

JOHANN WOLFGANG VON GOETHE
UNIVERSITÄT FRANKFURT

INSTITUT FÜR THEORETISCHE PHYSIK

BACHELOR THESIS

Some Polyakov Loop
Expectation Values with an
Effective Lattice Theory

Author:
Andreas HALSCH

Supervisor:
Prof. Owe PHILIPSEN



October 20, 2015

Abstract

In this work several expectation values of gauge invariant objects on the lattice called Polyakov loops are calculated analytically using an effective lattice theory. For this calculation the technique of Linked Cluster Expansion is briefly presented as well as an integration technique to solve gauge integrals over Polyakov loops. Finally a comparison between analytical results and numerical computations of the expectation values is made. At the end a correlation function of two cubed Polyakov loops is discussed.

In dieser Arbeit werden verschiedene Erwartungswerte eichinvarianter Objekte der Gitter QCD genannt Polyakov loops mit Hilfe einer effektiven Theorie auf dem Gitter analytisch berechnet. Weiterhin wird die Integration von Polyakov loops mithilfe des Haarschen Integrationsmaß erläutert. Zusätzlich wird zur Berechnung der Erwartungswerte der Formalismus der linked cluster expansion einführend erläutert. Die erlangten analytischen Ergebnisse werden mit numerischen Berechnungen der Erwartungswerte verglichen. Abschließend wird die Korrelationsfunktion zweier Polyakov loops zur dritten Potenz diskutiert.

Contents

Introduction	3
1 Gauge Theory on the Lattice	5
1.1 Gauge Transformations on the Lattice	5
1.2 The Haar Measure	8
1.3 The Polyakov Loop	9
2 The Effective Model	11
2.1 Definition of the Effective Action	11
2.2 Gauge Part	12
2.3 Fermionic Part	13
2.4 The Effective Partition Function	16
3 Gauge Integrals over Polyakov Loops	17
3.1 Gauge Integrals over Polyakov Loops - Using Traces of Wilson Lines	17
3.2 Gauge Integrals over Polyakov Loops - Using an Integration Measure for Polyakov Loops	20
4 Linked Cluster Expansion	21
4.1 Definitions	21
4.2 Introduction: Linked Cluster Expansion of the Free Energy . .	22
4.3 Linked Cluster Expansion of Expectation Values	25
4.4 Linked Cluster Expansion of Connected Two Point Correla- tion Functions	29
5 Expectation Values of Polyakov Loops	33
5.1 The L Expectation Value	33
5.2 The L^\dagger Expectation Value	42
5.3 The L^3 Expectation Value	46
5.4 The $(L^\dagger)^3$ Expectation Value	50
6 The $\langle L^3 L^3 \rangle - \langle L^3 \rangle \langle L^3 \rangle$ Correlation Function	55
Outlook	57

Introduction

Quantumchromodynamics is the fundamental theory describing interactions between quarks. The corresponding interaction is called strong interaction and is formulated in terms of quantum field theory meaning quantized quark fields interact by exchanging force carrier called gluons.

Quarks are spin $\frac{1}{2}$ particles which are described by a Lagrange density including the Dirac operator. Furthermore Quantumchromodynamics is a gauge theory so its corresponding Lagrangian needs to be invariant under local gauge transformations. The introduced gauge fields can be interpreted as gluon fields. Hence the Lagrange density of QCD is written as a superposition of the quark part coupling to gauge fields with help of the covariant derivative D_μ and a term mediating gauge field dynamics.

$$\mathcal{L}_{QCD} = \bar{q}(i\gamma^\mu D_\mu - m)q - \frac{1}{2}F_{\mu\nu}^a F_a^{\mu\nu}$$

In contrast to Quantumelectrodynamics the force carrier of QCD can self interact. As a result of this self interaction quarks can not be isolated at low temperatures. This phenomena is known as confinement and is the reason that quarks can only be found in connected bound states as baryons or mesons for low temperatures.

For low energies the strong self interaction of gluons manifests itself in large values of the coupling constant in QCD. Therefore perturbation theory is not possible for low energy QCD since higher orders can not be neglected. As a result new formulations of QCD models as effective theories need to be found.

A successful formulation of non-perturbative QCD is the lattice formulation. Instead of calculating in continuous space, spacetime is replaced by an Euclidean cubic lattice with constant lattice spacing a . Hence lattice points describe points in spacetime.

Lattice QCD is not only very successful when solving QCD with the help of numerical computation but also analytical calculations can help theoretical understanding. In this work analytical calculations of expectation values of gauge invariant objects on the lattice called Polyakov loops are presented and compared to numerical computed results. The calculations are done with help of an effective lattice theory of QCD where spatial gauge parts were already integrated out.

1 Gauge Theory on the Lattice

The principle of gauge invariance meaning the invariance of a Lagrangian under a local gauge transformation appears to be one of the most successful principles in physics when describing fundamental interactions. In fact three of the four fundamental interactions namely the electromagnetic, the weak and the strong interaction are formulated in terms of gauge theory.

The well known continuum gauge theory needs to be reformulated when discretizing spacetime. Therefore a gauge transformation on the lattice needs to be introduced.

1.1 Gauge Transformations on the Lattice

On the lattice we claim as well as in the continuum that the fermion spinors transform in the following way

$$\psi(x) \rightarrow \psi'(x) = \Lambda(x)\psi(x) \quad \Lambda(x) \in SU(3) \quad (1)$$

The introduced $\Lambda(x)$ are elements of the compact Lie-group $SU(3)$.

In the (naive) fermion action of an arbitrary lattice \mathcal{G} the continuum derivatives of the Dirac operator are replaced by numerical derivatives due to discretized spacetime [1].

$$S_F^0[\psi, \bar{\psi}] = a^4 \sum_{x \in \mathcal{G}} \bar{\psi}(x) \left(\sum_{\mu=1}^4 \gamma_\mu \frac{\psi(x + \hat{\mu}) - \psi(x - \hat{\mu})}{2a} + m\psi(x) \right) \quad (2)$$

$$\partial_\mu \psi(x) \rightarrow \frac{1}{2a}(\psi(x + \hat{\mu}) - \psi(x - \hat{\mu})) \quad (3)$$

With $x + \hat{\mu}$ denoting a neighboring lattice point to x .

Applying a local gauge transformation (1) does not lead to an invariance of the Lagrangian.

$$\bar{\psi}(x)\psi(x + \hat{\mu}) \rightarrow \bar{\psi}'(x)\psi'(x + \hat{\mu}) = \bar{\psi}(x)\Lambda(x)^\dagger \Lambda(x + \hat{\mu})\psi(x + \hat{\mu}) \quad (4)$$

1.1 Gauge Transformations on the Lattice

That is why a so called gauge field $U_\mu(x)$ is introduced.

$$\begin{aligned}\bar{\psi}'(x)U'_\mu(x)\psi'(x + \hat{\mu}) &= \bar{\psi}(x)\Lambda(x)^\dagger U'_\mu(x)\Lambda(x + \hat{\mu})\psi(x + \hat{\mu}) \\ &= \bar{\psi}(x)U_\mu(x)\psi(x + \hat{\mu})\end{aligned}\quad (5)$$

The gauge transformation of the gauge field is defined in such way that the Action S is kept invariant under a local gauge transformation.

$$U_\mu(x) \rightarrow U'_\mu(x) = \Lambda(x)U_\mu(x)\Lambda(x + \hat{\mu})^\dagger \quad (6)$$

The $U_\mu(x)$ are referred to as gauge link variables and can be interpreted as links connecting lattice points x and $x + \hat{\mu}$. They are often written as $U_\mu(x) = U(x, x + \hat{\mu})$.

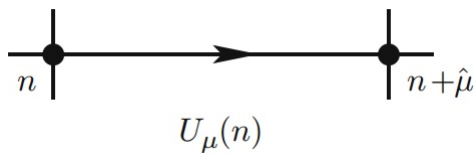


Figure 1: Gauge link connecting lattice points x and $x + \hat{\mu}$

Physical equivalent objects in continuum gauge theory are the so called gauge transporters, further details given in [1].

Yet the introduced gauge fields have no internal dynamics so a further dynamics term needs to be added to the action. Since every term of the action has to be gauge invariant the dynamics term has to fulfill this property.

1.1 Gauge Transformations on the Lattice

A simple gauge invariant object on the lattice is a closed Loop \mathcal{L} constructed from gauge links. Connecting four gauge links in a circle constructs the smallest closed loop called Plaquette U_p .

$$U_p = U(x, x + \hat{\mu})U(x + \hat{\mu}, x + \hat{\mu} + \hat{\nu})U(x + \hat{\mu} + \hat{\nu}, x + \hat{\nu})U(x + \hat{\nu}, x) \quad (7)$$

It can easily be checked that the trace of a Plaquette is a gauge invariant object.

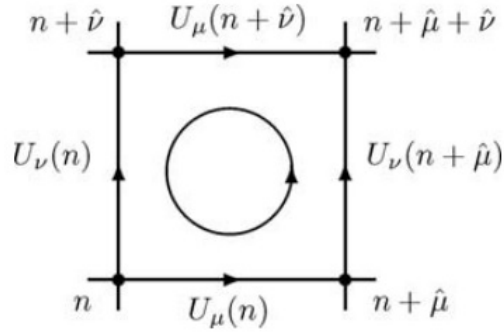


Figure 2: Plaquette constructed out of four gauge links

With help of the Plaquette a gauge invariant term describing gauge field dynamics can be defined. It is defined as a sum over all Plaquettes on the lattice and referred to as Wilson Gauge Action.

$$-S_g = \frac{\beta}{2N_c} \sum_p [\text{tr } U_p + \text{tr } U_p^\dagger] \quad (8)$$

With the inverse gauge coupling β and the number of colors N_c . For the SU(3) gauge theory of QCD N_c is given as $N_c = 3$ [1] [2].

$$\beta = \frac{6}{g^2} \quad (9)$$

1.2 The Haar Measure

Lattice QCD is formulated by making use of the Euclidean path integral. With its help the partition function of QCD can be written.

$$Z = \int \exp(-S_f[\psi, \bar{\psi}, U] - S_g[U]) [d\bar{\psi}][d\psi][dU] \quad (10)$$

The fermion Spinors ψ and $\bar{\psi}$ are integrated using Grassman integration. On the opposite a special group integration measure is needed for the gauge fields U.

As elements of the SU(3) group the gauge fields U need to be integrated over the group manifold of SU(3). The integration measure affording this is called Haar measure.

As a group integration measure the Haar measure is defined to be invariant under gauge transformation or more general under multiplication with a different group element. Let \mathcal{G} be an arbitrary SU(N) group, an integration measure of the group should then fulfill the following property.

$$dU = d(U \cdot V) = d(V \cdot U) \quad U, V \in \mathcal{G} \quad (11)$$

On top of that a normalization constraint is needed. Therefor the integral over the whole group is normalized to one per definition.

$$\int [dU] = 1 \quad (12)$$

With these two definitions the Haar measure over compact Lie groups is already completely defined.

With help of equation (11) one can prove the basic tool to analyse group integrals over compact Lie groups \mathcal{G} . [1]

$$\int f(U) [dU] = \int f(V \cdot U) [dU] = \int f(U \cdot W) [dU] \quad U, V, W \in \mathcal{G} \quad (13)$$

With f being an arbitrary function of group elements U.

1.3 The Polyakov Loop

Physical observables always have to be gauge invariant objects. The simplest gauge invariant object, the Plaquette has already been discussed in section 1.1. As mentioned a proper way to define gauge invariant objects on the lattice is to order gauge links in a closed loop \mathcal{L} .

For this work the Polyakov loop has a particular importance. To determine the Polyakov loop it is convenient to define the Wilson loop \mathcal{W} at first.

A Wilson loop consists of two spatial Wilson lines $S(\vec{x}, \vec{y}, t)$, $S(\vec{x}, \vec{y}, 0)$ and two temporal Wilson lines $T(\vec{x}, t)$, $T(\vec{y}, t)$ [1].

A spatial Wilson line connects two lattice points x and y along some arbitrary path $\mathcal{C}_{\vec{x}, \vec{y}}$ only consisting of spatial gauge links.

$$S(\vec{x}, \vec{y}, t) = \prod_{(\vec{k}, j) \in \mathcal{C}_{\vec{x}, \vec{y}}} U_j(\vec{k}, t) \quad (14)$$

The temporal Wilson line is a straight line of temporal gauge links situated on a fixed spatial position. It describes a timelike propagation from 0 to t along the temporal lattice extent.

$$T(\vec{x}, t) = \prod_{j=0}^{t-1} U_0(\vec{x}, j) \quad (15)$$

The Wilson loop is defined by attaching two spatial and two temporal lines in a closed loop and taking the trace.

$$\mathcal{W}[U] = \text{tr}[S(\vec{x}, \vec{y}, 0)T(\vec{y}, t)S(\vec{x}, \vec{y}, t)^\dagger T(\vec{x}, t)^\dagger] = \text{tr} \left[\prod_{(k, \mu) \in \mathcal{L}} U_\mu(k) \right] \quad (16)$$

For a Polyakov loop the temporal extent of the lattice needs to be maximized until it is as large as possible on the chosen lattice. N_T denotes the total number of lattice points in time direction. Furthermore periodic boundary conditions are used in temporal direction.

As a result the spatial pieces of the Wilson loop sit on top of each other oriented in opposite direction. It is possible to gauge the spatial links of the loop to 1 [1].

So the loop reduces to two disconnected temporal Wilson lines located at \vec{x} and \vec{y} with opposite orientation and maximized temporal extent N_T .

1.3 The Polyakov Loop

Taking the trace of these temporal Wilson lines individually defines the Polyakov loop [1].

$$L_{\vec{x}} = \text{tr} T(\vec{x}, N_T) = \text{tr} \left[\prod_{j=0}^{N_T-1} U_0(\vec{x}, j) \right] \quad (17)$$

This is a gauge invariant loop due to periodic boundary conditions in time direction. It is important to notice that the Polyakov loop is a timelike loop located at a fixed spatial position.

With the Haar measure and the Polyakov loop two fundamental quantities needed in this work are introduced. The next step is to formulate an effective QCD lattice theory whereby analytical calculations can be done.

2 The Effective Model

Effective theories are a successful approach to describe complicated physical phenomena. Often calculations are getting much simpler if they are done with help of an effective theory. The now introduced effective model has the advantage that analytical gauge integrations can be done.

2.1 Definition of the Effective Action

At first the effective action of the model is defined. Starting point is the well known Euclidean partition function of lattice QCD with its fermion action S_f and gauge action S_g [4].

$$Z = \int \exp(-S_f[\psi, \bar{\psi}, U] - S_g[U]) [d\bar{\psi}][d\psi][dU] \quad (18)$$

The Grassmann integration of the fermionic part leads to the so called fermion determinant $\det[Q^f]$ with f being the number of quark flavors [1].

For simplicity f is set to 1 in this work so there is only one quark flavor.

$$Z = \int \exp(-S_g[U]) \cdot \det(Q) [dU] \quad (19)$$

The effective action is now defined by integrating out all spatial link variables U_j [4].

$$Z = \int \exp(-S_{eff}) [dU_0] \quad (20)$$

with

$$\exp(-S_{eff}) = \int \exp(-S_g[U]) \cdot \det(Q) [dU_j] \quad (21)$$

As a consequence the defined effective partition function of the effective model only depends on temporal gauge links. Hence it only depends on temporal Wilson lines or their traces the Polyakov loops.

2.2 Gauge Part

The Yang Mills action in the partition function is expanded in terms of the character expansion [3].

$$\exp\left(\frac{\beta}{2N_c}[tr U_p + tr U_p^\dagger]\right) = c_0(\beta) \left[1 + \sum_{r \neq 0} d_r a_r(\beta) \chi_r(U_p)\right] \quad (22)$$

The coefficients $c_0(\beta)$ and $a_r(\beta) := u$ are functions of the inverse gauge coupling β and r denotes an arbitrary representation of the group. The traces are defined as $\chi_r(U_p) = tr U_p^r$.

The prefactor $c_0(\beta)$ cancels when calculating expectation values due to its independence of gauge links so it can be neglected.

To leading order the action is approximated by a chain of N_T Plaquettes winding around the lattice and closing via periodic boundary conditions [3].

$$\int tr (V \cdot U) tr (U^\dagger \cdot V) [dU] = \frac{1}{N_c} tr (V \cdot W) \quad (23)$$

The overlapping links of the Plaquettes can be integrated out using (23) [1]. So the chain of Plaquettes reduces to two nearest neighbor lines of N_T gauge links hence temporal Wilson lines. Taking the trace of these lines leads to Polyakov loops. As a result the gauge contribution to leading order is given by an interaction of two nearest neighbor Polyakov loops [3].

$$\exp\left(\frac{\beta}{2N_c}[tr U_p + tr U_p^\dagger]\right) \sim \exp\left(\lambda(u, N_t) \sum_{\langle \vec{x}\vec{y} \rangle} (L_{\vec{x}} L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}})\right) \quad (24)$$

The coupling $\lambda(u, N_t)$ is a function of the character coefficients $u(\beta)$. In [3] it is shown that λ can be expressed as a polynomial of the character coefficients $u(\beta)$ which are itself a polynomial of the inverse gauge coupling β . Roughly speaking λ replaces the inverse gauge coupling β .

Since all calculations in this work are done up to first order interaction (24) can be rewritten when neglecting higher orders.

$$\exp\left(\lambda \sum_{\langle \vec{x}\vec{y} \rangle} (L_{\vec{x}} L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}})\right) \sim 1 + \lambda \prod_{\langle \vec{x}\vec{y} \rangle} (L_{\vec{x}} L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}}) + o(\lambda^2) \quad (25)$$

2.3 Fermionic Part

As discussed in 2.1 the fermionic part in the partition function is replaced by the quark determinant with one quark flavor after Grassmann integration. A common way to deal with the quark determinant in lattice QCD is to use a special expansion technique named hopping expansion [1].

To apply hopping expansion the quark determinant needs to be rewritten.

$$\det(Q) = \det(C(\mathbb{1} - \kappa H)) \quad (26)$$

The prefactor C has no importance here it can be absorbed by redefining the quark fields. The matrix H is referred to as hopping matrix and collects all nearest neighbor terms in the Dirac operator Q . The parameter κ is a real number and referred to as hopping parameter.

Using a well known property of determinants and the series of the logarithm leads to the following form of the determinant.

$$\det[Q] = \det[\mathbb{1} - \kappa H] = \exp(\text{tr} [\ln(\mathbb{1} - \kappa H)]) = \exp\left(-\sum_{j=1}^{\infty} \frac{1}{j} \kappa^j \text{tr}[H^j]\right) \quad (27)$$

Since the hopping matrix collects all nearest neighbor terms in the Dirac operator, increasing the order of the hopping parameter κ is equivalent to consider interaction between quarks in further distances. For example first order in κ is equivalent to a nearest neighbor hop, second order already next to nearest neighbor etc.

When applying hopping expansion it is useful to split up the Dirac operator Q into temporal hops T and spatial hops S .

$$Q = 1 - T - S \quad (28)$$

Neglecting spatial hops leads to the static quark determinant $\det(Q_{stat})$ describing propagation in temporal direction only.

$$\det(Q_{stat}) = \det(1 - T) \quad (29)$$

2.3 Fermionic Part

After calculating the space and spin determinant the final result up to first order in hopping parameter κ can be written as [4]

$$\det(Q_{stat}) = \prod_{\vec{x}} [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2. \quad (30)$$

Since spatial links were integrated out the static quark determinant only depends on connected temporal gauge links respectively their traces, the Polyakov loops.

The couplings h_1 and \bar{h}_1 are functions of the hopping parameter κ .

$$h_1(\mu) = (2\kappa e^{a\mu})^{N_t} = \bar{h}_1(-\mu) \quad (31)$$

With chemical potential μ and the temporal extent of the lattice N_T . In short h_1 is the effective one point coupling constant to leading order [4]. After integration terms coupling with h_1 can be interpreted as quark contributions and terms coupling with \bar{h}_1 as antiquark contributions. More details can be found in [4].

Since spatial hops were neglected defining the static quark determinant the next step is to include interaction mediated by spatial hopping. The quark determinant is split up into a static and a kinetic part with the kinetic part mediated by spatial hops.

$$\det(Q) = \det(Q_{stat}) \det(Q_{kin}) \quad (32)$$

Since analytical calculations of expectation values up to first order nearest neighbor interaction are of interest it is sufficient to focus on a nearest neighbor two point interaction between spatial lattice sites.

A hop from one lattice site to a nearest neighbor site is equal to first order in κ . Hence an interaction between neighbored lattice sites can be described by hopping out and back in leading to κ^2 .

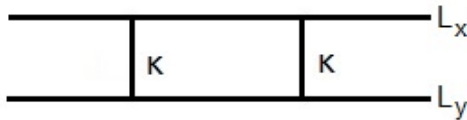


Figure 3: hopping between neighbored Polyakov loops

2.3 Fermionic Part

The calculation for a two point interaction between lattice sites with the used effective model can be found in [4] leading to the following expression.

$$-2h_2 \sum_{\langle \vec{x}, \vec{y} \rangle} \left[\left(tr \frac{h_1 W_{\vec{x}}}{1 + h_1 W_{\vec{x}}} - tr \frac{\bar{h}_1 W_{\vec{x}}^\dagger}{1 + \bar{h}_1 W_{\vec{x}}^\dagger} \right) \left(tr \frac{h_1 W_{\vec{y}}}{1 + h_1 W_{\vec{y}}} - tr \frac{\bar{h}_1 W_{\vec{y}}^\dagger}{1 + \bar{h}_1 W_{\vec{y}}^\dagger} \right) \right] \quad (33)$$

With \vec{x} and \vec{y} being nearest neighbor points on the lattice.

The two point interaction (33) is a function of temporal Wilson lines $W_{\vec{x}}$ and can be rewritten in terms of Polyakov loops.

$$\begin{aligned} -2h_2 \prod_{\langle \vec{x}, \vec{y} \rangle} & \frac{h_1 L_{\vec{x}} + 2h_1^2 L_{\vec{x}}^\dagger + 3h_1^3}{1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3} \frac{h_1 L_{\vec{y}} + 2h_1^2 L_{\vec{y}}^\dagger + 3h_1^3}{1 + h_1 L_{\vec{y}} + h_1^2 L_{\vec{y}}^\dagger + h_1^3} \\ & \frac{\bar{h}_1 L_{\vec{x}}^\dagger + 2\bar{h}_1^2 L_{\vec{x}} + 3\bar{h}_1^3}{1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3} \frac{\bar{h}_1 L_{\vec{y}}^\dagger + 2\bar{h}_1^2 L_{\vec{y}} + 3\bar{h}_1^3}{1 + \bar{h}_1 L_{\vec{y}}^\dagger + \bar{h}_1^2 L_{\vec{y}} + \bar{h}_1^3} \\ & - 2 \frac{h_1 L_{\vec{x}} + 2h_1^2 L_{\vec{x}}^\dagger + 3h_1^3}{1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3} \frac{\bar{h}_1 L_{\vec{y}}^\dagger + 2\bar{h}_1^2 L_{\vec{y}} + 3\bar{h}_1^3}{1 + \bar{h}_1 L_{\vec{y}}^\dagger + \bar{h}_1^2 L_{\vec{y}} + \bar{h}_1^3} \end{aligned} \quad (34)$$

It is useful to introduce a shortened notation for expression (34) by defining

$$\begin{aligned} M_{\vec{x}} & := \frac{h_1 L_{\vec{x}} + 2h_1^2 L_{\vec{x}}^\dagger + 3h_1^3}{1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3} \\ \bar{M}_{\vec{x}} & := \frac{\bar{h}_1 L_{\vec{x}}^\dagger + 2\bar{h}_1^2 L_{\vec{x}} + 3\bar{h}_1^3}{1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3} \end{aligned} \quad (35)$$

Now (34) can be written as

$$-2h_2 \prod_{\langle \vec{x}, \vec{y} \rangle} M_{\vec{x}} M_{\vec{y}} + \bar{M}_{\vec{x}} \bar{M}_{\vec{y}} - 2M_{\vec{x}} \bar{M}_{\vec{y}}. \quad (36)$$

The coupling h_2 is as mentioned a function of κ^2 .

$$h_2 = \frac{\kappa^2 N_T}{N_c} \quad (37)$$

With the number of colors $N_c = 3$ and the temporal lattice extent N_T .

So the final result for the Quark determinant up to first order interaction is achieved.

$$\det(Q) = \det(Q_{stat}) \cdot \prod_{\langle \vec{x}, \vec{y} \rangle} [1 - 2h_2 (M_{\vec{x}} M_{\vec{y}} + \bar{M}_{\vec{x}} \bar{M}_{\vec{y}} - 2M_{\vec{x}} \bar{M}_{\vec{y}})]. \quad (38)$$

2.4 The Effective Partition Function

For the calculation of expectation values of Polyakov loops a partition function is needed. The partition function of the full effective theory was given in 2.1.

$$Z = \int \exp(-S_g[U_0]) \cdot \det(Q) [dU_0] \quad (39)$$

In section 2.2 the gauge part was expanded in terms of an inverse gauge coupling up to first order. Further in section 2.3 the quark determinant was expanded by using hopping expansion up to a first order two point interaction.

Inserting the results (25) and (38) in (39) leads to the effective partition function of nearest neighbor two point interaction based on an effective QCD lattice theory.

$$Z = \int \det(Q_{stat}) \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 + \lambda(L_{\vec{x}}L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}}) \right] \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 - 2h_2(M_{\vec{x}}M_{\vec{y}} + \bar{M}_{\vec{x}}\bar{M}_{\vec{y}} - 2M_{\vec{x}}\bar{M}_{\vec{y}}) \right] [dU_0] \quad (40)$$

Whereby the following short notation was used.

$$\det(Q_{stat}) = \prod_{\vec{x}} [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3] [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 \quad (41)$$

$$M_{\vec{x}} := \frac{h_1 L_{\vec{x}} + 2h_1^2 L_{\vec{x}}^\dagger + 3h_1^3}{1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3} \quad \bar{M}_{\vec{x}} := \frac{\bar{h}_1 L_{\vec{x}}^\dagger + 2\bar{h}_1^2 L_{\vec{x}} + 3\bar{h}_1^3}{1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3} \quad (42)$$

This partition function is the starting point for the calculation of expectation values of Polyakov loops. On top of that a correlation function of two cubed nearest neighbor Polyakov loops is going to be calculated using (40).

3 Gauge Integrals over Polyakov Loops

In section 1 lattice gauge theory and its corresponding measure, the Haar measure has been introduced. Since spatial gauge links are integrated out in the effective model the effective partition function (40) is a function of Polyakov loops. Therefore a technique for the integration of Polyakov loops using the Haar measure is needed.

3.1 Gauge Integrals over Polyakov Loops - Using Traces of Wilson Lines

Since all spatial links are already integrated out defining the effective partition function the integration over temporal gauge links $[dU_0]$ remains.

In the used effective model the degrees of freedom are the temporal Wilson lines T . Therefore the measure can be written as

$$\int f(T, T^\dagger) [dU_0] = \int f(T, T^\dagger) [dT] \quad (43)$$

without changing the value of the integral [4].

The effective partition function is a function of Polyakov loops. Polyakov loops were defined as traces of temporal Wilson lines with maximized temporal extent N_T closed via periodic boundary condition in time direction. So the following integral needs to be solved

$$\int f(L, L^\dagger) [dT] = \int f(\text{tr}(T), \text{tr}(T^\dagger)) [dT]. \quad (44)$$

The calculation of (44) can be simplified by remembering the multiplets of $SU(3)$.

For example in $SU(3)$ the following multiplets are well known [7]

$$\begin{aligned} 3 \otimes \bar{3} &= 1 \oplus 8 \\ 3 \otimes 3 \otimes 3 &= 1 \oplus 8 \oplus 8 \oplus 10 \end{aligned} \quad (45)$$

For the calculation of (44) all multiplets containing a singlet as for example the ones in (45) are important. The reason is that only the integral over combinations of L and L^\dagger with a singlet in the corresponding $SU(3)$ multiplet has a non vanishing result.

3.1 Gauge Integrals over Polyakov Loops - Using Traces of Wilson Lines

For example

$$\begin{aligned} \int L_{\vec{x}} L_{\vec{x}}^\dagger [dT] = 1 &\Leftrightarrow 3 \otimes \bar{3} = 1 \oplus 8 \\ \int L_{\vec{x}}^3 [dT] = 1 &\Leftrightarrow 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10 \end{aligned} \quad (46)$$

The integral over all other combinations of L and L^\dagger with no singlet vanishes. For example

$$\int L_{\vec{x}}^2 [dT] = 0 \Leftrightarrow 3 \otimes 3 = 0 \quad (47)$$

Therefore only a few combinations of Polyakov loops give a non vanishing result. All other combinations can be neglected.

Looking at the partition function (40) and setting $h_2 = 0$ and $\lambda = 0$ leads to the integral over the static quark determinant.

$$\int \det(Q_{stat}) [dT] = \int \prod_{\vec{x}} [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 [dT] \quad (48)$$

All lattice points are equal so the product and the integral can be exchanged. On top of that the integral has the same result for every lattice point.

$$\begin{aligned} \prod_{\vec{x}} \int [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 [dT] \\ = \left(\int [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 [dT] \right)^{N_s^3} \end{aligned} \quad (49)$$

With N_s^3 being the lattice Volume.

To calculate (49) it is sufficient to solve

$$\int [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 [dT]. \quad (50)$$

Multiplying it out leads to several combinations of L and L^\dagger . As explained all combinations without a singlet in the corresponding SU(3) multiplet can be neglected.

3.1 Gauge Integrals over Polyakov Loops - Using Traces of Wilson Lines

Only a few combinations are remaining as for example

$$\int L_{\vec{x}} L_{\vec{x}}^\dagger [dT]. \quad (51)$$

As explained the Polyakov loop is the trace of a temporal Wilson line with maximized temporal extent so it is possible to rewrite (51) in terms of traces of temporal Wilson lines.

$$\int L_{\vec{x}} L_{\vec{x}}^\dagger [dT] = \int \text{tr}(T(\vec{x})) \text{tr}(T(\vec{x})^\dagger) [dT] = \int \sum_{a,b=1}^3 T(\vec{x})_{aa} T(\vec{x})_{bb}^\dagger [dT] \quad (52)$$

This integral has the same result for every index a, b so sum and integral can be exchanged.

$$\sum_{a,b} \int T(\vec{x})_{aa} T(\vec{x})_{bb}^\dagger [dT] = \frac{1}{N_c} \sum_{a,b=1}^3 \delta_{ab} \delta_{ab} = 1 \quad (53)$$

Using $N_c = 3$ number of colors in SU(3) gauge theory. The integral (53) is a well known integral in lattice gauge theory and can be found in [1].

All other remaining combinations from (50) can be treated in an equivalent way.

Never the less the presented technique indeed works well for combinations of Polyakov loops with small exponents. Combinations with large exponents like $L^7 L^\dagger$ often lead to very complicated gauge integrals. For these combinations a different method is much more efficient which is going to be presented in the following.

3.2 Gauge Integrals over Polyakov Loops - Using an Integration Measure for Polyakov Loops

The integration measure is rewritten in terms of Polyakov loops. As a result the Polyakov loops in the effective partition function can be integrated without rewriting them as traces over temporal Wilson lines.

$$\int [dW] \rightarrow \int e^V [dL] \quad (54)$$

For the integration a parameterization of the Polyakov loops is needed. It can be shown that traces of elements of the SU(3) group can be parameterized in terms of two angles θ and ϕ . Then the Polyakov loop as such a traced value can be written in a diagonal form [6].

$$L(\theta, \phi) = e^{i\phi} + e^{i\theta} + e^{-i(\theta+\phi)} \quad (55)$$

Hence the integration measure $[dL]$ can be rewritten as an integration over the two angles θ and ϕ [4].

$$\int e^V [dL] = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{2V} [d\phi] [d\theta] \quad (56)$$

The term e^{2V} is called potential term and can be interpreted as a Jacobian appearing when changing the integration measure. For SU(3) it can be found in [4].

$$V = \frac{1}{2} \ln(27 - 18|L|^2 + 8\text{Re}(L^3) - |L|^4) \quad (57)$$

Due to the rewritten Haar measure and the diagonalized Polyakov loops the integration becomes an integration over two parameters θ and ϕ . It can easily be solved with the help of analytical integration programs as for example mathematica.

Using the presented technique leads to the result for the integral over the static quark determinant (48).

$$\begin{aligned} \int \det(Q_{stat}) [dT] &= [1 + 4\bar{h}_1^3 + 4h_1^3 + \bar{h}_1^6 + h_1^6 + 4h_1\bar{h}_1 + 6h_1\bar{h}_1^4] \quad (58) \\ &\quad + 10h_1^2\bar{h}_1^2 + 6h_1^2\bar{h}_1^5 + 20h_1^3\bar{h}_1^3 + 4h_1^3\bar{h}_1^6 + 6h_1^4\bar{h}_1 \\ &\quad + 10h_1^4\bar{h}_1^4 + 6h_1^5\bar{h}_1^2 + 4h_1^5\bar{h}_1^5 + 4h_1^6\bar{h}_1^3 + h_1^6\bar{h}_1^6]^{N_s^3} \\ &:= z_0^{N_s^3} \end{aligned}$$

A list of all integrals over combinations of Polyakov loops appearing in this work can be found in the Appendix.

4 Linked Cluster Expansion

The technique of Linked Cluster Expansion which is briefly presented in the following chapter is a powerful tool for the calculation of expectation values or correlation functions on the lattice.

With its help connected correlators and expectation values can be calculated without thinking about combinatorial aspects of interaction on the lattice.

4.1 Definitions

Linked Cluster Expansion is most successfully formulated as a graphical expansion hence it is useful to introduce a graphical notation.

In this work expectation values and correlation functions are calculated up to first order nearest neighbor interaction that is why higher order interaction will not be discussed in this section. A more general introduction to Linked Cluster Expansion can be found in [5].

Graphically an arbitrary lattice point is represented by

$$\text{lattice point} \rightarrow \bullet. \tag{59}$$

An Operator as a function of Polyakov loops will graphically be represented by a so called root.

$$\text{Polyakov loop} \rightarrow \circ \tag{60}$$

The interaction between two graphs is represented by a simple line between the dots. In our case only first order nearest neighbor interaction proportional to $h_2 \propto \kappa^2$ is of interest.

A lattice point has more than one nearest neighbor so a number counting all nearest neighbor points is needed. This number is a characteristic number for the geometry of a lattice and is called coordination number k . The used cubic lattice has a coordination number $k = 6$.

4.2 Introduction: Linked Cluster Expansion of the Free Energy

The generating functional W in common quantum field theory defined by taking the logarithm of Z is used for the calculation of connected Feynman graphs. Linked cluster expansion is analogous an expansion of $\ln(Z)$ leading to connected graphs for example for the free energy, expectation values or correlation functions. For Linked Cluster Expansion Z is denoting a partition function.

The partition function of the used effective model was given in section 2.4.

$$Z = \int det(Q_{stat}) \prod_{\langle \vec{x}, \vec{y} \rangle} [1 - 2h_2(M_{\vec{x}}M_{\vec{y}} + \bar{M}_{\vec{x}}\bar{M}_{\vec{y}} - 2M_{\vec{x}}\bar{M}_{\vec{y}})] [dU_0] \quad (61)$$

For more simplicity λ has been set to zero. It should be mentioned that the λ part can be treated in an equivalent way.

Further the Linked Cluster Expansion of the free energy is now presented up to first order in h_2 coupling since higher orders are neglected in later calculations.

It is useful to rewrite the partition function in an exponentiated form to apply linked cluster expansion.

$$Z = \int det(Q_{stat}) \cdot exp \left[-2h_2 \sum_{\langle \vec{x}, \vec{y} \rangle} (M_{\vec{x}}M_{\vec{y}} + \bar{M}_{\vec{x}}\bar{M}_{\vec{y}} - 2M_{\vec{x}}\bar{M}_{\vec{y}}) \right] [dU_0] \quad (62)$$

This can be easily done here since higher order terms are neglected therefore the series of the exponential leads to (61).

Taking the logarithm of Z defines the free energy.

$$W = \ln \int det(Q_{stat}) \cdot exp \left[-2h_2 \sum_{\langle \vec{x}, \vec{y} \rangle} (M_{\vec{x}}M_{\vec{y}} + \bar{M}_{\vec{x}}\bar{M}_{\vec{y}} - 2M_{\vec{x}}\bar{M}_{\vec{y}}) \right] [dU_0] \quad (63)$$

It can be expanded as a series in h_2 . This expansion is referred to as Linked Cluster Expansion of the free energy.

4.2 Introduction: Linked Cluster Expansion of the Free Energy

First the coupling h_2 is set to 0. Furthermore so called source terms are introduced defined as exponentials including the interaction terms of (63). These terms are mathematically needed for the formalism of linked cluster expansion but have no physical interpretation.

$$\ln \int \det(Q_{stat}) \cdot \exp \left[\sum_{\vec{x}} \alpha(\vec{x}) M_{\vec{x}} + \sum_{\vec{x}} \bar{\alpha}(\vec{x}) \bar{M}_{\vec{x}} \right] [dU_0] \quad (64)$$

The expansion is formulated as a series in h_2 coupling. The corresponding interaction terms are calculated by applying functional derivatives in (64) with respect to α respectively $\bar{\alpha}$. This works like calculations in quantum field theory using a generating functional.

For simplicity the coupling $\bar{\alpha}$ is set to zero in the following but the neglected terms can be treated equally.

As explained W can now be written as a series in h_2 [5].

$$W = \sum_{\vec{x}} M_0^0 \vec{x} + \frac{1}{2} \sum_{\langle \vec{x}, \vec{y} \rangle} M_1^0(\vec{x}) v(\vec{x}, \vec{y}) M_1^0(\vec{y}) + o(h_2^2) \quad (65)$$

The first term corresponds to order h_2^0 so no interaction terms are involved. It is simply the logarithm of the integral over the static quark determinant.

$$\sum_{\vec{x}} M_0^0(\vec{x}) = \ln \int \det(Q_{stat}) [dU_0] \quad (66)$$

The second term corresponds to order h_2 . The term M_1^0 can be found with the help of a functional derivative with respect to α .

$$\begin{aligned} M_1^0(\vec{x}) &= \left(\frac{\delta}{\delta \alpha(\vec{x})} \ln \int \det(Q_{stat}) \left[\sum_{\vec{x}} \alpha(\vec{x}) M_{\vec{x}} \right] [dU_0] \right) \Big|_{\alpha=0} \\ &= \frac{\int \det(Q_{stat}) M_{\vec{x}} [dU_0]}{\int \det(Q_{stat}) [dU_0]} \end{aligned} \quad (67)$$

The term $M_1^0(\vec{y})$ has the same form due to the equality of all lattice points.

The term $v(\vec{x}, \vec{y})$ is given by the coupling $-2h_2$ between nearest neighbor lattice points coming from (63). So M_1^0 can be interpreted as the contribution from nearest neighbor interaction between two lattice points.

4.2 Introduction: Linked Cluster Expansion of the Free Energy

Since all lattice points are equal (67) has the same result for arbitrary lattice points. Therefore the sum over all nearest neighbors in (65) can be written as the number of nearest neighbor points on the lattice $k = 6$ times the lattice volume N_s^3 .

$$\sum_{\langle \vec{x}, \vec{y} \rangle} \rightarrow 6 \cdot N_s^3 \quad (68)$$

So (65) can be written as

$$W = \ln \int \det(Q_{stat}) [dU_0] - 6 \cdot N_s^3 \cdot h_2 \left(\frac{\int \det(Q_{stat}) M_{\vec{x}} [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right)^2 + o(h_2^2). \quad (69)$$

This is the free energy W up to first order in h_2 calculated with help of Linked Cluster Expansion. The corresponding partition function can easily be found by exponentiating the expression.

As mentioned the contributions from antiquark interaction and the mixing part in (61) can be calculated using the same technique. The following final result for W is achieved.

$$\begin{aligned} W = \ln \int \det(Q_{stat}) [dU_0] & \quad (70) \\ - 6 \cdot N_s^3 \cdot h_2 \left[-2 \left(\frac{\int \det(Q_{stat}) M_{\vec{x}} [dU_0] \int \det(Q_{stat}) \bar{M}_{\vec{x}} [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) \right. \\ \left. + \left(\frac{\int \det(Q_{stat}) M_{\vec{x}} [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right)^2 + \left(\frac{\int \det(Q_{stat}) \bar{M}_{\vec{x}} [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right)^2 \right] \end{aligned}$$

A huge advantage of the linked cluster expansion is its simple graphical representation. Especially higher order terms in h_2 can be found with the help of graph theory. The graphs can be translated to mathematical formulas by using simple rules [5].

$$W = \bullet + \bullet \text{---} \bullet + o(h_2^2)$$

Figure 4: Graphical representation of the free energy

With \bullet representing the M_0^0 contribution for order h_2^0 and M_1^0 for order h_2 .

4.3 Linked Cluster Expansion of Expectation Values

The line is given as $v(\vec{x}, \vec{y}) = -2h_2$. Summing over all lattice points easily leads to (65).

A more detailed introduction to linked cluster expansion also dealing with higher orders can be found in [5].

4.3 Linked Cluster Expansion of Expectation Values

The presented technique for the free energy can be extended to calculations of expectation values. Again the basic idea is to introduce source terms now not only for the interaction terms but also for the operator which expectation value should be calculated. Again the expectation value can be calculated as a series in h_2 with the help of functional derivatives.

In the effective model the expectation value of an operator is defined as

$$\begin{aligned} \langle O_{\vec{x}} \rangle = & \frac{1}{Z} \int O_{\vec{x}} \cdot \det(Q_{stat}) \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 + \lambda(L_{\vec{x}}L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}}) \right] \\ & \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 - 2h_2(M_{\vec{x}}M_{\vec{y}} + \bar{M}_{\vec{x}}\bar{M}_{\vec{y}} - 2M_{\vec{x}}\bar{M}_{\vec{y}}) \right] [dU_0] \end{aligned} \quad (71)$$

With the Operator O being a function of Polyakov loops.

For simplicity λ and \bar{h}_1 is set to zero. It should be mentioned that calculations of these quantities follow the same strategy.

The partition function is rewritten by introducing source terms including the interaction term and the operator.

$$M_{(0,0)}^0 = \ln \int \det(Q_{stat}) \cdot \exp \left(\sum_{\vec{x}} \alpha_{\vec{x}} \cdot M_{\vec{x}} + \sum_{\vec{x}} \beta_{\vec{x}} \cdot O_{\vec{x}} \right) [dU_0] \Big|_{\alpha, \beta=0} \quad (72)$$

The expectation value is given as a series in h_2 .

$$\langle O_{\vec{x}} \rangle = M_{(0,1)}^0(\vec{x}) + \sum_{\langle \vec{x}, \vec{y} \rangle} M_{(1,1)}^0(\vec{x})v(\vec{x}, \vec{y})M_{(1,0)}(\vec{y}) + o(h_2^2) \quad (73)$$

The operator O is located on a fixed spatial position \vec{x} therefore $M_{(0,1)}^0$ is not summed over the entire lattice [5].

4.3 Linked Cluster Expansion of Expectation Values

As mentioned in 4.1 these terms are referred to as roots writing \circ . Roots are always at fixed spatial positions and thus not summed over the lattice.

$M_{(0,1)}^0$ can easily be found by differentiating $M_{(0,0)}^0$ (72) with respect to β .

$$M_{(0,1)}^0 = \left(\frac{\delta}{\delta\beta_{\vec{x}}} \ln \int \det(Q_{stat}) \cdot \exp \left(\sum_{\vec{x}} \alpha_{\vec{x}} M_{\vec{x}} + \sum_{\vec{x}} \beta_{\vec{x}} O_{\vec{x}} \right) [dU_0] \right) \Big|_{\alpha, \beta=0} \quad (74)$$

$$M_{(0,1)}^0 = \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \quad (75)$$

It can be interpreted as a spatially fixed operator located on an arbitrary lattice point \vec{x} without interaction.

The indices of $M_{(0,1)}^0$ are chosen to show that no functional derivative with respect to α but one with respect to β should be taken.

For order h_2 the interaction terms read:

$$M_{(1,1)}^0 = \left(\frac{\delta^2}{\delta\alpha_{\vec{x}} \cdot \delta\beta_{\vec{x}}} \ln \int \det(Q_{stat}) \cdot \exp \left(\sum_{\vec{x}} \alpha_{\vec{x}} \cdot M_{\vec{x}} + \sum_{\vec{x}} \beta_{\vec{x}} \cdot O_{\vec{x}} \right) [dU_0] \right) \Big|_{\alpha, \beta=0} \quad (76)$$

$$M_{(1,1)}^0 = \frac{\int O_{\vec{x}} \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} - \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \quad (77)$$

This describes an interaction with a spatially fixed operator. It should be mentioned that the form of this term is referred to as Cumulant.

In general the mathematical formalism of Moments and Cumulants is closely related to the formalism of Linked Cluster Expansion. Further details can be found in [2].

4.3 Linked Cluster Expansion of Expectation Values

The $M_{(1,0)}^0$ term is equivalent to the M_1^0 term from the previous chapter and describes interaction with an arbitrary lattice point.

$$M_{(1,0)}^0(\vec{x}) = \frac{\int \det(Q_{stat}) M_{\vec{x}} [dU_0]}{\int \det(Q_{stat}) [dU_0]} \quad (78)$$

With help of Linked Cluster Expansion the expectation value of an operator $O_{\vec{x}}(L)$ up to first order nearest neighbor interaction is then given as

$$\begin{aligned} \langle O_{\vec{x}} \rangle = & \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \\ & - 2 \cdot h_2 \cdot 6 \cdot \frac{\int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \left(\frac{\int O_{\vec{x}} \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\ & \left. - \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) + o(h_2^2). \end{aligned} \quad (79)$$

Since all lattice points are equal the sum over all nearest neighbors in (73) was replaced by the coordination number of the lattice $k = 6$ counting all nearest neighbors of the root.

It is important to notice that no lattice volume factor is needed since $M_{(1,1)}^0$ describes a root fixed at a spatial position which is not summed over the entire lattice.

This can be understood more properly by looking at the graphical representation of the expectation value.

$$\langle \mathbf{O} \rangle = \bigcirc + \bigcirc \text{---} \bullet + o(h_2^2)$$

Figure 5: Graphical representation of the expectation value of an operator

As for the free energy the graphical expressions can be identified with the calculated M terms. The first term \circ is given by $M_{(0,1)}^0$ the second \circ by $M_{(1,1)}^0$ and the last \bullet by $M_{(1,0)}^0$. The line is given as $-2 \cdot 6 \cdot h_2$. So the graphical representation directly leads to (73) respectively (79).

4.3 Linked Cluster Expansion of Expectation Values

As mentioned all terms \bar{M} in (71) can be treated in an equivalent way. The final result for the expectation value of an operator up to first order nearest neighbor interaction is then given as

$$\begin{aligned}
\langle O_{\vec{x}} \rangle = & \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \quad (80) \\
& - 2 \cdot h_2 \cdot 6 \cdot \left[\frac{\int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \left(\frac{\int O_{\vec{x}} \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \right. \\
& \quad \left. \left. - \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) \right. \\
& + \frac{\int \bar{M}_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \left(\frac{\int O_{\vec{x}} \cdot \bar{M}_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\
& \quad \left. - \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \int \bar{M}_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) \\
& - \frac{\int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \left(\frac{\int O_{\vec{x}} \cdot \bar{M}_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\
& \quad \left. - \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \int \bar{M}_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) \\
& - \frac{\int \bar{M}_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \left(\frac{\int O_{\vec{x}} \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\
& \quad \left. - \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) \Bigg].
\end{aligned}$$

Since the λ part will not be needed in later calculations it is left out but a short outlook will be given at the end of the work.

Further details on Linked Cluster Expansion of expectation values can be found in [5]. Especially a formalism for higher orders is presented by expanding with the help of graph theory and translating the graphs to mathematical formulas using simple rules and the presented results.

4.4 Linked Cluster Expansion of Connected Two Point Correlation Functions

The last quantity needed in later calculations is the connected two point correlator of two operators located on nearest neighbor positions on a cubic lattice.

For the used effective model the connected two point correlation function of two Operators is given as

$$\begin{aligned}
 \langle O_{\vec{x}} \cdot O_{\vec{y}} \rangle - \langle O_{\vec{x}} \rangle \langle O_{\vec{y}} \rangle &= \frac{1}{Z} \int O_{\vec{x}} \cdot O_{\vec{y}} \cdot \det(Q_{stat}) \quad (81) \\
 \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 + \lambda(L_{\vec{x}} L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}}) \right] \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 - 2h_2(M_{\vec{x}} M_{\vec{y}} + \bar{M}_{\vec{x}} \bar{M}_{\vec{y}} - 2M_{\vec{x}} \bar{M}_{\vec{y}}) \right] [dU_0] \\
 - \frac{1}{Z^2} \int O_{\vec{x}} \cdot \det(Q_{stat}) \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 + \lambda(L_{\vec{x}} L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}}) \right] \\
 \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 - 2h_2(M_{\vec{x}} M_{\vec{y}} + \bar{M}_{\vec{x}} \bar{M}_{\vec{y}} - 2M_{\vec{x}} \bar{M}_{\vec{y}}) \right] [dU_0] \\
 \int O_{\vec{y}} \cdot \det(Q_{stat}) \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 + \lambda(L_{\vec{x}} L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}}) \right] \\
 \prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 - 2h_2(M_{\vec{x}} M_{\vec{y}} + \bar{M}_{\vec{x}} \bar{M}_{\vec{y}} - 2M_{\vec{x}} \bar{M}_{\vec{y}}) \right] [dU_0].
 \end{aligned}$$

Again the Operators have to be functions of Polyakov loops.

For simplicity \bar{M} is neglected and λ is set to zero.

The two Operators are located on fixed neighboring positions somewhere on the lattice. They can be represented by two graphical roots \circ .

$$\langle \mathbf{O}_x \mathbf{O}_y \rangle - \langle \mathbf{O}_x \rangle \langle \mathbf{O}_y \rangle = \text{---} \circ \text{---} \circ + \circ (\mathbf{h}_1^2)$$

Figure 6: Graphical representation of a connected correlator

The graph can be translated into formulas as explained in section 4.3. The roots \circ are representing $M_{(1,1)}^0$ terms. As mentioned rooted graphs are not summed over the lattice thus no extra prefactors are needed.

4.4 Linked Cluster Expansion of Connected Two Point Correlation Functions

It is also important to keep in mind that the connected correlator is calculated. Therefore the first contributing term is a nearest neighbor interaction between the two Operators. The reason for that is that the non interacting terms cancel when subtracting $\langle O \rangle^2$.

$$\langle O_{\vec{x}} \cdot O_{\vec{y}} \rangle - \langle O_{\vec{x}} \rangle \langle O_{\vec{y}} \rangle = M_{(1,1)}^0(\vec{x}) v(\vec{x}, \vec{y}) M_{(1,1)}^0(\vec{y}) \quad (82)$$

With \vec{x} and \vec{y} being nearest neighbors.

The $M_{(1,1)}^0$ terms are the same rooted terms as in the previous chapter and are calculated in the same way.

$$M_{(1,1)}^0 = \frac{\int O_{\vec{x}} \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} - \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \quad (83)$$

The coupling term $v(\vec{x}, \vec{y})$ is given as $-2 \cdot h_2$. Thus the correlator can be written as

$$\begin{aligned} \langle O_{\vec{x}} \cdot O_{\vec{y}} \rangle - \langle O_{\vec{x}} \rangle \langle O_{\vec{y}} \rangle = & \quad (84) \\ & -2 \cdot h_2 \left[\frac{\int O_{\vec{x}} \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\ & \quad \left. - \frac{\int O_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right] \\ & \left[\frac{\int O_{\vec{y}} \cdot M_{\vec{y}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\ & \quad \left. - \frac{\int O_{\vec{y}} \cdot \det(Q_{stat}) [dU_0] \int M_{\vec{y}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right]. \end{aligned}$$

4.4 Linked Cluster Expansion of Connected Two Point Correlation Functions

The full correlator including antiquark terms \bar{M} can be calculated using the same method.

$$\begin{aligned}
\langle O_{\bar{x}} \cdot O_{\bar{y}} \rangle - \langle O_{\bar{x}} \rangle \langle O_{\bar{y}} \rangle = & \quad (85) \\
- 2 \cdot h_2 \left[\left(\frac{\int O_{\bar{x}} \cdot M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \right. & \\
& \left. \left. - \frac{\int O_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) \right. & \\
& \left(\frac{\int O_{\bar{y}} \cdot M_{\bar{y}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. & \\
& \left. - \frac{\int O_{\bar{y}} \cdot \det(Q_{stat}) [dU_0] \int M_{\bar{y}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) & \\
+ \left(\frac{\int O_{\bar{x}} \cdot \bar{M}_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. & \\
& \left. - \frac{\int O_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \int \bar{M}_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) & \\
& \left(\frac{\int O_{\bar{y}} \cdot \bar{M}_{\bar{y}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. & \\
& \left. - \frac{\int O_{\bar{y}} \cdot \det(Q_{stat}) [dU_0] \int \bar{M}_{\bar{y}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) & \\
- 2 \left(\frac{\int O_{\bar{x}} \cdot M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. & \\
& \left. - \frac{\int O_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) & \\
& \left(\frac{\int O_{\bar{y}} \cdot \bar{M}_{\bar{y}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. & \\
& \left. - \frac{\int O_{\bar{y}} \cdot \det(Q_{stat}) [dU_0] \int \bar{M}_{\bar{y}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) & \\
& \left. \right] + o(h_2^2)
\end{aligned}$$

The λ part is left out since it will not be needed in later calculations. A short comment on gauge contributions will be made at the end of the work in an outlook. After finding a general expression for the expectation value of an operator as well as an expression for the connected correlator of two neighboring operators it is possible to calculate these values for certain operators.

5 Expectation Values of Polyakov Loops

At first several expectation values of Polyakov loops as $\langle L \rangle$, $\langle L^\dagger \rangle$, $\langle L^3 \rangle$ and $\langle (L^\dagger)^3 \rangle$ are calculated up to first order nearest neighbor interaction.

To shorten the calculation the method of Linked Cluster Expansion presented in the previous section 4 is used. Furthermore the gauge action part is neglected setting the coupling $\lambda = 0$. As mentioned the influence of gauge interaction will be shortly discussed in an outlook at the end of the work.

5.1 The L Expectation Value

For the calculation of the $\langle L \rangle$ expectation value the operator in expression (80) from section 4.3 is set to $O_{\vec{x}} = L_{\vec{x}}$.

All integrals appearing in expression (80) need to be calculated. For the calculation of these integrals the method presented in section 3 is used to solve the integrals over Polyakov loops.

Remembering (58) the integral over the static quark determinant was given as:

$$\begin{aligned}
 \int det(Q_{stat}) [dU_0] &= [1 + 4\bar{h}_1^3 + 4h_1^3 + \bar{h}_1^6 + h_1^6 + 4h_1\bar{h}_1 + 6h_1\bar{h}_1^4 \quad (86) \\
 &\quad + 10h_1^2\bar{h}_1^2 + 6h_1^2\bar{h}_1^5 + 20h_1^3\bar{h}_1^3 + 4h_1^3\bar{h}_1^6 + 6h_1^4\bar{h}_1 \\
 &\quad + 10h_1^4\bar{h}_1^4 + 6h_1^5\bar{h}_1^2 + 4h_1^5\bar{h}_1^5 + 4h_1^6\bar{h}_1^3 + h_1^6\bar{h}_1^6]^{N_s^3} \\
 &:= z_0^{N_s^3}.
 \end{aligned}$$

To shorten the notation it is useful to introduce symbols for integral expressions as done here by introducing z_0 .

The following integrals can all be found in expression (80) when replacing $O_{\vec{x}} \rightarrow L_{\vec{x}}$.

5.1 The L Expectation Value

The numerator of the $M_{(0,1)}^0$ term is given as:

$$\begin{aligned}
& \int L_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \tag{87} \\
&= \int L_{\vec{x}} \prod_{\vec{x}} [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^{-2} L_{\vec{x}} + \bar{h}_1^{-3}]^2 [dU_0] \\
&= \int L_{\vec{x}} [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^{-2} L_{\vec{x}} + \bar{h}_1^{-3}]^2 [dU_0] \\
&\quad \times \prod_{\{\vec{y}\} \setminus \vec{x}} \int [1 + h_1 L_{\vec{y}} + h_1^2 L_{\vec{y}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{y}}^\dagger + \bar{h}_1^{-2} L_{\vec{y}} + \bar{h}_1^{-3}]^2 [dU_0] \\
&= z_0^{N_s^3 - 1} \cdot \int L_{\vec{x}} [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^{-2} L_{\vec{x}} + \bar{h}_1^{-3}]^2 [dU_0] \\
&= z_0^{N_s^3 - 1} \cdot [3h_1^2 + 8h_1 \bar{h}_1^{-2} + 15h_1^4 \bar{h}_1^{-2} + 20h_1^2 \bar{h}_1^{-3} + 3\bar{h}_1^{-4} + 20h_1^3 \bar{h}_1^{-4} + 3h_1^6 \bar{h}_1^{-4} \\
&\quad + 2h_1^5 + 4h_1 \bar{h}_1^{-5} + 8h_1^4 \bar{h}_1^{-5} + 3h_1^2 \bar{h}_1^{-6} + 12h_1^3 \bar{h}_1^{-1} + 2h_1^6 \bar{h}_1^{-1} + 12h_1^5 \bar{h}_1^{-3} + 2h_1^5 \bar{h}_1^{-6}] \\
&:= z_0^{N_s^3 - 1} z_0^L
\end{aligned}$$

In the notation z_0^L the index 0 specifies the order of h_2 here h_2^0 and the exponent L shows that the operator $L_{\vec{x}}$ has been included in the integral.

The notation $\prod_{\{\vec{y}\} \setminus \vec{x}}$ is chosen to show that the product over all lattice points $\{\vec{y}\}$ should be taken except from one lattice point \vec{x} .

The numerator of $M_{(1,1)}^0$ is given as:

$$\begin{aligned}
& \int L_{\vec{x}} \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \tag{88} \\
&= \int L_{\vec{x}} \cdot \frac{h_1 L_{\vec{x}} + 2h_1^2 L_{\vec{x}}^\dagger + 3h_1^3}{1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3} \\
&\quad \times \prod_{\vec{x}} [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^{-2} L_{\vec{x}} + \bar{h}_1^{-3}]^2 [dU_0] \\
&= \int L_{\vec{x}} \cdot [h_1 L_{\vec{x}} + 2h_1^2 L_{\vec{x}}^\dagger + 3h_1^3] \\
&\quad [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3] [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^{-2} L_{\vec{x}} + \bar{h}_1^{-3}]^2 [dU_0] \\
&\quad \times \prod_{\{\vec{z}\} \setminus \vec{x}} \int [1 + h_1 L_{\vec{z}} + h_1^2 L_{\vec{z}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{z}}^\dagger + \bar{h}_1^{-2} L_{\vec{z}} + \bar{h}_1^{-3}]^2 [dU_0]
\end{aligned}$$

5.1 The L Expectation Value

Solving the integrals as in section 3 gives:

$$\begin{aligned}
& \int L_{\bar{x}} \cdot M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \\
&= z_0^{N_s^3-1} \cdot [3h_1^2 + 5h_1^5 + 18h_1^3\bar{h}_1 + 6h_1^6\bar{h}_1 + 4h_1\bar{h}_1^2 + 30h_1^4\bar{h}_1^2 + 20h_1^2\bar{h}_1^3 \\
&\quad + 30h_1^5\bar{h}_1^3 + 30h_1^3\bar{h}_1^4 + 9h_1^6\bar{h}_1^4 + 2h_1\bar{h}_1^5 + 16h_1^4\bar{h}_1^5 + 3h_1^2\bar{h}_1^6 + 5h_1^5\bar{h}_1^6] \\
&:= z_0^{N_s^3-1} \cdot z_1^L
\end{aligned} \tag{89}$$

The antiquark interaction is given as:

$$\begin{aligned}
& \int L_{\bar{x}} \cdot \bar{M}_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \\
&= \int L_{\bar{x}} \cdot [\bar{h}_1 L_{\bar{x}}^\dagger + 2\bar{h}_1^2 L_{\bar{x}} + 3\bar{h}_1^3] \\
&\quad [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3] [dU_0] \\
&\quad \times \prod_{\{\bar{z}\} \setminus \bar{x}} \int [1 + h_1 L_{\bar{z}} + h_1^2 L_{\bar{z}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{z}}^\dagger + \bar{h}_1^2 L_{\bar{z}} + \bar{h}_1^3]^2 [dU_0] \\
&= z_0^{N_s^3-1} \cdot [\bar{h}_1 + 6h_1^3\bar{h}_1 + h_1^6\bar{h}_1 + 8h_1\bar{h}_1^2 + 15h_1^4\bar{h}_1^2 + 30h_1^2\bar{h}_1^3 + 18h_1^5\bar{h}_1^3 \\
&\quad + 6\bar{h}_1^4 + 40h_1^3\bar{h}_1^4 + 6h_1^6\bar{h}_1^4 + 10h_1\bar{h}_1^5 + 20h_1^4\bar{h}_1^5 + 9h_1^2\bar{h}_1^6 + 6h_1^5\bar{h}_1^6] \\
&:= z_0^{N_s^3-1} \cdot \bar{z}_1^L
\end{aligned} \tag{90}$$

The bar over z is showing whether an antiquark interaction term is involved.

For $M_{(1,1)}^0$ the integrals without the operator $L_{\bar{x}}$ still need to be calculated. These are the same integrals as the missing numerator of $M_{(1,0)}^0$.

$$\begin{aligned}
& \int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \\
&= \int [h_1 L_{\bar{x}} + 2h_1^2 L_{\bar{x}}^\dagger + 3h_1^3] \\
&\quad [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3] [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
&\quad \times \prod_{\{\bar{z}\} \setminus \bar{x}} \int [1 + h_1 L_{\bar{z}} + h_1^2 L_{\bar{z}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{z}}^\dagger + \bar{h}_1^2 L_{\bar{z}} + \bar{h}_1^3]^2 [dU_0] \\
&= z_0^{N_s^3-1} \cdot [6h_1^3 + 3h_1^6 + 2h_1\bar{h}_1 + 12h_1^4\bar{h}_1 + 10h_1^2\bar{h}_1^2 + 15h_1^5\bar{h}_1^2 + 30h_1^3\bar{h}_1^3 \\
&\quad + 12h_1^6\bar{h}_1^3 + 3h_1\bar{h}_1^4 + 20h_1^4\bar{h}_1^4 + 6h_1^2\bar{h}_1^5 + 10h_1^5\bar{h}_1^5 + 6h_1^3\bar{h}_1^6 + 3h_1^6\bar{h}_1^6] \\
&:= z_0^{N_s^3-1} \cdot z_1
\end{aligned} \tag{91}$$

5.1 The L Expectation Value

$$\begin{aligned}
& \int \bar{M}_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \tag{92} \\
&= \int [\bar{h}_1 L_{\bar{x}}^\dagger + 2\bar{h}_1^2 L_{\bar{x}} + 3\bar{h}_1^3] \\
&\quad [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3] [dU_0] \\
&\quad \times \prod_{\{\bar{z}\} \setminus \bar{x}} \int [1 + h_1 L_{\bar{z}} + h_1^2 L_{\bar{z}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{z}}^\dagger + \bar{h}_1^2 L_{\bar{z}} + \bar{h}_1^3]^2 [dU_0] \\
&= z_0^{N_s^3-1} \cdot [2h_1 \bar{h}_1 + 3h_1^4 \bar{h}_1 + 10h_1^2 \bar{h}_1^2 + 6h_1^5 \bar{h}_1^2 + 6\bar{h}_1^3 + 30h_1^3 \bar{h}_1^3 + 6h_1^6 \bar{h}_1^3 \\
&\quad + 12h_1 \bar{h}_1^4 + 20h_1^4 \bar{h}_1^4 + 15h_1^2 \bar{h}_1^5 + 10h_1^5 \bar{h}_1^5 + 3\bar{h}_1^6 + 12h_1^3 \bar{h}_1^6 + 3h_1^6 \bar{h}_1^6] \\
&:= z_0^{N_s^3-1} \cdot \bar{z}_1
\end{aligned}$$

Using all the results the expression for $\langle L \rangle$ can be found. Therefore the terms in (80) are replaced by the results of the calculated integrals.

For example the first term in (80) corresponding to order h_2^0 is given as:

$$\begin{aligned}
& \frac{\int O_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \rightarrow \frac{\int L_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \\
& \frac{\int L_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} = \frac{z_0^{N_s^3-1} \cdot z_0^L}{z_0^{N_s^3}} = \frac{z_0^L}{z_0} \tag{93}
\end{aligned}$$

The first term corresponding to order h_2 in (80) is given as

$$\begin{aligned}
& \frac{\int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \tag{94} \\
& \quad \times \left(\frac{\int O_{\bar{x}} \cdot M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\
& \quad \left. - \frac{\int O_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) \\
& \rightarrow \frac{\int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \\
& \quad \times \left(\frac{\int L_{\bar{x}} \cdot M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\
& \quad \left. - \frac{\int L_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right)
\end{aligned}$$

5.1 The L Expectation Value

Inserting the results of the calculated integrals leads to

$$\begin{aligned}
& \frac{\int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \tag{95} \\
& \quad \times \left(\frac{\int L_{\bar{x}} \cdot M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} \right. \\
& \quad \left. - \frac{\int L_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \int M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} \right) \\
& = \frac{z_0^{N_s^3-1} \cdot z_1}{z_0^{N_s^3}} \times \left(\frac{z_0^{N_s^3-1} \cdot z_1^L}{z_0^{N_s^3}} - \frac{z_0^{2 \cdot N_s^3-2} \cdot z_0^L \cdot z_1}{z_0^{2 \cdot N_s^3}} \right) = \frac{z_1}{z_0} \times \left(\frac{z_1^L}{z_0} - \frac{z_0^L \cdot z_1}{z_0^2} \right)
\end{aligned}$$

Continuing this for every term in (80) finally leads to the full expectation value up to order h_2

$$\begin{aligned}
\langle L_{\bar{x}} \rangle = & \frac{z_0^L}{z_0} - 2 \cdot 6 \cdot h_2 \left[\left(\frac{z_1 \cdot z_1^L}{z_0^2} - \frac{z_0^L \cdot z_1^2}{z_0^3} \right) + \left(\frac{\bar{z}_1 \cdot \bar{z}_1^L}{z_0^2} - \frac{z_0^L \cdot \bar{z}_1^2}{z_0^3} \right) \right. \tag{96} \\
& \left. - \left(\frac{\bar{z}_1 \cdot z_1^L}{z_0^2} - \frac{z_0^L \cdot \bar{z}_1 \cdot z_1}{z_0^3} \right) - \left(\frac{z_1 \cdot \bar{z}_1^L}{z_0^2} - \frac{z_0^L \cdot \bar{z}_1 \cdot z_1}{z_0^3} \right) \right] + o(h_2^2)
\end{aligned}$$

It shows that the expectation value (96) can also be written in an exponentiated form. This is a property of the Linked Cluster Expansion for example discussed in [2].

Finding an exponentiated form for (96) leads to the expression

$$\begin{aligned}
\langle L_{\bar{x}} \rangle = & \frac{z_0^L}{z_0} \cdot \exp \left(-2 \cdot 6 \cdot h_2 \left[\frac{z_1}{z_0} \left(\frac{z_1^L}{z_0^L} - \frac{z_1}{z_0} \right) + \frac{\bar{z}_1}{z_0} \left(\frac{\bar{z}_1^L}{z_0^L} - \frac{\bar{z}_1}{z_0} \right) \right. \right. \tag{97} \\
& \left. \left. - \frac{z_1}{z_0} \left(\frac{\bar{z}_1^L}{z_0^L} - \frac{\bar{z}_1}{z_0} \right) - \frac{\bar{z}_1}{z_0} \left(\frac{z_1^L}{z_0^L} - \frac{z_1}{z_0} \right) \right] \right).
\end{aligned}$$

Using a simple Taylor expansion of the exponential function and neglecting all higher order terms than first order in h_2 directly leads back to expression (96).

For illustration the analytically calculated results of the expectation value are compared with numerical computations of $\langle L \rangle$ using the general expression (71) but neglecting λ .

5.1 The L Expectation Value

At first the expectation value without antiquark contributions is plotted. This is equivalent to setting $\bar{h}_1 = 0$. The analytical expectation value then simplifies to

$$\begin{aligned} \langle L_{\bar{x}} \rangle &= \frac{z_0^L}{z_0} - 2 \cdot 6 \cdot h_2 \left[\frac{z_1 \cdot z_1^L}{z_0^2} - \frac{z_0^L \cdot z_1^2}{z_0^3} \right] \\ \langle L_{\bar{x}} \rangle &= \frac{z_0^L}{z_0} \cdot \exp \left(-2 \cdot 6 \cdot h_2 \left[\frac{z_1}{z_0} \left(\frac{z_1^L}{z_0^L} - \frac{z_1}{z_0} \right) \right] \right). \end{aligned} \quad (98)$$

The values of $\langle L \rangle$ as function of h_2 are calculated using mathematica. The coupling h_1 is kept at a fixed value $h_1 = 0.8$. Further $N_T = 2$ and $N_s^3 = 8$ are chosen. Of course the calculation is done only for one quark flavor $N_f = 1$ with three colors $N_c = 3$.

Since h_2 is a function of κ^2 and h_2 increases, κ has to increase with $N_T = 2$ and $N_c = 3$ fixed. On the other side $h_1 = 0.8 = \text{const}$. Therefore the chemical Potential μ in the definition (31) of h_1 compensates the raising κ in that way that h_1 is kept constant.

The numerical data of $\langle L \rangle$ is calculated using Monte Carlo simulation with 50000 samples starting with $\kappa = 0.0387298$ up to order κ^2 .

5.1 The L Expectation Value

The plot of $\langle L \rangle$ without antiquark contribution.

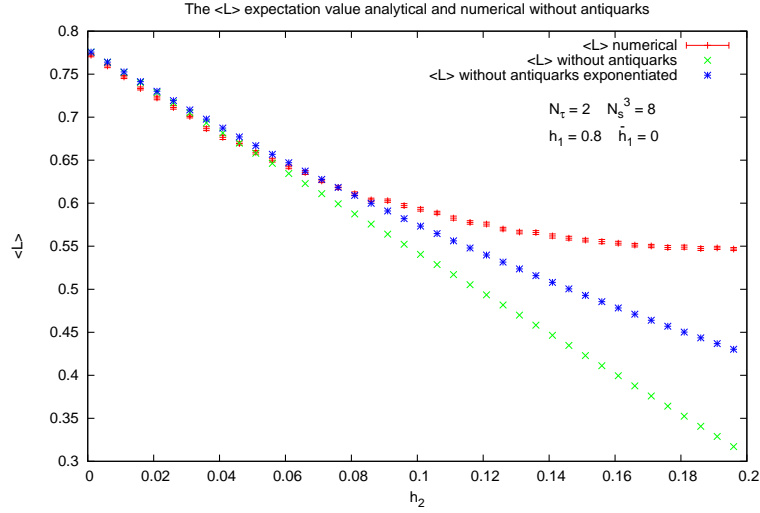


Figure 7: $\langle L \rangle$ without antiquark contribution

It can be noticed that the analytical results of $\langle L \rangle$ are overlapping quite good with the numerical computation for small h_2 . Especially the exponentiated form shows a good correspondance up to $h_2 \approx 0.08$.

As mentioned the numerical computation is done using the general expression (71) up to κ^2 without applying Linked Cluster Expansion. Therefore we expect that for large h_2 the analytical results differ from the numerical computed data since the Linked Cluster series as an approximation is getting more and more inaccurate. This behavior can clearly be noticed in the plot.

The exponentiated form includes approximated higher orders of h_2 . It was found based on the first order expectation value. Comparing it with the numerical data shows that the approximation leads to a better convergence towards the numerical solution.

For the last value the difference between numerical and analytical solution without exponentiation is about $\Delta \langle L \rangle \approx 0.3$. The difference between the numerical and the exponentiated solution is only about $\Delta \langle L \rangle \approx 0.15$.

5.1 The L Expectation Value

Another illustration for the full expectation value including antiquark contributions is done. The same values for $N_T = 2$ and $N_s^3 = 8$ are chosen and h_1 is kept fixed at $h_1 = 0.8$.

As explained h_2 increases as a function of κ^2 . Keeping h_1 at a fixed value for running κ leads to running \bar{h}_1 since \bar{h}_1 is a function of κ .

The couplings h_1 and \bar{h}_1 where given in section 2.3 as

$$h_1(\mu) = (2\kappa e^{a\mu})^{N_T} = \bar{h}_1(-\mu). \quad (99)$$

Since h_1 is fixed and κ is running μ is completely defined by h_1 . By using $N_T = 2$, \bar{h}_1 can be expressed in terms of h_2 .

$$h_1(\mu) = (2\kappa e^{a\mu})^{N_T} \rightarrow e^{-a\mu} = \frac{2\kappa}{\sqrt{h_1}} \quad (100)$$

The coupling h_2 is given as

$$h_2 = \frac{N_T}{N_c} \kappa^2. \quad (101)$$

Thus \bar{h}_1 can be written as a function of h_2 with $h_1 = 0.8$, $N_T = 2$ and $N_c = 3$.

$$\bar{h}_1 = \left(\frac{6h_2}{\sqrt{h_1}} \right)^{N_T} = \left(\frac{6h_2}{\sqrt{0.8}} \right)^2 \quad (102)$$

For the numerical computation Monte Carlo simulation with 50000 samples at fixed $h_1 = 0.8$ is used. Now of course without neglecting \bar{h}_1 .

5.1 The L Expectation Value

The analytically calculated data and the numerical data can be plotted as a function of h_2 .

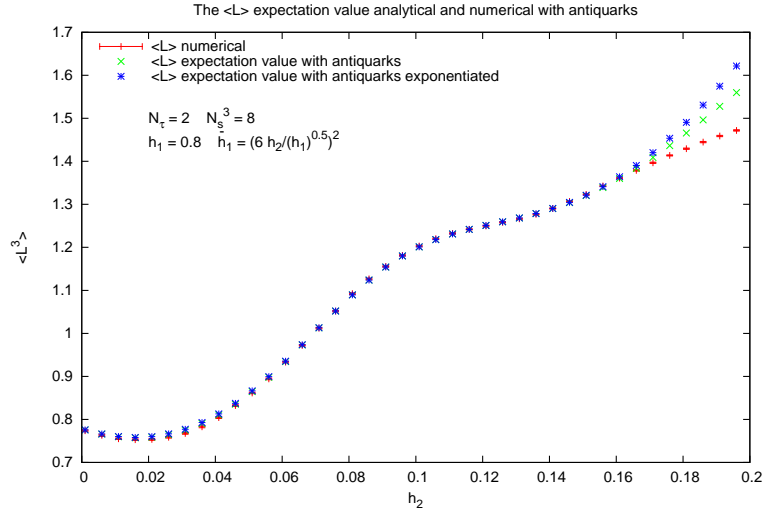


Figure 8: $\langle L \rangle$ with antiquark contribution

It can be noticed that even for large values of h_2 the analytical and the numerical results are overlapping quite good. Though starting with $h_2 \approx 0.16$ the analytical results differ from the numerical ones. This can be understood as a consequence of the approximation when applying Linked Cluster Expansion compared to the full expression (71) of the expectation value.

More interesting is that the exponentiated form of $\langle L \rangle$ with its higher order h_2 approximation is less convergent to the numerical result. So the exponentiated approximation seems to be not very precisely when including antiquarks. For a better convergence higher order corrections need to be calculated explicitly.

Further the maximal discrepancy between the last numerical and the last analytical value of the exponentiated form is about $\Delta \langle L \rangle \approx 0.13$. The discrepancy between the last numerical and the last analytical value with no exponentiation is even smaller $\Delta \langle L \rangle \approx 0.05$. So both discrepancies are smaller than in the previous case with neglected antiquark coupling.

Brief the analytical expectation value shows better convergence towards the numerical data without neglecting antiquark contributions.

5.2 The L^\dagger Expectation Value

Next the $\langle L^\dagger \rangle$ expectation value is calculated. Therefor the operator in expression (80) has to be replaced by $O_{\bar{x}} \rightarrow L_{\bar{x}}^\dagger$.

The resulting integrals can be calculated in the same way as explained in section 5.1 for the L expectation value. In fact the integrals z_0 , z_1 and \bar{z}_1 also appear in the calculation of $\langle L^\dagger \rangle$ and have the same results.

The integral corresponding to order h_2^0 in the numerator of $M_{(0,1)}^0$ is given as:

$$\begin{aligned}
 & \int L_{\bar{x}}^\dagger \cdot \det(Q_{stat}) [dU_0] & (103) \\
 &= \int L_{\bar{x}}^\dagger [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
 & \quad \times \prod_{\{\bar{y}\} \setminus \bar{x}} \int [1 + h_1 L_{\bar{y}} + h_1^2 L_{\bar{y}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{y}}^\dagger + \bar{h}_1^2 L_{\bar{y}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot \int L_{\bar{x}}^\dagger [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot [2h_1 + 3h_1^4 + 8h_1^2 \bar{h}_1 + 4h_1^5 \bar{h}_1 + 3\bar{h}_1^2 + 20h_1^3 \bar{h}_1^2 + 3h_1^6 \bar{h}_1^2 + 12h_1 \bar{h}_1^3 \\
 & \quad + 20h_1^4 \bar{h}_1^3 + 15h_1^2 \bar{h}_1^4 + 8h_1^5 \bar{h}_1^4 + 2\bar{h}_1^5 + 12h_1^3 \bar{h}_1^5 + 2h_1^6 \bar{h}_1^5 + 2h_1 \bar{h}_1^6 + 3h_1^4 \bar{h}_1^6] \\
 &:= z_0^{N_s^3-1} \cdot z_0^{L^\dagger}
 \end{aligned}$$

And the missing integrals of $M_{(1,1)}^0$ corresponding to order h_2 are given as:

$$\begin{aligned}
 & \int L_{\bar{x}}^\dagger \cdot M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] & (104) \\
 &= z_0^{N_s^3-1} \cdot \int L_{\bar{x}}^\dagger \cdot \frac{h_1 L_{\bar{x}} + 2h_1^2 L_{\bar{x}}^\dagger + 3h_1^3}{1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3} \\
 & \quad \times [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot [h_1 + 6h_1^4 + 8h_1^2 \bar{h}_1 + 10h_1^5 \bar{h}_1 + 30h_1^3 \bar{h}_1^2 + 9h_1^6 \bar{h}_1^2 + 6h_1 \bar{h}_1^3 \\
 & \quad + 40h_1^4 \bar{h}_1^3 + 15h_1^2 \bar{h}_1^4 + 20h_1^5 \bar{h}_1^4 + 18h_1^3 \bar{h}_1^5 + 6h_1^6 \bar{h}_1^5 + h_1 \bar{h}_1^6 + 6h_1^4 \bar{h}_1^6] \\
 &:= z_0^{N_s^3-1} \cdot z_1^{L^\dagger}
 \end{aligned}$$

5.2 The L^\dagger Expectation Value

$$\begin{aligned}
& \int L_{\bar{x}}^\dagger \cdot \bar{M}_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \tag{105} \\
&= z_0^{N_s^3-1} \cdot \int L_{\bar{x}}^\dagger \cdot \frac{\bar{h}_1 L_{\bar{x}}^\dagger + 2\bar{h}_1^2 L_{\bar{x}} + 3\bar{h}_1^3}{1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3} \\
&\quad \times [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
&= z_0^{N_s^3-1} \cdot [4h_1^2 \bar{h}_1 + 2h_1^5 \bar{h}_1 + 3\bar{h}_1^2 + 20h_1^3 \bar{h}_1^2 + 3h_1^6 \bar{h}_1^2 + 18h_1 \bar{h}_1^3 + 30h_1^4 \bar{h}_1^3 \\
&\quad + 30h_1^2 \bar{h}_1^4 + 16h_1^5 \bar{h}_1^4 + 5\bar{h}_1^5 + 30h_1^3 \bar{h}_1^5 + 5h_1^6 \bar{h}_1^5 + 6h_1 \bar{h}_1^6 + 9h_1^4 \bar{h}_1^6] \\
&:= z_0^{N_s^3-1} \cdot \bar{z}_1^{L^\dagger}
\end{aligned}$$

Since the expectation value $\langle L^\dagger \rangle$ is also based on expression (80) it has the same structure as the $\langle L \rangle$ expectation value. Only the results of the integrals $z_0^{L^\dagger}$, $z_1^{L^\dagger}$ and $\bar{z}_1^{L^\dagger}$ are different.

$$\begin{aligned}
\langle L_{\bar{x}}^\dagger \rangle &= \frac{z_0^{L^\dagger}}{z_0} - 2 \cdot 6 \cdot h_2 \left[\left(\frac{z_1 \cdot z_1^{L^\dagger}}{z_0^2} - \frac{z_0^{L^\dagger} \cdot z_1^2}{z_0^3} \right) + \left(\frac{\bar{z}_1 \cdot \bar{z}_1^{L^\dagger}}{z_0^2} - \frac{z_0^{L^\dagger} \cdot \bar{z}_1^2}{z_0^3} \right) \right. \\
&\quad \left. - \left(\frac{\bar{z}_1 \cdot z_1^{L^\dagger}}{z_0^2} - \frac{z_0^{L^\dagger} \cdot \bar{z}_1 \cdot z_1}{z_0^3} \right) - \left(\frac{z_1 \cdot \bar{z}_1^{L^\dagger}}{z_0^2} - \frac{z_0^{L^\dagger} \cdot \bar{z}_1 \cdot z_1}{z_0^3} \right) \right] + o(h_2^2) \tag{106}
\end{aligned}$$

$$\begin{aligned}
\langle L_{\bar{x}}^\dagger \rangle &= \frac{z_0^{L^\dagger}}{z_0} \cdot \exp \left(-2 \cdot 6 \cdot h_2 \left[\frac{z_1}{z_0} \left(\frac{z_1^{L^\dagger}}{z_0^{L^\dagger}} - \frac{z_1}{z_0} \right) + \frac{\bar{z}_1}{z_0} \left(\frac{\bar{z}_1^{L^\dagger}}{z_0^{L^\dagger}} - \frac{\bar{z}_1}{z_0} \right) \right. \right. \\
&\quad \left. \left. - \frac{z_1}{z_0} \left(\frac{\bar{z}_1^{L^\dagger}}{z_0^{L^\dagger}} - \frac{\bar{z}_1}{z_0} \right) - \frac{\bar{z}_1}{z_0} \left(\frac{z_1^{L^\dagger}}{z_0^{L^\dagger}} - \frac{z_1}{z_0} \right) \right] \right) \tag{107}
\end{aligned}$$

Again $\langle L^\dagger \rangle$ without antiquark contributions is illustrated in comparison to a numerical computation. The calculations are done for fixed $h_1 = 0.8$.

Setting $\bar{h}_1 = 0$ leads to the expectation value without antiquarks.

$$\langle L_{\bar{x}}^\dagger \rangle = \frac{z_0^{L^\dagger}}{z_0} - 2 \cdot 6 \cdot h_2 \left[\frac{z_1 \cdot z_1^{L^\dagger}}{z_0^2} - \frac{z_0^{L^\dagger} \cdot z_1^2}{z_0^3} \right] \tag{108}$$

$$\langle L_{\bar{x}}^\dagger \rangle = \frac{z_0^{L^\dagger}}{z_0} \cdot \exp \left(-2 \cdot 6 \cdot h_2 \left[\frac{z_1}{z_0} \left(\frac{z_1^{L^\dagger}}{z_0^{L^\dagger}} - \frac{z_1}{z_0} \right) \right] \right) \tag{109}$$

5.2 The L^\dagger Expectation Value

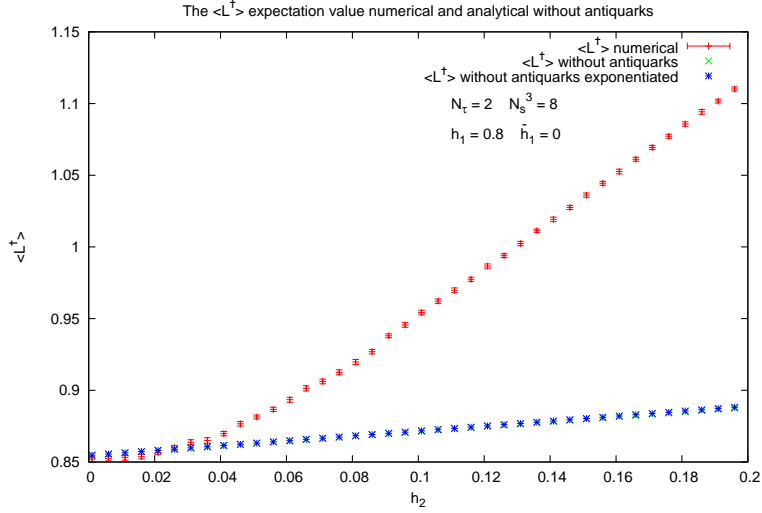


Figure 9: $\langle L^\dagger \rangle$ without antiquarks

The same conditions as for $\langle L \rangle$ are chosen to find the analytical and the numerical data.

It can be noticed that even for small values of h_2 beginning with $h_2 \approx 0.04$ the numerical simulated values of $\langle L^\dagger \rangle$ differ from the analytical results. Since the analytical expectation value of $\langle L^\dagger \rangle$ has the same structure as $\langle L \rangle$ it seems that higher order terms of the Linked Cluster series are needed for a better convergence.

The graph corresponding to the exponentiated form of the expectation value has no better convergence towards the numerical data although it contains approximated higher order terms. It appears that higher order terms need to be calculated explicitly to achieve a better convergence.

However when looking at the maximal discrepancy between the numerical data and the analytical results it shows that it is only about $\Delta \langle L^\dagger \rangle \approx 0.2$ since the scale of the y-axis is rather small. So in fact the discrepancy is smaller than the discrepancy of the non exponentiated form in the $\langle L \rangle$ case.

Nevertheless the analytical results differ from the numerical result even for small values of h_2 .

5.2 The L^\dagger Expectation Value

On top of that the full expectation value involving antiquarks is plotted and compared to numerical computed results using the same conditions as before.

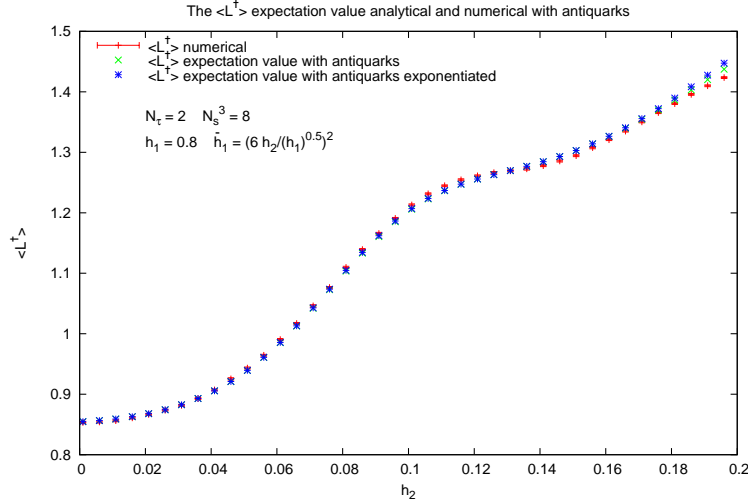


Figure 10: $\langle L^\dagger \rangle$ with antiquarks

Taking care of antiquarks leads to a much better convergence. The analytical calculated results fit very good to the numerical data. Even for large values of h_2 the analytical solutions run closely to the numerical solution.

For $\langle L \rangle$ it was possible to observe that the exponentiated form with its approximation of higher orders leads to less precision when including antiquark contributions. This effect still can be noticed but only very little in the $\langle L^\dagger \rangle$ case. The values of $\langle L^\dagger \rangle$ calculated using the exponentiated form are a little lower than the values of the non exponentiated expectation value compared to the numerical data. Of course this can only be noticed for large values of $h_2 \approx 0.18$.

On top of that the maximal discrepancy between numerical and analytical data is much smaller when including antiquark contributions compared to neglecting them.

Again the analytical expectation value with non vanishing antiquark contributions seems to be better convergent towards a numerical computation.

5.3 The L^3 Expectation Value

In the same way the expectation value of a cubed Polyakov loop can be calculated. Due to the higher power of the operator higher orders of Polyakov loops appear in the integrals. Therefore the prefactors in front of h_1 and \bar{h}_1 coupling increase.

The first step is to replace the Operator in (80) by a cubed Polyakov loop $O_{\vec{x}} \rightarrow L_{\vec{x}}^3$. As before the integrals z_0 , z_1 and \bar{z}_1 have the same result.

The first new integral corresponding to order h_2^0 is the numerator of $M_{(0,1)}^0$.

$$\begin{aligned}
 & \int L_{\vec{x}}^3 \cdot \det(Q_{stat}) [dU_0] & (110) \\
 &= \int L_{\vec{x}}^3 [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 [dU_0] \\
 & \quad \times \prod_{\{\vec{y}\} \setminus \vec{x}} \int [1 + h_1 L_{\vec{y}} + h_1^2 L_{\vec{y}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{y}}^\dagger + \bar{h}_1^2 L_{\vec{y}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot \int L_{\vec{x}}^3 [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot [1 + 8h_1^3 + h_1^6 + 12h_1\bar{h}_1 + 24h_1^4\bar{h}_1 + 45h_1^2\bar{h}_1^2 + 24h_1^5\bar{h}_1^2 + 8\bar{h}_1^3 \\
 & \quad + 72h_1^3\bar{h}_1^3 + 8h_1^6\bar{h}_1^3 + 22h_1\bar{h}_1^4 + 45h_1^4\bar{h}_1^4 + 22h_1^2\bar{h}_1^5 + 12h_1^5\bar{h}_1^5 \\
 & \quad + \bar{h}_1^6 + 8h_1^3\bar{h}_1^6 + h_1^6\bar{h}_1^6] \\
 &:= z_0^{N_s^3-1} \cdot z_0^{L^3}
 \end{aligned}$$

Further for order h_2 a new contribution is given by the numerator of $M_{(1,1)}^0$.

$$\begin{aligned}
 & \int L_{\vec{x}}^3 \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] & (111) \\
 &= z_0^{N_s^3-1} \cdot \int L_{\vec{x}}^3 \cdot \frac{h_1 L_{\vec{x}} + 2h_1^2 L_{\vec{x}}^\dagger + 3h_1^3}{1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3} \\
 & \quad \times [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot [12h_1^3 + 3h_1^6 + 6h_1\bar{h}_1 + 48h_1^4\bar{h}_1 + 45h_1^2\bar{h}_1^2 + 60h_1^5\bar{h}_1^2 + 108h_1^3\bar{h}_1^3 \\
 & \quad + 24h_1^6\bar{h}_1^3 + 11h_1\bar{h}_1^4 + 90h_1^4\bar{h}_1^4 + 22h_1^2\bar{h}_1^5 + 30h_1^5\bar{h}_1^5 + 12h_1^3\bar{h}_1^6 + 3h_1^6\bar{h}_1^6] \\
 &:= z_0^{N_s^3-1} \cdot z_1^{L^3}
 \end{aligned}$$

5.3 The L^3 Expectation Value

$$\begin{aligned}
& \int L_{\vec{x}}^3 \cdot \bar{M}_{\vec{x}} \cdot \det(Q_{stat}) [dU_0] \quad (112) \\
&= z_0^{N_s^3-1} \cdot \int L_{\vec{x}}^3 \cdot \frac{\bar{h}_1 L_{\vec{x}}^\dagger + 2\bar{h}_1^2 L_{\vec{x}} + 3\bar{h}_1^3}{1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3} \\
&\quad \times [1 + h_1 L_{\vec{x}} + h_1^2 L_{\vec{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\vec{x}}^\dagger + \bar{h}_1^2 L_{\vec{x}} + \bar{h}_1^3]^2 [dU_0] \\
&= z_0^{N_s^3-1} \cdot [6h_1\bar{h}_1 + 12h_1^4\bar{h}_1 + 45h_1^2\bar{h}_1^2 + 24h_1^5\bar{h}_1^2 + 12\bar{h}_1^3 + 108h_1^3\bar{h}_1^3 + 12h_1^6\bar{h}_1^3 \\
&\quad + 44h_1\bar{h}_1^4 + 90h_1^4\bar{h}_1^4 + 55h_1^2\bar{h}_1^5 + 30h_1^5\bar{h}_1^5 + 3\bar{h}_1^6 + 24h_1^3\bar{h}_1^6 + 3h_1^6\bar{h}_1^6] \\
&:= z_0^{N_s^3-1} \cdot \bar{z}_1^{L^3}
\end{aligned}$$

Hence the analytical expectation value up to first order nearest neighbor interaction can be formulated.

$$\begin{aligned}
\langle L_{\vec{x}}^3 \rangle &= \frac{z_0^{L^3}}{z_0} - 2 \cdot 6 \cdot h_2 \left[\left(\frac{z_1 \cdot z_1^{L^3}}{z_0^2} - \frac{z_0^{L^3} \cdot z_1^2}{z_0^3} \right) + \left(\frac{\bar{z}_1 \cdot \bar{z}_1^{L^3}}{z_0^2} - \frac{z_0^{L^3} \cdot \bar{z}_1^2}{z_0^3} \right) \right. \\
&\quad \left. - \left(\frac{\bar{z}_1 \cdot z_1^{L^3}}{z_0^2} - \frac{z_0^{L^3} \cdot \bar{z}_1 \cdot z_1}{z_0^3} \right) - \left(\frac{z_1 \cdot \bar{z}_1^{L^3}}{z_0^2} - \frac{z_0^{L^3} \cdot \bar{z}_1 \cdot z_1}{z_0^3} \right) \right] + o(h_2^2) \quad (113)
\end{aligned}$$

$$\begin{aligned}
\langle L_{\vec{x}}^3 \rangle &= \frac{z_0^{L^3}}{z_0} \cdot \exp \left(-2 \cdot 6 \cdot h_2 \left[\frac{z_1}{z_0} \left(\frac{z_1^{L^3}}{z_0^{L^3}} - \frac{z_1}{z_0} \right) + \frac{\bar{z}_1}{z_0} \left(\frac{\bar{z}_1^{L^3}}{z_0^{L^3}} - \frac{\bar{z}_1}{z_0} \right) \right. \right. \\
&\quad \left. \left. - \frac{z_1}{z_0} \left(\frac{\bar{z}_1^{L^3}}{z_0^{L^3}} - \frac{\bar{z}_1}{z_0} \right) - \frac{\bar{z}_1}{z_0} \left(\frac{z_1^{L^3}}{z_0^{L^3}} - \frac{z_1}{z_0} \right) \right] \right) \quad (114)
\end{aligned}$$

Setting $\bar{h}_1 = 0$ leads to a result without antiquark contributions.

$$\langle L_{\vec{x}}^3 \rangle = \frac{z_0^{L^3}}{z_0} - 2 \cdot 6 \cdot h_2 \left[\frac{z_1 \cdot z_1^{L^3}}{z_0^2} - \frac{z_0^{L^3} \cdot z_1^2}{z_0^3} \right] \quad (115)$$

$$\langle L_{\vec{x}}^3 \rangle = \frac{z_0^{L^3}}{z_0} \cdot \exp \left(-2 \cdot 6 \cdot h_2 \left[\frac{z_1}{z_0} \left(\frac{z_1^{L^3}}{z_0^{L^3}} - \frac{z_1}{z_0} \right) \right] \right) \quad (116)$$

5.3 The L^3 Expectation Value

The analytical results of $\langle L^3 \rangle$ are plotted with the same conditions as before to compare them with a numerical computation.

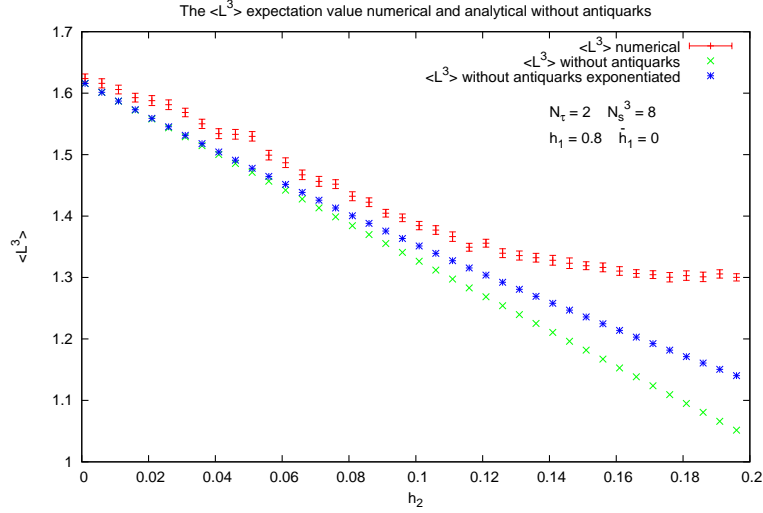


Figure 11: $\langle L^3 \rangle$ without antiquarks

It can clearly be noticed that the numerical results of the expectation value are constantly a little higher than the analytical calculated values. There could be two reasons for that.

On one side the prefactors in front of h_1 coupling coming from $z_1^{L^3}$ and $\bar{z}_1^{L^3}$ are larger than in the previous cases since L appears in higher powers due to multiplying with a cubed Polyakov loop. Thus differences between the numerical solution using expression (71) and the analytic Linked Cluster series carry more weight even for small h_2 values.

On the other side it can be noticed that the numerical data seems to be noisy or more precisely that the errors of the numerical values might be underrated. Therefore the difference could have its origin in the numerical computation of the expectation value.

Never the less it can be noticed that the exponentiated form of $\langle L^3 \rangle$ with its higher order h_2 approximation fits a little better to the numerical results.

5.3 The L^3 Expectation Value

Further the full expression of $\langle L^3 \rangle$ is plotted as a function of h_2 and compared to a numerical computation.

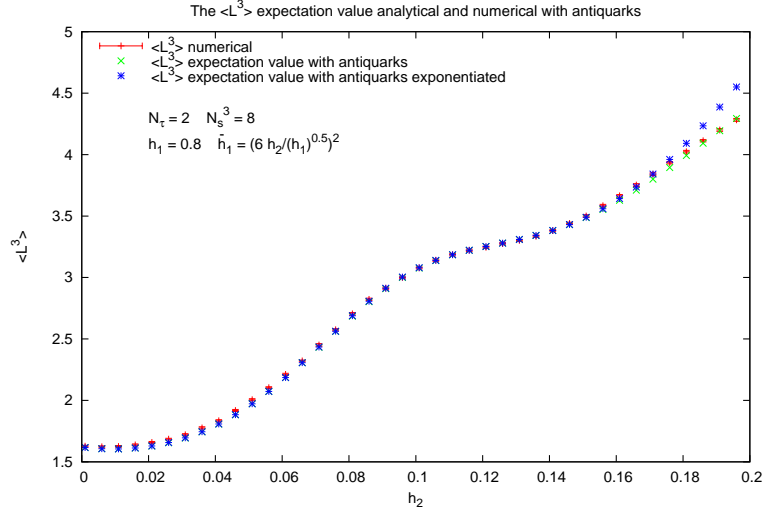


Figure 12: $\langle L^3 \rangle$ with antiquarks

Again the numerical and analytical results seem to fit much better when involving antiquarks. On top of that the numerical data shows less noise than in the previous case. Even for large values of h_2 numerical and analytical data have a good correspondence.

Starting with $h_2 \approx 0.17$ both analytical solutions differ from the numerical one. A possible way to fix it would be a calculation of higher order corrections to the Linked Cluster series. In contrast to the previous cases both analytical solutions differ from the numerical data in the same way so the exponentiated form does not lead to a less convergence in this case.

On top of that the maximal discrepancy between the solutions with antiquarks is of the same order as the maximal discrepancy between the numerical data and the exponentiated form when neglecting antiquarks.

The expectation value of a cubed Polyakov loop can physically be interpreted as a baryon on the lattice. In general a cubed Polyakov loop as operator in the used effective model can be interpreted as a baryonic contribution.

Combining a Polyakov loop and a daggered Polyakov loop leads to mesons.

5.4 The $(L^\dagger)^3$ Expectation Value

Finally the expectation value of a cubed and daggered Polyakov loop can be calculated.

The Operator in (80) is replaced by a cubed and daggered Polyakov loop $O_{\bar{x}} \rightarrow (L_{\bar{x}}^\dagger)^3$. Of course the integrals for z_0 , z_1 and \bar{z}_1 give the same results.

Corresponding to order h_1^0 the following integral appears in the numerator of $M_{(0,1)}^0$.

$$\begin{aligned}
 & \int (L_{\bar{x}}^\dagger)^3 \cdot \det(Q_{stat}) [dU_0] \tag{117} \\
 &= \int (L_{\bar{x}}^\dagger)^3 [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
 & \quad \times \prod_{\{\bar{y}\} \setminus \bar{x}} \int [1 + h_1 L_{\bar{y}} + h_1^2 L_{\bar{y}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{y}}^\dagger + \bar{h}_1^2 L_{\bar{y}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot \int L_{\bar{x}}^3 [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot [1 + 8h_1^3 + h_1^6 + 12h_1\bar{h}_1 + 22h_1^4\bar{h}_1 + 45h_1^2\bar{h}_1^2 + 22h_1^5\bar{h}_1^2 \\
 & \quad + 8\bar{h}_1^3 + 72h_1^3\bar{h}_1^3 + 8h_1^6\bar{h}_1^3 + 24h_1\bar{h}_1^4 + 45h_1^4\bar{h}_1^4 + 24h_1^2\bar{h}_1^5 \\
 & \quad + 12h_1^5\bar{h}_1^5 + \bar{h}_1^6 + 8h_1^3\bar{h}_1^6 + h_1^6\bar{h}_1^6] \\
 &= z_0^{N_s^3-1} \cdot z_0 (L_{\bar{x}}^\dagger)^3
 \end{aligned}$$

For order h_2 the following integrals can be found in the numerator of $M_{(1,1)}^0$.

$$\begin{aligned}
 & \int (L_{\bar{x}}^\dagger)^3 \cdot M_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \tag{118} \\
 &= z_0^{N_s^3-1} \cdot \int (L_{\bar{x}}^\dagger)^3 \cdot \frac{h_1 L_{\bar{x}} + 2h_1^2 L_{\bar{x}}^\dagger + 3h_1^3}{1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3} \\
 & \quad \times [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
 &= z_0^{N_s^3-1} \cdot [12h_1^3 + 3h_1^6 + 6h_1\bar{h}_1 + 44h_1^4\bar{h}_1 + 45h_1^2\bar{h}_1^2 + 55h_1^5\bar{h}_1^2 + 108h_1^3\bar{h}_1^3 \\
 & \quad + 24h_1^6\bar{h}_1^3 + 12h_1\bar{h}_1^4 + 90h_1^4\bar{h}_1^4 + 24h_1^2\bar{h}_1^5 + 30h_1^5\bar{h}_1^5 + 12h_1^3\bar{h}_1^6 + 3h_1^6\bar{h}_1^6] \\
 &:= z_0^{N_s^3-1} \cdot z_1 (L_{\bar{x}}^\dagger)^3
 \end{aligned}$$

5.4 The $(L^\dagger)^3$ Expectation Value

$$\begin{aligned}
& \int \left(L_{\bar{x}}^\dagger\right)^3 \cdot \bar{M}_{\bar{x}} \cdot \det(Q_{stat}) [dU_0] \tag{119} \\
&= z_0^{N_s^3-1} \cdot \int \left(L_{\bar{x}}^\dagger\right)^3 \cdot \frac{\bar{h}_1 L_{\bar{x}}^\dagger + 2\bar{h}_1^2 L_{\bar{x}} + 3\bar{h}_1^3}{1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3} \\
&\quad \times [1 + h_1 L_{\bar{x}} + h_1^2 L_{\bar{x}}^\dagger + h_1^3]^2 [1 + \bar{h}_1 L_{\bar{x}}^\dagger + \bar{h}_1^2 L_{\bar{x}} + \bar{h}_1^3]^2 [dU_0] \\
&= z_0^{N_s^3-1} \cdot [6h_1\bar{h}_1 + 11h_1^4\bar{h}_1 + 45h_1^2\bar{h}_1^2 + 22h_1^5\bar{h}_1^2 + 12\bar{h}_1^3 + 108h_1^3\bar{h}_1^3 + 12h_1^6\bar{h}_1^3 \\
&\quad + 48h_1\bar{h}_1^4 + 90h_1^4\bar{h}_1^4 + 65h_1^2\bar{h}_1^5 + 30h_1^5\bar{h}_1^5 + 3\bar{h}_1^6 + 24h_1^3\bar{h}_1^6 + 3h_1^6\bar{h}_1^6] \\
&:= z_0^{N_s^3-1} \cdot \bar{z}_1 \left(L_{\bar{x}}^\dagger\right)^3
\end{aligned}$$

With help of these results the expectation value can be written down.

$$\begin{aligned}
& \langle \left(L_{\bar{x}}^\dagger\right)^3 \rangle = \tag{120} \\
& \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3}{z_0} - 2 \cdot 6 \cdot h_2 \left[\left(\frac{z_1 \cdot z_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^2} - \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3 \cdot z_1^2}{z_0^3} \right) + \left(\frac{\bar{z}_1 \cdot \bar{z}_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^2} - \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3 \cdot \bar{z}_1^2}{z_0^3} \right) \right. \\
& \left. - \left(\frac{\bar{z}_1 \cdot z_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^2} - \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3 \cdot \bar{z}_1 \cdot z_1}{z_0^3} \right) - \left(\frac{z_1 \cdot \bar{z}_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^2} - \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3 \cdot \bar{z}_1 \cdot z_1}{z_0^3} \right) \right] + o(h_2^2)
\end{aligned}$$

$$\begin{aligned}
& \langle \left(L_{\bar{x}}^\dagger\right)^3 \rangle = \tag{121} \\
& \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3}{z_0} \cdot \exp \left(-2 \cdot 6 \cdot h_2 \left[\frac{z_1}{z_0} \left(\frac{z_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^3} - \frac{z_1}{z_0} \right) + \frac{\bar{z}_1}{z_0} \left(\frac{\bar{z}_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^3} - \frac{\bar{z}_1}{z_0} \right) \right. \right. \\
& \quad \left. \left. - \frac{z_1}{z_0} \left(\frac{\bar{z}_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^3} - \frac{\bar{z}_1}{z_0} \right) - \frac{\bar{z}_1}{z_0} \left(\frac{z_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^3} - \frac{z_1}{z_0} \right) \right] \right)
\end{aligned}$$

Setting $\bar{h}_1 = 0$ leads to the expectation value without antiquarks.

$$\langle \left(L_{\bar{x}}^\dagger\right)^3 \rangle = \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3}{z_0} - 2 \cdot 6 \cdot h_2 \left[\frac{z_1 \cdot z_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^2} - \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3 \cdot z_1^2}{z_0^3} \right] + o(h_2^2) \tag{122}$$

$$\langle \left(L_{\bar{x}}^\dagger\right)^3 \rangle = \frac{z_0 \left(L_{\bar{x}}^\dagger\right)^3}{z_0} \cdot \exp \left(-2 \cdot 6 \cdot h_2 \left[\frac{z_1}{z_0} \left(\frac{z_1 \left(L_{\bar{x}}^\dagger\right)^3}{z_0^3} - \frac{z_1}{z_0} \right) \right] \right) \tag{123}$$

5.4 The $(L^\dagger)^3$ Expectation Value

The analytical results of $\langle (L^\dagger)^3 \rangle$ are compared with the numerical computation keeping again $h_1 = 0.8$ constant. At first without involving antiquarks.

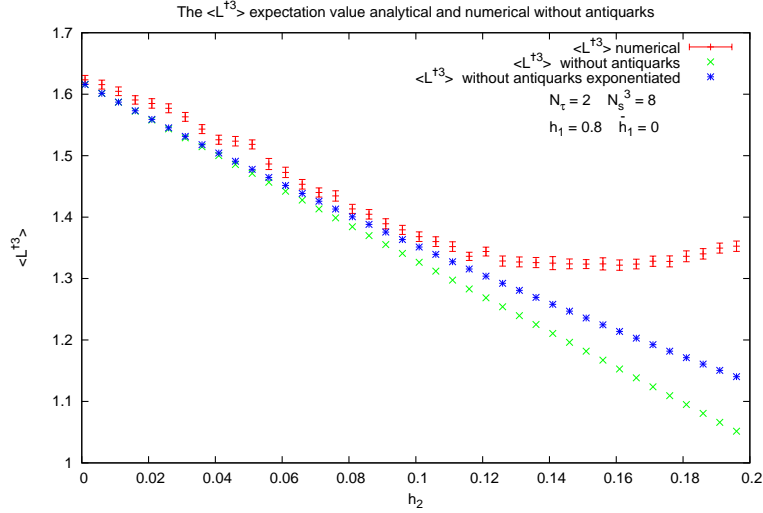


Figure 13: $\langle (L^\dagger)^3 \rangle$ without antiquarks

For $\langle (L^\dagger)^3 \rangle$ without antiquarks the same problem as for $\langle L^3 \rangle$ without antiquarks may be noticed. The numerical data is constantly above the analytical results.

Again this could have its origin in the bigger susceptibility of the analytical solution for discrepancies between the analytical and numerical values due to larger factors in front of h_1 coupling in the analytic solution.

Besides the numerical data shows larger fluctuations than the assumed errors could compensate.

The behavior of the exponentiated form fits to the previously observed behavior when neglecting antiquark contributions. The data from it is better convergent to the numerical data than the data of the non exponentiated form.

Nevertheless the analytical solutions and the numerical solution move apart starting with $h_2 \approx 0.12$. This problem might be fixed by including higher order corrections to the Linked Cluster series of the expectation value.

5.4 The $\langle (L^\dagger)^3 \rangle$ Expectation Value

Finally the full analytical expression of $\langle (L^\dagger)^3 \rangle$ is compared with a numerical computation of the expectation value.

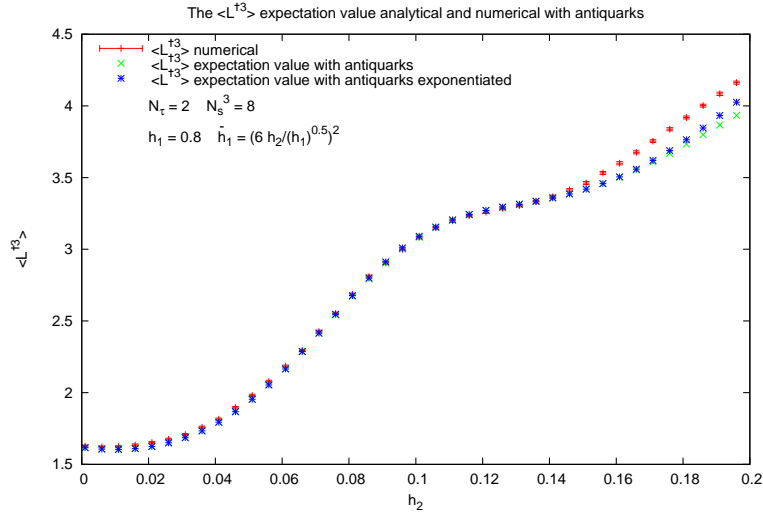


Figure 14: $\langle (L^\dagger)^3 \rangle$ with antiquarks

As in all previous cases the comparison of numerical and analytical results shows the least differences for the expectation value involving antiquarks. Indeed for $\langle (L^\dagger)^3 \rangle$ the numerical results differ from the analytical results already for lower h_2 values than in the previous cases starting with $h_2 \approx 0.15$. Nevertheless the comparison shows a good correspondence.

In this case the approximated exponentiated form of the expectation value is a little better convergent towards the numerical solution than the non exponentiated form when involving antiquarks. But again this convergence could surely be improved by including higher order terms in the linked cluster series.

The discussed cases show that analytical calculations of Polyakov loop expectation values up to first order nearest neighbor interaction fit good to numerical computed results.

Especially if antiquarks are involved in the partition function a great correspondence could have been noticed. Further the success of Linked Cluster Expansion could be shown since it leads to good results by applying a straight forward mathematical formalism.

6 The $\langle L^3 L^3 \rangle - \langle L^3 \rangle \langle L^3 \rangle$ Correlation Function

The final calculation in this work is a connected correlation function of two nearest neighbor cubed Polyakov loops on the lattice. For simplicity the gauge interaction proportional to λ is neglected and calculations are done up to first order in h_2 . A short comment on λ will be given in an Outlook at the end.

Starting point is the expression (85) for connected correlation functions of nearest neighbor objects from section 4.4. As done in section 5 the two operators in expression (85) are replaced by two cubed Polyakov loops $O_{\vec{x}} \rightarrow L_{\vec{x}}^3$, $O_{\vec{y}} \rightarrow L_{\vec{y}}^3$.

The integrals of (85) are all well known from the previous section 5.3.

$$\frac{\int L_{\vec{x}}^3 \cdot M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{\int \det(Q_{stat}) [dU_0]} = \frac{z_0^{N_s^3-1} \cdot z_1^{L^3}}{z_0^{N_s^3}} = \frac{z_1^{L^3}}{z_0} \quad (124)$$

$$\frac{\int L_{\vec{x}}^3 \cdot \det(Q_{stat}) [dU_0] \int M_{\vec{x}} \cdot \det(Q_{stat}) [dU_0]}{(\int \det(Q_{stat}) [dU_0])^2} = \frac{z_0^{2N_s^3-2} \cdot z_0^{L^3} \cdot z_1}{z_0^{2N_s^3}} = \frac{z_0^{L^3} \cdot z_1}{z_0^2} \quad (125)$$

Analogous for the antiquark contribution and the mixing terms.

Using all the results the full correlation function can be written down.

$$\begin{aligned} \langle L_{\vec{x}}^3 L_{\vec{y}}^3 \rangle - \langle L_{\vec{x}}^3 \rangle \langle L_{\vec{y}}^3 \rangle &= -2 \cdot h_2 \left[\left(\frac{z_1^{L^3}}{z_0} - \frac{z_0^{L^3} \cdot z_1}{z_0^2} \right) \right. \\ &\times \left(\frac{z_1^{L^3}}{z_0} - \frac{z_0^{L^3} \cdot z_1}{z_0^2} \right) + \left(\frac{\bar{z}_1^{L^3}}{z_0} - \frac{z_0^{L^3} \cdot \bar{z}_1}{z_0^2} \right) \times \left(\frac{\bar{z}_1^{L^3}}{z_0} - \frac{z_0^{L^3} \cdot \bar{z}_1}{z_0^2} \right) \\ &\left. - 2 \left(\frac{z_1^{L^3}}{z_0} - \frac{z_0^{L^3} \cdot z_1}{z_0^2} \right) \times \left(\frac{\bar{z}_1^{L^3}}{z_0} - \frac{z_0^{L^3} \cdot \bar{z}_1}{z_0^2} \right) \right] + o(h_2^2) \end{aligned} \quad (126)$$

It should be mentioned that no exponentiated expression could have been found for the connected correlation function.

The analytical result is compared to a numerical computation of two nearest neighbor cubed Polyakov loops on the lattice.

The plot of the full analytical correlator including antiquarks and the corresponding numerical data is done by keeping $h_1 = 0.8$ fixed and plotting in terms of h_2 as in section 5.

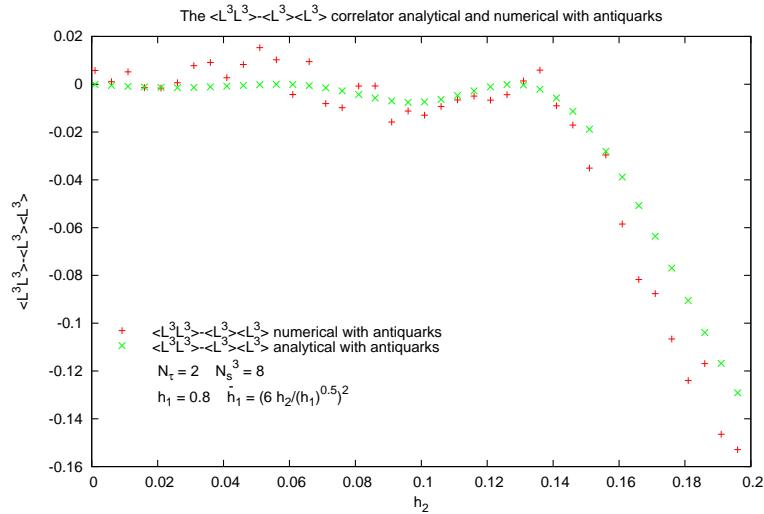


Figure 15: $\langle L^3 L^3 \rangle - \langle L^3 \rangle \langle L^3 \rangle$ with antiquarks

Although a numerical computation of the correlation function does not lead to proper results a comparison between the shape of the analytical graph and the trend of the numerical data shows a correspondence.

The numerical computation has been done with the same conditions as in section 5. A Monte Carlo simulation with 50000 samples starting with $\kappa = 0.0387298$ was used.

Since the numerical calculations were not subject of this work the question how the numerical computation could be improved is left open. It seems like an improved numerical computation could possibly fit with the analytical data.

A more detailed calculation of correlation functions for example including higher orders could be subject of future projects.

Outlook

As mentioned in sections 5 and 6 all gauge contributions according to gauge coupling λ were neglected. The method of linked cluster expansion presented in section 4 can also be used for the calculation of gauge interaction. It is possible to treat the gauge part in the partition function (40) namely

$$\prod_{\langle \vec{x}, \vec{y} \rangle} \left[1 + \lambda(L_{\vec{x}} L_{\vec{y}}^\dagger + L_{\vec{x}}^\dagger L_{\vec{y}}) \right]$$

in an analogous way to the fermionic part. Linked Cluster Expansion then leads to a series in λ by introducing source terms and applying functional derivatives. Afterwards the main calculation is solving the gauge integrals over the appearing powers of Polyakov loops.

For first order nearest neighbor interaction not only higher order terms corresponding to one coupling constant are neglected, also all mixing partitions as $\lambda \cdot h_2$ are neglected. Therefore the expectation value respectively the correlator is given as a superposition of the gauge and fermionic interaction part.

$$\langle O_{\vec{x}} \rangle = \text{"no coupling term"} + \text{"}\lambda \text{ term"} + \text{"}h_2 \text{ term"} + o(\lambda \cdot h_2) \quad (127)$$

The partition corresponding to h_2 has been calculated in section 5 for several expectation values and in section 6 for a $L^3 L^3$ -correlator. The next step would be to calculate the λ partition.

Furthermore a solution including higher orders of h_2 , λ and especially mixing terms would be of interest. Calculations for non vanishing λ have been left out in this work since the couplings h_1 , \bar{h}_1 and h_2 would not only be functions of the hopping parameter κ but also functions of the gauge coupling λ . This phenomena is referred to as gauge corrections to the coupling constants further details can be found in [4].

Finally higher order terms and larger distances corresponding to the $\langle L^3 L^3 \rangle - \langle L^3 \rangle \langle L^3 \rangle$ correlation function would be of interest. As mentioned the expectation value of a cubed Polyakov loop can physically be interpreted as barionic contribution within the used effective model. It can be shown that the connected correlator of two cubed Polyakov loops is related to the potential between two baryons located on the lattice. Indeed for proper results the correlator needs to be calculated as a function of distance between the baryons. Therefore higher order terms are needed.

Appendix

List of gauge integrals over Polyakov loops

$$\int [dU_0] = 1$$

$$\int L_{\vec{x}} L_{\vec{x}}^\dagger [dU_0] = 1$$

$$\int L_{\vec{x}}^2 L_{\vec{x}}^{\dagger 2} [dU_0] = 2$$

$$\int L_{\vec{x}}^3 L_{\vec{x}}^{\dagger 3} [dU_0] = 6$$

$$\int L_{\vec{x}}^4 L_{\vec{x}}^{\dagger 4} [dU_0] = 23$$

$$\int L_{\vec{x}}^3 [dU_0] = 1$$

$$\int L_{\vec{x}}^4 L_{\vec{x}}^\dagger [dU_0] = 3$$

$$\int L_{\vec{x}}^5 L_{\vec{x}}^{\dagger 2} [dU_0] = 11$$

$$\int L_{\vec{x}}^6 L_{\vec{x}}^{\dagger 3} [dU_0] = 47$$

$$\int L_{\vec{x}}^6 [dU_0] = 5$$

$$\int L_{\vec{x}}^7 L_{\vec{x}}^\dagger [dU_0] = 21$$

$$\int L_{\vec{x}}^9 [dU_0] = 42$$

References

- [1] C. Gattringer and Christian B. Lang, *Quantum Chromodynamics on the lattice: An Introductory Presentation*, Lect. Notes Phys. 788, Springer Berlin Heidelberg (2010), DOI: 10.1007/978-642-01850-3
- [2] I. Montvay and G. Münster, *Quantum fields on a lattice*, Cambridge monographs on mathematical physics, Cambridge Univ. Pr. (1994), ISBN: 0-521-40432-0
- [3] J. Langelage, S. Lottini and O. Philipsen, *Centre symmetric 3d effective actions for thermal $SU(N)$ Yang-Mills from strong coupling series*, JHEP **02** (2011) 057 [*Erratum ibid.* **1107** (2011) 014] [arXiv:1010.0951] [INSPIRE]
- [4] J. Langelage, M. Neuman and O. Philipsen, *Heavy dense QCD and nuclear matter from an effective lattice theory*, JHEP **09** (2014) 131, [arXiv:1403.4162] [INSPIRE]
- [5] M. Wortis, *Linked Cluster Expansion* (1980), in C. Domb, M.S. Green: *Phase Transitions and Critical Phenomena*, Vol.3, 113-180 (Academic Press, New York)
- [6] M. Gross, J. Bartholomew and D. Hochberg, *$SU(N)$ deconfinement transition and the N -state clock model*, Report No. EFI-83-35-CHICAGO (1983)
- [7] T. P. Ling and L. F. Li, *Gauge theory of elementary particle physics*, Oxford Univ. Pr. (1988), p. 86-119, ISBN: 978-0198519614

Pictures

- [Figure 1] C. Gattringer and Christian B. Lang, *Quantum Chromodynamics on the lattice: An Introductory Presentation*, Lect. Notes Phys. 788, Springer Berlin Heidelberg (2010), p. 34, DOI: 10.1007/978-642-01850-3
- [Figure 2] C. Gattringer and Christian B. Lang, *Quantum Chromodynamics on the lattice: An Introductory Presentation*, Lect. Notes Phys. 788, Springer Berlin Heidelberg (2010), p. 38, DOI: 10.1007/978-642-01850-3

Selbstständigkeitserklärung

Gemäß §30 (12) der Ordnung des Fachbereichs Physik an der Johann Wolfgang von Goethe Universität für den Bachelor- und Masterstudiengang Physik vom 24.04.2013 versichere ich, dass ich die vorliegende Arbeit selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel verfasst habe. Ferner erkläre ich, dass diese Arbeit, auch nicht auszugsweise, für eine andere Prüfung oder Studienleistung verwendet worden ist.

Frankfurt am Main 26.09.2015

Andreas Halsch