), a(\mathbf{k})$ are creation and annihilation operators for particles of momentum $\mathbf{k}$.

In analogy to the harmonic oscillator, the procedure can be continued to higher states. One easily checks that

$$: \hat{P}^\mu : \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle = (k_1^\mu + k_2^\mu)\hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle, \tag{3.32}$$

and so the state

$$|\mathbf{k}_2, \mathbf{k}_1\rangle = \frac{1}{\sqrt{2!}}\hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle \tag{3.33}$$

is a two-particle state (the factorial is there to have it normalised in the same way as the one-particle state), and so on for higher Fock states.

At the long last we can now see how the field in our free quantum field theory is related to particles. A particle of momentum $\mathbf{k}$ corresponds to an excited Fourier mode of a field. Since the field is a superpositon of all possible Fourier modes, one field is enough to describe all possible configurations representing one or many particles of the same kind in any desired momentum state.

Let us investigate what happens under interchange of the two particles. Since $[\hat{a}^\dagger(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k})] = 0$ for all $\mathbf{k}_1, \mathbf{k}_2$, we see that

$$|\mathbf{k}_2, \mathbf{k}_1\rangle = |\mathbf{k}_1, \mathbf{k}_2\rangle, \tag{3.34}$$

and hence the state is symmetric under interchange of the two particles. Thus, the particles described by the scalar field are bosons.

Finally we complete the analogy to the harmonic oscillator by introducing a number operator

$$\hat{N}(\mathbf{k}) = \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \quad \hat{\mathcal{N}} = \int d^3k\ \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \tag{3.35}$$

which gives us the number of bosons described by a particular Fock state,

$$\hat{\mathcal{N}}|0\rangle = 0, \quad \hat{\mathcal{N}}|\mathbf{k}\rangle = |\mathbf{k}\rangle, \quad \hat{\mathcal{N}}|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = n|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle. \tag{3.36}$$

Of course the normal-ordered Hamiltonian can now simply be given in terms of this operator,

$$: \hat{H} := \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} E(\mathbf{k})\hat{N}(\mathbf{k}), \tag{3.37}$$

i.e. when acting on a Fock state it simply sums up the energies of the individual particles to give

$$: \hat{H} : |\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = \left(E(\mathbf{k}_1) + \ldots E(\mathbf{k}_n)\right)|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle. \tag{3.38}$$

This concludes the quantisation of our free scalar field theory. We have followed the canonical quantisation procedure familiar from quantum mechanics. Due to the infinite

number of degrees of freedom, we encountered a divergent vacuum energy, which we had to renormalise. The renormalised Hamiltonian and the Fock states that we constructed describe free relativistic, uncharged spin zero particles of mass m, such as neutral pions, for example.

If we want to describe charged pions as well, we need to introduce complex scalar fields, the real and imaginary parts being necessary to describe opposite charges. For particles with spin we need still more degrees of freedom and use vector or spinor fields, which have the appropriate rotation and Lorentz transformation properties. Moreover, for fermions there is the Pauli principle prohibiting identical particles with the same quantum numbers to occupy the same state, so the state vectors have to be anti-symmetric under interchange of two particles. This is achieved by imposing anti-commutation relations, rather than commutation relations, on the corresponding field operators. Apart from these complications which account for the nature of the particles, the formalism and quantisation procedure is the same as for the simpler scalar fields, to which we shall stick for this reason.

Problems

3.1 Using the expressions for $\hat{\phi}$ and $\hat{\pi}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, show that the unequal time commutator $\left[\hat{\phi}(x), \hat{\pi}(x')\right]$ is given by

$$\left[\hat{\phi}(x), \hat{\pi}(x')\right] = \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left(e^{ip\cdot(x-x')} + e^{-ip\cdot(x-x')}\right).$$

Show that for $t = t'$ one recovers the equal time commutator

$$\left[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)\right] = i\delta^3(\mathbf{x} - \mathbf{x}').$$

3.2 Being time-dependent Heisenberg operators, the operators $\hat{O} = \hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}, t)$ of scalar field theory obey the Heisenberg equation

$$i\frac{\partial}{\partial t}\hat{O} = [\hat{O}, \hat{H}].$$

In analogy to what you did in problem 1.5, demonstrate the equivalence of this equation with the Klein-Gordon equation.

3.3 Express the Hamiltonian

$$\hat{H} = \frac{1}{2} \int d^3x \left\{\partial_0\hat{\phi})^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\right\}$$

of the quantised free scalar field theory in terms of creation and annihilation operators and show that it is given by

$$\hat{H} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})} E(\mathbf{p}) \left\{\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\right\}.$$

3.3 Prove the commutator relation

$$\Big[: \hat{P}^\mu :, \hat{a}^\dagger(\mathbf{k}) \Big] = k^\mu \hat{a}^\dagger(\mathbf{k})$$

to show that

$$: \hat{P}^\mu : \, \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle = (k_1^\mu + k_2^\mu)\, \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle. \tag{3.39}$$

Interpret the physics of this result.

3.4 Prove by induction that

$$\int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\, \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \; \underbrace{|\mathbf{k}, \ldots, \mathbf{k}\rangle}_{n \text{ momenta}} = n \; \underbrace{|\mathbf{k}, \ldots, \mathbf{k}\rangle}_{n \text{ momenta}} .$$

[**Hint:** induction proceeds in two steps. *i)* show that the statement is true for some starting value of n; *ii)* show that if the statement holds for some general n, then it also holds for $n+1$.]

4 Interacting scalar fields

From now on we shall always discuss quantised real scalar fields. It is then convenient to drop the "hats" on the operators that we have considered up to now. So far we have only discussed free fields without any interaction between them, which we could solve exactly in terms of plane waves. As this does not make for a very interesting theory, let us now add an interaction Lagrangian $\mathcal{L}_{\text{int}}$. The full Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \tag{4.1}$$

where

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 \tag{4.2}$$

is the free Lagrangian density discussed before. The Hamiltonian density of the interaction is related to $\mathcal{L}_{\text{int}}$ simply by

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}, \tag{4.3}$$

which follows from its definition. We shall leave the details of $\mathcal{L}_{\text{int}}$ unspecified for the moment. What we will be concerned with mostly are scattering processes, in which two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter, thereby producing a number of particles in the final state, characterised by momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$. This is schematically shown in Fig. 4. Our task is to find a description of such a scattering process in terms of the underlying quantum field theory.

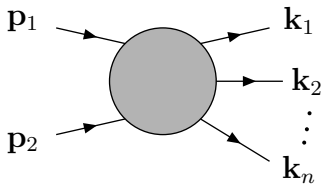


Figure 4: Scattering of two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ into n particles with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$ in the final state.

4.1 The S-matrix

The timescales over which interactions happen are extremely short. The scattering (interaction) process takes place during a short interval around some particular time t with $-\infty \ll t \ll \infty$. Long before t, the incoming particles evolve independently and freely. They are described by a field operator $\phi_{\rm in}$ defined through

$$\lim_{t\to-\infty} \phi(x) = \phi_{\rm in}(x), \tag{4.4}$$

which acts on a corresponding basis of $|{\rm in}\rangle$ states. Long after the collision the particles in the final state evolve again like in the free theory, and the corresponding operator is

$$\lim_{t\to+\infty} \phi(x) = \phi_{\rm out}(x), \tag{4.5}$$

acting on states $|{\rm out}\rangle$. The fields $\phi_{\rm in}$, $\phi_{\rm out}$ are the asymptotic limits of the Heisenberg operator ϕ. They both satisfy the free Klein-Gordon equation, i.e.

$$(\Box + m^2)\phi_{\rm in}(x) = 0, \qquad (\Box + m^2)\phi_{\rm out}(x) = 0. \tag{4.6}$$

Operators describing free fields can be expressed as a superposition of plane waves (see Eq. (3.6)). Thus, for $\phi_{\rm in}$ we have

$$\phi_{\rm in}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} \left({\rm e}^{ik\cdot x} a^\dagger_{\rm in}(\mathbf{k}) + {\rm e}^{-ik\cdot x} a_{\rm in}(\mathbf{k}) \right), \tag{4.7}$$

with an entirely analogous expression for $\phi_{\rm out}(x)$. Note that the operators $a^\dagger$ and a also carry subscripts "in" and "out".

We can now use the creation operators $a^\dagger_{\rm in}$ and $a^\dagger_{\rm out}$ to build up Fock states from the vacuum. For instance

$$a^\dagger_{\rm in}(\mathbf{p}_1)\, a^\dagger_{\rm in}(\mathbf{p}_2)|0\rangle = |\mathbf{p}_1, \mathbf{p}_2; {\rm in}\rangle, \tag{4.8}$$

$$a^\dagger_{\rm out}(\mathbf{k}_1) \cdots a^\dagger_{\rm out}(\mathbf{k}_n)|0\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n; {\rm out}\rangle. \tag{4.9}$$

We must now distinguish between Fock states generated by $a^\dagger_{\rm in}$ and $a^\dagger_{\rm out}$, and therefore we have labelled the Fock states accordingly. In eqs. (4.8) and (4.9) we have assumed that there is a stable and unique vacuum state:

$$|0\rangle = |0; {\rm in}\rangle = |0; {\rm out}\rangle. \tag{4.10}$$

Mathematically speaking, the $a^\dagger_{\rm in}$'s and $a^\dagger_{\rm out}$'s generate two different bases of the Fock space. Since the physics that we want to describe must be independent of the choice of basis, expectation values expressed in terms of "in" and "out" operators and states must satisfy

$$\langle \text{in}|\, \phi_{\rm in}(x)\, |\text{in}\rangle = \langle \text{out}|\, \phi_{\rm out}(x)\, |\text{out}\rangle\,. \tag{4.11}$$

Here $|\text{in}\rangle$ and $|\text{out}\rangle$ denote generic "in" and "out" states. We can relate the two bases by introducing a unitary operator S such that

$$\phi_{\rm in}(x) = S\,\phi_{\rm out}(x)\,S^\dagger \tag{4.12}$$

$$|\text{in}\rangle = S\,|\text{out}\rangle\,, \quad |\text{out}\rangle = S^\dagger\,|\text{in}\rangle\,, \quad S^\dagger S = 1. \tag{4.13}$$

S is called the S-matrix or S-operator. Note that the plane wave solutions of $\phi_{\rm in}$ and $\phi_{\rm out}$ also imply that

$$a^\dagger_{\rm in} = S\,a^\dagger_{\rm out}\,S^\dagger, \qquad \hat{a}_{\rm in} = S\,\hat{a}_{\rm out}\,S^\dagger. \tag{4.14}$$

By comparing "in" with "out" states one can extract information about the interaction – this is the very essence of detector experiments, where one tries to infer the nature of the interaction by studying the products of the scattering of particles that have been collided with known energies. As we will see below, this information is contained in the elements of the S-matrix.

By contrast, in the absence of any interaction, i.e. for $\mathcal{L}_{\rm int} = 0$ the distinction between $\phi_{\rm in}$ and $\phi_{\rm out}$ is not necessary. They can thus be identified, and then the relation between different bases of the Fock space becomes trivial, $S = 1$, as one would expect.

What we are ultimately interested in are transition amplitudes between an initial state i of, say, two particles of momenta $\mathbf{p}_1, \mathbf{p}_2$, and a final state f, for instance n particles of unequal momenta. The transition amplitude is then given by

$$\langle f,\, \text{out}|\, i,\, \text{in}\rangle = \langle f,\, \text{out}|\, S\,|i,\, \text{out}\rangle = \langle f,\, \text{in}|\, S\,|i,\, \text{in}\rangle \equiv S_{\rm fi}. \tag{4.15}$$

The S-matrix element $S_{\rm fi}$ therefore describes the transition amplitude for the scattering process in question. The scattering cross section, which is a measurable quantity, is then proportional to $|S_{\rm fi}|^2$. All information about the scattering is thus encoded in the S-matrix, which must therefore be closely related to the interaction Hamiltonian density $\mathcal{H}_{\rm int}$. However, before we try to derive the relation between S and $\mathcal{H}_{\rm int}$ we have to take a slight detour.

4.2 More on time evolution: Dirac picture

The operators $\phi(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$ which we have encountered are Heisenberg fields and thus time-dependent. The state vectors are time-independent in the sense that they do not satisfy a non-trivial equation of motion. Nevertheless, state vectors in the Heisenberg picture can carry a time label. For instance, the "in"-states of the previous subsection are defined at $t = -\infty$. The relation of the Heisenberg operator $\phi_H(x)$ with its counterpart ϕ_S in the Schrödinger picture is given by

$$\phi_H(\mathbf{x}, t) = \mathrm{e}^{iHt}\,\phi_S\,\mathrm{e}^{-iHt}, \qquad H = H_0 + H_{\rm int}, \tag{4.16}$$

Note that this relation involves the *full* Hamiltonian $H = H_0 + H_{\rm int}$ in the interacting theory. We have so far found solutions to the Klein-Gordon equation in the free theory, and so we know how to handle time evolution in this case. However, in the interacting case the Klein-Gordon equation has an extra term,

$$(\Box + m^2)\phi(x) + \frac{\delta V_{\rm int}(\phi)}{\delta\phi} = 0, \tag{4.17}$$

due to the potential of the interactions. Apart from very special cases of this potential, the equation cannot be solved anymore in closed form, and thus we no longer know the time evolution. It is therefore useful to introduce a new quantum picture for the interacting theory, in which the time dependence is governed by H_0 only. This is the so-called Dirac or Interaction picture. The relation between fields in the Interaction picture, ϕ_I, and in the Schrödinger picture, ϕ_S, is given by

$$\phi_I(\mathbf{x}, t) = \mathrm{e}^{iH_0 t}\, \phi_S\, \mathrm{e}^{-iH_0 t}. \tag{4.18}$$

At $t = -\infty$ the interaction vanishes, i.e. $H_{\rm int} = 0$, and hence the fields in the Interaction and Heisenberg pictures are identical, i.e. $\phi_H(\mathbf{x}, t) = \phi_I(\mathbf{x}, t)$ for $t \to -\infty$. The relation between ϕ_H and ϕ_I can be worked out easily:

$$\begin{aligned}
\phi_H(\mathbf{x}, t) &= \mathrm{e}^{iHt}\, \phi_S\, \mathrm{e}^{-iHt} \\
&= \mathrm{e}^{iHt}\, \mathrm{e}^{-iH_0 t}\, \underbrace{\mathrm{e}^{iH_0 t} \phi_S\, \mathrm{e}^{-iH_0 t}}_{\phi_I(\mathbf{x},t)}\, \mathrm{e}^{iH_0 t}\, \mathrm{e}^{-iHt} \\
&= U^{-1}(t)\, \phi_I(\mathbf{x}, t)\, U(t),
\end{aligned} \tag{4.19}$$

where we have introduced the unitary operator $U(t)$

$$U(t) = \mathrm{e}^{iH_0 t}\, \mathrm{e}^{-iHt}, \qquad U^\dagger U = 1. \tag{4.20}$$

The field $\phi_H(\mathbf{x}, t)$ contains the information about the interaction, since it evolves over time with the full Hamiltonian. In order to describe the "in" and "out" field operators, we can now make the following identifications:

$$t \to -\infty \;:\; \phi_{\rm in}(\mathbf{x}, t) = \phi_I(\mathbf{x}, t) = \phi_H(\mathbf{x}, t), \tag{4.21}$$

$$t \to +\infty \;:\; \phi_{\rm out}(\mathbf{x}, t) = \phi_H(\mathbf{x}, t). \tag{4.22}$$

Furthermore, since the fields ϕ_I evolve over time with the free Hamiltonian H_0, they always act in the basis of "in" vectors, such that

$$\phi_{\rm in}(\mathbf{x}, t) = \phi_I(\mathbf{x}, t), \qquad -\infty < t < \infty. \tag{4.23}$$

The relation between ϕ_I and ϕ_H at any time t is given by

$$\phi_I(\mathbf{x}, t) = U(t)\, \phi_H(\mathbf{x}, t)\, U^{-1}(t). \tag{4.24}$$

As $t \to \infty$ the identifications of eqs. (4.22) and (4.23) yield

$$\phi_{\rm in} = U(\infty)\, \phi_{\rm out}\, U^\dagger(\infty). \tag{4.25}$$

From the definition of the S-matrix, Eq. (4.12) we then read off that

$$\lim_{t\to\infty} U(t) = S. \tag{4.26}$$

We have thus derived a formal expression for the S-matrix in terms of the operator $U(t)$, which tells us how operators and state vectors deviate from the free theory at time t, measured relative to $t_0 = -\infty$, i.e. long before the interaction process.

An important boundary condition for $U(t)$ is

$$\lim_{t\to-\infty} U(t) = 1. \tag{4.27}$$

What we mean here is the following: the operator U actually describes the evolution relative to some initial time t_0, which we will normally suppress, i.e. we write $U(t)$ instead of $U(t, t_0)$. We regard t_0 merely as a time label and fix it at $-\infty$, where the interaction vanishes. Equation (4.27) then simply states that U becomes unity as $t \to t_0$, which means that in this limit there is no distinction between Heisenberg and Dirac fields.

Using the definition of $U(t)$, Eq. (4.20), it is an easy exercise to derive the equation of motion for $U(t)$:

$$i\frac{d}{dt}\,U(t) = H_{\rm int}(t)\,U(t), \qquad H_{\rm int}(t) = \mathrm{e}^{iH_0 t}\,H_{\rm int}\,\mathrm{e}^{-iH_0 t}. \tag{4.28}$$

The time-dependent operator $H_{\rm int}(t)$ is defined in the interaction picture, and depends on the fields $\phi_{\rm in}$, $\pi_{\rm in}$ in the "in" basis. Let us now solve the equation of motion for $U(t)$ with the boundary condition $\lim_{t\to-\infty} U(t) = 1$. Integrating Eq. (4.28) gives

$$\begin{aligned}
\int_{-\infty}^{t} \frac{d}{dt_1} U(t_1)\,dt_1 &= -i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1 \\
U(t) - U(-\infty) &= -i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1 \\
\Rightarrow\; U(t) &= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1.
\end{aligned} \tag{4.29}$$

The rhs. still depends on U, but we can substitute our new expression for $U(t)$ into the integrand, which gives

$$\begin{aligned}
U(t) &= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)\left\{1 - i\int_{-\infty}^{t_1} H_{\rm int}(t_2)\,U(t_2)\,dt_2\right\}dt_1 \\
&= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)dt_1 - \int_{-\infty}^{t} dt_1\,H_{\rm int}(t_1)\int_{-\infty}^{t_1} dt_2\,H_{\rm int}(t_2)\,U(t_2),
\end{aligned} \tag{4.30}$$

where $t_2 < t_1 < t$. This procedure can be iterated further, so that the nth term in the sum is

$$(-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n\,H_{\rm int}(t_1)\,H_{\rm int}(t_2)\cdots H_{\rm int}(t_n). \tag{4.31}$$

This iterative solution could be written in much more compact form, were it not for the fact that the upper integration bounds were all different, and that the ordering $t_n <$

$t_{n-1} < \ldots < t_1 < t$ had to be obeyed. Time ordering is an important issue, since one has to ensure that the interaction Hamiltonians act at the proper time, thereby ensuring the causality of the theory. By introducing the time-ordered product of operators, one can use a compact notation, such that the resulting expressions still obey causality. The time-ordered product of two fields $\phi(t_1)$ and $\phi(t_2)$ is defined as

$$\begin{aligned} T\left\{\phi(t_1)\,\phi(t_2)\right\} &= \begin{cases} \phi(t_1)\phi(t_2) & t_1 > t_2 \\ \phi(t_2)\phi(t_1) & t_1 < t_2 \end{cases} \\ &\equiv \theta(t_1 - t_2)\,\phi(t_1)\phi(t_2) + \theta(t_2 - t_1)\,\phi(t_2)\phi(t_1), \end{aligned} \tag{4.32}$$

where θ denotes the step function. The generalisation to products of n operators is obvious. Using time ordering for the nth term of Eq. (4.31) we obtain

$$\frac{(-i)^n}{n!}\prod_{i=1}^{n}\int_{-\infty}^{t} dt_i\; T\left\{H_{\rm int}(t_1)\,H_{\rm int}(t_2)\cdots H_{\rm int}(t_n)\right\}, \tag{4.33}$$

and since this looks like the nth term in the series expansion of an exponential, we can finally rewrite the solution for $U(t)$ in compact form as

$$U(t) = T\,\exp\left\{-i\int_{-\infty}^{t} H_{\rm int}(t')\,dt'\right\}, \tag{4.34}$$

where the "T" in front ensures the correct time ordering.

4.3 S-matrix and Green's functions

The S-matrix, which relates the "in" and "out" fields before and after the scattering process, can be written as

$$S = 1 + iT, \tag{4.35}$$

where T is commonly called the T-matrix. The fact that S contains the unit operator means that also the case where none of the particles scatter is encoded in S. On the other hand, the non-trivial case is described by the T-matrix, and this is what we are interested in. However, the S-matrix is not easily usable for practical calculations. As it stands now, it is a rather abstract concept, and we still have to relate it to the field operators appearing in our Lagrangian. This is achieved by establishing a general relation between S-matrix elements and n-point Green's functions,

$$G^n(x_1, \ldots x_n) = \langle 0|T(\phi(x_1)\ldots\phi(x_n))|0\rangle. \tag{4.36}$$

Once this step is completed, then for any given Lagrange density we may compute the Green's functions of the fields, which will in turn give us the S-matrix elements providing the link to experiment. In order to achieve this, we have to express the "in/out"-states in terms of creation operators $a^\dagger_{\rm in/out}$ and the vacuum, then express the creation operators by the fields $\phi_{\rm in/out}$, and finally use the time evolution to connect those with the fields ϕ in our Lagrangian.

Let us consider again the scattering process depicted in Fig. 4. The S-matrix element in this case is

$$\begin{aligned} S_{\rm fi} &= \Big\langle \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n; \text{out} \Big| \mathbf{p}_1, \mathbf{p}_2; \text{in} \Big\rangle \\ &= \Big\langle \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n; \text{out} \Big| a^\dagger_{\rm in}(\mathbf{p}_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle, \end{aligned} \tag{4.37}$$

where $a^\dagger_{\rm in}$ is the creation operator pertaining to the "in" field $\phi_{\rm in}$. Our task is now to express $a^\dagger_{\rm in}$ in terms of $\phi_{\rm in}$, and repeat this procedure for all other momenta labelling our Fock states.

The following identities will prove useful

$$\begin{aligned} a^\dagger(\mathbf{p}) &= i \int d^3x \left\{ \left(\partial_0 \, \mathrm{e}^{-iq\cdot x}\right) \phi(x) - \mathrm{e}^{-iq\cdot x} \, \left(\partial_0 \phi(x)\right) \right\} \\ &\equiv -i \int d^3x \, \mathrm{e}^{-iq\cdot x} \overleftrightarrow{\partial_0} \, \phi(x), \end{aligned} \tag{4.38}$$

$$\begin{aligned} \hat{a}(\mathbf{p}) &= -i \int d^3x \left\{ \left(\partial_0 \, \mathrm{e}^{iq\cdot x}\right) \phi(x) - \mathrm{e}^{iq\cdot x} \, \left(\partial_0 \phi(x)\right) \right\} \\ &\equiv i \int d^3x \, \mathrm{e}^{iq\cdot x} \overleftrightarrow{\partial_0} \, \phi(x). \end{aligned} \tag{4.39}$$

The S-matrix element can then be rewritten as

$$\begin{aligned} S_{\rm fi} &= -i \int d^3x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \overleftrightarrow{\partial_0} \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \Big| \phi_{\rm in}(x_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle \\ &= -i \lim_{t_1 \to -\infty} \int d^3x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \overleftrightarrow{\partial_0} \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \Big| \phi(x_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle, \end{aligned} \tag{4.40}$$

where in the last line we have used Eq. (4.4) to replace $\phi_{\rm in}$ by ϕ. We can now rewrite $\lim_{t_1 \to -\infty}$ using the following identity, which holds for an arbitrary, differentiable function $f(t)$, whose limit $t \to \pm\infty$ exists:

$$\lim_{t \to -\infty} f(t) = \lim_{t \to +\infty} f(t) - \int_{-\infty}^{+\infty} \frac{df}{dt} \, dt. \tag{4.41}$$

The S-matrix element then reads

$$\begin{aligned} S_{\rm fi} &= -i \lim_{t_1 \to +\infty} \int d^3x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \overleftrightarrow{\partial_0} \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \Big| \phi(x_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle \\ &\quad + i \int_{-\infty}^{+\infty} dt_1 \, \frac{\partial}{\partial t_1} \left\{ \int d^3x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \overleftrightarrow{\partial_0} \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \Big| \phi(x_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle \right\}. \end{aligned} \tag{4.42}$$

The first term in this expression involves $\lim_{t_1 \to +\infty} \phi = \phi_{\rm out}$, which gives rise to a contribution

$$\propto \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \Big| a^\dagger_{\rm out}(\mathbf{p}_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle. \tag{4.43}$$

This is non-zero only if $\mathbf{p}_1$ is equal to one of $\mathbf{k}_1, \ldots, \mathbf{k}_n$. This, however, means that the particle with momentum $\mathbf{p}_1$ does not scatter, and hence the first term does not contribute to the T-matrix of Eq. (4.35). We are then left with the following expression for $S_{\rm fi}$:

$$S_{\rm fi} = -i \int d^4x_1 \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \Big| \partial_0 \left\{ \left(\partial_0 \mathrm{e}^{-ip_1\cdot x_1}\right) \phi(x_1) - \mathrm{e}^{-ip_1\cdot x_1} \left(\partial_0 \phi(x_1)\right) \right\} \Big| \mathbf{p}_2; \text{in} \Big\rangle. \tag{4.44}$$

The time derivatives in the integrand can be worked out:

$$\begin{aligned}
&\partial_0 \left\{ \left(\partial_0 \mathrm{e}^{-ip_1\cdot x_1}\right) \phi(x_1) - \mathrm{e}^{-ip_1\cdot x_1} \left(\partial_0 \phi(x_1)\right) \right\} \\
&= -\left[E(\mathbf{p}_1)\right]^2 \, \mathrm{e}^{-ip_1\cdot x_1} \, \phi(x_1) - \mathrm{e}^{-ip_1\cdot x_1} \, \partial_0^2 \phi(x_1) \\
&= -\left\{ \left(\left(-\nabla^2 + m^2\right) \, \mathrm{e}^{-ip_1\cdot x_1} \right) \phi(x_1) + \mathrm{e}^{-ip_1\cdot x_1} \, \partial_0^2 \, \phi(x_1) \right\} , \qquad (4.45)
\end{aligned}$$

where we have used that $-\nabla^2 \mathrm{e}^{-ip_1\cdot x_1} = \mathbf{p}_1^2 \, \mathrm{e}^{-ip_1\cdot x_1}$. For the S-matrix element one obtains

$$\begin{aligned}
S_{\mathrm{fi}} &= i \int d^4x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \mathrm{out} \Big| \left(\partial_0^2 - \nabla^2 + m^2\right) \phi(x_1) \Big| \mathbf{p}_2; \mathrm{in} \Big\rangle \\
&= i \int d^4x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \, \left(\Box_{x_1} + m^2\right) \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \mathrm{out} \Big| \phi(x_1) \Big| \mathbf{p}_2; \mathrm{in} \Big\rangle . \qquad (4.46)
\end{aligned}$$

What we have obtained after this rather lengthy step of algebra is an expression in which the field operator is sandwiched between Fock states, one of which has been reduced to a one-particle state. We can now successively eliminate all momentum variables from the Fock states, by repeating the procedure for the momentum $\mathbf{p}_2$, as well as the n momenta of the "out" state. The final expression for S_{fi} is

$$\begin{aligned}
S_{\mathrm{fi}} &= (i)^{n+2} \int d^4x_1 \int d^4x_2 \int d^4y_1 \cdots \int d^4y_n \, \mathrm{e}^{(-ip_1\cdot x_1 - ip_2\cdot x_2 + ik_1\cdot y_1 + \cdots + k_n\cdot y_n)} \\
&\quad \times \left(\Box_{x_1} + m^2\right) \left(\Box_{x_2} + m^2\right) \left(\Box_{y_1} + m^2\right) \cdots \left(\Box_{y_n} + m^2\right) \\
&\quad \times \Big\langle 0; \mathrm{out} \Big| T\{\phi(y_1) \cdots \phi(y_n) \phi(x_1) \phi(x_2)\} \Big| 0; \mathrm{in} \Big\rangle , \qquad (4.47)
\end{aligned}$$

where the time-ordering inside the vacuum expectation value (VEV) ensures that causality is obeyed. The above expression is known as the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. It relates the formal definition of the scattering amplitude to a vacuum expectation value of time-ordered fields. Since the vacuum is uniquely the same for "in/out", the VEV in the LSZ formula for the scattering of two initial particles into n particles in the final state is recognised as the $(n+2)$-point Green's function:

$$G_{n+2}(y_1, y_2, \ldots, y_n, x_1, x_2) = \Big\langle 0 \Big| T\{\phi(y_1) \cdots \phi(y_n) \phi(x_1) \phi(x_2)\} \Big| 0 \Big\rangle . \qquad (4.48)$$

You will note that we still have not calculated or evaluated anything, but merely rewritten the expression for the scattering matrix elements. Nevertheless, the LSZ formula is of tremendous importance and a central piece of QFT. It provides the link between fields in the Lagrangian and the scattering amplitude S_{fi}^2, which yields the cross section, measurable in an experiment. Up to here no assumptions or approximations have been made, so this connection between physics and formalism is rather tight. It also illustrates a profound phenomenon of QFT and particle physics: the scattering properties of particles, in other words their interactions, are encoded in the vacuum structure, i.e. the vacuum is non-trivial!

4.4 How to compute Green's functions

Of course, in order to calculate cross sections, we need to compute the Green's functions. Alas, for any physically interesting and interacting theory this cannot be done exactly,

contrary to the free theory discussed earlier. Instead, approximation methods have to be used in order to simplify the calculation, while hopefully still giving reliable results. Or one reformulates the entire QFT as a lattice field theory, which in principle allows to compute Green's functions without any approximations (in practice this still turns out to be a difficult task for physically relevant systems). This is what many theorists do for a living. But the formalism stands, and if there are discrepancies between theory and experiments, one "only" needs to check the accuracy with which the Green's functions have been calculated or measured, before approving or discarding a particular Lagrangian.

In the next section we shall discuss how to compute the Green's function of scalar field theory in perturbation theory. Before we can tackle the actual computation, we must take a further step. Let us consider the n-point Green's function

$$G_n(x_1, \ldots, x_n) = \langle 0 \,| T\{\phi(x_1) \cdots \phi(x_n)\} |\, 0 \rangle \,. \tag{4.49}$$

The fields ϕ which appear in this expression are Heisenberg fields, whose time evolution is governed by the full Hamiltonian $H_0 + H_{\rm int}$. In particular, the ϕ's are *not* the $\phi_{\rm in}$'s. We know how to handle the latter, because they correspond to a free field theory, but not the former, whose time evolution is governed by the interacting theory, whose solutions we do not know. Let us thus start to isolate the dependence of the fields on the interaction Hamiltonian. Recall the relation between the Heisenberg fields $\phi(t)$ and the "in"-fields[2]

$$\phi(t) = U^{-1}(t)\, \phi_{\rm in}(t)\, U(t). \tag{4.50}$$

We now assume that the fields are properly time-ordered, i.e. $t_1 > t_2 > \ldots > t_n$, so that we can forget about writing $T(\cdots)$ everywhere. After inserting Eq. (4.50) into the definition of G_n one obtains

$$\begin{aligned} G_n = \langle 0 | U^{-1}(t_1) \phi_{\rm in}(t_1) U(t_1)\, U^{-1}(t_2) \phi_{\rm in}(t_2) U(t_2) \cdots \\ \times U^{-1}(t_n) \phi_{\rm in}(t_n) U(t_n) | 0 \rangle. \end{aligned} \tag{4.51}$$

Now we introduce another time label t such that $t \gg t_1$ and $-t \ll t_1$. For the n-point function we now obtain

$$\begin{aligned} G_n = \Big\langle 0 \Big| U^{-1}(t) \Big\{ U(t) U^{-1}(t_1) \phi_{\rm in}(t_1) U(t_1)\, U^{-1}(t_2) \phi_{\rm in}(t_2) U(t_2) \cdots \\ \times U^{-1}(t_n) \phi_{\rm in}(t_n) U(t_n) U^{-1}(-t) \Big\} U(-t) \Big| 0 \Big\rangle. \end{aligned} \tag{4.52}$$

The expression in curly braces is now time-ordered by construction. An important observation at this point is that it involves pairs of U and its inverse, for instance

$$U(t) U^{-1}(t_1) \equiv U(t, t_1). \tag{4.53}$$

One can easily convince oneself that $U(t, t_1)$ provides the net time evolution from t_1 to t. We can now write G_n as

$$G_n = \Big\langle 0 \Big| U^{-1}(t)\, T \Big\{ \phi_{\rm in}(t_1) \cdots \phi_{\rm in}(t_n) \underbrace{U(t, t_1)\, U(t_1, t_2) \cdots U(t_n, -t)}_{U(t, -t)} \Big\} U(-t) \Big| 0 \Big\rangle. \tag{4.54}$$

[2]Here and in the following we suppress the spatial argument of the fields for the sake of brevity.

Let us now take $t \to \infty$. The relation between $U(t)$ and the S-matrix Eq. (4.26), as well as the boundary condition Eq. (4.27) tell us that

$$\lim_{t\to\infty} U(-t) = 1, \qquad \lim_{t\to\infty} U(t,-t) = S, \tag{4.55}$$

which can be inserted into the above expression. We still have to work out the meaning of $\langle 0|U^{-1}(\infty)$ in the expression for G_n. In a paper by Gell-Mann and Low it was argued that the time evolution operator must leave the vacuum invariant (up to a phase), which justifies the ansatz

$$\langle 0|U^{-1}(\infty) = K\langle 0|, \tag{4.56}$$

with K being the phase. Multiplying this relation with $|0\rangle$ from the right gives

$$\langle 0|U^{-1}(\infty)|0\rangle = K\langle 0|0\rangle = K. \tag{4.57}$$

Furthermore, Gell-Mann and Low showed that

$$\langle 0|U^{-1}(\infty)|0\rangle = \frac{1}{\langle 0|U(\infty)|0\rangle}, \tag{4.58}$$

which implies

$$K = \frac{1}{\langle 0|S|0\rangle}. \tag{4.59}$$

After inserting all these relations into the expression for G_n we obtain

$$G_n(x_1,\ldots,x_n) = \frac{\langle 0|\,T\left\{\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)\,S\right\}|0\rangle}{\langle 0|S|0\rangle}. \tag{4.60}$$

The S-matrix is given by

$$S = T\,\exp\left\{-i\int_{-\infty}^{+\infty} H_{\rm int}(t)\,dt\right\}, \quad H_{\rm int} = H_{\rm int}(\phi_{\rm in},\pi_{\rm in}), \tag{4.61}$$

and thus we have finally succeeded in expressing the n-point Green's function exclusively in terms of the "in"-fields. This completes the derivation of a relation between the general definition of the scattering amplitude $S_{\rm fi}$ and the VEV of time-ordered "in"-fields. The link between the scattering amplitude and the underlying field theory is provided by the n-point Green's function.

Problems

4.1 Using the definition $U(t) = e^{iH_0t}\,e^{-iHt}$, derive the evolution equation for $U(t)$:

$$i\frac{d}{dt}\,U(t) = H_{\rm int}(t)\,U(t),$$

where

$$H_{\rm int}(t) = e^{iH_0t}\,H_{\rm int}\,e^{-iH_0t}.$$

4.2 Given that $\phi_{\rm in}$ is a free field, obeying the Heisenberg equation of motion

$$i\dot{\phi}_{\rm in} = [H_0(\phi_{\rm in}, \pi_{\rm in}),\, \phi_{\rm in}]\,,$$

show that $\phi_{\rm out}$ is also a free field, which obeys

$$i\dot{\phi}_{\rm out} = [H_0(\phi_{\rm out}, \pi_{\rm out}),\, \phi_{\rm out}]\,.$$

[**Hint:** use $\phi_{\rm out} = S^\dagger \phi_{\rm in} S$ and $\pi_{\rm out} = S^\dagger \pi_{\rm in} S$. Keep in mind that the S-matrix has no explicit time dependence.]

5 Perturbation Theory

In this section we are going to calculate the Green's functions of scalar quantum field theory explicitly. We will specify the interaction Lagrangian in detail and use an approximation known as perturbation theory. At the end we will derive a set of rules, which represent a systematic prescription for the calculation of Green's functions, and can be easily generalised to apply to other, more complicated field theories. These are the famous Feynman rules.

We start by making a definite choice for the interaction Lagrangian $\mathcal{L}_{\rm int}$. Although one may think of many different expressions for $\mathcal{L}_{\rm int}$, one has to obey some basic principles: firstly, $\mathcal{L}_{\rm int}$ must be chosen such that the potential it generates is bounded from below – otherwise the system has no ground state. Secondly, our interacting theory should be renormalisable. Despite being of great importance, the second issue will not be addressed in these lectures. The requirement of renormalisability arises because the non-trivial vacuum, much like a medium, interacts with particles to modify their properties. Moreover, if one computes quantities like the energy or charge of a particle, one typically obtains a divergent result[3]. There are classes of quantum field theories, called renormalisable, in which these divergences can be removed by suitable redefinitions of the fields and the parameters (masses and coupling constants).

For our theory of a real scalar field in four space-time dimensions, it turns out that the only interaction term which leads to a renormalisable theory must be quartic in the fields. Thus we choose

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{4!}\phi^4(x), \tag{5.1}$$

where the coupling constant λ describes the strength of the interaction between the scalar fields, much like, say, the electric charge describing the strength of the interaction between photons and electrons. The full Lagrangian of the theory then reads

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{5.2}$$

[3]This is despite the subtraction of the vacuum energy discussed earlier.

and the explicit expressions for the interaction Hamiltonian and the S-matrix are

$$\mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int}, \quad H_{\rm int} = \frac{\lambda}{4!}\int d^3x\, \phi^4_{\rm in}(\mathbf{x},t)$$
$$S = T\,\exp\left\{-i\frac{\lambda}{4!}\int d^4x\,\phi^4_{\rm in}(x)\right\}. \tag{5.3}$$

The n-point Green's function is

$$G_n(x_1,\ldots,x_n) = \frac{\displaystyle\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r\frac{1}{r!}\left\langle 0\left|T\left\{\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)\left(\int d^4y\,\phi^4_{\rm in}(y)\right)^r\right\}\right|0\right\rangle}{\displaystyle\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r\frac{1}{r!}\left\langle 0\left|T\left(\int d^4y\,\phi^4_{\rm in}(y)\right)^r\right|0\right\rangle}. \tag{5.4}$$

This expression cannot be dealt with as it stands. In order to evaluate it we must expand G_n in powers of the coupling λ and truncate the series after a finite number of terms. This only makes sense if λ is sufficiently small. In other words, the interaction Lagrangian must act as a small perturbation on the system. As a consequence, the procedure of expanding Green's functions in powers of the coupling is referred to as perturbation theory.

5.1 Wick's Theorem

The n-point Green's function in Eq. (5.4) involves the time-ordered product over at least n fields. There is a method to express VEV's of n fields, i.e. $\langle 0|T\{\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)\}|0\rangle$ in terms of VEV's involving two fields only. This is known as Wick's theorem.

Let us for the moment ignore the subscript "in" and return to the definition of normal-ordered fields. The normal-ordered product $:\phi(x_1)\phi(x_2):$ differs from $\phi(x_1)\phi(x_2)$ by the vacuum expectation value, i.e.

$$\phi(x_1)\phi(x_2) \;=:\phi(x_1)\phi(x_2):+\langle 0|\phi(x_1)\phi(x_2)|0\rangle. \tag{5.5}$$

We are now going to combine normal-ordered products with time ordering. The time-ordered product $T\{\phi(x_1)\phi(x_2)\}$ is given by

$$\begin{aligned} T\{\phi(x_1)\phi(x_2)\} &= \phi(x_1)\phi(x_2)\theta(t_1-t_2)+\phi(x_2)\phi(x_1)\theta(t_2-t_1)\\ &= :\phi(x_1)\phi(x_2):\,\Big(\theta(t_1-t_2)+\theta(t_2-t_1)\Big)\\ &\quad +\langle 0|\phi(x_1)\phi(x_2)\theta(t_1-t_2)+\phi(x_2)\phi(x_1)\theta(t_2-t_1)|0\rangle. \end{aligned} \tag{5.6}$$

Here we have used the important observation that

$$:\phi(x_1)\phi(x_2):\;=:\phi(x_2)\phi(x_1):, \tag{5.7}$$

which means that normal-ordered products of fields are automatically time-ordered.[4] Equation (5.6) is Wick's theorem for the case of two fields:

$$T\{\phi(x_1)\phi(x_2)\} \;=:\phi(x_1)\phi(x_2):+\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle. \tag{5.8}$$

[4]The reverse is, however, not true!

For the case of three fields, Wick's theorem yields

$$T\{\phi(x_1)\phi(x_2)\phi(x_3)\} = :\phi(x_1)\phi(x_2)\phi(x_3): + :\phi(x_1):\langle 0|T\{\phi(x_2)\phi(x_3)\}|0\rangle + :\phi(x_2):\langle 0|T\{\phi(x_1)\phi(x_3)\}|0\rangle + :\phi(x_3):\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle \quad (5.9)$$

At this point the general pattern becomes clear: any time-ordered product of fields is equal to its normal-ordered version plus terms in which pairs of fields are removed from the normal-ordered product and sandwiched between the vacuum to form 2-point functions. Then one sums over all permutations. Without proof we give the expression for the general case of n fields (n even):

$$\begin{aligned} T\{\phi(x_1)\cdots\phi(x_n)\} = & \\ & :\phi(x_1)\cdots\phi(x_n): \\ & +:\phi(x_1)\cdots\widehat{\phi(x_i)}\cdots\widehat{\phi(x_j)}\cdots\phi(x_n):\langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle + \text{perms.} \\ & +:\phi(x_1)\cdots\widehat{\phi(x_i)}\cdots\widehat{\phi(x_j)}\cdots\widehat{\phi(x_k)}\cdots\widehat{\phi(x_l)}\cdots\phi(x_n): \\ & \quad \times\langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle\langle 0|T\{\phi(x_k)\phi(x_l)\}|0\rangle + \text{perms.} \\ & +\ldots+ \\ & +\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle\langle 0|T\{\phi(x_3)\phi(x_4)\}|0\rangle\cdots\langle 0|T\{\phi(x_{n-1})\phi(x_n)\}|0\rangle \\ & \quad + \text{perms.}. \end{aligned} \quad (5.10)$$

The symbol $\widehat{\phi(x_i)}$ indicates that $\phi(x_i)$ has been removed from the normal-ordered product.

Let us now go back to $\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle$. If we insert Wick's theorem, then we find that only the contribution in the last line of Eq. (5.10) survives: by definition the VEV of a normal-ordered product of fields vanishes, and it is precisely the last line of Wick's theorem in which no normal-ordered products are left. The only surviving contribution is that in which all fields have been paired or "contracted". Sometimes a contraction is represented by the notation:

$$\phi\,\underbrace{(x_i)\phi}(x_j) \equiv \langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle, \quad (5.11)$$

i.e. the pair of fields which is contracted is joined by the braces. Wick's theorem can now be rephrased as

$$\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle \;=\; \text{sum of all possible contractions of } n \text{ fields.} \quad (5.12)$$

Let us look at a few examples. The first is the 4-point function

$$\begin{aligned} \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle = & \phi\,\underbrace{(x_1)\phi}(x_2)\phi\,\underbrace{(x_3)\phi}(x_4) \\ & +\phi\,\underbrace{(x_1)\phi\overbrace{(x_2)\phi}(x_3)\phi}(x_4) + \phi\underbrace{(x_1)\phi\overbrace{(x_2)\phi}(x_3)\phi}(x_4) \end{aligned} \quad (5.13)$$

The second example is again a 4-point function, where two of the fields are also normal-ordered:

$$\begin{aligned} \langle 0|T\{\phi(x_1)\phi(x_2):\phi(x_3)\phi(x_4):\}|0\rangle = & \phi\,\underbrace{(x_1)\phi}(x_2):\phi\,\underbrace{(x_3)\phi}(x_4): \\ & +\phi\,\underbrace{(x_1)\phi\overbrace{(x_2):\phi}(x_3)\phi}(x_4): +\phi(x_1)\underbrace{\phi\overbrace{(x_2):\phi}(x_3)\phi}(x_4): \end{aligned} \quad (5.14)$$

In this example, though, the contraction of $:\phi(x_3)\phi(x_4):$ vanishes by construction, so only the last two terms survive! As a general rule, contractions which only involve fields inside a normal-ordered product vanish. Such contractions contribute only to the vacuum. Normal ordering can therefore simplify the calculation of Green's functions quite considerably, as we shall see explicitly below.

5.2 The Feynman propagator

Using Wick's Theorem one can relate any n-point Green's functions to an expression involving only 2-point functions. Let us have a closer look at

$$G_2(x,y) = \langle 0|T\{\phi_{\rm in}(x)\phi_{\rm in}(y)\}|0\rangle. \tag{5.15}$$

We can now insert the solution for ϕ in terms of $\hat{a}$ and $\hat{a}^\dagger$. If we assume $t_x > t_y$ then $G_2(x,y)$ can be written as

$$\begin{aligned} G_2(x,y) &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})} \\ &\quad \times \left\langle 0 \left| \left(\hat{a}^\dagger(\mathbf{p})\, \mathrm{e}^{ip\cdot x} + \hat{a}(\mathbf{p})\, \mathrm{e}^{-ip\cdot x}\right) \left(\hat{a}^\dagger(\mathbf{q})\, \mathrm{e}^{iq\cdot y} + \hat{a}(\mathbf{q})\, \mathrm{e}^{-iq\cdot y}\right) \right| 0 \right\rangle \\ &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})}\, \mathrm{e}^{-ip\cdot x+iq\cdot y} \left\langle 0 \left| \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q}) \right| 0 \right\rangle . \end{aligned} \tag{5.16}$$

This shows that G_2 can be interpreted as the amplitude for a meson which is created at y and destroyed again at point x. We can now replace $\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q})$ by its commutator:

$$\begin{aligned} G_2(x,y) &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})}\, \mathrm{e}^{-ip\cdot x+iq\cdot y} \left\langle 0 \left| \left[\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{q})\right] \right| 0 \right\rangle \\ &= \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\, \mathrm{e}^{-ip\cdot(x-y)}, \end{aligned} \tag{5.17}$$

and the general result, after restoring time-ordering, reads

$$G_2(x,y) = \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})} \left(\mathrm{e}^{-ip\cdot(x-y)}\theta(t_x-t_y) + \mathrm{e}^{ip\cdot(x-y)}\theta(t_y-t_x)\right). \tag{5.18}$$

Furthermore, using contour integration one can show that this expression can be rewritten as a 4-dimensional integral

$$G_2(x,y) = i\int \frac{d^4p}{(2\pi)^4}\, \frac{\mathrm{e}^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon}, \tag{5.19}$$

where ϵ is a small parameter which ensures that G_2 does not develop a pole. This calculation has established that $G_2(x,y)$ actually depends only on the difference $(x-y)$. Equation (5.19) is called the Feynman propagator $G_F(x-y)$:

$$G_F(x-y) \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i\int \frac{d^4p}{(2\pi)^4}\, \frac{\mathrm{e}^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon}. \tag{5.20}$$

The Feynman propagator is a Green's function of the Klein-Gordon equation, i.e. it satisfies

$$\left(\Box_x + m^2\right) G_F(x-y) = -i\delta^4(x-y), \tag{5.21}$$

and describes the propagation of a meson between the space-time points x and y.

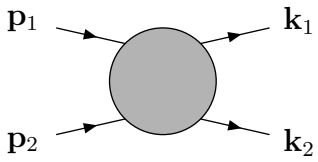


Figure 5: Scattering of two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ into 2 particles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$.

5.3 Two-particle scattering to O(λ)

Let us now consider a scattering process in which two incoming particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter into two outgoing ones with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, as shown in Fig. 5. The S-matrix element in this case is

$$\begin{aligned} S_{\rm fi} &= \langle \mathbf{k}_1, \mathbf{k}_2; \text{out} | \mathbf{p}_1, \mathbf{p}_2; \text{in} \rangle \\ &= \langle \mathbf{k}_1, \mathbf{k}_2; \text{in} | S | \mathbf{p}_1, \mathbf{p}_2; \text{in} \rangle, \end{aligned} \tag{5.22}$$

and $S = 1 + iT$. The LSZ formula Eq. (4.47) tells us that we must compute G_4 in order to obtain $S_{\rm fi}$. Let us work out G_4 in powers of λ using Wick's theorem. To make life simpler, we shall introduce normal ordering into the definition of S, i.e.

$$S = T \, \exp\left\{ -i\frac{\lambda}{4!} \int d^4x \, :\phi^4_{\rm in}(x): \right\} \tag{5.23}$$

Suppressing the subscripts "in" from now on, the expression we have to evaluate order by order in λ is

$$\begin{aligned} &G_n(x_1, \ldots, x_n) \\ &= \frac{\displaystyle\sum_{r=0}^{\infty} \left(-\frac{i\lambda}{4!}\right)^r \frac{1}{r!} \left\langle 0 \left| T\left\{ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \left(\int d^4y \, :\phi^4(y): \right)^r \right\} \right| 0 \right\rangle}{\displaystyle\sum_{r=0}^{\infty} \left(-\frac{i\lambda}{4!}\right)^r \frac{1}{r!} \left\langle 0 \left| T\left(\int d^4y \, :\phi^4(y): \right)^r \right| 0 \right\rangle}. \end{aligned} \tag{5.24}$$

Starting with the denominator, we note that for $r = 0$ one finds

$$r = 0: \qquad \text{denominator} = 1. \tag{5.25}$$

If $r = 1$, then the expression in the denominator only involves fields which are normal-ordered. Following the discussion at the end of section 5.1 we conclude that these contributions must vanish, hence

$$r = 1: \qquad \text{denominator} = 0. \tag{5.26}$$

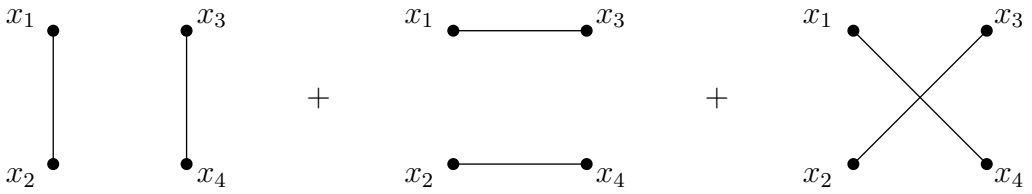
The contribution for $r = 2$, however, is non-zero. But then the case of $r = 2$ corresponds already to O(λ^2), which is higher than the order which we are working to. Therefore

$$\text{denominator} = 1 \text{ to order } \lambda. \tag{5.27}$$

Turning now to the numerator, we start with $r = 0$ and apply Wick's theorem, which gives

$$\begin{aligned} r = 0: \quad & \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle \\ = \; & G_F(x_1 - x_2)\, G_F(x_3 - x_4) + G_F(x_1 - x_3)\, G_F(x_2 - x_4) \\ & + G_F(x_1 - x_4)\, G_F(x_2 - x_3), \end{aligned} \tag{5.28}$$

which can be graphically represented as

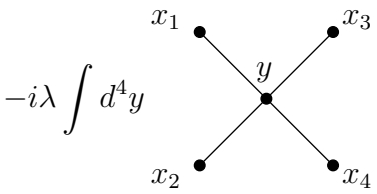


But this is the same answer as if we had set $\lambda = 0$, so $r = 0$ in the numerator does not describe scattering and is hence not a contribution to the T-matrix.

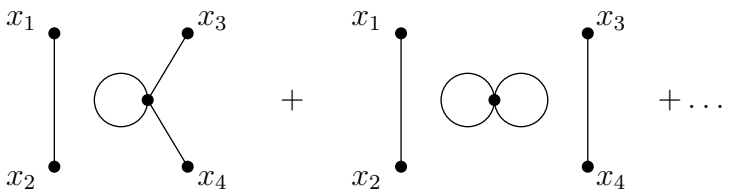
For $r = 1$ in the numerator we have to evaluate

$$\begin{aligned} r = 1: \quad & -\frac{i\lambda}{4!} \left\langle 0 \left| T \left\{ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : \int d^4y\, \phi^4(y) : \right\} \right| 0 \right\rangle \\ = \; & -\frac{i\lambda}{4!} \int d^4y\; 4!\; G_F(x_1 - y) G_F(x_2 - y) G_F(x_3 - y) G_F(x_4 - y), \end{aligned} \tag{5.29}$$

where we have taken into account that contractions involving two fields inside $: \cdots :$ vanish. The factor 4! inside the integrand is a combinatorial factor: it is equal to the number of permutations which must be summed over according to Wick's theorem and cancels the 4! in the denominator of the interaction Lagrangian. Graphically this contribution is represented by



where the integration over y denotes the sum over all possible locations of the interaction point y. Without normal ordering we would have encountered the following contributions for $r = 1$:



Such contributions are corrections to the vacuum and are *cancelled* by the denominator. This demonstrates how normal ordering simplifies the calculation by automatically subtracting terms which do not contribute to the actual scattering process.

To summarise, the final answer for the scattering amplitude to O(λ) is given by Eq. (5.29).

5.4 Graphical representation of the Wick expansion: Feynman rules

We have already encountered the graphical representation of the expansion of Green's functions in perturbation theory after applying Wick's theorem. It is possible to formulate a simple set of rules which allow to draw the graphs directly without using Wick's theorem and to write down the corresponding algebraic expressions.

We again consider a neutral scalar field whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{5.30}$$

Suppose now that we want to compute the O(λ^m) contribution to the n-point Green's function $G_n(x_1, \ldots, x_n)$. This is achieved by going through the following steps:

(1) Draw all <u>distinct</u> diagrams with n external lines and m 4-fold vertices:

- Draw n dots and label them $x_1, \ldots, x_n$ (external points)
- Draw m dots and label them $y_1, \ldots, y_m$ (vertices)
- Join the dots according to the following rules:
  - only one line emanates from each x_i
  - exactly four lines run into each y_j
  - the resulting diagram must be connected, i.e. there must be a continuous path between any two points.

(2) Assign a factor $-\dfrac{i\lambda}{4!}\displaystyle\int d^4y_i$ to the vertex at y_i

(3) Assign a factor $G_F(x_i - y_j)$ to the line joining x_i and y_j

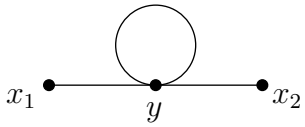
(4) Multiply by the number of contractions $\mathcal{C}$ from the Wick expansion which lead to the same diagram.

These are the Feynman rules for scalar field theory in <u>position space</u>.

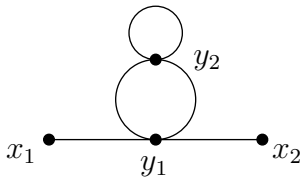
Let us look at an example, namely the 2-point function. According to the Feynman rules the contributions up to order λ^2 are as follows:

O(1): 

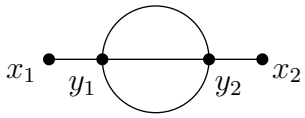
$= G_F(x_1 - x_2)$

O(λ): 

"tadpole diagram";
(cancelled by normal ordering)

$O(\lambda^2)$: 

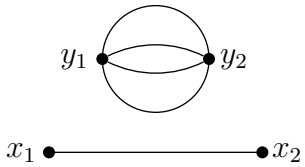
"tadpole diagram";
(cancelled by normal ordering)

$O(\lambda^2)$: 

$$= \mathcal{C}\left(-\frac{i\lambda}{4!}\right)^2 \int d^4y_1 d^4y_2\, G_F(x_1-y_1)\,[G_F(y_1-y_2)]^3\, G_F(y_2-x_2)$$

The combinatorial factor for this contribution is worked out as $\mathcal{C} = 4 \cdot 4!$. Note that the same graph, but with the positions of y_1 and y_2 interchanged is topologically distinct. Numerically it has the same value as the above graph, and so the corresponding expression has to be multiplied by a factor 2.

Another contribution at order λ^2 is

$O(\lambda^2)$: 

vacuum contribution;
not connected

This contribution must be discarded, since not all of the points are connected via a continuous line.

Let us end this discussion with a small remark on the tadpole diagrams encountered above. These contributions to the 2-point function are cancelled if the interaction term is normal-ordered. However, unlike the case of the 4-point function, the corresponding diagrams satisfy the Feynman rules listed above. In particular, the diagrams are connected and are not simply vacuum contributions. They must hence be included in the expression for the 2-point function.

5.5 Feynman rules in momentum space

It is often simpler to work in momentum space, and hence we will discuss the derivation of Feynman rules in this case. If one works in momentum space, the Green's functions are related to those in position space by a Fourier transform

$$G_n(x_1,\ldots,x_n) = \int \frac{d^4p_1}{(2\pi)^4}\cdots\int \frac{d^4p_n}{(2\pi)^4}\, \mathrm{e}^{ip_1\cdot x_1+\ldots+ip_n\cdot x_n}\, \widetilde{G}_n(p_1,\ldots,p_n). \tag{5.31}$$

The Feynman rules then serve to compute the Green's function $\widetilde{G}_n(p_1,\ldots,p_n)$ order by order in the coupling.

In every scattering process the overall momentum must be conserved, and hence

$$\sum_{i=1}^{n} p_i = 0. \tag{5.32}$$

This can be incorporated into the definition of the momentum space Green's function one is trying to compute:

$$\widetilde{G}_n(p_1,\ldots,p_n) = (2\pi)^4\delta^4\left(\sum_{i=1}^{n} p_i\right)\,\mathcal{G}_n(p_1,\ldots,p_n). \tag{5.33}$$

Here we won't be concerned with the exact derivation of the momentum space Feynman rules, but only list them as a recipe.

Feynman rules (momentum space)

(1) Draw all distinct diagrams with n external lines and m 4-fold vertices:

- Assign momenta $p_1,\ldots,p_n$ to the external lines
- Assign momenta k_j to the internal lines

(2) Assign to each external line a factor

$$\frac{i}{p_k^2 - m^2 + i\epsilon}$$

(3) Assign to each internal line a factor

$$\int \frac{d^4k_j}{(2\pi)^4}\,\frac{i}{k_j^2 - m^2 + i\epsilon}$$

(4) Each vertex contributes a factor

$$-\frac{i\lambda}{4!}(2\pi)^4\delta^4\left(\sum \text{momenta}\right),$$

(the delta function ensures that momentum is conserved at each vertex).

(5) Multiply by the combinatorial factor $\mathcal{C}$, which is the number of contractions leading to the same momentum space diagram (note that $\mathcal{C}$ may be different from the combinatorial factor for the same diagram considered in position space!)

5.6 S-matrix and truncated Green's functions

The final topic in these lectures is the derivation of a simple relation between the S-matrix element and a particular momentum space Green's function, which has its external legs amputated: the so-called truncated Green's function. This further simplifies the calculation of scattering amplitudes using Feynman rules.

Let us return to the LSZ formalism and consider the scattering of m initial particles (momenta $\mathbf{p}_1,\ldots,\mathbf{p}_m$) into n final particles with momenta $\mathbf{k}_1,\ldots,\mathbf{k}_n$. The LSZ formula

tells us that the S-matrix element is given by

$$\begin{aligned}
&\Big\langle \mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|\mathbf{p}_1,\ldots,\mathbf{p}_m;\mathrm{in}\Big\rangle \\
&\quad = \quad (i)^{n+m}\int\prod_{i=1}^{m}d^4x_i\int\prod_{j=1}^{n}d^4y_j\exp\left\{-i\sum_{i=1}^{m}p_i\cdot x_i+i\sum_{j=1}^{n}k_j\cdot y_j\right\} \\
&\qquad\times\prod_{i=1}^{m}\left(\Box_{x_i}+m^2\right)\prod_{j=1}^{n}\left(\Box_{y_j}+m^2\right)\,G_{n+m}(x_1,\ldots,x_m,y_1,\ldots,y_n). \qquad (5.34)
\end{aligned}$$

Let us have a closer look at $G_{n+m}(x_1,\ldots,x_m,y_1,\ldots,y_n)$. As shown in Fig. 6 it can be split into Feynman propagators, which connect the external points to the vertices at $z_1,\ldots,z_{n+m}$, and a remaining Green's function $\overline{G}_{n+m}$, according to

$$G_{n+m}=\int d^4z_1\cdots d^4z_{n+m}\,G_F(x_1-z_1)\cdots G_F(y_n-z_{n+m})\,\overline{G}_{n+m}(z_1,\ldots,z_{n+m}), \qquad (5.35)$$

where, perhaps for obvious reasons, $\overline{G}_{n+m}$ is called the truncated Green's function.

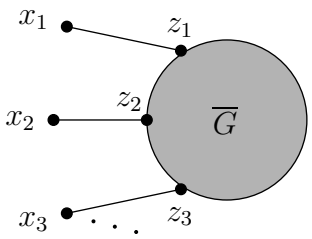


Figure 6: The construction of the truncated Green's function in position space.

Putting Eq. (5.35) back into the LSZ expression for the S-matrix element, and using that

$$\left(\Box_{x_i}+m^2\right)\,G_F(x_i-z_i)=-i\delta^4(x_i-z_i) \qquad (5.36)$$

one obtains

$$\begin{aligned}
&\Big\langle \mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|\mathbf{p}_1,\ldots,\mathbf{p}_m;\mathrm{in}\Big\rangle \\
&\quad = \quad (i)^{n+m}\int\prod_{i=1}^{m}d^4x_i\int\prod_{j=1}^{n}d^4y_j\exp\left\{-i\sum_{i=1}^{m}p_i\cdot x_i+i\sum_{j=1}^{n}k_j\cdot y_j\right\} \qquad (5.37) \\
&\qquad\times(-i)^{n+m}\int d^4z_1\cdots d^4z_{n+m}\,\delta^4(x_1-z_1)\cdots\delta^4(y_n-z_{n+m})\,\overline{G}_{n+m}(z_1,\ldots,z_{n+m}).
\end{aligned}$$

After performing all the integrations over the z_k's, the final relation becomes

$$\begin{aligned}
&\Big\langle \mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|\mathbf{p}_1,\ldots,\mathbf{p}_m;\mathrm{in}\Big\rangle \\
&\quad = \quad \int\prod_{i=1}^{m}d^4x_i\prod_{j=1}^{n}d^4y_j\exp\left\{-i\sum_{i=1}^{m}p_i\cdot x_i+i\sum_{j=1}^{n}k_j\cdot y_j\right\} \\
&\qquad\times\,\overline{G}_{n+m}(x_1,\ldots,x_m,y_1,\ldots,y_n) \\
&\quad \equiv \quad \overline{\mathcal{G}}_{n+m}(p_1,\ldots,p_m,k_1,\ldots,k_n), \qquad (5.38)
\end{aligned}$$

where $\overline{\mathcal{G}}_{n+m}$ is the truncated $n+m$-point function in momentum space. This result shows that the scattering matrix element is directly given by the truncated Green's function in momentum space. The latter can be obtained using the Feynman rules without the expression for the external legs.

Problems

5.1 Verify that

$$: \phi(x_1)\phi(x_2) : = : \phi(x_2)\phi(x_1) :$$

Hint: write $\phi = \phi^+ + \phi^-$, where ϕ^+ and ϕ^- are creation and annihilation components of ϕ.

5.2 Verify that

$$G_F(x-y) = i \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{e}^{ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$

is a Green's function of $(\partial^\mu \partial_\mu + m^2)$ as $\epsilon \to 0$ (where $\partial_\mu \equiv \partial/\partial x^\mu$).

5.3 Find the expressions corresponding to the following *momentum space* Feynman diagrams

Integrate out all the δ-functions but do not perform the remaining integrals.

6 Concluding remarks

Although we have missed out on many important topics in Quantum Field Theory, we got to the point where we established contact between the underlying formalism of Quantum Field Theory and the Feynman rules, which are widely used in perturbative calculations. The main concepts of the formulation were discussed: we introduced field operators, multi-particle states that live in Fock spaces, creation and annihilation operators, the connections between particles and fields as well as that between n-point Green's functions and scattering matrix elements. Besides slight complications in accounting for the additional degrees of freedom, the same basic ingredients can be used to formulate a quantum theory for electrons, photons or any other fields describing particles in the Standard Model and beyond. Starting from relativistic wave equations, this is discussed in the lectures by Nick Evans at this school. Renormalisation is a topic which is not so easily discussed in a relatively short period of time, and hence I refer the reader to standard textbooks on Quantum Field Theory, which are listed below. The same applies to the method of quantisation via path integrals.

Acknowledgements

I am indebted to Hartmut Wittig, on whose 2003 account of these lectures the current version is based. In particular Sections 4 and 5 have been taken over with little change. I would like to thank Tim Greenshaw for running the school so successfully, as well as Margaret Evans for her friendly and efficient organisation. Many thanks go to my fellow lecturers and the tutors for the pleasant and entertaining collaboration, and to all the students for their interest and questions, which made for a lively and inspiring atmosphere.

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A Notation and conventions

4-vectors:

$$
\begin{aligned}
&x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x}) \\
&x_\mu = g_{\mu\nu}\, x^\nu = (x^0, -\mathbf{x}) = (t, -\mathbf{x}) \\
&\text{Metric tensor:} \quad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

Scalar product:

$$
\begin{aligned}
x^\mu x_\mu &= x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 \\
&= t^2 - \mathbf{x}^2
\end{aligned}
$$

Gradient operators:

$$
\begin{aligned}
&\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right) \\
&\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right) \\
&\text{d'Alembertian:} \quad \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box
\end{aligned}
$$

Momentum operator:

$$
\hat{p}^\mu = i\hbar\partial^\mu = \left(i\hbar\frac{\partial}{\partial t},\, -i\hbar\nabla\right) = \left(\hat{E},\, \hat{\mathbf{p}}\right) \quad \text{(as it should be)}
$$

δ-functions:

$$
\begin{aligned}
&\int d^3p\, f(\mathbf{p})\, \delta^3(\mathbf{p} - \mathbf{q}) = f(\mathbf{q}) \\
&\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3 \delta^3(\mathbf{p}) \\
&\int \frac{d^3p}{(2\pi)^3}\, e^{-i\mathbf{p}\cdot\mathbf{x}} = \delta^3(\mathbf{x}) \\
&\text{(similarly in four dimensions)}
\end{aligned}
$$

Note:

$$
\begin{aligned}
\delta(x^2 - x_0^2) &= \delta\{(x - x_0)(x + x_0)\} \\
&= \frac{1}{2x}\left\{\delta(x - x_0) + \delta(x + x_0)\right\}
\end{aligned}
$$