

# A Mille-feuille Universe

Roland Triay<sup>1</sup> and Víctor M. Villalba<sup>2</sup>

Received June 17, 1999

---

Motivated by seeking kinetic origins for seeds of the large-scale structure formation in the universe, we investigate the properties of a generalized axisymmetric Bianchi IX type model. Such a model, which is supposed to describe non-interacting radiation of dust-like matter posterior to decoupling, has the advantage of behaving asymptotically as a close FRW model with a cosmological constant. It shows a vanishing vorticity, a decreasing shear of matter with time and a FRW chronology, which are properties that make it likely when compared to observational constraints. The formula which provides us with the redshifts of sources is derived.

---

KEY WORDS : Generalized Bianchi type IX model

## 1. INTRODUCTION

The standard world model of the Universe assumes a Robertson-Walker (RW) metric and a Friedmann chronology for the matter-dominated era, dust matter being the only source of the gravitational field [22,32]. Although such a simple picture could be questioned because large-scale structures (LSS) are observed in the distribution galaxies in space (see e.g., Refs. 19,13,27), a rigorous proof shows that this approach is however still valid [31]. The main reason is that the derivation of the RW metric can be obtained simply from a geometric interpretation of the Cosmic Microwave Background (CMB) isotropy, regardless of Einstein equations [25]. There-

---

<sup>1</sup> Centre de Physique Théorique C.N.R.S., Luminy Case 907, F-13288 Marseille Cedex 9, France and Université de Provence. E-mail: triay@cpt.univ-mrs.fr

<sup>2</sup> Centro de Física, Instituto Venezolano de Investigaciones Científicas IVIC, Apdo. 21827, Caracas 1020-A Venezuela. E-mail: villalba@ivic.ivic.ve

fore, one can safely ascertain that there is necessarily a scale up to which the isotropy and the homogeneity of the space are ensured despite the presence of LSS. On the other hand, in order to interpret the CMB isotropy and the smallness of the spatial curvature, this standard picture was improved by assuming an inflation era at a primordial epoch [1]. While the initial motivation [2] for such a scenario was different from today's, inflation is now understood as a necessary ingredient to overcome inherent problems with the FRW model and the origin of the LSS formation from quantum fluctuations. However, such an approach can be questioned [31], when it turns out that a spatially closed geometry provides us with another issue [26,29,30]. There is indeed observational evidence from supernovae for an accelerating universe and a cosmological constant [24,10], and for a short extragalactic distance scale [18,23,12,15,21], which makes such an alternative highly likely, although the origin of LSS remains an open problem. We may envisage kinematic origins for seeds responsible for the LSS formation during primordial era, such as local instabilities producing swirl (which may explain the rotation of spiral galaxies). It is clear however that the more vanishing the anisotropy the more reliable the model, in particular at decoupling. Analysis of isotropization effects has already been performed [16,5,3] but with different motivations, and more recently with the aim of investigating the initial conditions allowing an inflationary epoch that make the anisotropy negligible [4]. In this paper we investigate the consequence of such a scenario in the observable large-scale velocity flows of galaxies. A natural question to ask is how to interpret observed deviation from the Hubble flow such as bulk flows or coherent large-scale streaming motions [20,14,6,17,33]. For the reasons given above, the candidate model is a Bianchi IX type model, since it is well known to behave asymptotically as a closed FRW model, and we investigate its residual kinematic from decoupling. To be compatible with previous investigations [9,7,28,11], we assume an axisymmetrical model. Units are chosen so that the speed of the light  $c = 1$  and Planck's constant  $\hbar = 1$ .

## 2. THE WORLD MODEL

For theoretical reasons [9,28,11], we assume an axisymmetric spatially closed world model which accounts for a non-interacting homogeneous distribution of matter and radiation. Let  $a$  denote the scale factor and  $\tau$  the angular distance from a pole, the candidate line element is given by a Bianchi IX type metric which reads

$$ds^2 = -dt^2 + a^2(t)(d\tau^2 + \sin^2(\tau)(d\mathfrak{D}^2 + \sin^2(\mathfrak{D})(d\varphi + \psi(t, \tau)dt)^2)), \quad (1)$$

where  $\vartheta$  accounts for the angular distance of the line element from the axis of anisotropy, and the function  $\psi(t, \tau)$  accounts for anisotropy properties. To have an idea of the related geometry, let us choose a parametrization of the hypersphere  $S^3$  by means of  $S^2$  concentric spheres, the riemanian angular radius of spheres  $\tau$  varies continuously from  $\tau = 0$  (the pole) to the antipode  $\tau = \pi$ . Hence, we understand that the only source of anisotropy corresponds to axisymmetric kinematic properties of the cosmological fluid. Note that a more general solution allows an additional degree of freedom, since we limit the  $S^2$  spheres to rotate (by varying the angle  $\varphi$ ) about the same axis (the axis orthogonal to the equator).

In order to analyze the physical properties, we proceed to separate any tensor into space and time parts corresponding to the way an observer moving with 4-velocity  $u^\mu$  would measure these fields. In order to achieve it, we use the projection tensor

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \tag{2}$$

which satisfies the following relations:

$$h^\mu{}_\mu = 3, \quad h^\mu{}_\nu h^\nu{}_\alpha = h^\mu{}_\alpha, \quad h_{\mu\nu} u^\nu = 0. \tag{3}$$

If  $u^\mu$  denotes the average velocity of the matter then the decomposition acquires an invariant significance. Indeed, the covariant derivative of the velocity can be broken down so that

$$u_{\mu;\nu} = \frac{\theta}{3} h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - \dot{u}_\mu u_\nu, \tag{4}$$

where

$$\omega_{\mu\nu} = u_{[\mu;\nu]} - \dot{u}_{[\mu} u_{\nu]} \tag{5}$$

is the *vorticity* tensor,

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} - \dot{u}_{(\mu} u_{\nu)} - \frac{1}{3}\theta h_{\mu\nu} \tag{6}$$

is the *shear*,

$$\theta = u^\mu{}_{;\mu} \tag{7}$$

is the *expansion rate* and the absolute acceleration  $\dot{u}^\mu$  is given by the vector field

$$\dot{u}^\mu = u^\mu{}_{;\nu} u^\nu, \quad \dot{u}^\mu u_\mu = 0. \tag{8}$$

We have the relativistic invariants

$$\omega = \left(\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}\right)^{1/2}, \quad \sigma = \left(\frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}\right)^{1/2}. \tag{9}$$

The splitting of the energy-momentum tensor is given with respect to the velocity field  $u^\mu$  and the projection tensor  $h_{\mu\nu}$  as follows:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu}, \quad (10)$$

with  $q_\mu u^\mu = 0$ ,  $\pi_{\mu\nu} u^\nu = 0$ ,  $\pi_{;\mu}^\mu = 0$ , where  $\rho$  is the *total energy density* measured by an observer moving with 4-velocity  $u^\mu$ ,  $q_\mu$  is the *energy flux* relative to  $u^\mu$ ,  $p$  is the *isotropic pressure*, and  $\pi_{\mu\nu}$  is the *traceless anisotropic pressure* due to a process such as viscosity.

In order to carry out the decomposition we choose a timelike vector defined by

$$u^\mu = (1, 0, 0, -\psi), \quad u_\mu = (-1, 0, 0, 0), \quad (11)$$

which is geodesic, and thus the absolute acceleration vanishes  $u^\mu = 0$ . The field equations of general relativity read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (12)$$

where  $G$  is the Newton's constant of gravitation,  $R_{\mu\nu}$  is the Ricci tensor, and  $R$  is the scalar curvature. By contracting the conservation equation  $T_{;\nu}^{\mu\nu} = 0$  with  $u_\mu$ , we obtain

$$\dot{\rho} + (\rho + p)\theta + \pi_{\mu\nu}\sigma^{\mu\nu} + q_{;\mu}^\mu + \dot{u}_\mu q^\mu = 0. \quad (13)$$

Let us assume the following (phenomenological) state equation:

$$\pi_{\mu\nu} = -\mu\sigma_{\mu\nu}, \quad (14)$$

where the coefficient of viscosity  $\mu \geq 0$ . Hence, since the velocity vector is geodesic and the heat flux  $q^\mu$  reads

$$q^0 = q^1 = q^2 = 0, \quad q^3 = \frac{1}{16\pi G} \psi \sigma^2 \quad (15)$$

with  $q_{;\mu}^\mu = 0$ , eq. (13) transforms into

$$\dot{\rho} + (\rho + p)\theta - 2\mu\sigma^2 = 0. \quad (16)$$

The kinematic quantities  $\sigma_{\mu\nu}$ ,  $\theta$  and  $\omega_{\mu\nu}$  are calculated according to eq. (11). Hence, the shear tensor is given by

$$\sigma_{\mu\nu} = \frac{1}{2} a^2 \psi' \sin^2(\tau) \sin^2(\mathfrak{D}) \begin{pmatrix} 0 & \psi & 0 & 0 \\ \psi & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (17)$$

the vorticity tensor  $\omega_{\mu\nu}$  is zero and the expansion factor is given by

$$\theta = 3 \frac{\dot{a}}{a}. \tag{18}$$

Therefore, according to eq. (17) and the simple form of  $\omega_{\mu\nu}$ , the invariant quantities given in eq. (9) transform into

$$\omega = 0, \quad \sigma = \frac{1}{\sqrt{2}} \psi' \sin(\tau) \sin(\mathfrak{S}). \tag{19}$$

We easily check that the Raychaudhuri's equation

$$R_{\mu\nu} u^\mu u^\nu = \dot{\theta} + \frac{1}{3} \theta^2 - \dot{u}^\mu_{;\mu} + 2(\sigma^2 - \omega^2) - \Lambda \tag{20}$$

is fulfilled.

In order to solve eq. (16), we need to specify an equation of state determining  $\rho$  from the thermodynamic variables. Hence, we assume that the source of gravity is a non-interacting mixture of radiation and dust matter with specific density

$$\rho = \rho_r + \rho_m. \tag{21}$$

By assuming a vanishing contribution of the dust to the pressure, the pressure of the fluid is only due to the radiation  $p = p_r$ , which satisfies the equation of state

$$\rho_r = 3p_r. \tag{22}$$

In order to calculate  $\rho_m$  it is convenient to compute the scalar curvature  $R$  associated with the line element given in eq. (1), and we obtain

$$R = -\sigma^2 - \frac{6}{a^2} (1 + \dot{a}^2 + \dot{a}\ddot{a}) = -\sigma^2 + R_{RW}, \tag{23}$$

where  $R_{RW}$  is the spatial curvature of the Robertson-Walker metric. From eqs. (10),(12),(22), we easily obtain

$$8\pi G \rho_m = \sigma^2 + \frac{6}{a^2} (1 + \dot{a}^2 + \dot{a}\ddot{a}) - 4\Lambda = \sigma^2 - R_{RW} - \Lambda \tag{24}$$

and the pressure of radiation can be computed from the relation

$$G_{22} = 8\pi G T_{22} + \Lambda g_{22}. \tag{25}$$

Since

$$G_{22} = \left[ \frac{\sigma^2}{2} a^2 + 1 + \dot{a}^2 + 2\dot{a}\ddot{a} \right] \sin^2(\tau) \quad (26)$$

and  $T_{22} = p g_{22}$ , we have

$$p_r = -\frac{1}{8\pi G} \left( \frac{\sigma^2}{2} + \frac{1 + \dot{a}^2 + 2\dot{a}\ddot{a}}{a^2} - \Lambda \right). \quad (27)$$

Then, from eqs. (22),(24),(27), we have explicit expressions of the pressure and densities in terms of the scale factor  $a$  and the scalar  $\sigma$ . The next step is to establish the dependence of  $\sigma$  on  $R$  as well as to obtain an explicit expression for the evolution of  $a$ . By using eq. (22) we find that eq. (16) reduces to

$$\dot{\rho}_m + 3\dot{p}_r + (\rho_m + 4p_r)\theta - 2\mu\sigma^2 = 0. \quad (28)$$

The scalar coefficient of viscosity  $\mu$  can be computed from the equation

$$G_{13} = 8\pi G T_{13} = -8\pi G \mu \sigma_{13}. \quad (29)$$

Then we obtain

$$3\frac{\dot{a}}{a} + \frac{\dot{\psi}'}{\psi'} = -8\pi G \mu, \quad (30)$$

which is a relation that  $\mu$  must satisfy. Then, by substituting eqs. (24),(27) into eq. (28), and by taking into account eq. (30), we find

$$\frac{1}{\sigma^2} \frac{d\sigma^2}{dt} + \frac{7}{2} \theta + 2\frac{\dot{\psi}'}{\psi'} = 0. \quad (31)$$

The solution of eq. (31) reads

$$\sigma^2(t, \tau, \mathfrak{D}) = \frac{1}{2} a^{-21/8} \sin^2(\tau) \sin^2(\mathfrak{D}) F^2(\tau). \quad (32)$$

Notice that the shear vanishes at the poles ( $\tau = 0, \pi$ ) and when the line element follows the axis of anisotropy ( $\mathfrak{D} = 0$ ).

Hence, the total density  $\rho$  can be written as a sum of three terms:

$$\rho = \frac{\rho_m(t_o)}{a^3} + \frac{\rho_r(t_o)}{a^4} + \frac{\sigma^2}{16\pi G}, \quad (33)$$

where  $\rho_m(t_o)$  and  $\rho_r(t_o)$  correspond respectively to specific densities today  $t = t_o$ .

To derive an equation governing the dependence of  $a(t)$  with time, from eqs. (24),(27),(21) we obtain

$$\frac{\dot{a}}{a} = \sqrt{P(a)}, \tag{34}$$

where

$$P(a) = \frac{\Lambda}{3} - \frac{1}{a^2} + \frac{8\pi G}{3} \frac{\rho_m}{a^3} + \frac{8\pi G}{3} \frac{\rho_r}{a^4}. \tag{35}$$

It is worth noticing that the expansion parameter is given by a Friedmann-Lemaître model-Gamov filled out with non-interacting dust and radiation.

### 3. REDSHIFT OF SOURCES

In order to interpret the redshift of sources we use the geometric optics approximation which says that light travels on null geodesic. The frequency of a light signal of wave vector  $k^\alpha$  measured by an observer with 4-velocity  $u^\alpha$  reads

$$v = -k_\alpha u^\alpha \tag{36}$$

where  $k_\alpha k^\alpha = 0$  and  $u_\alpha u^\alpha = -1$ . Hence, since the vector  $k^\alpha$  is geodesic, we have

$$\frac{dv}{ds} = -u_{\alpha;b} k^\alpha k^\beta = -\left( \sigma_{\alpha\beta} + \frac{1}{3} \theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta \right) k^\alpha k^\beta, \tag{37}$$

where  $s$  is an affine parameter along the geodesic. Since  $\dot{u}_\alpha = 0$ , and  $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ , we readily obtain

$$\frac{dv}{ds} = -\sigma_{\alpha\beta} k^\alpha k^\beta - \frac{\dot{a}}{a} v^2, \tag{38}$$

and the integration turns out to be quite cumbersome. For small deviations from FRW geodesic, so that  $ds \approx adt = da/\sqrt{P(a)}$ , the wavelength  $\lambda = 1/v$  of the photon changes according to

$$d\lambda = -\frac{dv}{v^2} = \sigma_{\alpha\beta} \frac{k^\alpha}{v} \frac{k^\beta}{v} \frac{da}{\sqrt{P(a)}} + da. \tag{39}$$

### ACKNOWLEDGEMENTS

This work has been supported by CNRS/CONICIT cooperation agreements.

## REFERENCES

1. Abbott, L. F., Pi, S.-Y. (1986). *Inflationary Cosmology* (World Scientific, Singapore).
2. Albrecht, A., Steinhardt, P. J. (1982). *Phys. Rev. Lett.* **48**, 1220.
3. Barrow, J. D., Juskiewicz, R., Sonoda, D. H. (1985). *Mon. Not. Roy. Astron. Soc.* **231**, 917.
4. Bergamini, R., Sedici, P., Verrocchio, P. (1997). *Phys. Rev.* **D55**, 1896.
5. Collins, C. B., and Hawking, S. W. (1973). *Mon. Not. Roy. Astron. Soc.* **162**, 307.
6. Dekel, A., et al. (1999). *Astrophys. J.* **522**, 1.
7. Desert, F. X., Schatzman, E. (1986). *Astron. Astrophys.* **158**, 135.
8. Ellis, G. F. R. (1971). *Relativistic Cosmology (Rendiconti Scuola Enrico Fermi, XLVII)* (Academic Press, New York).
9. Fliche, H. H., Souriau, J. M., and Triay, R. (1982). *Astron. Astrophys.* **108**, 256.
10. Filippenko, A. V., Riess, A. G. (1998). Preprint astro-ph/9807008, to appear in *Proc. 3rd Int. Symposium on Sources and Detection of Dark Matter in the Universe (DM98)(February 1998)*, D. Cline, ed.
11. Fliche, H. H., Souriau, J. M. (1990). *Astron. Astrophys.* **233**, 317.
12. Freedman, W. L., et al. (1994). *Nature* **371**, 757.
13. Geller, M. J., Huchra, J. P. (1989). *Science* **246**, 897.
14. Giovanelli, R., Haynes, M. P., Wegner, G., da Costa, L. N., Freudling, W., Salzer, J. J. (1996). *Astrophys. J. Lett.* **464**, L99.
15. Harris, W. E., Durrell, P. R., Pierce, M. J., Secker, J. (1998). *Nature* **395**, 45.
16. Hawking, S. W., Collins, C. B. (1969). *Mon. Not. Roy. Astron. Soc.* **142**, 129.
17. Hudson, M. J., Smith, R. J., Lucey, J. R., Schlegel, D. J., Davies, R. L. (1999). *Astrophys. J. Lett.* **512**, L79.
18. Jacoby, G. H., et al. (1992). *Pub. Astron. Soc. Pac.* **104**, 599.
19. de Lapparent, V., Geller, M. J., Huchra, J. P. (1986). *Astrophys. J.* **302**, L1.
20. Lauer, T. R., Postman, M. (1994). *Astrophys. J.* **425**, 418.
21. Nevalainen, J., Roos, M. (1998). *Astron. Astrophys.* **339**, 7.
22. Peebles, P. J. E. (1993). *Principles of Physical Cosmology* (Princeton University Press, Princeton).
23. Pierce, M. J. (1994). *Nature* **371**, 385.
24. Riess, A. G. et al. (1998). *Astron. J.* **116**, 1009.
25. Souriau, J. M. Coll. Internationaux CNRS **237**,59 (1974)
26. Souriau, J. M., Triay, R. (1997). *Gravit. Cosmol.* **3**, 51.
27. Strauss, M. A., Willick, J. A. (1995). *Phys. Rep.* **261**, 271.
28. Triay, R. (1989). In *Large Scale Structures and Peculiar Motions in the Universe*, D. W. Latham and L. N. da Costa, eds. (ASP Conference Series).
29. Triay, R. (1995). Preprint CPT-95/P .3227.
30. Triay, R. (1997). *Gravit. Cosmol.* **3**, 54.
31. Triay, R. (1997). *Cont. Math.* **203**, 227.
32. Weinberg, S. (1972). *Gravitation and Cosmology* (Wiley, New York).
33. Willick, J. A. (1999). *Astrophys. J.* **522**, 647.