Hydrodynamics and Magnetohydrodynamics: Exercises

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Listed below are the exercises that have been assigned during the course and collected according to the lecture in which they were assigned. These exercises can be solved independently or together during the exercise time. Some of these questions could be part of the oral exam.

Lecture I

1. Prove the Newtonian H-theorem, that is,
$$\frac{\partial f_0}{\partial t} = \Gamma(f_0) = 0\,. \tag{1}$$
where f_0 is the *equilibrimum distribution function*. In particular, show that the condition (1) is fully equivalent to the condition
$$f_0(\vec{\boldsymbol{u}}_2')f_0(\vec{\boldsymbol{u}}_1') - f_0(\vec{\boldsymbol{u}}_2)f_0(\vec{\boldsymbol{u}}_1) = 0\,, \tag{2}$$
where $f_{1,2} := f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{u}}_{1,2})$, $f'_{1,2} := f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{u}}'_{1,2})$ are the distribution functions before and after the collision at time t and position $\vec{\boldsymbol{x}}$ (The subscripts "1" and "2" refer to the particles undergoing the collision, while unprimed and primed variables refer to quantities before and after the collision.).

2. Starting from the transport equation
$$\frac{\partial(n\langle\psi\rangle)}{\partial t} + \frac{\partial(n\langle u_i\psi\rangle)}{\partial x_i} - n\left\langle u_i\frac{\partial\psi}{\partial x_i}\right\rangle - \frac{n}{m}\left\langle F_i\frac{\partial\psi}{\partial u_i}\right\rangle - \frac{n}{m}\left\langle \frac{\partial F_i}{\partial u_i}\psi\right\rangle = 0\,, \tag{3}$$
show that it is possible to obtain the hydrodynamic equations
$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0\,, \tag{4}$$
$$\frac{\partial(\rho v_j)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_i} + \frac{\partial P_{ij}}{\partial x_i} - \frac{\rho}{m}F_j = 0\,, \tag{5}$$
$$\frac{\partial(\rho\epsilon)}{\partial t} + \frac{\partial(\rho\epsilon v_i)}{\partial x_i} + \frac{\partial q_i}{\partial x_i} + P_{ij}\Lambda^{ij} = 0\,. \tag{6}$$
when using as collisional invariant $\psi = m, mu_j$ and $\frac{1}{2}m|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2$, respectively

3. ***Optional.*** Show that Eqs. (4)– (6) can also be written as
$$\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i} + \frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i} - \frac{1}{m}F_j = 0\,, \tag{7}$$
$$\frac{\partial\epsilon}{\partial t} + v_i\frac{\partial\epsilon}{\partial x_i} + \frac{1}{\rho}\frac{\partial q_i}{\partial x_i} + \frac{1}{\rho}P_{ij}\Lambda^{ij} = 0\,. \tag{8}$$

Lecture II

1. Using the following ansatz for the Maxwell-Boltzmann (equilibrium) distribution function
$$\ln(f_0(\vec{\boldsymbol{u}})) = -A(\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0)^2 + \ln C\,, \tag{9}$$

and the definition of the specific internal energy

$$\epsilon := \frac{1}{2}\langle|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle = \frac{1}{2n}\int |\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 f\, d^3u\,. \tag{10}$$

prove that the constant A and C are given by

$$A = \frac{3}{4\epsilon}\,, \qquad\qquad C = n\left(\frac{3}{4\pi\epsilon}\right)^{3/2}\,. \tag{11}$$

2. Recalling that for a classical monoatomic fluid the specific internal energy is given by

$$\epsilon = \frac{3}{2}\frac{k_{\mathrm{B}}T}{m}\,, \tag{12}$$

show that the explicit expression for the Maxwell-Boltzmann distribution function is

$$f_0(\vec{\boldsymbol{u}}) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left(-\frac{m(\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}})^2}{2k_{\mathrm{B}}T}\right)\,. \tag{13}$$

3. Using the definition of the Maxwell-Boltzmann distribution function for the velocity norm u for a fluid with zero macroscopic velocity (*i.e.*, $\vec{\boldsymbol{v}} = 0$)

$$f_0(u) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left(-\frac{mu^2}{2k_{\mathrm{B}}T}\right)\,, \tag{14}$$

show that the average speed is

$$v = \left(\frac{8k_{\mathrm{B}}T}{\pi m}\right)^{1/2}\,. \tag{15}$$

4. ***Optional.*** Show that the most probable speed is

$$v = \left(\frac{2k_{\mathrm{B}}T}{m}\right)^{1/2}\,. \tag{16}$$

Lecture III

1. Show that the scalar quantity d^3p/p_0 is a Lorentz invariant, where $\boldsymbol{p} = cm\boldsymbol{u}$ is the four-momentum and $\boldsymbol{u}$ the four-velocity. [*Hint*: exploit the normalization condition of the four-velocity].

2. Show that the conservation equation for the total energy density

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon + p\right)\vec{\boldsymbol{v}}\right] = \frac{\rho}{m}\vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}}\,, \qquad (17)$$

can also be written as

$$\frac{D}{Dt}\left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon\right) + \left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon + p\right)\vec{\nabla}\cdot\vec{\boldsymbol{v}} = \rho\vec{\boldsymbol{v}}\cdot\left(\frac{\vec{\boldsymbol{F}}}{m} - \frac{1}{\rho}\vec{\nabla}p\right)\,. \qquad (18)$$

where D/Dt is the Lagrangian derivative.

3. ***Optional.*** Consider a two-dimensional flow in which two fluids of the same type have uniform velocity in opposite direction and are subject to an external gravitational potential with uniform acceleration g and uniform pressure p. Determine the evolution of the fluid when perturbed; compare your results with the properties of the Kelvin-Helmholtz instability. [*Hint*: Use a Cartesian coordinate system in which the fluids have velocities $\vec{\boldsymbol{v}}_1 = (v^x, 0)$, $\vec{\boldsymbol{v}}_2 = (-v^x, 0)$, and introduce perturbations in velocity and pressure, *i.e.*, $\vec{\boldsymbol{v}}_1 \to \vec{\boldsymbol{v}}_1 = (v^x + \delta v^x, \delta v^y)$, $\vec{\boldsymbol{v}}_2 \to \vec{\boldsymbol{v}}_2 = (-v^x + \delta v^x, \delta v^y)$. Study the space of solutions of the linearized equations].

Lecture IV-V

1. If ϵ, T, s, p, and ρ are respectively the specific internal energy, the temperature, the specific entropy, the pressure and the rest-mass density, show that the first law of thermodynamics

$$d\epsilon = Tds - pd\left(\frac{1}{\rho}\right)\,, \qquad (19)$$

can be writen alternatively as

$$dp = \rho dh - \rho T ds\,, \qquad (20)$$

$$de = h d\rho + \rho T ds\,, \qquad (21)$$

where $h = (e+p)/\rho$ is the specific enthalpy.

2. Show that the first law of thermodynamics (19) can alternatively be written as

$$dp = \frac{n}{N}\left(dH - TdS\right)\,, \qquad (22)$$

$$de = \frac{1}{N}\left(Hdn + nTdS\right)\,, \qquad (23)$$

where

$$H := Nmh = \frac{N(e+p)}{n} = V(e+p) \qquad (24)$$

is the *enthalpy*.

3. Prove that for the ideal-fluid equation of state $p = \rho\epsilon(\gamma - 1)$ and for the polytropic equations of state $p = K\rho^\Gamma$, the sound speeds

$$c_s^2 \coloneqq \left(\frac{\partial p}{\partial e}\right)_s \tag{25}$$

are given respectively by

$$c_s^2 = \frac{\gamma\epsilon(\gamma-1)}{c^2+\gamma\epsilon} = \left(\frac{h-c^2}{h}\right)(\gamma-1) = \frac{\gamma p}{\rho h}\,, \tag{26}$$

$$c_s^2 = \frac{\Gamma p}{\rho h} = \frac{\Gamma(\Gamma-1)p}{\rho(\Gamma-1)+\Gamma p} = \left(\frac{1}{\Gamma K\rho^{\Gamma-1}} + \frac{1}{\Gamma-1}\right)^{-1}\,. \tag{27}$$

Lecture VI

1. Show that if the vorticity tensor, the shear tensor and the expansion scalar are defined as

$$\omega_{\mu\nu} \coloneqq h^\alpha{}_\mu h^\beta{}_\nu \nabla_{[\beta}u_{\alpha]}\,, \tag{28}$$

$$\sigma_{\mu\nu} \coloneqq h^\alpha{}_\mu h^\beta{}_\nu \nabla_{(\beta}u_{\alpha)}\,, \tag{29}$$

$$\Theta \coloneqq h^{\mu\nu}\nabla_\nu u_\mu\,. \tag{30}$$

where $\boldsymbol{h}$ is the projector orthogonal to $\boldsymbol{u}$, their explicit expressions are

$$\omega_{\mu\nu} = \nabla_{[\nu}u_{\mu]} + a_{[\mu}u_{\nu]}\,, \tag{31}$$

$$\sigma_{\mu\nu} = \nabla_{(\mu}u_{\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{3}\Theta h_{\mu\nu}\,, \tag{32}$$

$$\Theta = \nabla_\mu u^\mu\,. \tag{33}$$

2. Show that for a perfect fluid with energy momentum tensor

$$T^{\mu\nu} = (e+p)u^\mu u^\nu + p g^{\mu\nu}\,, \tag{34}$$

the following projection

$$L_\mu \coloneqq -h^\alpha{}_\mu u^\beta T_{\alpha\beta}\,, \tag{35}$$

$$\tag{36}$$

is identically zero. Explain why.

3. ***Optional.*** Show that starting by the following definitions

$$L_{\mu\nu} \coloneqq h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta}\,, \tag{37}$$

$$L_\mu \coloneqq -h^\alpha{}_\mu u^\beta T_{\alpha\beta}\,, \tag{38}$$

$$e \coloneqq u^\alpha u^\beta T_{\alpha\beta}\,, \tag{39}$$

the following identity is true

$$T_{\mu\nu} = e u_\mu u_\nu + 2\, u_{(\mu} L_{\nu)} + L_{\mu\nu}\,. \tag{40}$$

Lecture VII

1. Show that the Newtonian limit of the relativistic continuity equation

$$u^\mu \nabla_\mu \rho + \rho \nabla_\mu u^\mu = 0\,, \tag{41}$$

is given by

$$\partial_t \rho + v^i \partial_i \rho + \rho \partial_i v^i = 0\,. \tag{42}$$

2. Show that the Newtonian limit of the relativistic equation of conservation of momentum

$$u^\mu \nabla_\mu \rho + \rho \nabla_\mu u^\mu = 0\,, \tag{43}$$

is given by

$$\partial_t v^i + v^j \partial_j v^i + \frac{1}{\rho} \partial_i p + \partial_i \phi = 0\,, \tag{44}$$

where ϕ is the potential of an external force.

3. ***Optional.*** Show that the Newtonian limit of the relativistic equation of conservation of energy

$$u^\mu \nabla_\mu e + \rho h \nabla_\mu u^\mu = 0\,, \tag{45}$$

is given by

$$\partial_t \left(\frac{1}{2} \rho v^i v_i + \rho \epsilon \right) + \partial_i \left[\left(\frac{1}{2} \rho (v^i v_i) + \rho \epsilon + p \right) v^i \right] + \rho v^i \partial_i \phi = 0\,. \tag{46}$$

Lecture VIII

1. Show that the relation between the vorticity tensor $\Omega_{\mu\nu}$ and the kinematic vorticity tensor is given by $\omega_{\mu\nu}$

$$\Omega_{\mu\nu} = 2h \left(\omega_{\mu\nu} - a_{[\mu} u_{\nu]} + u_{[\mu} \nabla_{\nu]} \ln h \right)\,. \tag{47}$$

Discuss the implications of Eq. (47).

2. Show that the Newtonian limit of the Carter–Lichnerowicz equation

$$\Omega_{\mu\nu}u^{\nu} = T\nabla_{\mu}s\,. \tag{48}$$

is given by

$$\frac{\partial\vec{\boldsymbol{v}}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}\vec{\boldsymbol{v}}^2 + \epsilon + \frac{p}{\rho} + \phi\right) - \vec{\boldsymbol{v}} \times (\vec{\nabla} \times \vec{\boldsymbol{v}}) = T\vec{\nabla}s\,, \tag{49}$$

which is also known as the *Crocco* equation of motion.

3. ***Optional.*** Show that the vorticity four-vector

$$\Omega^{\mu} := {}^{*}\Omega^{\mu\nu}u_{\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\Omega_{\alpha\beta}u_{\nu}\,, \tag{50}$$

and the kinematic vorticity four-vector

$$\omega^{\mu} := {}^{*}\omega^{\mu\nu}u_{\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\omega_{\alpha\beta}u_{\nu}\,, \tag{51}$$

are related as

$$\Omega^{\mu} = 2h\omega^{\mu}\,. \tag{52}$$

Lecture IX

1. Assuming for simplicity that the flow is one-dimensional (*i.e.*, for $\mu = 0, 1$) and the spacetime flat, we rewrite the conservation equations for energy and linear momentum

$$\nabla_{\mu}T^{\mu\nu} = 0\,. \tag{53}$$

can be written in a Cartesian coordinate system as

$$\partial_t\left[\left(e + pv^2\right)W^2\right] + \partial_x\left[\left(e + p\right)W^2v\right] = 0\,, \tag{54}$$

$$\partial_t\left[\left(e + p\right)W^2v\right] + \partial_x\left[\left(ev^2 + p\right)W^2\right] = 0\,, \tag{55}$$

where $u^{\mu} = W(1, v)$ and $W = (1 - v^2)^{-1/2}$ is the Lorentz factor.

2. Linearize Eqs. (54)–(55) by introducing perturbations of the type

$$e = e_0 + \delta e\,, \qquad p = p_0 + \delta p\,, \qquad v = v_0 + \delta v = \delta v\,, \tag{56}$$

Show that the resulting equations satisfy a wave equation

$$\Box\delta e = 0\,. \tag{57}$$

What are the assumptions needed to derive Eq. (57)? What is the speed of propagation of these waves?

3. ***Optional.*** The continuity and momentum equations can be written as

$$\partial_t(\rho W) + \partial_x(\rho W v) = 0\,, \tag{58}$$

$$W\partial_t(Wv) + Wv\partial_x(Wv) = -\frac{1}{\rho h}\left[\partial_x p + W^2 v \partial_t p + W^2 v^2 \partial_x p\right]\,. \tag{59}$$

Show that these partial differential equations (that you can try to derive or take as given) can be written as the following ordinary differential equations

$$(v-\xi)\frac{d\rho}{d\xi} + \rho[W^2 v(v-\xi)+1]\frac{dv}{d\xi} = 0\,, \tag{60}$$

$$\rho h W^2 (v-\xi)\frac{dv}{d\xi} + (1-v\xi)\frac{dp}{d\xi} = 0\,, \tag{61}$$

after introducing the similarity variable $\xi \coloneqq x/t$ and the following differential operators

$$\partial_t = -\left(\frac{\xi}{t}\right)\frac{d}{d\xi}\,, \qquad\qquad \partial_x = \left(\frac{1}{t}\right)\frac{d}{d\xi}\,. \tag{62}$$

Lecture X

1. Using the Rankine-Hugoniot conditions expressing the conservation of rest mass, energy and momentum across a shock wave

$$[\![\rho u^\mu]\!]\, n_\mu = 0\,, \tag{63}$$

$$[\![T^{\mu\nu}]\!]\, n_\nu = 0\,, \tag{64}$$

derive the expression for the Taub adiabat

$$[\![h^2]\!] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b}\right)[\![p]\!]\,. \tag{65}$$

Show that its Newtonian equivalent is given by the Hugoniot adiabat

$$\left[\!\!\left[\epsilon + \frac{p}{\rho}\right]\!\!\right] = \frac{1}{2}\left(\frac{1}{\rho_a} + \frac{1}{\rho_b}\right)[\![p]\!]\,. \tag{66}$$

2. Using the junction conditions

$$v_a^2 = \frac{(p_a - p_b)(e_b + p_a)}{(e_a - e_b)(e_a + p_b)}\,, \tag{67}$$

$$v_b^2 = \frac{(p_a - p_b)(e_a + p_b)}{(e_a - e_b)(e_b + p_a)}\,, \tag{68}$$

and under the assumption of a highly relativistic shock, a cold fluid ahead of the shock and an ultrarelativistic one behind the shock, *i.e.*,

$$W_a \gg 1, \quad p_a \approx 0, \quad e_a \approx \rho_a, \quad p_b = \frac{e_b}{3}, \tag{69}$$

show that the energy density in the shocked fluid scales like the square of the Lorentz factor of the shock front (with respect to the unshocked fluid).

$$e_b = 2W_a^2 e_a \,. \tag{70}$$

This is a result often used in astrophysical relativistic shocks.

3. ***Optional.*** Using the junction condition

$$J := \rho_a W_a W_{\scriptscriptstyle S}(V_{\scriptscriptstyle S} - v_a) = \rho_b W_b W_{\scriptscriptstyle S}(V_{\scriptscriptstyle S} - v_b)\,, \tag{71}$$

compute the mass flux J such that the shock velocity V_s is twice the velocity v_a in the unshocked region. Compare it with the corresponding Newtonian mass flux. Which of the two is larger for the same value of v_a?

Lecture XI

1. Verify that the double brackets satisfy the following identities:
   (i) $\alpha\,[\![A]\!] = [\![\alpha A]\!]$, if and only if $[\![\alpha]\!] = 0$;
   (ii) $[\![A + B]\!] = [\![A]\!] + [\![B]\!]$;
   (iii) $[\![A\,B]\!] \neq [\![A]\!]\,[\![B]\!]$;
   (iv) $[\![A]\!]\,[\![B]\!] = [\![B]\!]\,[\![A]\!]$;
   (v) $[\![A^2]\!] \neq [\![A]\!]^2$.

2. In the case of an ultrarelativistic fluid with $p = e/3$ and $c_s = 1/\sqrt{3}$, the following identitiies can be derived

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b}\right) v_b\,, \qquad\qquad v_a = \frac{1}{3v_b}\,. \tag{72}$$

Using them show that

$$W_a^2 = \frac{3}{8}\left(\frac{3e_a + e_b}{e_a}\right)\,, \tag{73}$$

$$W_b^2 = \frac{3}{8}\left(\frac{3e_b + e_a}{e_b}\right)\,, \tag{74}$$

$$W_{ab}^2 = \frac{(3e_a + e_b)(3e_b + e_a)}{16e_1e_2} = \frac{4}{9}W_a^2 W_b^2\,, \tag{75}$$

3. ***Optional.*** Show that the following identity is true [*Hint*: exploit the expressions proven above]

$$W_a^2 - 2W_{ab}^2 + W_b^2 = 1\,. \tag{76}$$

Lecture XII

1. Compute the gyrofrequencies ω_c and Larmor radii r_L for an electron (e) and a proton (p) under the following physical conditions:

	$n_e = n_p[\mathrm{cm}^{-3}]$	$T_e = T_p[\mathrm{K}]$	$B[\mathrm{G}]$
fusion machine	10^{16}	10^{7}	10^{4}
Earth's magnetosphere	10^{4}	10^{3}	10^{-2}
center of the Sun	10^{26}	$10^{7.2}$	10^{6}
solar corona	10^{8}	10^{6}	1
solar wind	10	10^{5}	10^{-5}
neutron star's atmosphere	10^{12}	10^{7}	10^{12}

   Compare the gyrofrequencies with the plasma frequencies

$$\omega^2_{P,e} := \frac{4\pi n_e e^2}{m_e}\,, \qquad\qquad \omega^2_{P,p} := \frac{4\pi n_p e^2}{m_p}\,. \tag{77}$$

2. Compute the drift velocity for a charged particle in uniform and static gravitational and magnetic fields. How does this compare with the motion in the presence of an electric field? Does the motion produce a net current? How large is it?

3. ***Optional.*** Study the motion of a charged particle in a time-varying magnetic field: is the motion with closed orbits? [*Hint*: recall that a time-varying magnetic field will produce a non-uniform electric field].

Lecture XIII

1. Study the motion of a charged particle in a time-varying magnetic field: is the motion with closed orbits? [*Hint*: recall that a time-varying magnetic field will produce a non-uniform electric field].

2. Derive the secular drift velocity for a charged particle in a non-uniform magnetic field. [*Hint*: recall that you are interested in the secular behaviour, which can be obtained after a time integration over a period].

3. ***Optional.*** The Earth's magnetic field in the equatorial plane is

$$B_{_E} = 3 \times 10^{-1} \left(\frac{R_{_E}}{r}\right)^3 \ \mathrm{G} \tag{78}$$

   where $R_{_E} = 6.37 \times 10^8$ cm. At about five Earth's radii, $r/R_{_E} = 5$ and in one of the Van Allen radiation belts, the electrons have an energy of 30 keV and the protons an energy of 1 eV.

(a) Calculate the total drift for both protons and electrons. Considering that the magnetic field has a north pole at the Earth's north pole, describe the sense of motion of these drifts.

(b) Calculate the ring current density when the plasma has a number density $n = 10^{-1}\,\mathrm{cm}^{-3}$.

(c) Calculate the time to drift once around the Earth.

Lecture XIV

1. Compute the Debye length $\lambda_{_D}$ and the plasma paramater Λ for an electron (e) and a proton (p) under the same physical conditions considered for Exercise 1. of Lecture XII.

2. If the collision frequency for an electron in a plasma at temperature T is given by
$$f_{\rm coll} := \frac{\sqrt{2}\omega_{P,e}^4}{64\pi n_e}\left(\frac{k_{_{\rm B}}T}{m_e}\right)^{-3/2}\ln\Lambda\,, \tag{79}$$
compute the collision frequency in the same physical conditions above and compare it with the plasma frequency $\omega_{P,e}$.

Lectures XV-XVI

1. Consider the induction equation in the ideal-MHD limit
$$\partial_t\vec{\boldsymbol{B}} = \nabla\times\left(\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}\right)\,, \tag{80}$$
and prove that the right-hand side can be decomposed into an *advection* term, an *expansion* term and a *stretching* term.

2. Consider the induction equation in the ideal-MHD limit and show that it satisfies the thesis of the frozen-flux theorem, that is
$$\frac{d}{dt}\Phi_{_{\vec{\boldsymbol{B}}}} = 0\,, \tag{81}$$
where
$$\Phi_{_{\vec{\boldsymbol{B}}}} := \int_\Sigma \vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{n}}\,ds\,, \tag{82}$$
is the flux of magnetic field across the open surface Σ of local norm $\vec{\boldsymbol{n}}$.

3. Consider the ideal-MHD limit and determine the additional terms that appear in the equation of conservation of the total energy as a result of a nonzero magnetic field.

4. ***Optional.*** Derive the expressions for the linearized MHD equations in the ideal-MHD limit.