Bachelor Thesis

Free Fermion Gases with Wilson and Perfect Actions

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Abstract

Within this thesis, thermodynamic quantities like the pressure and particle density will be calculated for *free perfect fermions*. For this purpose a perfect lattice action has to be constructed by applying a *renormalisation group transformation* to the continuum action [1, page 4]. Then the resulting integrals will be solved on a lattice by a C-code. Therefor the massless and the massive case will be considered and at the end, the calculated results can be compared graphically with Wilson-fermions and the continuum case.

Contents

1 Introduction

This thesis will concentrate on a thermodynamic system of free fermions, particles which have half-integer spin. The fundamental class of these particles contain the six types of leptons (electron muon, tau and their neutrino partners) and the six different flavors of quarks (up , down, strange, charm, top and bottom) [2]. In order to describe this particles, we will make use of a quantum-field-theory of the *standard model* (the most successful model to describe fundamental matter until today) called *Quantum Chromodynamics* (QCD). For leptons is this possible, because we consider free particles. Otherwise it would not be possible, because they do not interact strongly.

The theory introduces three different color-charges, which are not only carried by the quarks, but also by the force-carriers called *gluons*. The gluons always carry color-charge and anti-color-charge (e.g red and anti-blue), while the quarks and anti-quarks only carry color-charge or anti-color-charge. But this color-charge has never been detected in an experiment, which allows the conclusion, that only color-singlets (colorless objects) occur in nature (confinement [3, page 226-227]). An exception of this is the quark-gluon-plasma, which can occur under extreme conditions, that can only be generated in the cores of neutron-stars or in the early universe, where quarks and gluons are quasi-free particles [4].

In order to describe a system of particles in a finite volume in contact with a heat bath (variable temperature), we also have to make use of statistical mechanics. There the system can be described through ensembles like the canonical- or grand canonical ensemble. In the following we will use the grand canonical ensemble, because it also allows particle exchange with the heat bath and the partition function can be calculated by quantum-field-theory methods.

For physicists, thermodynamic quantities like the pressure and particle density are interesting observables of a system, as described above. In order to calculate them, we will come to formulas, which are no longer solvable in closed form, so we use *brute force* (nummerical discretisation schemes, which will be explained later). Most of these schemes (e.g the Wilsons QCD-action) are perturbative to the lattice spacing a (nearest neighbour difference), so we have to use very fine lattices to keep the cutt-off effects small. But with finer lattices the computing time rises like a^{-6} or sometimes a^{-10} [5, page 1-2] and the continuum limit ($\mathrm{a} \to 0$) is hard to reach. So the question arises, whether there are schemes, that are non-perturbative in a? The answer is yes, there are. They go under the name of *perfect actions* and have some fascinating properties like the reproduction of continuum symmetries (e.g poincare invariance). Because of this properties, the scheme is also of interest for the practical aspect of reducing the computational effort in simulations [5, page 2].

2 Theory

This chapter will give a short recap of the basic knowledge of QCD in the continuum case and on the lattice. Also the relation between statistical mechanics and QCD will be mentioned at the end. Also sum-convention is used in this thesis!

2.1 QCD in the continuum

QCD is the fundamental theory of the strong interaction, which describes color-charged particles as quarks and gluons. The action of QCD in Euclidean space can be separated in a fermionic and a gluonic part [6, page 27-30]:

$$\mathrm{S}[\psi,\bar{\psi},\mathrm{A}] = \mathrm{S_F}[\psi,\bar{\psi},\mathrm{A}] + \mathrm{S_G}[\mathrm{A}] \tag{2.1}$$

$$\mathrm{S_F}[\psi,\bar{\psi},\mathrm{A}] = \sum_{\mathrm{f}=1}^{\mathrm{N_f}} \int \mathrm{d}^4\mathrm{x}\, \bar{\psi}^{(\mathrm{f})}(\gamma_\mu(\partial_\mu + \mathrm{i}\mathrm{A}_\mu(\mathrm{x})) + \mathrm{m}^{(\mathrm{f})})\psi^{(\mathrm{f})}(\mathrm{x}) \tag{2.2}$$

$$\mathrm{S_G}[\mathrm{A}] = \frac{1}{2\mathrm{g}^2} \int \mathrm{d}^4\mathrm{x}\, \mathrm{tr}[\mathrm{F}_{\mu\nu}(\mathrm{x})\mathrm{F}_{\mu\nu}(\mathrm{x})] \tag{2.3}$$

The quarks and antiquarks are described by 4-spinors $\psi, \bar{\psi}$ and the gluons by the gauge fields A_μ. The mass of the quarks is $\mathrm{m}^{(\mathrm{f})}$, γ_μ are the Euclidean gamma matrices und g is the coupling strength of the strong interaction. One should notice, that there is no difference between covariant und contravariant indices, so only lower indices will be used. $\mathrm{F}_{\mu\nu}$ describes the field strength tensor and is defined via:

$$\mathrm{F}_{\mu\nu}(\mathrm{x}) = \partial_\mu \mathrm{A}_\nu(\mathrm{x}) - \partial_\nu \mathrm{A}_\mu(\mathrm{x}) + \mathrm{i}[\mathrm{A}_\mu(\mathrm{x}), \mathrm{A}_\nu(\mathrm{x})] \tag{2.4}$$

We notice, that $\mathrm{S_F}$ contains the propagation of quarks and the interaction with gluons, while $\mathrm{S_G}$ only describes gluon propagation and gluon-gluon interaction.

In the continuum, expectation values of operators can be expressed with path integrals [6, page 20]:

$$\left\langle \hat{\mathrm{O}}_2(\mathrm{t})\hat{\mathrm{O}}_1(0) \right\rangle = \frac{1}{\mathrm{Z}} \int \mathcal{D}[\psi,\bar{\psi}]\mathcal{D}[\mathrm{A}]\, \mathrm{O}_2[\psi(.,\mathrm{t}),\bar{\psi}(.,\mathrm{t}),\mathrm{A}(.,\mathrm{t})]\mathrm{O}_1[\psi(.,0),\bar{\psi}(.,0),\mathrm{A}(.,0)]\mathrm{e}^{-\mathrm{S}[\psi,\bar{\psi},\mathrm{A}]} \tag{2.5}$$

$$\mathrm{Z} = \int \mathcal{D}[\psi,\bar{\psi}]\mathcal{D}[\mathrm{A}]\, \mathrm{e}^{-\mathrm{S}[\psi,\bar{\psi},\mathrm{A}]} \tag{2.6}$$

$$\mathcal{D}[\psi,\bar{\psi}] = \prod_{\mathrm{x}\in\mathbb{R}^4} \prod_{\mathrm{f},\alpha,\mathrm{c}} \mathrm{d}\psi^{(\mathrm{f})}_{\alpha,\mathrm{c}}(\mathrm{x})\mathrm{d}\bar{\psi}^{(\mathrm{f})}_{\alpha,\mathrm{c}}(\mathrm{x}); \quad \mathcal{D}[\mathrm{A}] = \prod_{\mathrm{x}\in\mathbb{R}^4} \prod_{\mu=1}^{4} \mathrm{dA}_\mu(\mathrm{x}) \tag{2.7}$$

The operators $\hat{O}_i$ contain field operators as in canonical quantisation, but under the path integral they are functionals of the fields. Z is here the partition function, $\mathcal{D}[\psi, \bar{\psi}]$ and $\mathcal{D}[A]$ are the measures of the path integral.

In generall we cannot solve such integrals in closed form, which is the reason for trying pertubation theory. But in low ernegy scales like for hadronic matter ($\Lambda_{QCD} \approx 200 - 400$MeV [3, page 224-225]) $\alpha_s(\Lambda_{QCD}) = \frac{g^2}{4\pi}$ diverges and one can no longer expand in powers of g! This leads to the lattice approach in the next section.

2.2 QCD on the lattice

Lattice QCD is a non-perturbative method to solve QCD in low energy scales. In the lattice formulation the Euclidean spacetime is replaced by a 4D lattice [6, page 332]:

$$\Lambda = \Big\{ n = (n_1, n_2, n_3, n_4);\ n_1, n_2, n_3 = 0, 1, ..., N_s - 1;\ n_4 = 0, 1, ..., N_t - 1 \Big\} \tag{2.8}$$

The vector $n \in \Lambda$ labels the points in the discrete spacetime separated by a lattice constant a, which has dimension length. The spinors live now on the lattice points:

$$\psi(n), \bar{\psi}(n),\ n \in \Lambda \tag{2.9}$$

The color, Dirac and flavour indices are suppressed and only the integer-valued 4-vector n is used to label the positions of the quarks, not the physical spacetime point $x = an$. Due to the finite size of the lattice, we simply choose periodic boundary conditions, which means point N is identified with point 0! The partial derivative is discretized by the symmetric expression:

$$\partial_\mu \psi(n) = \frac{\psi(n + \hat{\mu}) - \psi(n - \hat{\mu})}{2a} + O(a^2) \tag{2.10}$$

Here $\hat{\mu}$ denotes the unit vector of length a in μ-direction. The integral is replaced by a sum over all lattice points and reads:

$$\int d^4x \rightarrow a^4 \sum_{n \in \Lambda} \tag{2.11}$$

2.2.1 Naive fermion discretisation

After applying 2.9,2.10 and 2.11 to 2.2, the lattice fermion action for free fermions ($A_\mu = 0$) reads [6, page 33]:

$$S_F^0[\psi, \bar{\psi}] = a^4 \sum_{f=1}^{N_f} \sum_{n \in \Lambda} \bar{\psi}^{(f)}(n) \Big(\gamma_\mu \frac{\psi(n + \hat{\mu}) - \psi(n - \hat{\mu})}{2a} + m^{(f)} \psi^{(f)}(n) \Big) \tag{2.12}$$

But this action is not invariant under SU(3) gauge transformations. Recall the transformation behaviour of the fermion fields:

$$\psi(\mathrm{n}) \rightarrow \psi'(\mathrm{n}) = \Omega(\mathrm{n})\psi(\mathrm{n});\ \bar{\psi(\mathrm{n})} \rightarrow \bar{\psi}'(\mathrm{n}) = \bar{\psi}(\mathrm{n})\Omega^{\dagger}(\mathrm{n}) \tag{2.13}$$

So we can see easily, that the discretized partial derivativ is not gauge invariant. In order to make this gauge invariant, a field $\mathrm{U}_{\mu}(\mathrm{n}) \in \mathrm{SU}(3)$ has to be introduced, so that:

$$\mathrm{U}_{\mu}(\mathrm{n}) \rightarrow \mathrm{U}'_{\mu}(\mathrm{n}) = \Omega(\mathrm{n})\mathrm{U}_{\mu}(\mathrm{n})\Omega(\mathrm{n}+\hat{\mu})^{\dagger} \tag{2.14}$$

These matrix-valued fields are orientated and connect the sites n and $\mathrm{n}+\hat{\mu}$, so they are called *link variables*. It is also possible to define a link variable that points in negative μ direction and connects n and $\mathrm{n}-\hat{\mu}$ [6, page 34]:

$$\mathrm{U}_{-\mu}(\mathrm{n}) = \mathrm{U}_{\mu}(\mathrm{n}-\hat{\mu})^{\dagger} \rightarrow \mathrm{U}'_{-\mu}(\mathrm{n}) = \Omega(\mathrm{n})\mathrm{U}_{-\mu}(\mathrm{n})\Omega(\mathrm{n}-\hat{\mu})^{\dagger} \tag{2.15}$$

After we have introduced the link variables to make the action invariant under gauge transformations, we can now write down the so-called *naive fermion action*:

$$\mathrm{S}_{\mathrm{F}}[\psi,\bar{\psi}] = \mathrm{a}^4 \sum_{\mathrm{f}=1}^{\mathrm{N_f}} \sum_{\mathrm{n}\in\Lambda} \bar{\psi}^{(\mathrm{f})}(\mathrm{n})\Big(\gamma_{\mu}\frac{\mathrm{U}_{\mu}(\mathrm{n})\psi(\mathrm{n}+\hat{\mu}) - \mathrm{U}_{-\mu}(\mathrm{n})\psi(\mathrm{n}-\hat{\mu})}{2\mathrm{a}} + \mathrm{m}^{(\mathrm{f})}\psi^{(\mathrm{f})}(\mathrm{n})\Big) \tag{2.16}$$

Since the action is bilinear in $\bar{\psi}$ and ψ, it can be written in the following form (matrix-vector notation for Dirac and color indices) [6, page 110]:

$$\mathrm{S}_{\mathrm{F}}[\psi,\bar{\psi}] = \mathrm{a}^4 \sum_{\mathrm{f}=1}^{\mathrm{N_f}} \sum_{\mathrm{n,m}\in\Lambda} \bar{\psi}^{(\mathrm{f})}(\mathrm{n})\mathrm{D}(\mathrm{n}|\mathrm{m})\psi^{(\mathrm{f})}(\mathrm{m}) \tag{2.17}$$

Here $\mathrm{D}(\mathrm{n}|\mathrm{m})$ is the naive Dirac operator on the lattice, which is given by:

$$\mathrm{D}(\mathrm{n}|\mathrm{m}) = \gamma_{\mu}\frac{\mathrm{U}_{\mu}(\mathrm{n})\delta_{\mathrm{n}+\hat{\mu},\mathrm{m}} - \mathrm{U}_{-\mu}(\mathrm{n})\delta_{\mathrm{n}-\hat{\mu},\mathrm{m}}}{2\mathrm{a}} + \mathrm{m}^{(\mathrm{f})}\delta_{\mathrm{n},\mathrm{m}} \tag{2.18}$$

2.2.2 The fermion doubling problem

The Fourier transform Dirac operator for trivial gauge field $\mathrm{U}_{\mu}(\mathrm{n}) = \mathbb{1}$ is given by [6, page 111]:

$$\begin{aligned}
\widetilde{\mathrm{D}}(\mathrm{p}|\mathrm{q}) &= \frac{1}{|\Lambda|}\sum_{\mathrm{n,m}\in\Lambda} \mathrm{e}^{-\mathrm{ip}\cdot\mathrm{na}}\mathrm{D}(\mathrm{n}|\mathrm{m})\mathrm{e}^{\mathrm{iq}\cdot\mathrm{ma}} \\
&= \frac{1}{|\Lambda|}\sum_{\mathrm{n,m}\in\Lambda} \mathrm{e}^{-\mathrm{i}(\mathrm{p}-\mathrm{q})\cdot\mathrm{na}}\Big(\gamma_{\mu}\frac{\mathrm{e}^{\mathrm{iq}_{\mu}\mathrm{a}} - \mathrm{e}^{-\mathrm{iq}_{\mu}\mathrm{a}}}{2\mathrm{a}} + \mathrm{m}\mathbb{1}\Big) \\
&= \delta(\mathrm{p}-\mathrm{q})\Big(\frac{\mathrm{i}}{\mathrm{a}}\gamma_{\mu}\sin(\mathrm{p}_{\mu}\mathrm{a}) + \mathrm{m}\mathbb{1}\Big) = \delta(\mathrm{p}-\mathrm{q})\widetilde{\mathrm{D}}(\mathrm{p})
\end{aligned} \tag{2.19}$$

Here $|\Lambda|$ is the total number of all lattice points. The inverse of the fourier-transformed Dirac operator is then:

$$\tilde{D}^{-1}(p) = \frac{m\mathbb{1} - \frac{i}{a}\gamma_\mu \sin(p_\mu a)}{m^2 + \frac{1}{a^2}\sin(p_\rho a)\cdot\sin(p_\rho a)} \tag{2.20}$$

Proof:

$$\begin{aligned}\tilde{D}^{-1}(p)\cdot\tilde{D}(p) &= \Big(\frac{m\mathbb{1} - \frac{i}{a}\gamma_\mu \sin(p_\mu a)}{m^2 + \frac{1}{a^2}\sin(p_\rho a)\cdot\sin(p_\rho a)}\Big)\cdot\Big(\frac{i}{a}\gamma_\nu\sin(p_\nu a) + m\mathbb{1}\Big)\\ &= \frac{m^2\mathbb{1} + \frac{1}{a^2}\gamma_\mu\gamma_\nu\sin(p_\mu a)\sin(p_\nu a)}{m^2 + \frac{1}{a^2}\sin(p_\rho a)\cdot\sin(p_\rho a)}\\ &= \frac{m^2\mathbb{1} + \frac{1}{a^2}\frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)\sin(p_\mu a)\sin(p_\nu a)}{m^2 + \frac{1}{a^2}\sin(p_\rho a)\cdot\sin(p_\rho a)}\\ &= \frac{m^2\mathbb{1} + \frac{1}{a^2}\frac{1}{2}(2\delta_{\mu\nu}\mathbb{1})\sin(p_\mu a)\sin(p_\nu a)}{m^2 + \frac{1}{a^2}\sin(p_\rho a)\cdot\sin(p_\rho a)}\\ &= \frac{m^2 + \frac{1}{a^2}\sin(p_\mu a)\sin(p_\mu a)}{m^2 + \frac{1}{a^2}\sin(p_\rho a)\cdot\sin(p_\rho a)}\mathbb{1} = \mathbb{1}\text{ q.e.d}\end{aligned} \tag{2.21}$$

Where the relation 6.12 for Euclidean gamma-matrices has been used. Now we can look at the poles of the propagator (inverse operator). Of interest is the case of massles quarks ($m = 0$) and it should be mentioned, that the propagator has the correct continuum limit ($a \to 0$) [6, page 112]:

$$\tilde{D}^{-1}(p)|_{m=0} = \frac{-\frac{i}{a}\gamma_\mu\sin(p_\mu a)}{\frac{1}{a^2}\sin(p_\rho a)\cdot\sin(p_\rho a)} \to \frac{-i\,\gamma_\mu\,p_\mu}{p^2} \tag{2.22}$$

Where the Taylor expansion of $\sin(x) \approx x;\ x \approx 0$ has been used. As we can see, the propagator in the continuum (right side of 2.22) has only one pole:

$$p_\mu = (0, 0, 0, 0) \tag{2.23}$$

But the lattice propagator(middel term) has additional poles, if all of the components of p_μ are 0 or $\frac{\pi}{a}$, there is a pole. This is possible, because the momentum space contains all momenta $p_\mu \in (-\frac{\pi}{a}, \frac{\pi}{a}]$. After all there are $\sum_{i=0}^{3}\binom{4}{4-i} = 15$ unwanted poles, called *doublers*. In order to remove these unwanted poles, Wilson has introduced an extra term in the Dirac operator, which will be discussed in the next chapter.

2.2.3 Wilson fermions

After adding the *Wilson term*, the Dirac operator in momentum space reads the following [6, page 112-113]:

$$\tilde{D}(p) = \frac{i}{a}\gamma_\mu\sin(p_\mu a) + \Big(m + \frac{1}{a}\sum_{\nu=1}^{4}(1 - \cos(p_\nu a))\Big)\mathbb{1} \tag{2.24}$$

We can easily see, that the Wilson term vanishes for $p_\mu = (0, 0, 0, 0)$, but for each component $p_\mu = \frac{\pi}{a}$ there is an extra contribution to the total mass of the doublers $\frac{2l}{a}$, where $l = 4 - i$ is the number of momentum components with $p_\mu = \frac{\pi}{a}$. In the continuum limit the doublers become infinitly heavy and are no longer physically relevant.

2.2.4 QCD lattice path integral

With all these things introduced, we can now formulate an expression for Euclidean corrolators in the lattice path integral formalism. It takes the form [6, page 39-40]:

$$\left\langle \hat{O}_2(t)\hat{O}_1(0)\right\rangle = \frac{1}{Z}\int \mathcal{D}[\psi,\bar{\psi}]\mathcal{D}[U]\, O_2[\psi(.,t),\bar{\psi}(.,t),U(.,t)]O_1[\psi(.,0),\bar{\psi}(.,0),U(.,0)]e^{-S[\psi,\bar{\psi},U]} \tag{2.25}$$

$$Z = \int \mathcal{D}[\psi,\bar{\psi}]\mathcal{D}[U]\, e^{-S[\psi,\bar{\psi},U]} \tag{2.26}$$

$$\mathcal{D}[\psi,\bar{\psi}] = \prod_{n\in\Lambda}\prod_{f,\alpha,c} d\psi^{(f)}_{\alpha,c}(n)d\bar{\psi}^{(f)}_{\alpha,c}(n); \quad \mathcal{D}[U] = \prod_{n\in\Lambda}\prod_{\mu=1}^{4} dU_\mu(n) \tag{2.27}$$

The difference to the continuum is obviously the fact, that the fields and link variables live now on the lattice and the continuous time has now changed to $t = n_4 a$, but the operaters are functionals of the fields as in the continuum case.

2.3 The relation between QCD and statistical mechanics

By comparing QCD on the lattice with statistical mechanics many equivalences can be found. In the canonical ensemble, i.e. for a system in a heat bath with temperature T, the probability $P(\alpha)$ for finding a system in a state α ist given by the Boltzmann distribution:

$$P(\alpha) = \frac{1}{Z}e^{-\beta E_\alpha} \tag{2.28}$$

Here $\beta = \frac{1}{k_B T}$, with the Boltzmann constant k_B and temperature T. The partition function Z is given by:

$$Z = \sum_\alpha e^{-\beta E_\alpha} = tr[e^{-\beta\hat{H}}] \tag{2.29}$$

Where the sum runs over all microstates α. An expectation value of an operator $\hat{O}$ is given by:

$$\left\langle \hat{O}\right\rangle = \frac{1}{Z}\sum_\alpha O_\alpha e^{-\beta E_\alpha} = \frac{tr[\hat{O}e^{-\beta\hat{H}}]}{tr[e^{-\beta\hat{H}}]} \tag{2.30}$$

If we compare this expression with 2.25, the similarity is obvious [6, page 22]. The Boltzmann factor $e^{-\beta E_\alpha}$ is replaced by $e^{-S[\psi,\bar{\psi},U]}$ and the sum over all states by an integral over the fields at any lattice point. To do calculations at finite temperature, N_t is now related to T by [6, page 302]:

$$\beta = \frac{1}{k_B T} = aN_t \tag{2.31}$$

We can see, that $\beta \to \infty$ is equal to $T \to 0$. We can interprete our lattice now as a system with finite spacial volume and finite (and fixed) temperature T. The continuum limit of such a system corresponds to $a \to 0$ and $aN_t = \frac{1}{T} = const$ (where natural units are used: $k_B = 1$).

2.3.1 The chemical potential and thermodynamic quantities

If we consider a system with non-vanishing fermion number density, an extra term including chemical potential μ must be added to the partition function. This leads to the grand canonical partition function [6, page 312]:

$$Z(T,\mu) = \mathrm{tr}[e^{-\beta(\hat{H}-\mu\hat{N}_f)}] \tag{2.32}$$

Using the fact, that the eigenvalues of the fermion number operator are integer values, it is possible to expand the grand canonical partition function in a power series of the fugacity $z = e^{\beta\mu}$:

$$Z(T,\mu) = \sum_n z^n Z_n(T) \tag{2.33}$$

Here $n \in \mathbb{Z}$ and the negative values of n correspond to a net number of antifermions. With the partition function we can calculate new expectation values of some observables [6, page 310-312]:

$$P = T\frac{\partial \ln Z(T,\mu)}{\partial V} = T\frac{\ln Z(T,\mu)}{V} \tag{2.34}$$

$$n_f = \frac{N_f}{V} = \frac{T}{V}\frac{\partial \ln Z(T,\mu)}{\partial \mu} = \frac{\partial P}{\partial \mu} \tag{2.35}$$

Where the linearity of $Z(T,\mu)$ in V was used in 2.34. Further on the lattice the volume takes the form $V = (aN_s)^3$. For Baryons, which are built up by three quarks, one have: $n_B = \frac{n_f}{3}$.

2.4 Construction of a perfect action for free lattice fermions with chemical potential

To construct lattice fermion field $\bar{\Psi}, \Psi$, the continuum field $\bar{\psi}, \psi$ is averaged over a hypercube $\mathrm{c_x}$ centered at the point x [1, page 4]:

$$\bar{\Psi}(\mathrm{x}) = \int_{\mathrm{c_x}} \mathrm{dy}^4\, \bar{\psi}(\mathrm{y}); \quad \Psi(\mathrm{x}) = \int_{\mathrm{c_x}} \mathrm{dy}^4\, \psi(\mathrm{y}) \tag{2.36}$$

From now on *all physical quantities will be measured in lattice units*! So the hypercubes have unit length in lattice units. In order to optain a perfect lattice action, all continuum degrees of freedom have to be integrated out (this is a *renormalisation group transformation* called $'$*blocking from the continuum*$'$) using 2.36 [1, page 4]:

$$\mathrm{e}^{-\mathrm{S}(\bar{\Psi},\Psi)} = \int \mathcal{D}[\bar{\psi}, \psi]\, \mathrm{e}^{-\mathrm{S}(\bar{\psi},\psi)} \prod_{\mathrm{x}} \delta\Big(\bar{\Psi}(\mathrm{x}) - \int_{\mathrm{c_x}} \mathrm{dy}^4\, \bar{\psi}(\mathrm{y})\Big)\, \delta\Big(\Psi(\mathrm{x}) - \int_{\mathrm{c_x}} \mathrm{dy}^4\, \psi(\mathrm{y})\Big) \tag{2.37}$$

Here $\mathrm{S}(\bar{\psi}, \psi)$ is the continuum action for free fermions given in 2.2 with $\mathrm{A}_\mu = 0$. We can transform this action in momentum space by Fourier transformation:

$$\begin{aligned}
\mathrm{S}(\bar{\psi}, \psi) &= \int \mathrm{d}^4\mathrm{x}\, \bar{\psi}(\mathrm{x})(\gamma_\mu \partial_\mu + \mathrm{m})\psi(\mathrm{x}) \\
&= \int \mathrm{d}^4\mathrm{x} \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \bar{\psi}(\mathrm{k})\mathrm{e}^{\mathrm{i}\mathrm{k}_\alpha \mathrm{x}_\alpha}(\gamma_\mu \partial_\mu + \mathrm{m}) \int \frac{\mathrm{d}^4\mathrm{k}'}{(2\pi)^4}\, \psi(\mathrm{k}')\mathrm{e}^{\mathrm{i}\mathrm{k}'_\beta \mathrm{x}_\beta} \\
&= \int \mathrm{d}^4\mathrm{x} \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \int \frac{\mathrm{d}^4\mathrm{k}'}{(2\pi)^4}\, \bar{\psi}(\mathrm{k})\mathrm{e}^{\mathrm{i}(\mathrm{k}_\alpha + \mathrm{k}'_\alpha)\mathrm{x}_\alpha}(\mathrm{i}\gamma_\mu \mathrm{k}'_\mu + \mathrm{m})\psi(\mathrm{k}') \\
&= \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \int \frac{\mathrm{d}^4\mathrm{k}'}{(2\pi)^4}\, \bar{\psi}(\mathrm{k})(2\pi)^4\delta^4(\mathrm{k}+\mathrm{k}')(\mathrm{i}\gamma_\mu \mathrm{k}'_\mu + \mathrm{m})\psi(\mathrm{k}') \\
&= \int \frac{\mathrm{d}^4\mathrm{k}'}{(2\pi)^4}\, \bar{\psi}(-\mathrm{k}')(\mathrm{i}\gamma_\mu \mathrm{k}'_\mu + \mathrm{m})\psi(\mathrm{k}') \\
&= \int \frac{\mathrm{d}^4\mathrm{k}'}{(2\pi)^4}\, \bar{\psi}(-\mathrm{k}')\delta^{\mathrm{f}}(\mathrm{k}')^{-1}\psi(\mathrm{k}')
\end{aligned} \tag{2.38}$$

Here the integral representation of the Dirac-distribution was used and the Dirac operator in momentum space has been defined as $\delta^{\mathrm{f}}(\mathrm{k}')^{-1}$. The lattice fermion field in momentum space can be obtained by rewriting the lattice field in coordinate space and compare it with the discrete Fourier transform lattice field:

$$\begin{aligned}
\Psi(\mathrm{x}) = \int_{\mathrm{c_x}} \mathrm{dy}^4\, \psi(\mathrm{y}) &= \int_{\mathrm{c_x}} \mathrm{dy}^4 \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \psi(\mathrm{k})\mathrm{e}^{\mathrm{i}\mathrm{k}_\alpha \mathrm{y}_\alpha} \\
&= \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \psi(\mathrm{k}) \int_{\mathrm{c_x}} \mathrm{dy}^4\, \mathrm{e}^{\mathrm{i}\mathrm{k}_\alpha \mathrm{y}_\alpha} \\
&= \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \psi(\mathrm{k})\mathrm{e}^{\mathrm{i}\mathrm{k}_\alpha \mathrm{x}_\alpha} \prod_{\mu=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{dy}_\mu\, \mathrm{e}^{\mathrm{i}\mathrm{k}_\mu \mathrm{y}_\mu}
\end{aligned} \tag{2.39}$$

The integral over y is a well known standard integral, which appears also in the Frauenhofer diffraction at the single slit. The integral over momenta can be replaced by a sum over all lattice points and an integral over the Brillouin zone [6, page 222-223]:

$$\begin{aligned}\Psi(x) &= \int \frac{d^4k}{(2\pi)^4}\, e^{ik_\alpha x_\alpha} \psi(k) \underbrace{\prod_{\mu=1}^{4} \frac{2\sin(\frac{k_\mu}{2})}{k_\mu}}_{=\Pi} \\ &= \int_B \frac{d^4k}{(2\pi)^4}\, e^{ik_\alpha x_\alpha} \sum_{l\in\mathbb{Z}^4} \psi(k+2\pi l)\Pi(k+2\pi l) = \int_B \frac{d^4k}{(2\pi)^4}\, e^{ik_\alpha x_\alpha} \Psi(k)\end{aligned} \tag{2.40}$$

In the last step a lattice Fourier transformation was applied (with $N_s = N_t \to \infty$ such that $\frac{1}{N_s^3 N_t}\sum_k \to \int_B \frac{dk^4}{(2\pi)^4}$). After comparing the left and the right side of the last line, we can write [1, page 4]:

$$\Psi(k) = \sum_{l\in\mathbb{Z}^4} \psi(k+2\pi l)\Pi(k+2\pi l) \tag{2.41}$$

To perform the integration over the continuum fields in 2.37, the Grassmann Dirac-distributions (Appendix) are represented by integrals over auxiliary fields $\bar{\eta}, \eta$. After changing to momentum space, we have [1, page 4]:

$$\begin{aligned}e^{-S(\bar{\Psi},\Psi)} = &\int \mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\bar{\eta},\eta] \exp\Big(-\int \frac{d^4k}{(2\pi)^4}\, \bar{\psi}(-k)\delta^f(k)^{-1}\psi(k)\Big) \\ &\exp\Big(\int_B \frac{d^4k}{(2\pi)^4}\, [\bar{\Psi}(-k) - \sum_{l\in\mathbb{Z}^4} \bar{\psi}(-k+2\pi l)\Pi(k+2\pi l)]\eta(k)\Big) \\ &\exp\Big(\int_B \frac{d^4k}{(2\pi)^4}\, \bar{\eta}(-k)[\Psi(k) - \sum_{l\in\mathbb{Z}^4} \psi(k+2\pi l)\Pi(k+2\pi l) + a\eta(k)]\Big)\end{aligned} \tag{2.42}$$

Here a chiral symmetry breaking mass term $a\bar{\eta}\eta$ was introduced to smear the Dirac-distribution to a Gaussian-distribution of width a. Now we notice, that the integral over the Brillouin zone and the sum over the integer vectors l (periodic boundary conditions) can be combined to an integral over the entire momentum space [1, page 5]:

$$\begin{aligned}e^{-S(\bar{\Psi},\Psi)} = &\int \mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\bar{\eta},\eta] \exp\Big(-\int \frac{d^4k}{(2\pi)^4}\, \bar{\psi}(-k)\delta^f(k)^{-1}\psi(k) \\ &+ \bar{\psi}(-k)\Pi(k)\eta(k) + \bar{\eta}(-k)\psi(k)\Pi(k)\Big) \\ &\exp\Big(\int_B \frac{d^4k}{(2\pi)^4}\, \bar{\Psi}(-k)\eta(k) + \bar{\eta}(-k)\Psi(k) + a\bar{\eta}(-k)\eta(k)]\Big)\end{aligned} \tag{2.43}$$

This expression is very similar to the expression of a *generating functional* with source terms $\Pi(\mathrm{k})\bar{\eta}(-\mathrm{k}), \Pi(\mathrm{k})\eta(\mathrm{k})$. After completing the square in the exponent, the Gaussian integral over $\bar{\psi}, \psi$ can be performed and the resulting determinant can be pulled into the normalising partition function. Then we are left with [1, page 5]:

$$\begin{aligned} \mathrm{e}^{-\mathrm{S}(\bar{\Psi},\Psi)} = \int \mathcal{D}[\bar{\eta},\eta] \exp\Big(\int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \bar{\eta}(-\mathrm{k})\Pi(\mathrm{k})^2\delta^{\mathrm{f}}(\mathrm{k})\eta(\mathrm{k})\Big) \\ \exp\Big(\int_{\mathrm{B}} \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \bar{\Psi}(-\mathrm{k})\eta(\mathrm{k}) + \bar{\eta}(-\mathrm{k})\Psi(\mathrm{k}) + \mathrm{a}\bar{\eta}(-\mathrm{k})\eta(\mathrm{k})]\Big) \end{aligned} \tag{2.44}$$

Now the integral over the entire momentum space will be splitted in the sum and the Brillouin integral in order to combine the exponentials [1, page 5]:

$$\begin{aligned} \mathrm{e}^{-\mathrm{S}(\bar{\Psi},\Psi)} = \int \mathcal{D}[\bar{\eta},\eta] \exp\Big(\int_{\mathrm{B}} \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \bar{\eta}(-\mathrm{k})[\sum_{\mathrm{l}\in\mathbb{Z}^4} \Pi(\mathrm{k}+2\pi\mathrm{l})^2\delta^{\mathrm{f}}(\mathrm{k}+2\pi\mathrm{l}) + \mathrm{a}]\eta(\mathrm{k}) \\ + \bar{\Psi}(-\mathrm{k})\eta(\mathrm{k}) + \bar{\eta}(-\mathrm{k})\Psi(\mathrm{k})]\Big) \end{aligned} \tag{2.45}$$

And again after completing the square in the exponent and performing the Gaussian integral we obtain the final result:

$$\mathrm{e}^{-\mathrm{S}(\bar{\Psi},\Psi)} = \exp\Big(-\int_{\mathrm{B}} \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \bar{\Psi}(-\mathrm{k})[\sum_{\mathrm{l}\in\mathbb{Z}^4} \Pi(\mathrm{k}+2\pi\mathrm{l})^2\delta^{\mathrm{f}}(\mathrm{k}+2\pi\mathrm{l}) + \mathrm{a}]^{-1}\Psi(\mathrm{k})\Big) \tag{2.46}$$

From 2.46 we can immediatly read off the perfect action and its propagator [1, page 6]:

$$\begin{aligned} \mathrm{S}(\bar{\Psi},\Psi) &= \int_{\mathrm{B}} \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\, \bar{\Psi}(-\mathrm{k})\Delta^{\mathrm{f}}(\mathrm{k})^{-1}\Psi(\mathrm{k}) \\ \Delta^{\mathrm{f}}(\mathrm{k}) &= \sum_{\mathrm{l}\in\mathbb{Z}^4} \Pi(\mathrm{k}+2\pi\mathrm{l})^2\delta^{\mathrm{f}}(\mathrm{k}+2\pi\mathrm{l}) + \mathrm{a} \end{aligned} \tag{2.47}$$

It should be mentioned, that the sum over l converges, because Π^2 suppresses contributions of large l by $\frac{1}{\mathrm{l}^2}$ and the continuum propagator also supresses by $\frac{1}{\mathrm{l}}$. It also turns out, that locality is optimal for [1, page 6]:

$$\mathrm{a} = \frac{\mathrm{e}^{\mathrm{m}} - 1 - \mathrm{m}}{\mathrm{m}^2} \tag{2.48}$$

In the massless case, we obtain the following limit:

$$\begin{aligned} \lim_{\mathrm{m}\to 0}\mathrm{a} &= \lim_{\mathrm{m}\to 0}\frac{\mathrm{e}^{\mathrm{m}} - 1 - \mathrm{m}}{\mathrm{m}^2} = \lim_{\mathrm{m}\to 0}\frac{\sum_{\mathrm{j}=0}^{\infty}\frac{1}{\mathrm{j}!}\mathrm{m}^{\mathrm{j}} - 1 - \mathrm{m}}{\mathrm{m}^2} \\ &= \lim_{\mathrm{m}\to 0}\frac{\sum_{\mathrm{j}=2}^{\infty}\frac{1}{\mathrm{j}!}\mathrm{m}^{\mathrm{j}}}{\mathrm{m}^2} = \lim_{\mathrm{m}\to 0}\sum_{\mathrm{j}=2}^{\infty}\frac{1}{\mathrm{j}!}\mathrm{m}^{\mathrm{j}-2} = \lim_{\mathrm{m}\to 0}(\frac{1}{2} + \frac{1}{6}\mathrm{m} + \mathrm{O}(\mathrm{m}^2)) \to \frac{1}{2} \end{aligned} \tag{2.49}$$

2.4.1 Properties of the perfect action

The propagator in Dirac-space is:

$$\Delta^f(k) = \sum_{l\in\mathbb{Z}^4} \Pi(k+2\pi l)^2\delta^f(k+2\pi l) + a = \sum_{l\in\mathbb{Z}^4} \Pi(k+2\pi l)^2(i\gamma_\mu(k_\mu + 2\pi l_\mu) + m\mathbb{1})^{-1} + a\mathbb{1}$$
$$= \sum_{l\in\mathbb{Z}^4} \frac{\Pi(k+2\pi l)^2(-i\gamma_\mu(k_\mu+2\pi l_\mu) + m\mathbb{1})}{m^2 + (k_\nu + 2\pi l_\nu)(k_\nu + 2\pi l_\nu)} + a\mathbb{1} \tag{2.50}$$

In the last step, the denominator was expanded by its complex conjugate and again the relation 6.12 for the Euclidean gamma-matrices has been used. In this representation we can simply read off the pole if we set $E = -i(k_4 + 2\pi l_4)$ [5, page 8]:

$$E^2(k) = m^2 + \sum_{j=1}^{3}(k_j + 2\pi l_j)^2 \tag{2.51}$$

This is the *exact continuum spectrum for* $l_j = 0$ (plus 2π periodic copies, which appears because of periodic boundary conditions). It should also be mentioned, that the propagator has the right continuum limit and is *free from doublers*! To show this, we consider the massless case without the smearing parameter a [6, page 226-227]:

$$\Delta^f(k) = \sum_{l\in\mathbb{Z}^4} \frac{-i\gamma_\mu(k_\mu + 2\pi l_\mu)\Pi(k+2\pi l)^2}{(k_\nu + 2\pi l_\nu)(k_\nu+2\pi l_\nu)} = -i\gamma_\mu v_\mu(k) \tag{2.52}$$

Now we have to show, that:

$$\Delta^f(k) \approx \begin{cases} -i\gamma_\mu \frac{k_\mu}{k^2} & \text{when } k_\mu \approx 0 \\ 0 & \text{when } k_\mu \approx \pm\pi \end{cases} \tag{2.53}$$

To look at the limits, we have to calculate what the function $\Pi(k)$ does in these cases:

$$\lim_{k_\mu\to 0}\Pi(k) = \lim_{k_\mu \to 0} \prod_{\mu=1}^{4} \frac{2\sin(\frac{k_\mu}{2})}{k_\mu} \to \prod_{\mu=1}^{4} \frac{2\frac{k_\mu}{2}}{k_\mu} = 1 \tag{2.54}$$

$$\lim_{k_\mu\to \pm\pi}\Pi(k) = \lim_{k_\mu \to \pm\pi} \prod_{\mu=1}^{4} \frac{2\sin(\frac{k_\mu}{2})}{k_\mu} \leq (\frac{2}{\pi})^4 \ll 1 \tag{2.55}$$

This result and (52) imply the required properties of (53) and the propagator diverges if and only if $k_\mu = 0$!

2.4.2 Implementation of the chemical potential

On the lattice for the free Wilson action a chemical potential μ can be introduced by replacing $\bar{\Psi}(x), \Psi(x)$ by [7, page 3]:

$$ {}^{\mu}\bar{\Psi}(x) = \bar{\Psi}(x)\Omega^{-1}(x) = \bar{\Psi}(x)e^{-\mu x_4}; \quad {}^{\mu}\Psi(x) = \Omega(x)\Psi(x) = e^{\mu x_4}\Psi(x) \tag{2.56} $$

Introducing it this way, lets the chemical potential appear as a pure, imaginary and constant Abelian gauge potential $A_4 = i\mu$ so that the phasetransformation is given as $\Omega(x) = e^{-iA_4 x_4}$ [3, page 3]. It is the question, how to incorporate this in momentum space. As an example we will do this only for the anti-fermionfield $\bar{\Psi}$. Therefore we consider a Fourier transformation of 2.56:

$$ {}^{\mu}\bar{\Psi}(k) = \sum_{x} e^{-ik_\alpha x_\alpha}\,{}^{\mu}\bar{\Psi}(x) = \sum_{x} e^{-ik_\alpha x_\alpha} e^{-\mu x_4}\bar{\Psi}(x) = \sum_{x} e^{-i(k_i x_i + x_4(k_4 - i\mu))}\bar{\Psi}(x) \tag{2.57} $$

So we have:

$$ \bar{\Psi}(k) \rightarrow {}^{\mu}\bar{\Psi}(k_j, k_4 - i\mu); \quad j = 1,2,3 \tag{2.58} $$

In perturbation theory, it is equivalent to leave the fermionic fields unchanged and to replace the propagator the following way [7, page 3]:

$$ \Delta^f(k) \rightarrow \Delta^f(k_j, k_4 + i\mu); \quad j = 1,2,3 \tag{2.59} $$

Applying this to the perfect action in 2.47 at $\mu > 0$, then it reads in momentum space:

$$ S({}^{\mu}\bar{\Psi}, {}^{\mu}\Psi) = \int_B \frac{d^4k}{(2\pi)^4}\,\bar{\Psi}(-k)\Delta^f(k_j, k_4 + i\mu)^{-1}\Psi(k); \quad j = 1,2,3 \tag{2.60} $$

2.4.3 Computation of the pressure

To compute the pressure of this perfect free fermion gas, we have to make use of 2.26 and 2.34:

$$ P = T\frac{\ln Z(T,\mu)}{V} = \frac{T}{V}\ln\Big(\int \mathcal{D}[\Psi,\bar{\Psi}]\,e^{-S[\Psi,\bar{\Psi}]}\Big) \tag{2.61} $$

Now T and V will be replaced by the number of lattice points and the Gaussian path integral will be performed via the *Matthews-Salam formula* [6, page 108]:

$$ P = \frac{1}{N_s^3 N_t}\ln\Big(\det(-\Delta^f(k_j, k_4 + i\mu)^{-1})\Big) \tag{2.62} $$

Now we use this theorem: The Dirac-operator is diagonal in momentum space, so the determinant in momentum space is simply the product of all momenta and we are left with the determinant in Dirac-space:

$$ P = \frac{1}{N_s^3 N_t}\ln\Big(\prod_k \det{}_d(-\Delta^f(k_j, k_4 + i\mu)^{-1})\Big) \tag{2.63} $$

Then, after using standard properties of the determinant and logarithm, we obtain:

$$P = \frac{1}{N_s^3 N_t} \sum_k -\ln\Big((-1)^4 \det_d(\Delta^f(k_j, k_4 + i\mu))\Big) \tag{2.64}$$

Now we take the zero temperature limit ($N_s = N_t \to \infty$), so that the sum changes to an continuous integration over the Brillouin zone:

$$P = \int_B \frac{d^4k}{(2\pi)^4} - \ln\Big(\det_d(\Delta^f(k_j, k_4 + i\mu))\Big) \tag{2.65}$$

And because we are only interested in the effects of the chemical potential, we substract the pressure without chemical potential and obtain the final result [7, page 5]:

$$P = \int_B \frac{d^4k}{(2\pi)^4} \ln\Big(\det_d(\Delta^f(k))\Big) - \ln\Big(\det_d(\Delta^f(k_j, k_4 + i\mu))\Big) \tag{2.66}$$

In the following, we will use this expression. But it should be mentioned, that we just looked at *one* fermionic degree of freedom. For completeness we correct this by introducing a spinfactor by hand (s is the value of the spin, e.g $s = \frac{1}{2}$ for quarks or leptons):

$$P = (2s+1) \int_B \frac{d^4k}{(2\pi)^4} \ln\Big(\det_d(\Delta^f(k))\Big) - \ln\Big(\det_d(\Delta^f(k_j, k_4 + i\mu))\Big) \tag{2.67}$$

2.4.4 Analytical attempt to compute the pressure

If we want to solve equation 2.66, we must find a way to recombine the sum over $\mathbb{Z}^4$ with the integral over the Brillouin zone to an integral over the entire momentum space. Therefore we first make use of the very popular formula for the expansion of the determinant to rewrite the integrand [6, page 335]:

$$\ln\Big(\det_d(\Delta^f(k))\Big) = \mathrm{tr}_d\Big(\ln(\Delta^f(k))\Big) \tag{2.68}$$

Now we rewrite this expression a bit to use the Taylor expansion of the logarithm:

$$\begin{aligned} \mathrm{tr}_d\Big(\ln(\Delta^f(k))\Big) &= \mathrm{tr}_d\Big(\ln(\mathbb{1} - (\mathbb{1} - \Delta^f(k)))\Big) \\ &= -\sum_{j=1}^{\infty} \frac{1}{j} \mathrm{tr}_d((\mathbb{1} - \Delta^f(k))^j) \end{aligned} \tag{2.69}$$

We have now rewritten the determinant in an infinite series of powers of traces of the propagator. If we take the trace of $\Delta^f(k)$, then we are left with:

$$\mathrm{tr}_d(\Delta^f(k)) = \sum_{l \in \mathbb{Z}^4} \frac{4m\,\Pi(k + 2\pi l)^2}{m^2 + (k_\nu + 2\pi l_\nu)(k_\nu + 2\pi l_\nu)} + 4a \tag{2.70}$$

Here we used the formulas $\mathrm{tr_d}(\mathbb{1}) = 4$ and $\mathrm{tr_d}(\gamma_\mu) = 0$. This result also implies, that we can combine the sum with the integral over the Brillouin zone up to the first order. But we have to check, whether this is also possible for higher orders in $\mathrm{tr_d}(\Delta^\mathrm{f}(\mathrm{k}))$:

$$\mathrm{tr_d}(\Delta^\mathrm{f}(\mathrm{k})^2) = \mathrm{tr_d}\Bigg(\sum_{\mathrm{l}\in\mathbb{Z}^4}\sum_{\mathrm{l}'\in\mathbb{Z}^4}\Big(\frac{\Pi(\mathrm{k}+2\pi\mathrm{l})^2(-\mathrm{i}\gamma_\mu(\mathrm{k}_\mu+2\pi\mathrm{l}_\mu)+\mathrm{m}\mathbb{1})}{\mathrm{m}^2+(\mathrm{k}_\nu+2\pi\mathrm{l}_\nu)(\mathrm{k}_\nu+2\pi\mathrm{l}_\nu)}+\mathrm{a}\mathbb{1}\Big)$$
$$\Big(\frac{\Pi(\mathrm{k}+2\pi\mathrm{l}')^2(-\mathrm{i}\gamma_\mu(\mathrm{k}_\mu+2\pi\mathrm{l}'_\mu)+\mathrm{m}\mathbb{1})}{\mathrm{m}^2+(\mathrm{k}_\nu+2\pi\mathrm{l}'_\nu)(\mathrm{k}_\nu+2\pi\mathrm{l}'_\nu)}+\mathrm{a}\mathbb{1}\Big)\Bigg) \tag{2.71}$$

We can see, that the two sums will not separate into terms, which contain only one sum. So we are not able to compare the sum and the integral for higher orders than linear. We have to solve 2.66 nummerically, which will be discussed in the next chapter.

3 Nummerical computation of the pressure

If we want to compute the pressure nummerically, we can not work with 2.66. We have to go back to 2.64 and substract the part without chemical potential. Then we come to the following:

$$P = \frac{1}{N_s^3 N_t} \sum_k \ln\Big(\det_d(\Delta^f(k))\Big) - \ln\Big(\det_d(\Delta^f(k_j, k_4 + i\mu))\Big) \tag{3.1}$$

The sum over k runs as given in the Appendix for discrete Fourier transformations including anti-periodic boundary conditions in time-direction.

3.1 Evaluation of the propagator

As given in 2.47, the propagator is a sum over all integer numbers in 4-dimensions. It is clear, that we have to truncate the sum at one point. Therefore we have to find an agreement between accuracy and runtime. The truncated propagator reads now:

$$\Delta^f(k, l_{max})_{trunc} = \sum_{-l_{max} \leq l \leq l_{max}} \frac{\Pi(k + 2\pi l)^2(-i\gamma_\mu(k_\mu + 2\pi l_\mu) + m\mathbb{1})}{m^2 + (k_\nu + 2\pi l_\nu)(k_\nu + 2\pi l_\nu)} + a\mathbb{1} \tag{3.2}$$

Now we want to study how fast the propagator converges. Therefore we plot for some arbitrary k_μ the relative differences (RD) for the $\ln(\det_d(\Delta^f(k, \mu, l_{max})_{trunc}))$ for different values of μ. We set $l_{max} = 20$ as the value, to which we want to calculate the relative difference. It should also be mentioned, that we only focus on the *real part* of $\ln(\det_d(\Delta^f(k, l_{max})_{trunc}))$:

$$RD(k, \mu, l_{max}) = \frac{|Re(\ln(\det_d(\Delta^f(k, \mu, l_{max} = 20)_{trunc}))) - Re(\ln(\det_d(\Delta^f(k, \mu, l_{max})_{trunc})))_{trunc})|}{|Re(\ln(\det_d(\Delta^f(k, \mu, l_{max} = 20)_{trunc})))|} \tag{3.3}$$

Here the chemical potential appears in the form given in 2.59! In the massless case, we have the following behaviour for different values of k_μ:

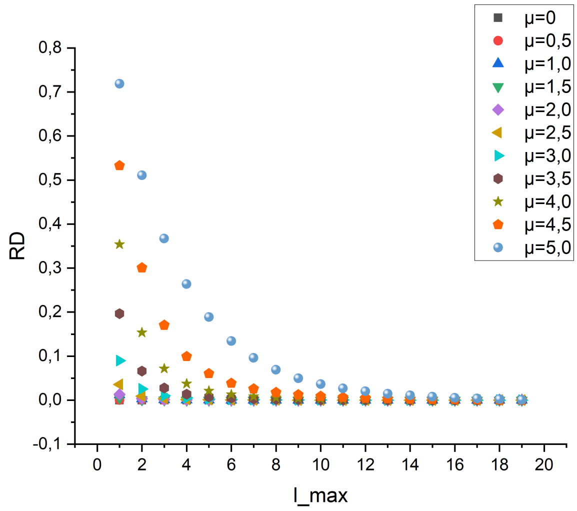

Figure 3.1: RD for m = 0 and $k_\mu = (0.1, 0.1, 0.1, 0.1)$

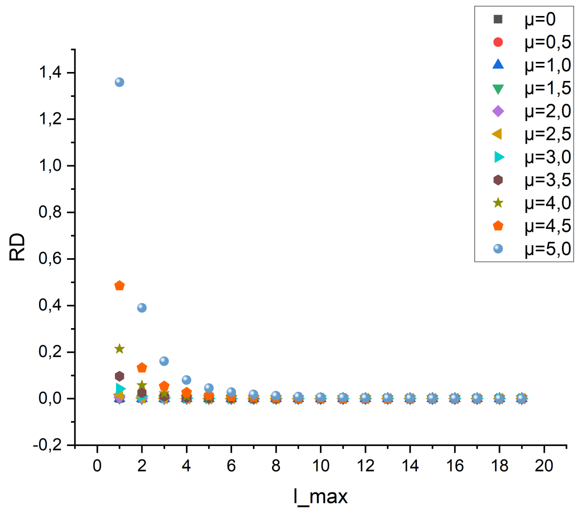

Figure 3.2: RD for m = 0 and $k_\mu = (1.6, 1.6, 1.6, 1.6)$

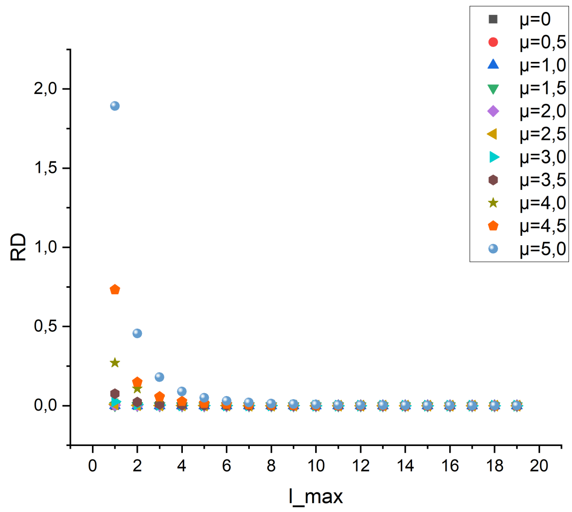

Figure 3.3: RD for m = 0 and $k_\mu = (3.1, 3.1, 3.1, 3.1)$

We see that the convergency is bad for huge values of μ and small values of k_μ. But in order to keep the runtime small, we will work with $l_{max} = 8$. In the massive case, e.g $m = 2.0$, we have:

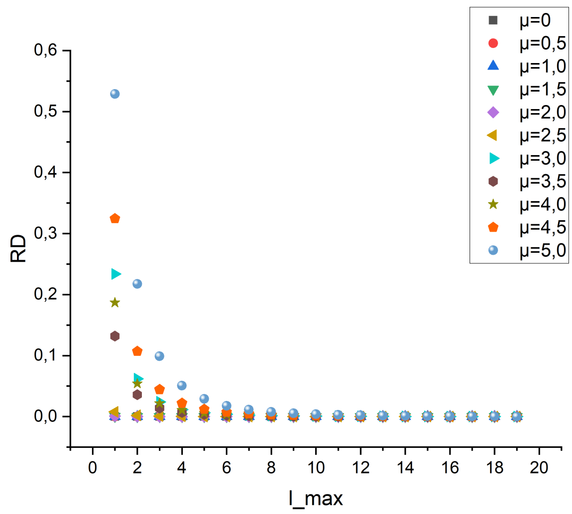

Figure 3.4: RD for m = 2.0 and $k_\mu = (0.1, 0.1, 0.1, 0.1)$

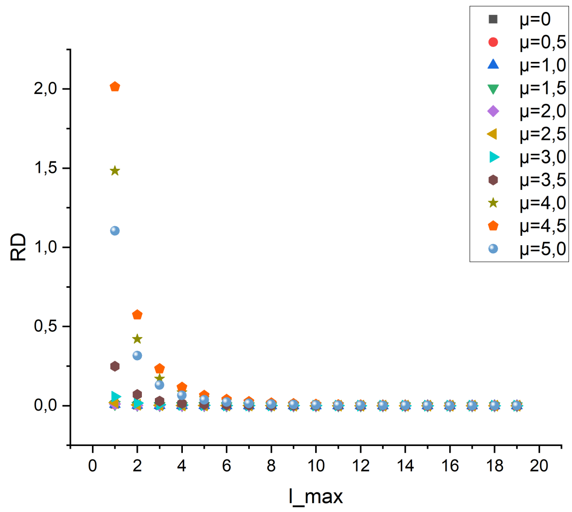

Figure 3.5: RD for m = 2.0 and $k_\mu = (1.6, 1.6, 1.6, 1.6)$

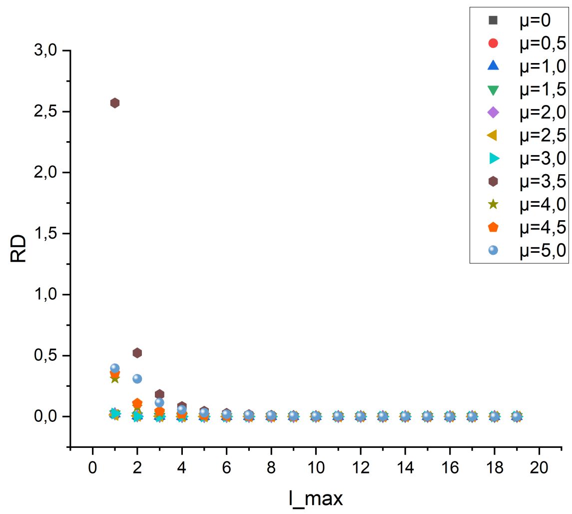

Figure 3.6: RD for m = 2.0 and $k_\mu = (3.1, 3.1, 3.1, 3.1)$

Here we can see that in the massive case RD converges faster. But for small l_{max} the inaccuracy is no longer ordered by μ, e.g $\mu = 5$ is no longer the curve with the biggest inaccuracy.

3.2 $\frac{P}{\mu^4}$ for massless fermions

Now we replace the propagator by the truncated one in 3.1 and carry out the sum over k vor different choices of $T = \frac{1}{N_t}$ (in lattice units) and fixed $N_s = 20$. After setting $m = 0$ and dividing 3.1 by μ^4, we end up with the final result. To compare it with the Wilson fermions and the continuum case, we use the results, obtained in [8, page 30]:

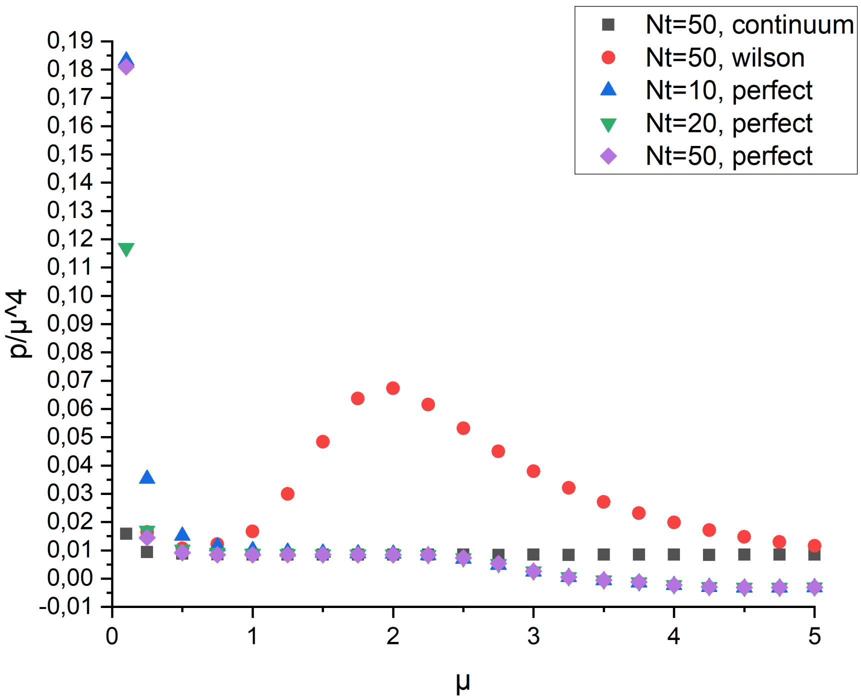

Figure 3.7: The Ratio $\frac{P}{\mu^4}$ as a function of the chemical potential μ for $m = 0$ and different $T = \frac{1}{N_t}$.

Here we see that if we lower the temperature (increase N_t), the value $\frac{P}{\mu^4}$ for the perfect fermions at small μ becomes better in the sense, that they are closer to the continuum value. For larger values of μ ($\mu > 2.5$) the perfect fermions are no longer physical, because the pressure can not be negativ. This effect is maybe caused by the perturbative introduction of the chemical potential in the propagator! But otherwise we are only interested in the continuum limit $\mu \to 0$ and there the perfect fermions fit well with exception of this finite temperature artefact, which can be suppressed by choosing larger lattices.

The divergency at $\mu = 0$ is caused by finite temperature effects as mentioned above. This means, that $\frac{P}{\mu^4} \sim \frac{T}{\mu^r}$; $0 < r < 4$, which diverges for $\mu \to 0$ if $T \neq 0$.

3.3 $\frac{\mathrm{P}}{\mu^4}$ for massive fermions

Now we consider massive fermions with $\mathrm{m} = 2.0$, $\mathrm{T} = \frac{1}{\mathrm{N_t}} = 0.02$ and $\mathrm{N_s} = 20$. Again we compare it with the Wilson fermions and the continuum case [8, page 32]:

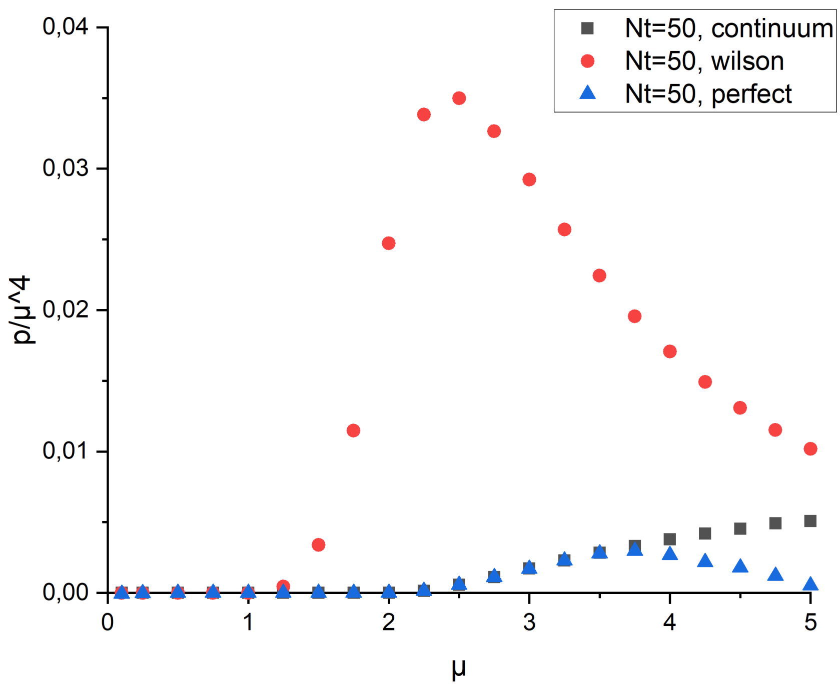

Figure 3.8: The Ratio $\frac{\mathrm{P}}{\mu^4}$ as a function of the chemical potential μ for $\mathrm{m} = 2.0$.

Through the mass, there is no longer a divergency at $\mu = 0$. All curves start at zero for $\mu = 0$, but the Wilson fermions begin to increase before $\mu > \mathrm{m}$, while the perfect fermions follow the continuum curve and they begin to increase when $\mu > \mathrm{m}$. The perfect fermions are very close to the continuum case up to $\mu \approx 3.5$.

4 Nummerical computation of the particle density

4.1 Computation of the particle density

The particle density can be computed by using 2.35 and 3.1:

$$\begin{aligned} n_f &= \frac{\partial P}{\partial \mu} = \frac{\partial}{\partial \mu}\int_B \frac{d^4k}{(2\pi)^4} \ln\Big(\det_d(\Delta^f(k))\Big) - \ln\Big(\det_d(\Delta^f(k_j, k_4 + i\mu))\Big) \\ &= -\int_B \frac{d^4k}{(2\pi)^4} \frac{\partial}{\partial \mu}\ln\Big(\det_d(\Delta^f(k_j, k_4 + i\mu))\Big) = -\int_B \frac{d^4k}{(2\pi)^4} \frac{\partial}{\partial \mu}\mathrm{tr}(\ln(\Delta^f(k_j, k_4 + i\mu))) \\ &= -\int_B \frac{d^4k}{(2\pi)^4} \mathrm{tr}\Big(\frac{\partial}{\partial \mu}\ln(\Delta^f(k_j, k_4 + i\mu))\Big) = -\int_B \frac{d^4k}{(2\pi)^4} \mathrm{tr}\Big(\Delta^f(k_j, k_4 + i\mu)^{-1}\frac{\partial}{\partial \mu}\Delta^f(k_j, k_4 + i\mu)\Big) \end{aligned} \tag{4.1}$$

Here j again runs from 1 to 3 and we used 2.68 in the second line. In the third line we used, that the trace is just the sum over the diagonal elements, so it commutes with the derivative because of the linearity of the derivativ. In the last line, we have to do a bit of work to see that its true. First the logarithm for complex matrices can be defined as the *inverse function* of the matrix exponential:

$$\ln(B) = A \iff B = e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k;\ A, B \in \mathbb{C}^{4\times 4};\ \det(B) \neq 0 \tag{4.2}$$

The determinant of the propagator is not zero by constuction, because otherwise we were not be able to invert the Dirac-operator and solve the path integrals in the construction of the perfect action! So everything is well defined and we can compute the parameter derivative of the matrix logarithm:

$$\frac{d}{dt}\ln\Big(B(t)\Big) \tag{4.3}$$

Therefore we use, that the parameter derivativ of the matrix exponential is:

$$\frac{d}{dt}e^{A(t)} = e^{A(t)}\Big(\frac{d}{dt}A(t)\Big) \tag{4.4}$$

This can be easily proved by inserting the taylorseries of the exponential. Now we look at the following expression:

$$\frac{d}{dt}\ln\Big(e^{A(t)}\Big) = \frac{d}{dt}A(t) \tag{4.5}$$

But 4.5 can also be computed by using the chain rule:

$$\frac{d}{dt}\ln\Big(e^{A(t)}\Big) = \ln'\Big(\underbrace{B(t)}_{=e^{A(t)}}\Big)\frac{d}{dt}e^{A(t)} = \ln'\Big(B(t)\Big)B(t)\Big(\frac{d}{dt}A(t)\Big) \tag{4.6}$$

After comparing 4.5 and 4.6, it is obviouse, that $\ln'(B(t)) = B(t)^{-1}$ and so the last step in 4.1 is proved and we can write down the particle density for baryons as:

$$n_B = -\frac{1}{3N_s^3N_t}\sum_k \operatorname{tr}\Big(\Delta^f(k_j, k_4 + i\mu)^{-1}\frac{\partial}{\partial\mu}\Delta^f(k_j, k_4 + i\mu)\Big) \tag{4.7}$$

The derivative of the propagator with respect to μ is the following:

$$\frac{\partial}{\partial\mu}\Delta^f(k_j, k_4 + i\mu) = \frac{\partial}{\partial\mu}\sum_{l\in\mathbb{Z}^4}\frac{\Pi(k + 2\pi l, \mu)^2(-i\sum_{\mu=1}^3\gamma_\mu(k_\mu + 2\pi l_\mu) + -i\gamma_4(k_4 + i\mu + 2\pi l_4) + m\mathbb{1})}{m^2 + \sum_{\nu=1}^3(k_\nu + 2\pi l_\nu)^2 + (k_4 + i\mu + 2\pi l_4)^2} + a\mathbb{1}$$

$$= \sum_{l\in\mathbb{Z}^4}\prod_{\mu=1}^3\Big(\frac{2\sin(\frac{k_\mu+2\pi l_\mu}{2})}{k_\mu + 2\pi l_\mu}\Big)^2\Bigg[\frac{4i\sin(\frac{k_4+i\mu+2\pi l_4}{2})(\cos(\frac{k_4+i\mu+2\pi l_4}{2}) - 2\sin(\frac{k_4+i\mu+2\pi l_4}{2}))}{(k_4 + i\mu + 2\pi l_4)^3}\cdot$$

$$\frac{(-i\sum_{\mu=1}^3\gamma_\mu(k_\mu + 2\pi l_\mu) + -i\gamma_4(k_4 + i\mu + 2\pi l_4) + m\mathbb{1})}{m^2 + \sum_{\nu=1}^3(k_\nu + 2\pi l_\nu)^2 + (k_4 + i\mu + 2\pi l_4)^2} + \Big(\frac{2\sin(\frac{k_4+i\mu+2\pi l_4}{2})}{k_4 + i\mu + 2\pi l_4}\Big)^2\cdot$$

$$\Big(\frac{\gamma_4}{(m^2 + \sum_{\nu=1}^3(k_\nu + 2\pi l_\nu)^2 + (k_4 + i\mu + 2\pi l_4)^2)} -$$

$$\frac{2i\,(k_4 + i\mu + 2\pi l_4)(-i\sum_{\mu=1}^3\gamma_\mu(k_\mu + 2\pi l_\mu) + -i\gamma_4(k_4 + i\mu + 2\pi l_4) + m\mathbb{1})}{(m^2 + \sum_{\nu=1}^3(k_\nu + 2\pi l_\nu)^2 + (k_4 + i\mu + 2\pi l_4)^2)^2}\Big)\Bigg] \tag{4.8}$$

4.2 $\frac{n_B}{\mu^3}$ for massless fermions

Again we replace the propagator by the truncated one in 4.7 and carry out the sum over k for $T = \frac{1}{N_t} = 0.02$ and $N_s = 20$. Then we divide the whole thing by μ^3 and are left with the following result, where the Wilson fermions and the continuum case are optained in [8, page 38]:

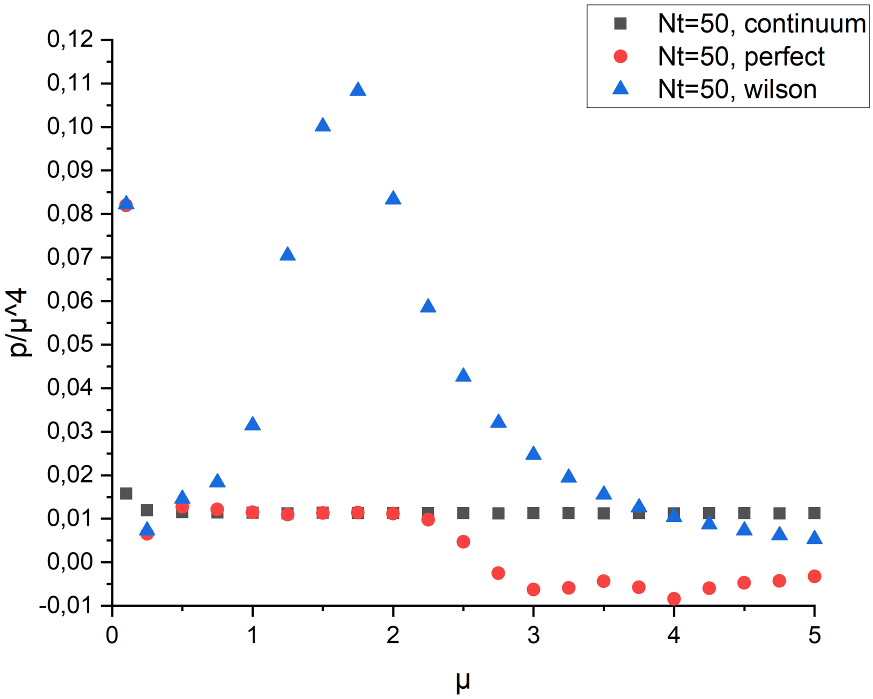

Figure 4.1: The Ratio $\frac{n_B}{\mu^3}$ as a function of the chemical potential μ for $m = 0$.

The divergency at $\mu = 0$ is similar to $\frac{P}{\mu^4}$ a finite temperature artefact, which would vanish in the case $T = 0$. We see, that the perfect fermions approximate the continuum very well up to $\mu \approx 2$. For larger μ the perfect fermions are no longer physical similar to the pressure. The reason is again the perturbative introduction of the chemical potential in the propagator.

4.3 $\frac{n_B}{\mu^3}$ for massive fermions

Again we consider massive fermions with $m = 2.0$, $T = \frac{1}{N_t} = 0.02$, $N_s = 20$ and compare it with the Wilson fermions and the continuum case [8, page 40]:

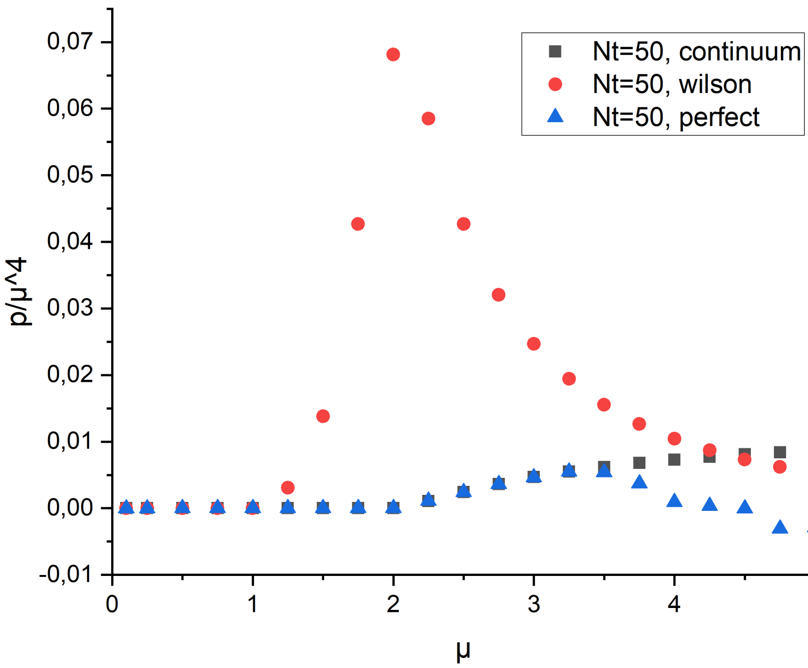

Figure 4.2: The Ratio $\frac{n_B}{\mu^3}$ as a function of the chemical potential μ for $m = 2.0$.

Similar to the the section 3.3, there is no divergency at $\mu = 0$. Again all curves start at zero for $\mu = 0$ and increase, when $\mu > m$ (again not the Wilson fermions, which increase before μ becomes larger then m). This behaviour can be explained, when we identify the chemical potential (derivative of the free energy with respect to the number of particles) at low temperatures ($T \to 0$) as the *Fermi energy*, which is the supremum for the energy states of a fermionic system at $T = 0$. For $\mu < m$ the Fermi energy is smaller then the rest energy of a particle, so there can not be a particle! Or with other words (if we use the defintion over the derivative): For $\mu < m$ there is not enough (free) energy to create a particle. If $\mu > m$, then there is enough energy to create a particle and the particle density rises. The perfect fermions follow the continuum curve up to $\mu \approx 3.5$.

5 Summary

In this Thesis an introduction in continuum and lattice QCD, *perfect fermions* and thermodynamic quantities is given. These thermodynamic quantities are calculated in dependency of the chemical potential and the mass of the fermions.

The results of the numeric calculations with massless and massive perfect fermions were then graphically compared to the Wilson fermions and the continuum case. We could see, that even for small latices like 20^4 or $20^3 \times 50$ the perfect fermions are a good approximation of the continuum to some value of μ even for massive fermions in contrast to the Wilson fermions. But also the perfect propagator can not represent the continuum for all μ. Especially for higher values of μ the values for the perfect fermions decrease in all cases. This effect is not only caused by the perturbativ introduction of the chemical potential, but we should also keep in mind, that we have truncated the propagator and worked at finite temperature in a finite volume.

It should be mentioned at the end, that the results of this thesis are not identical with the results of e.g. [5] or [7], because they all use *perfect hypercube fermions*. The essential difference to this thesis is, that they do not work in momentum space after they have derived the perfect action. They go back to position space and calculate the couplings in a hypercube (to keep the action as local as possible) and use these couplings in larger volumes [5, page 10]. This technique was maybe used, because the brute force method is very computation-intensive and in the late 90s the computers were not fast enough for this.

This thesis has shown, that perfect actions give better results for the pressure or the particle density then the Wilson action (in relation to the continuum case). A further bachelor thesis could have a look at other thermodynamic quantities like the energy density or the entropy for perfect fermions.

6 Appendix

Here a few formulas are given, which are used in this Thesis:

6.1 Fourier transformation on the lattice Λ

Toroidal boundary conditions are imposed, so that for an arbitrary function on the lattice the following condition holds [6, page 332]:

$$f(n + \hat{\mu}N_\mu) = e^{i2\pi\Theta_\mu} f(n) \tag{6.1}$$

We see immediatly, that directions with $\Theta_\mu = 0$ have periodic boundary conditions and directions with $\Theta_\mu = \frac{1}{2}$ have anti-periodic boundary conditions.

The corresponding momentum space $\tilde{\Lambda}$ for fermions with anti-periodic boundary conditions in time-direction [9, page 2] take the form:

$$\begin{aligned} \tilde{\Lambda} = \Big\{ k = (k_1, k_2, k_3, k_4);\ k_1, k_2, k_3 = \frac{2\pi}{aN_s} n_s;\ n_s = -\frac{N_s}{2} + 1, ..., \frac{N_s}{2}; \\ k_4 = \frac{2\pi}{aN_t}(n_t + \frac{1}{2});\ n_t = -\frac{N_t}{2}, ..., \frac{N_t}{2} - 1 \Big\} \end{aligned} \tag{6.2}$$

The discret Fourier transformation then reads [6, page 333]:

$$\Psi(n) = \frac{1}{|\Lambda|} \sum_{k \in \tilde{\Lambda}} \Psi(k) e^{ikna};\ \ \Psi(k) = \sum_{n \in \Lambda} \Psi(n) e^{-ikna} \tag{6.3}$$

The Dirac-distribution in discret Fourier-Representation becomes a 4-Kronecker-delta:

$$\frac{1}{|\Lambda|} \sum_{n \in \Lambda} e^{i(k-k')na} = \delta_{k_1 k_1'} \delta_{k_2 k_2'} \delta_{k_3 k_3'} \delta_{k_4 k_4'} \tag{6.4}$$

For completness, the Fourier transformation in the continuum is given by:

$$\psi(x) = \int \frac{d^4k}{(2\pi)^4} \psi(k) e^{ik_\alpha x_\alpha};\ \ \psi(k) = \int d^4x\, \psi(x) e^{-ik_\alpha x_\alpha} \tag{6.5}$$

The Dirac-distribution in Fourier-Representation:

$$\delta^4(k - k') = \int \frac{d^4x}{(2\pi)^4} e^{i(k_\alpha - k_\alpha')x_\alpha} \tag{6.6}$$

6.2 The Grassmann Dirac-distribution

Let $\psi, \bar{\psi}$ be grassmannvalued variables. The properties can be found at e.g. [6, page 105-109]! Because of the anticommuting property, $\psi, \bar{\psi}$ are nilpotent and functions of $\psi, \bar{\psi}$ can only be linear:

$$\mathrm{f}(\psi) = \mathrm{a} + \mathrm{b}\psi; \quad \mathrm{a}, \psi \in \mathbb{G}; \mathrm{b} \in \mathbb{C} \tag{6.7}$$

The Dirac-distribution then reads [10, page 48]:

$$\delta_{\mathrm{g}}(\psi) = \psi \tag{6.8}$$

Proof of (90):

$$\begin{aligned} &\int \mathrm{d}\psi' \delta_{\mathrm{g}}(\psi' - \psi)\mathrm{f}(\psi') = \int \mathrm{d}\psi' (\psi' - \psi)(\mathrm{a} + \mathrm{b}\psi') \\ &= \int \mathrm{d}\psi' [\psi' \mathrm{a} + \psi' \mathrm{b}\psi' - \psi \mathrm{a} - \psi \mathrm{b}\psi'] \end{aligned} \tag{6.9}$$

Now the properties of normalisation, nilpotency and anticommutation are used:

$$\begin{aligned} &\int \mathrm{d}\psi' \delta_{\mathrm{g}}(\psi' - \psi)\mathrm{f}(\psi') = \underbrace{\int \mathrm{d}\psi' [\psi'}_{=1} \mathrm{a} + \underbrace{\psi' \mathrm{b}\psi'}_{=0} - \psi \mathrm{a} - \psi \mathrm{b}\psi'] \\ &= \mathrm{a} - \psi \mathrm{a} \underbrace{\int \mathrm{d}\psi'}_{=0} - \int \mathrm{d}\psi' \psi \mathrm{b}\psi' = \mathrm{a} + \psi \mathrm{b} \underbrace{\int \mathrm{d}\psi' \psi'}_{=1} \\ &= \mathrm{a} + \mathrm{b}\psi = \mathrm{f}(\psi) \text{ q.e.d} \end{aligned} \tag{6.10}$$

Using all this properties, we can rewrite the Grassmann Dirac-distribution the following way:

$$\delta_{\mathrm{g}}(\psi' - \psi) = (\psi' - \psi) = \int \mathrm{d}\bar{\psi}\, 1 + \bar{\psi}(\psi' - \psi) = \int \mathrm{d}\bar{\psi} \sum_{\mathrm{i}=0}^{\infty} \frac{(\bar{\psi}(\psi' - \psi))^{\mathrm{i}}}{\mathrm{i}!} = \int \mathrm{d}\bar{\psi} \mathrm{e}^{\bar{\psi}(\psi' - \psi)} \tag{6.11}$$

6.3 The γ-matrices

The γ-matrices in Euclidean space can be looked up in [6, page 330-331] The γ-matrices fulfill the Clifford-Algebra and are hermitean and self-inverse:

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \mathbb{1} \tag{6.12}$$

$$\gamma_\mu = \gamma_\mu^\dagger = \gamma_\mu^{-1} \tag{6.13}$$

List of Figures

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